



Research article

Generalized multiplicatively fractional Hermite–Hadamard-type inequalities with applications to special functions and probability theory

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Abstract: In this manuscript, we established new (Hermite–Hadamard) $\mathcal{HHI}$ -type inequalities for multiplicatively superquadratic functions, along with corresponding inequalities for the product and quotient of multiplicatively superquadratic, subquadratic, convex, and concave functions via Katugampola fractional integral operators. A notable advantage of these operators is their flexibility: By setting the parameter $\rho = 1$, the derived results naturally reduce to those involving the (Riemann–Liouville) $\mathcal{RL}$ fractional integral operators. The validity and efficiency of the proposed results were demonstrated through graphical and numerical verifications. Furthermore, we explored applications of the established inequalities to special means, functions, and moments of random variables associated with the uniform probability distribution function (pdf). The developed $\mathcal{HHI}$ -type inequalities offer precise bounds for fractional integral expressions that frequently appear in nonlinear boundary value problems (BVPs). These bounds help control the behavior of solution operators. Consequently, they support the analysis of existence, uniqueness, and stability of solutions in both classical and fractional BVPs.

Keywords: modified Bessel function of type-1; superquadratic function; multiplicative superquadratic function; $\mathcal{HHI}$ -type inequality; multiplicative Katugampola fractional integral operators; moment of random variables

Mathematics Subject Classification: 26D15, 26E60, 26A33, 60E05, 33F05

1. Introduction

This notion of inequality in mathematics goes far beyond the initial concept of the comparison of magnitude and encompasses a sophisticated vocabulary of inequalities, bounds, and constraints. This vocabulary is found to be absolutely indispensable in all areas of the mathematical spectrum, ranging from convergence theory in calculus, through the accuracy of approximations in numerical mathematics, topological properties in functional analysis, and physical limitations in engineering sciences. The fundamental role of inequalities in quantifying error bounds has been the driving force behind continued research into various numerical integration techniques, with special emphasis on Newton–Cotes quadrature methods and their generalizations. Modern applied mathematics views inequalities not only as tools but as fundamental concepts that shape the way we pose problems and solve them. The profoundest connection between inequalities and mathematics is through the theory of convex functions, in which the geometric properties of these functions lead to powerful inequalities such as Jensen's inequality, $\mathbb{HHI}$ bounds, and their latest generalizations, while inequalities, in turn, lead to the theory of convex functions. This connection is constantly evolving, as seen in the latest developments in fractional calculus, interval mathematics, and non-convex functions [1–5].

Definition 1.1. [1] A function $\Psi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is convex, if

$$\Psi(\beta\vartheta + (1 - \beta)v) \leq \beta\Psi(\vartheta) + (1 - \beta)\Psi(v) \quad (1.1)$$

holds $\forall \beta \in [0, 1]$ and $\vartheta, v \in I$.

Remark 1.2. If inequality (1.1) is flipped, then we attain the definition of a concave function.

The $\mathbb{HHI}$ inequality, discovered independently by Hermite and Hadamard [2], provides lower and upper bounds for the integral average of a convex function on a compact interval, serving as a fundamental tool in convex analysis with applications in numerical integration, optimization, and approximation. These inequalities can be expressed as if $\Psi : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $\vartheta, v \in I$ with $\vartheta < v$,

$$\Psi\left(\frac{\vartheta + v}{2}\right) \leq \frac{1}{v - \vartheta} \int_{\vartheta}^v \Psi(W) dW \leq \frac{\Psi(\vartheta) + \Psi(v)}{2}. \quad (1.2)$$

Details regarding the concept of multiplicatively convex functions can be found in [1], where it is defined as follows:

Definition 1.3. The positive function $\Psi : I \rightarrow \mathbb{R}$ is called multiplicatively convex, if

$$\Psi(\beta\vartheta + (1 - \beta)v) \leq [\Psi(\vartheta)]^\beta [\Psi(v)]^{(1-\beta)}, \quad (1.3)$$

holds, $\forall \beta \in [0, 1]$ and $\vartheta, v \in I$.

Remark 1.4. If inequality (1.3) is flipped, then we attain the definition for a multiplicatively concave function.

Among the most fascinating areas of research in the theory of inequalities is the refinement and modification of classical inequalities involving convex functions. These are usually done in an attempt

to improve such inequalities using the detailed analysis of generalized convexity. Of all the extensions that have been developed in this area, the theory of superquadratic functions is perhaps the most important and powerful tool in deriving sharper bounds and estimates.

Abramovich, Jameson, and Sinnamon [6] are the pioneers of the idea of superquadratics, and their work on this powerful generalization has greatly enriched the theory of integral inequalities. Superquadratics provide tighter bounds than are possible with the use of convexity and fill an important gap in applied mathematics. In many applications, there is a need to improve the accuracy of approximation in mathematical modeling and optimization problems.

A map $\Psi : [0, \infty) \rightarrow \mathfrak{R}$ is said to be superquadratic, if it fulfills the below-mentioned inequality:

$$\Psi(\varkappa) \geq \Psi(\mathcal{W}) + \mathfrak{C}_{\mathcal{W}}(\varkappa - \mathcal{W}) + \Psi(|\varkappa - \mathcal{W}|), \quad (1.4)$$

for all $\varkappa \geq 0$, with a constant $\mathfrak{C}_{\mathcal{W}} \in \mathfrak{R}$. Moreover the function Ψ is subquadratic if the inequality (1.4) is reversed.

More specifically, a superquadratic function chosen at random meets the three extra conditions as stated in the following lemma:

Lemma 1.5. *If Ψ is a superquadratic function, then*

- $\Psi(0) \leq 0$.
- *If Ψ is differentiable and $\Psi(0) = \Psi'(0) = 0$, then $\Psi'(\mathcal{W}) = \mathfrak{C}_{\mathcal{W}}$.*
- *If $\Psi'(0) = \Psi(0) = 0$ and Ψ is positive then Ψ is convex.*

The following lemma elucidates the fundamental relationship between superquadraticity and convexity.

Lemma 1.6. *If Ψ' is convex and $\Psi'(0) = \Psi(0) = 0$, then Ψ is superquadratic.*

The line-of-support characterization of superquadraticity is presented in (1.4). Alternatively, Abramovich et al. [6] introduced an equivalent formulation of superquadraticity, which is stated as follows:

Definition 1.7. *If Ψ is a superquadratic function, then*

$$\begin{aligned} \Psi((1 - \beta)\mathcal{W}_1 + \beta\mathcal{W}_2) &\leq (1 - \beta)\Psi(\mathcal{W}_1) + \beta\Psi(\mathcal{W}_2) - \beta\Psi((1 - \beta)|\mathcal{W}_1 - \mathcal{W}_2|) \\ &\quad - (1 - \beta)\Psi(\beta|\mathcal{W}_1 - \mathcal{W}_2|) \end{aligned} \quad (1.5)$$

holds for every $\beta \in (0, 1)$ and $\mathcal{W}_1, \mathcal{W}_2 \geq 0$.

Remark 1.8. *The function Ψ is referred to as subquadratic when the inequality sign in (1.5) is reversed.*

The inequalities of Jensen-type and HHH-type are two core results that have played a pivotal role in the development of superquadraticity. These findings occupy a central position in the study of superquadratic functions and have been extensively employed in various theoretical and applied investigations.

Theorem 1.9. *If Ψ is a superquadratic function, then*

$$\sum_{i=1}^n \beta_i \Psi(\mathcal{W}_i) \geq \Psi(\overline{\mathcal{W}}) + \sum_{i=1}^n \beta_i \Psi(|\mathcal{W}_i - \overline{\mathcal{W}}|) \quad (1.6)$$

holds $\forall \mathcal{W}_i \geq 0$ and $0 \leq \beta_i \leq 1$, where $\sum_{i=1}^n \beta_i = 1$ and $\overline{\mathcal{W}} = \sum_{i=1}^n \beta_i \mathcal{W}_i$.

In the framework of superquadraticity, Banić et al. [7] established the following $\mathbb{H}\mathbb{H}$ -type inequalities.

Theorem 1.10. *If Ψ is a superquadratic function, then*

$$\begin{aligned} \Psi\left(\frac{\vartheta + \nu}{2}\right) + \frac{1}{\nu - \vartheta} \int_{\vartheta}^{\nu} \Psi\left(\left|\mathcal{W} - \frac{\vartheta + \nu}{2}\right|\right) d\mathcal{W} &\leq \frac{1}{\nu - \vartheta} \int_{\vartheta}^{\nu} \Psi(\mathcal{W}) d\mathcal{W} \\ &\leq \frac{\Psi(\vartheta) + \Psi(\nu)}{2} - \frac{1}{(\nu - \vartheta)^2} \int_{\vartheta}^{\nu} [(\nu - \mathcal{W})\Psi(\mathcal{W} - \vartheta) + (\mathcal{W} - \vartheta)\Psi(\nu - \mathcal{W})] d\mathcal{W}. \end{aligned} \quad (1.7)$$

The following is the fractional version of the inequalities of $\mathbb{H}\mathbb{H}$ -type established by Li and Chen [8] via $\mathbb{R}\mathbb{L}$ fractional operators.

Theorem 1.11. *If the function Ψ is superquadratic on $[\vartheta, \nu]$, then*

$$\begin{aligned} \Psi\left(\frac{\vartheta + \nu}{2}\right) + \frac{\alpha}{2(\nu - \vartheta)^{\alpha}} \int_{\vartheta}^{\nu} \Psi\left(\left|\frac{\vartheta + \nu}{2} - \mathcal{W}\right|\right) ((\nu - \mathcal{W})^{\alpha-1} + (\mathcal{W} - \vartheta)^{\alpha-1}) d\mathcal{W} \\ \leq \frac{\Gamma(1 + \alpha)}{2(\nu - \vartheta)^{\alpha}} (\mathfrak{J}_{\vartheta+}^{\alpha} \Psi(\nu) + \mathfrak{J}_{\nu-}^{\alpha} \Psi(\vartheta)) \\ \leq \frac{\Psi(\vartheta) + \Psi(\nu)}{2} - \frac{\alpha}{2(\nu - \vartheta)^{\alpha}} \int_{\vartheta}^{\nu} \left(\left(\frac{\mathcal{W} - \vartheta}{\nu - \vartheta}\right) \Psi(\nu - \mathcal{W}) + \left(\frac{\nu - \mathcal{W}}{\nu - \vartheta}\right) \Psi(\mathcal{W} - \vartheta) \right) \\ \times ((\nu - \mathcal{W})^{\alpha-1} + (\mathcal{W} - \vartheta)^{\alpha-1}) d\mathcal{W} \end{aligned} \quad (1.8)$$

holds, where $\mathfrak{J}_{\vartheta+}^{\alpha}$ and $\mathfrak{J}_{\nu-}^{\alpha}$ are given by:

$$\mathfrak{J}_{\vartheta+}^{\alpha} \Psi(\mathcal{W}) = \frac{1}{\Gamma(\alpha)} \int_{\vartheta}^{\mathcal{W}} (\mathcal{W} - \kappa)^{\alpha-1} \Psi(\kappa) d\kappa \quad (\mathcal{W} > \vartheta)$$

and

$$\mathfrak{J}_{\nu-}^{\alpha} \Psi(\mathcal{W}) = \frac{1}{\Gamma(\alpha)} \int_{\mathcal{W}}^{\nu} (\kappa - \mathcal{W})^{\alpha-1} \Psi(\kappa) d\kappa \quad (\mathcal{W} < \nu),$$

where $\Gamma(\alpha) = \int_0^{\infty} \kappa^{\alpha-1} e^{-\kappa} d\kappa$.

Recent developments in the theory of superquadratic functions have witnessed substantial attention, especially in connection with their profound applications to information theory [9]. Further progress was achieved by Alomari et al. [10], who proposed the notion of h -superquadraticity, explored its key properties, and delineated a new subclass in the wider theory. Butt and Khan [11] considered the same notion and derived the $\mathbb{H}\mathbb{H}$ and Fejér-type inequalities. Within this line of investigation, Mario Krnić and his co-authors further advanced the theoretical framework by formulating the notion of a multiplicatively superquadratic function in [12].

Definition 1.12. *A map $\Psi : \mathbb{I} \subseteq [0, \infty) \rightarrow (0, \infty)$ is called multiplicatively superquadratic if*

$$\Psi((1 - \beta)\vartheta + \beta\nu) \leq \frac{\Psi^{1-\beta}(\vartheta)\Psi^{\beta}(\nu)}{\Psi^{\beta}((1 - \beta)|\vartheta - \nu|)\Psi^{1-\beta}(\beta|\vartheta - \nu|)} \quad (1.9)$$

holds $\forall \vartheta, v \in I$ and $\beta \in [0, 1]$.

Remark 1.13. If the “ $\leq$ ” in (1.9) reverses, then the function Ψ is called multiplicatively subquadratic.

Butt and Khan [13] took into account the notion of multiplicatively subquadraticity and derived the $\mathbb{H}\mathbb{H}$ -type inequalities via multiplicative calculus.

Further developments in the theory of superquadraticity have led to the introduction of several generalized notions in diverse mathematical settings. These include the center-radius interval-valued superquadratic function [14], superquadratic interval-valued functions [15], and fuzzy superquadratic interval-valued functions [16]. In addition, various parametric extensions such as P -superquadratic functions [17], m -superquadratic functions [18], and (P, m) -superquadratic functions [19] have been proposed. Moreover, the concept has also been explored within stochastic and fractal frameworks, leading to the notions of superquadratic stochastic processes [20] and fractal superquadratic functions [21].

Integration and differentiation are fundamental operations in mathematical analysis, representing continuous analogues of addition and subtraction, respectively. An alternative framework, known as multiplicative calculus, introduces modified notions of differentiation and integration in which multiplication and division replace the classical roles of addition and subtraction. This approach is also commonly referred to as non-Newtonian calculus. Multiplicative calculus, in our opinion, offers a powerful framework that could greatly enhance theoretical and applied work in economics and finance.

To illustrate the relevance of multiplicative calculus, let us begin with a simple financial example:

Suppose an amount ϑ is deposited and grows to v in one year. The corresponding yearly growth factor is $\frac{v}{\vartheta}$. If this increase is spread evenly across the months, and we let p stand for the month-over-month growth factor, then the relation across twelve months becomes:

$$v = \vartheta(p)^{12}.$$

Solving for p , we find:

$$p = \left(\frac{v}{\vartheta}\right)^{\frac{1}{12}}.$$

Let us extend the scenario to more frequent compounding, such as when the deposit is updated each second, each minute, or each hour. Define $\Psi(\beta)$ as the deposit value at time β . Under these circumstances, the proper notion of the rate of change is not captured by the additive derivative, but by the multiplicative derivative, defined as:

$$\lim_{\delta\beta \rightarrow 0} \left(\frac{\Psi(\beta + \delta\beta)}{\Psi(\beta)} \right)^{\frac{1}{\delta\beta}}. \quad (1.10)$$

Unlike the classical derivative, which is given by:

$$\Psi'(\beta) = \lim_{\delta\beta \rightarrow 0} \frac{\Psi(\beta + \delta\beta) - \Psi(\beta)}{\delta\beta}, \quad (1.11)$$

the multiplicative derivative substitutes subtraction with division and linear scaling with exponentiation, thereby measuring proportional change. It is most useful in settings where relative growth dominates, including finance, biology, and exponential systems.

Throughout history, mathematical analysis has held a leading position within mathematics, and inequalities have been central to its progress. From the 18th to 19th centuries, many distinguished mathematicians contributed to laying the theoretical framework for inequalities, motivated by their growing influence across mathematics. The subject gained further momentum in the 20th century, particularly after the groundbreaking 1934 treatise of Hardy, Littlewood, and Pólya, which established inequalities as a fundamental area of contemporary research.

Recently, Ali et al. [22] demonstrated an $\mathcal{HHI}$ -type inequality in the setting of multiplicative calculus, which they formulated as:

Theorem 1.14. *Let the function $\Psi : [\vartheta, \nu] \rightarrow \mathfrak{R}^+$ exhibit multiplicative convexity, then*

$$\Psi\left(\frac{\vartheta + \nu}{2}\right) \leq \left(\int_{\vartheta}^{\nu} (\Psi(\mathcal{W}))^{d\mathcal{W}}\right)^{\frac{1}{\nu-\vartheta}} \leq G(\Psi(\vartheta), \Psi(\nu)). \quad (1.12)$$

Following the pioneering work of Ali et al. [22], considerable progress has been made in extending integral inequalities within multiplicative calculus. Multiplicative inequalities have been proposed for several function classes, including multiplicative harmonic convexity [23], multiplicative preinvex P-convexity [24], and multiplicatively strong convexity [25]. Researchers have also examined other types of integer-order multiplicative inequalities: For instance, Xie et al. [26] established inequalities of $\mathcal{HHI}$ -type for $*$ -differentiable functions, while Khan and Budak [27] obtained $\mathcal{HHI}$ -type inequalities for $**$ -differentiable functions in the frame of multiplicative integrals. Further contributions on Ostrowski-, Simpson-, and Maclaurin-type integral inequalities are reported in [28, 29] and related studies.

Though integer-order inequalities for multiplicative calculus have been well-explored, the study for the fractional case is not fully developed. However, a significant advancement was made in the field when Budak and Özcelçelik [30] introduced $\mathcal{HHI}$ -type inequalities for multiplicative $\mathcal{RL}$ -fractional integrals in the year 2020. The study triggered the investigation of various aspects of the field, including the work by Fu et al. [31], who extended the results to multiplicatively convex functions using the multiplicative tempered fractional integrals. Further, Peng and Du [32] applied the operators to differentiable m -preinvex functions and (s, m) -preinvex functions. In the year 2024, Peng et al. [33] introduced multiplicative fractional integrals with exponential kernels. The study resulted in the derivation of new upper bounds. Further, the study was extended by Kashuri et al. [34] using multiplicative Sarikaya fractional integrals. Merad et al. [35] provided symmetric Maclaurin-type inequalities for functions with bounded and convex multiplicative derivatives. Thus, the study has greatly contributed to the development of the field of multiplicative calculus. Samuel et al. developed new fixed-point theorems in orthogonal $\mathcal{S}$ -multiplicative metric spaces and applied them to multiplicative and fractional integral equations, contributing to the development of multiplicative and fractional analysis [36]. In addition to the above studies, various recent studies including the ones discussed in the papers cited in [37, 38] provide valuable contributions to the development of the multiplicative fractional calculus.

Motivated by recent developments in the study of multiplicative convexities and superquadratic structures, this work focuses on establishing novel $\mathcal{HHI}$ -type inequalities for multiplicatively superquadratic functions, as well as for the quotient and product of multiplicative convex, concave, subquadratic, and superquadratic functions, within the framework of Katugampola fractional integral operators. The significance of our approach lies in the fact that the obtained results not only generalize

existing inequalities but also unify them, since they naturally reduce to those involving RL- fractional integral operators as special cases. Furthermore, the reliability of the derived results is illustrated graphically, and their applications are explored in terms of special means, functional inequalities, and the moments of random variables associated with the uniform probability distribution.

2. Preliminaries

In this section, we review certain definitions, properties, and concepts related to multiplicative calculus.

Definition 2.1. [38] *It is possible to express the derivative of the positive real-valued function Ψ in the domain of multiplicative calculus as given below:*

$$\Psi^*(\mathcal{W}) = \lim_{h \rightarrow 0} \left(\frac{\Psi(\mathcal{W} + h)}{\Psi(\mathcal{W})} \right)^{\frac{1}{h}}.$$

Definition 2.2. [38] *The relationship between ordinary and multiplicative derivatives is provided by the following statement.*

$$\Psi^*(\mathcal{W}) = \exp\{((\ln \circ \Psi)(\mathcal{W}))'\} = \exp\left\{\frac{\Psi'(\mathcal{W})}{\Psi(\mathcal{W})}\right\}.$$

Theorem 2.3. [38] *With a constant c and multiplicative differentiable functions Ψ_1 and Ψ_2 , we obtain the subsequent relations:*

1. $(c\Psi_1)^*(\mathcal{W}) = \Psi_1^*(\mathcal{W}),$
2. $(\Psi_1\Psi_2)^*(\mathcal{W}) = \Psi_1^*(\mathcal{W})\Psi_2^*(\mathcal{W}),$
3. $(\Psi_1 + \Psi_2)^*(\mathcal{W}) = \Psi_1^*(\mathcal{W})^{\frac{\Psi_1(\mathcal{W})}{\Psi_1(\mathcal{W}) + \Psi_2(\mathcal{W})}} \cdot \Psi_2^*(\mathcal{W})^{\frac{\Psi_2(\mathcal{W})}{\Psi_1(\mathcal{W}) + \Psi_2(\mathcal{W})}},$
4. $\left(\frac{\Psi_1}{\Psi_2}\right)^*(\mathcal{W}) = \frac{\Psi_1^*(\mathcal{W})}{\Psi_2^*(\mathcal{W})},$
5. $(\Psi_1^{\Psi_2})^*(\mathcal{W}) = \Psi_1^*(\mathcal{W})^{\Psi_2(\mathcal{W})}\Psi_1(\mathcal{W})^{\Psi_2'(\mathcal{W})}.$

Proposition 2.4. [38] *To get the definite multiplicative integral over $[\vartheta, v]$ for the positive real-valued and Riemann integrable function Ψ , we have*

$$\int_{\vartheta}^v (\Psi(\mathcal{W}))^{d\mathcal{W}} = \exp\left\{\int_{\vartheta}^v \ln(\Psi(\mathcal{W}))d\mathcal{W}\right\}.$$

Proposition 2.5. [38] *The relations for multiplicative integrable functions Ψ_1 and Ψ_2 are as follows:*

- (1) $\int_{\vartheta}^v ((\Psi_1(\mathcal{W}))^q)^{d\mathcal{W}} = \int_{\vartheta}^v ((\Psi_1(\mathcal{W}))^{d\mathcal{W}})^q,$
- (2) $\int_{\vartheta}^v (\Psi_1(\mathcal{W})\Psi_2(\mathcal{W}))^{d\mathcal{W}} = \int_{\vartheta}^v (\Psi_1(\mathcal{W}))^{d\mathcal{W}} \cdot \int_{\vartheta}^v (\Psi_2(\mathcal{W}))^{d\mathcal{W}},$
- (3) $\int_{\vartheta}^v \left(\frac{\Psi_1(\mathcal{W})}{\Psi_2(\mathcal{W})}\right)^{d\mathcal{W}} = \frac{\int_{\vartheta}^v (\Psi_1(\mathcal{W}))^{d\mathcal{W}}}{\int_{\vartheta}^v (\Psi_2(\mathcal{W}))^{d\mathcal{W}}},$

$$(4) \int_{\vartheta}^{\nu} (\Psi_1(\mathcal{W}))^{d\mathcal{W}} = \int_{\vartheta}^c (\Psi_1(\mathcal{W}))^{d\mathcal{W}} \cdot \int_c^{\nu} (\Psi_1(\mathcal{W}))^{d\mathcal{W}}, \vartheta \leq c \leq \nu,$$

$$(5) \int_{\vartheta}^{\vartheta} (\Psi_1(\mathcal{W}))^{d\mathcal{W}} = 1, \int_{\vartheta}^{\nu} (\Psi_1(\mathcal{W}))^{d\mathcal{W}} = \left(\int_{\vartheta}^{\vartheta} (\Psi_1(\mathcal{W}_1))^{d\mathcal{W}} \right)^{-1}.$$

The integration-by-parts formula within the framework of multiplicative calculus is presented in the following theorem:

Theorem 2.6. [38] Let $\Psi_1 : [\vartheta, \nu] \rightarrow \mathfrak{R}$ possess multiplicative differentiability and $\Psi_2 : [\vartheta, \nu] \rightarrow \mathfrak{R}$ and $h : I \subseteq \mathfrak{R} \rightarrow [\vartheta, \nu]$ possess differentiability. Then we attain

$$\int_{\vartheta}^{\nu} \left(\Psi_1^*(h(\mathcal{W}))^{\Psi_2(\mathcal{W})h'(\mathcal{W})} \right)^{d\mathcal{W}} = \frac{\Psi_1(h(\nu))^{\Psi_2(\nu)}}{\Psi_1(h(\vartheta))^{\Psi_2(\vartheta)}} \cdot \frac{1}{\int_{\vartheta}^{\nu} (\Psi_1(h(\mathcal{W}))^{\Psi_2'(\mathcal{W})})^{d\mathcal{W}}}.$$

To conclude this section, we now proceed to introduce the multiplicative Katugampola fractional integral operators [39].

Definition 2.7. Let $\Psi : [\vartheta^{\rho}, \nu^{\rho}] \rightarrow \mathfrak{R}^+$, and then the left- and right-sided multiplicative Katugampola fractional integral operators, denoted by ${}^{\rho}K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W})$ and ${}^{\rho}K_{\nu^-}^{\alpha} \Psi(\mathcal{W})$, of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho > 0$ are defined as follows:

$${}^{\rho}K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W}^{\rho}) = \exp\{{}^{\rho}K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W}^{\rho})\} = \exp\left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\vartheta}^{\mathcal{W}} \frac{\beta^{\rho-1}}{(\mathcal{W}^{\rho} - \beta^{\rho})^{1-\alpha}} (\ln \circ \Psi)(\beta^{\rho}) d\beta \right\}, \quad \mathcal{W} > \vartheta,$$

and

$${}^{\rho}K_{\nu^-}^{\alpha} \Psi(\mathcal{W}^{\rho}) = \exp\{{}^{\rho}K_{\nu^-}^{\alpha} \Psi(\mathcal{W}^{\rho})\} = \exp\left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\mathcal{W}}^{\nu} \frac{\beta^{\rho-1}}{(\beta^{\rho} - \mathcal{W}^{\rho})^{1-\alpha}} (\ln \circ \Psi)(\beta^{\rho}) d\beta \right\}, \quad \mathcal{W} < \nu,$$

where $\Gamma(\alpha) = \int_0^{\infty} \beta^{\alpha-1} e^{-\beta} d\beta$ is termed as the gamma function.

Remark 2.8. If $\rho = 1$ is put in Definition 2.7, we attain the definition for multiplicatively $\mathbb{RL}$ fractional integrals.

$${}^*K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W}) = \exp\{{}^*K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W})\} = \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\vartheta}^{\mathcal{W}} (\mathcal{W} - \beta)^{\alpha-1} (\ln \circ \Psi)(\beta) d\beta \right\}, \quad \mathcal{W} > \vartheta,$$

and

$${}^*K_{\nu^-}^{\alpha} \Psi(\mathcal{W}) = \exp\{{}^*K_{\nu^-}^{\alpha} \Psi(\mathcal{W})\} = \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\mathcal{W}}^{\nu} (\beta - \mathcal{W})^{\alpha-1} (\ln \circ \Psi)(\beta) d\beta \right\}, \quad \mathcal{W} < \nu.$$

Remark 2.9. If $\rho \rightarrow 0^+$ in the Definition 2.7, we attain the definition for multiplicatively Hadamard fractional integrals.

$${}^*K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W}) = \exp\{{}^*K_{\vartheta^+}^{\alpha} \Psi(\mathcal{W})\} = \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\vartheta}^{\mathcal{W}} \left(\ln \left(\frac{\mathcal{W}}{\beta} \right) \right)^{\alpha-1} \frac{(\ln \circ \Psi)(\beta)}{\beta} d\beta \right\}, \quad \mathcal{W} > \vartheta,$$

and

$${}^*K_{\nu^-}^{\alpha} \Psi(\mathcal{W}) = \exp\{{}^*K_{\nu^-}^{\alpha} \Psi(\mathcal{W})\} = \exp\left\{ \frac{1}{\Gamma(\alpha)} \int_{\mathcal{W}}^{\nu} \left(\ln \left(\frac{\beta}{\mathcal{W}} \right) \right)^{\alpha-1} \frac{(\ln \circ \Psi)(\beta)}{\beta} d\beta \right\}, \quad \mathcal{W} < \nu.$$

3. Main results

This section presents integral inequalities of $\mathbb{HH}$ type established through multiplicative Katugampola fractional integrals for multiplicative functions, including superquadratic, subquadratic, convex, and concave cases. Throughout the article, the notation $G(.,.)$ stands for the geometric mean and $\mathcal{H} = (\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}$.

Theorem 3.1. *Let $\Psi : [\vartheta^\rho, \nu^\rho] \rightarrow \mathfrak{R}$ be a $\ast$ -integrable function with $0 \leq \vartheta < \nu$. If Ψ is a multiplicatively superquadratic function on $[\vartheta^\rho, \nu^\rho]$, then*

$$\begin{aligned} & \Psi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \left(\int_{\vartheta}^{\nu} \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)^{\mathcal{H}} d\mathcal{W} \right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq \left[{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho) \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{\left[\int_{\vartheta}^{\nu} \left((\Psi(|\nu^\rho - \mathcal{W}^\rho|))^{\mathcal{H}} (\Psi(|\mathcal{W}^\rho - \vartheta^\rho|))^{\mathcal{H}} (\nu^\rho - \mathcal{W}^\rho) \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha + 1}}} \end{aligned} \quad (3.1)$$

holds $\forall \alpha > 0$ and $\rho > 0$.

Proof. Since Ψ is a positive and multiplicative superquadratic function, then

$$\begin{aligned} (\ln \circ \Psi)\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) &= (\ln \circ \Psi)\left(\frac{\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho + \beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho}{2}\right) \\ &\leq \frac{1}{2} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) + \frac{1}{2} (\ln \circ \Psi)(\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) \\ &\quad - \frac{1}{2} (\ln \circ \Psi)\left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2}\right) - \frac{1}{2} (\ln \circ \Psi)\left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2}\right). \end{aligned}$$

Multiplying through by $\beta^{\alpha\rho-1}$ and integrating the inequality w.r.t β on $[0, 1]$, we arrive at

$$\begin{aligned} & \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) d\beta \leq \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho \\ & \quad + (1 - \beta^\rho) \nu^\rho) d\beta + \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \\ & \quad - \frac{1}{2} \left(\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)\left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2}\right) d\beta \right. \\ & \quad \left. + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)\left(\frac{|(2\beta^\rho - 1)(\vartheta^\rho - \nu^\rho)|}{2}\right) d\beta \right). \end{aligned}$$

After straightforward calculations and an appropriate change of variables, we arrive at

$$\begin{aligned} (\ln \circ \Psi)\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) &\leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) \right] \\ &\quad - \frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left(\int_{\vartheta}^{\nu} (\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} (\ln \circ \Psi)\left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right|\right) d\mathcal{W} \right. \end{aligned}$$

$$+ \int_{\vartheta}^{\nu} (\nu^{\rho} - \mathcal{W}^{\rho})^{\alpha-1} (\ln \circ \Psi) \left(\left| \frac{\vartheta^{\rho} + \nu^{\rho}}{2} - \mathcal{W}^{\rho} \right| \right) d\mathcal{W}.$$

Thus, we have

$$\begin{aligned} & \exp \left\{ (\ln \circ \Psi) \left(\frac{\vartheta^{\rho} + \nu^{\rho}}{2} \right) \right\} \\ & \leq \exp \left\{ \frac{\rho^{\alpha} \Gamma(\alpha+1)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}} \left[{}^{\rho}K_{\vartheta^{+}}^{\alpha} (\ln \circ \Psi)(\nu^{\rho}) + {}^{\rho}K_{\nu^{-}}^{\alpha} (\ln \circ \Psi)(\vartheta^{\rho}) \right] \right. \\ & \quad \left. - \frac{\alpha\rho}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}} \int_{\vartheta}^{\nu} \left[(\mathcal{W}^{\rho} - \vartheta^{\rho})^{\alpha-1} + (\nu^{\rho} - \mathcal{W}^{\rho})^{\alpha-1} \right] (\ln \circ \Psi) \left(\left| \frac{\vartheta^{\rho} + \nu^{\rho}}{2} - \mathcal{W}^{\rho} \right| \right) d\mathcal{W} \right\}. \end{aligned}$$

From Propositions 2.4 and 2.5, we attain

$$\Psi \left(\frac{\vartheta^{\rho} + \nu^{\rho}}{2} \right) \leq \frac{[{}^{\rho}K_{\vartheta^{+}}^{\alpha} \Psi(\nu^{\rho}) \cdot {}^{\rho}K_{\nu^{-}}^{\alpha} \Psi(\vartheta^{\rho})]^{\frac{\rho^{\alpha} \Gamma(\alpha+1)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}}}}{\left(\int_{\vartheta}^{\nu} (\Psi(|\frac{\vartheta^{\rho} + \nu^{\rho}}{2} - \mathcal{W}^{\rho}|))^{((\mathcal{W}^{\rho} - \vartheta^{\rho})^{\alpha-1} + (\nu^{\rho} - \mathcal{W}^{\rho})^{\alpha-1})} d\mathcal{W} \right)^{\frac{\alpha\rho}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}}}},$$

or

$$\begin{aligned} & \Psi \left(\frac{\vartheta^{\rho} + \nu^{\rho}}{2} \right) \left(\int_{\vartheta}^{\nu} \left(\Psi \left(\left| \frac{\vartheta^{\rho} + \nu^{\rho}}{2} - \mathcal{W}^{\rho} \right| \right) \right)^{((\mathcal{W}^{\rho} - \vartheta^{\rho})^{\alpha-1} + (\nu^{\rho} - \mathcal{W}^{\rho})^{\alpha-1})} d\mathcal{W} \right)^{\frac{\alpha\rho}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}}} \\ & \leq [{}^{\rho}K_{\vartheta^{+}}^{\alpha} \Psi(\nu^{\rho}) \cdot {}^{\rho}K_{\nu^{-}}^{\alpha} \Psi(\vartheta^{\rho})]^{\frac{\rho^{\alpha} \Gamma(\alpha+1)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}}}. \end{aligned} \quad (3.2)$$

We have thereby proved the first part of inequality (3.1). For the second part, we analyze the right-hand side of inequality (3.2). Employing the definition of multiplicative superquadraticity, we arrive at:

$$\begin{aligned} & [{}^{\rho}K_{\vartheta^{+}}^{\alpha} \Psi(\nu^{\rho}) \cdot {}^{\rho}K_{\nu^{-}}^{\alpha} \Psi(\vartheta^{\rho})]^{\frac{\rho^{\alpha} \Gamma(\alpha+1)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}}} \\ & = \exp \left\{ \frac{\rho^{\alpha} \Gamma(\alpha+1)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}} \left[{}^{\rho}K_{\vartheta^{+}}^{\alpha} (\ln \circ \Psi)(\nu^{\rho}) + {}^{\rho}K_{\nu^{-}}^{\alpha} (\ln \circ \Psi)(\vartheta^{\rho}) \right] \right\} \\ & = \exp \left\{ \frac{\alpha\rho}{2} \left[\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^{\rho} \vartheta^{\rho} + (1 - \beta^{\rho}) \nu^{\rho}) d\beta \right. \right. \\ & \quad \left. \left. + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^{\rho} \nu^{\rho} + (1 - \beta^{\rho}) \vartheta^{\rho}) d\beta \right] \right\} \leq \exp \left\{ \frac{\alpha\rho}{2} \left[\frac{\ln(\Psi(\vartheta^{\rho}) \Psi(\nu^{\rho}))}{\alpha\rho} \right. \right. \\ & \quad \left. \left. - \left[\int_0^1 \beta^{\rho(\alpha-1)} [\beta^{\rho} (\ln \circ \Psi)((1 - \beta^{\rho}) |\nu^{\rho} - \vartheta^{\rho}|) + (1 - \beta^{\rho}) (\ln \circ \Psi)(\beta^{\rho} |\nu^{\rho} - \vartheta^{\rho}|)] \beta^{\rho-1} d\beta \right. \right. \right. \\ & \quad \left. \left. \left. + \int_0^1 \beta^{\rho(\alpha-1)} [\beta^{\rho} (\ln \circ \Psi)((1 - \beta^{\rho}) |\nu^{\rho} - \vartheta^{\rho}|) + (1 - \beta^{\rho}) (\ln \circ \Psi)(\beta^{\rho} |\nu^{\rho} - \vartheta^{\rho}|)] \beta^{\rho-1} d\beta \right] \right] \right\}. \end{aligned}$$

Setting $\mathcal{W}^{\rho} = \beta^{\rho} \vartheta^{\rho} + (1 - \beta^{\rho}) \nu^{\rho}$ in the first integral term and $\mathcal{W}^{\rho} = \beta^{\rho} \nu^{\rho} + (1 - \beta^{\rho}) \vartheta^{\rho}$ in the second integral term, and after simple calculation, we attain:

$$[{}^{\rho}K_{\vartheta^{+}}^{\alpha} \Psi(\nu^{\rho}) \cdot {}^{\rho}K_{\nu^{-}}^{\alpha} \Psi(\vartheta^{\rho})]^{\frac{\rho^{\alpha} \Gamma(\alpha+1)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}}} \leq G(\Psi(\vartheta^{\rho}), \Psi(\nu^{\rho})) \exp \left\{ \frac{-\alpha\rho}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha\rho+1}} \right\}$$

$$\begin{aligned}
& \times \int_{\vartheta}^v (\mathcal{H}) ((\mathcal{W}^{\rho} - \vartheta^{\rho})(\ln \circ \Psi)(v^{\rho} - \mathcal{W}^{\rho}) + (v^{\rho} - \mathcal{W}^{\rho})(\ln \circ \Psi)(\mathcal{W}^{\rho} - \vartheta^{\rho})) d\mathcal{W} \Big\} \\
& = \frac{G(\Psi(\vartheta^{\rho}), \Psi(v^{\rho}))}{\left[\int_{\vartheta}^v \left(\left((\Psi(|v^{\rho} - \mathcal{W}^{\rho}|))^{\mathcal{W}^{\rho} - \vartheta^{\rho}} (\Psi(|\mathcal{W}^{\rho} - \vartheta^{\rho}|))^{\mathcal{W}^{\rho} - v^{\rho}} \right)^{\mathcal{H}} \right)^{d\mathcal{W}} \right]^{\frac{\alpha \rho}{2(v^{\rho} - \vartheta^{\rho})^{\alpha \rho} + 1}}}. \quad (3.3)
\end{aligned}$$

Merging (3.2) and (3.3), we achieve the required result. $\square$

Remark 3.2. If the inequalities are reversed in Theorem 3.1, we attain the result for a multiplicatively subquadratic function.

Remark 3.3. If $\rho = 1$ is put in Theorem 3.1, we achieve the result involving $\mathbb{RL}$ operators.

$$\begin{aligned}
& \Psi\left(\frac{\vartheta + v}{2}\right) \left(\int_{\vartheta}^v \left(\Psi\left(\left|\frac{\vartheta + v}{2} - \mathcal{W}\right|\right) \right)^{((\mathcal{W} - \vartheta)^{\alpha-1} + (v - \mathcal{W})^{\alpha-1})} d\mathcal{W} \right)^{\frac{\alpha}{2(v - \vartheta)^{\alpha}}} \\
& \leq [{}_{*}K_{\vartheta^{+}}^{\alpha} \Psi(v) \cdot {}_{*}K_{v^{-}}^{\alpha} \Psi(\vartheta)]^{\frac{\Gamma(\alpha+1)}{2(v - \vartheta)^{\alpha}}} \\
& \leq \frac{G(\Psi(\vartheta), \Psi(v))}{\left[\int_{\vartheta}^v \left(\left((\Psi(|v - \mathcal{W}|))^{\mathcal{W} - \vartheta} (\Psi(|\mathcal{W} - \vartheta|))^{\mathcal{W} - v} \right)^{((v - \mathcal{W})^{\alpha-1} + (\mathcal{W} - \vartheta)^{\alpha-1})} \right)^{d\mathcal{W}} \right]^{\frac{\alpha}{2(v - \vartheta)^{\alpha} + 1}}}.
\end{aligned}$$

Remark 3.4. As a particular case, setting $\alpha = 1$ in Theorem 3.1 yields

$$\begin{aligned}
& \Psi\left(\frac{\vartheta^{\rho} + v^{\rho}}{2}\right) \left(\int_{\vartheta}^v \left(\Psi\left(\left|\frac{\vartheta^{\rho} + v^{\rho}}{2} - \mathcal{W}^{\rho}\right|\right) \right)^{d\mathcal{W}} \right)^{\frac{\rho}{(v^{\rho} - \vartheta^{\rho})}} \leq \left(\int_{\vartheta}^v (\Psi(\mathcal{W}^{\rho}))^{d\mathcal{W}} \right)^{\frac{\rho}{v^{\rho} - \vartheta^{\rho}}} \\
& \leq \frac{G(\Psi(\vartheta^{\rho}), \Psi(v^{\rho}))}{\left[\int_{\vartheta}^v \left((\Psi(|v^{\rho} - \mathcal{W}^{\rho}|))^{\mathcal{W}^{\rho} - \vartheta^{\rho}} \cdot (\Psi(|\mathcal{W}^{\rho} - \vartheta^{\rho}|))^{\mathcal{W}^{\rho} - v^{\rho}} \right)^{d\mathcal{W}} \right]^{\frac{\rho}{(v^{\rho} - \vartheta^{\rho})^{1+\rho}}}}.
\end{aligned}$$

Remark 3.5. If $\rho = 1$ and $\alpha = 1$ are put in Theorem 3.1, we achieve

$$\begin{aligned}
& \Psi\left(\frac{\vartheta + v}{2}\right) \left(\int_{\vartheta}^v \left(\Psi\left(\left|\frac{\vartheta + v}{2} - \mathcal{W}\right|\right) \right)^{d\mathcal{W}} \right)^{\frac{1}{(v - \vartheta)}} \leq \left(\int_{\vartheta}^v (\Psi(\mathcal{W}))^{d\mathcal{W}} \right)^{\frac{1}{v - \vartheta}} \\
& \leq \frac{G(\Psi(\vartheta), \Psi(v))}{\left[\int_{\vartheta}^v \left((\Psi(|v - \mathcal{W}|))^{\mathcal{W} - \vartheta} \cdot (\Psi(|\mathcal{W} - \vartheta|))^{\mathcal{W} - v} \right)^{d\mathcal{W}} \right]^{\frac{1}{(v - \vartheta)^2}}}.
\end{aligned}$$

Subsequently, several examples are provided to illustrate and verify Theorem 3.1 using 2D and 3D visualizations together with tabular representation. First of all, we consider the 2D graphs, where the values for ρ are fixed while the values for α vary over the interval $(0, 1)$.

Example 3.6. We take into account the function $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ defined on $(0, 1)$. We have Table 1 and Figure 1, illustrating Theorem 3.1 for $\vartheta = 0$, $v = 1$, $\rho = 0.5$, and $\alpha \in (0, 1)$. Their left-, middle- and right- terms are designated by the symbols L , M , and R , respectively.

Table 1. Numerical demonstration of Theorem 3.1 via $\Psi(W) = \exp\{W^4\}$ for $\vartheta = 0$, $\nu = 1$, $\rho = 0.5$, and $\alpha \in (0, 1)$.

α	R	M	L
0.1	2.6901	1.5233	1.0905
0.2	2.6708	1.4362	1.0849
0.3	2.6575	1.3739	1.0809
0.4	2.6482	1.3286	1.0780
0.5	2.6417	1.2952	1.0758
0.6	2.6371	1.2705	1.0742
0.7	2.6339	1.2520	1.0730
0.8	2.6316	1.2384	1.0722
0.9	2.6301	1.2285	1.0716

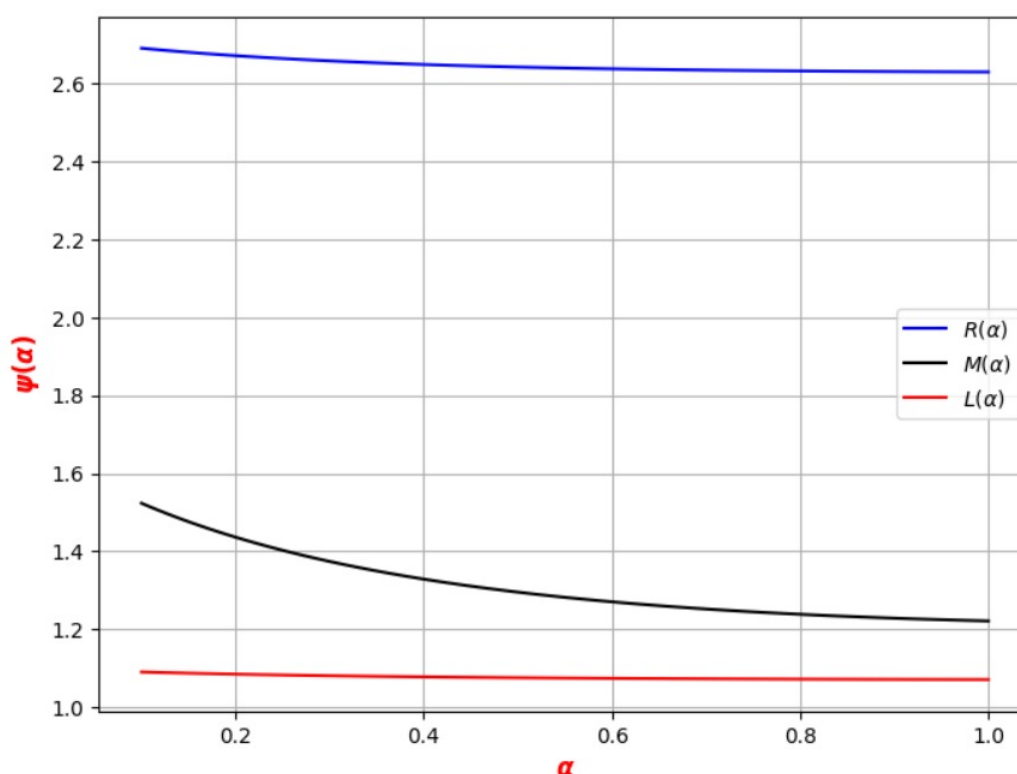


Figure 1. Graphical depiction of Theorem 3.1 via $\Psi(W) = \exp\{W^4\}$ for $\vartheta = 0$, $\nu = 1$, $\rho = 0.5$ and $\alpha \in (0, 1)$.

Example 3.7. Taking the same function as defined in Example 3.6, we the Table 2 and Figure 2, illustrating Theorem 3.1 for $\vartheta = 0$, $\nu = 1$, $\rho = 0.6$, and $\alpha \in (0, 1)$. Their left-, middle- and right- terms are designated by the symbols L , M , and R , respectively.

Table 2. Numerical demonstration of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\rho = 0.6$, and $\alpha \in (0, 1)$.

α	R	M	L
0.1	2.6851	1.5233	1.0063
0.2	2.6621	1.4362	1.0122
0.3	2.6460	1.3739	1.0179
0.4	2.6346	1.3286	1.0232
0.5	2.6266	1.2952	1.0281
0.6	2.6208	1.2705	1.0326
0.7	2.6168	1.2520	1.0368
0.8	2.6139	1.2384	1.0406
0.9	2.6120	1.2285	1.0474

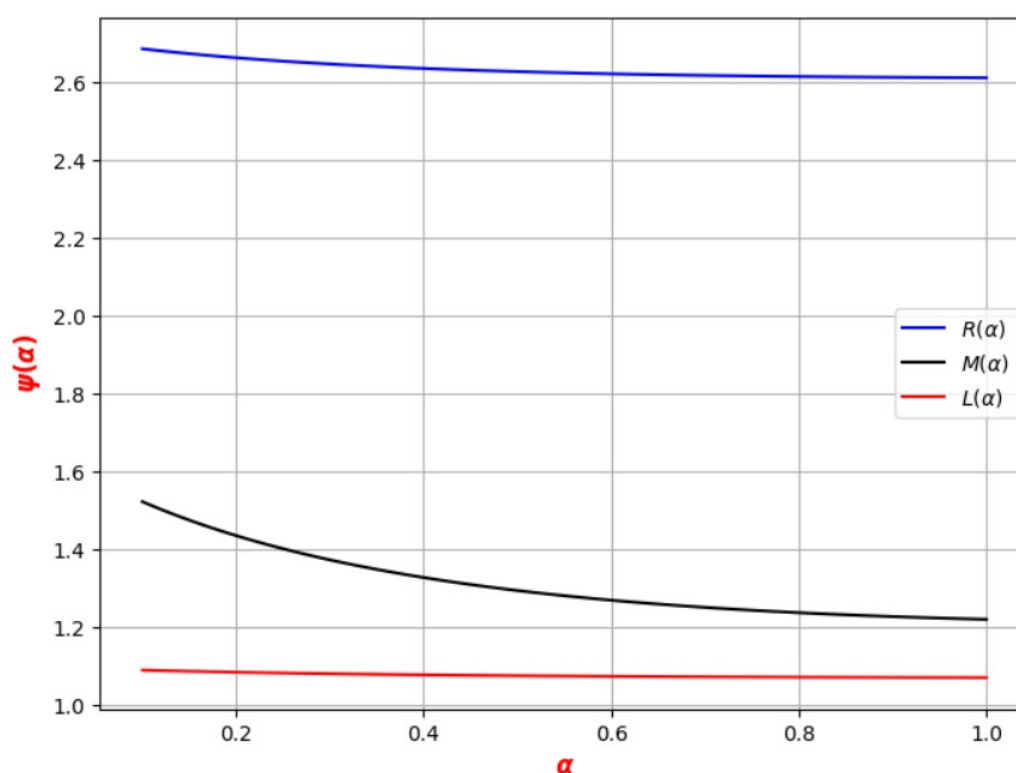


Figure 2. Graphical depiction of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\rho = 0.6$, and $\alpha \in (0, 1)$.

Example 3.8. Taking the same function as given in Example 3.6, we have Table 3 and Figure 3, illustrating Theorem 3.1 for $\vartheta = 0$, $\nu = 1$, $\rho = 0.9$, and $\alpha \in (0, 1)$. Their left-, middle- and right-terms are denoted by the symbols L , M , and R , respectively.

Table 3. Numerical demonstration of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\rho = 0.9$, and $\alpha \in (0, 1)$.

α	R	M	L
0.1	2.6683	1.5233	0.9720
0.2	2.6341	1.4362	0.9888
0.3	2.6102	1.3739	1.0016
0.4	2.5935	1.3286	1.0120
0.5	2.5817	1.2952	1.0207
0.6	2.5734	1.2705	1.0281
0.7	2.5675	1.2520	1.0346
0.8	2.5634	1.2384	1.0402
0.9	2.5606	1.2285	1.0452

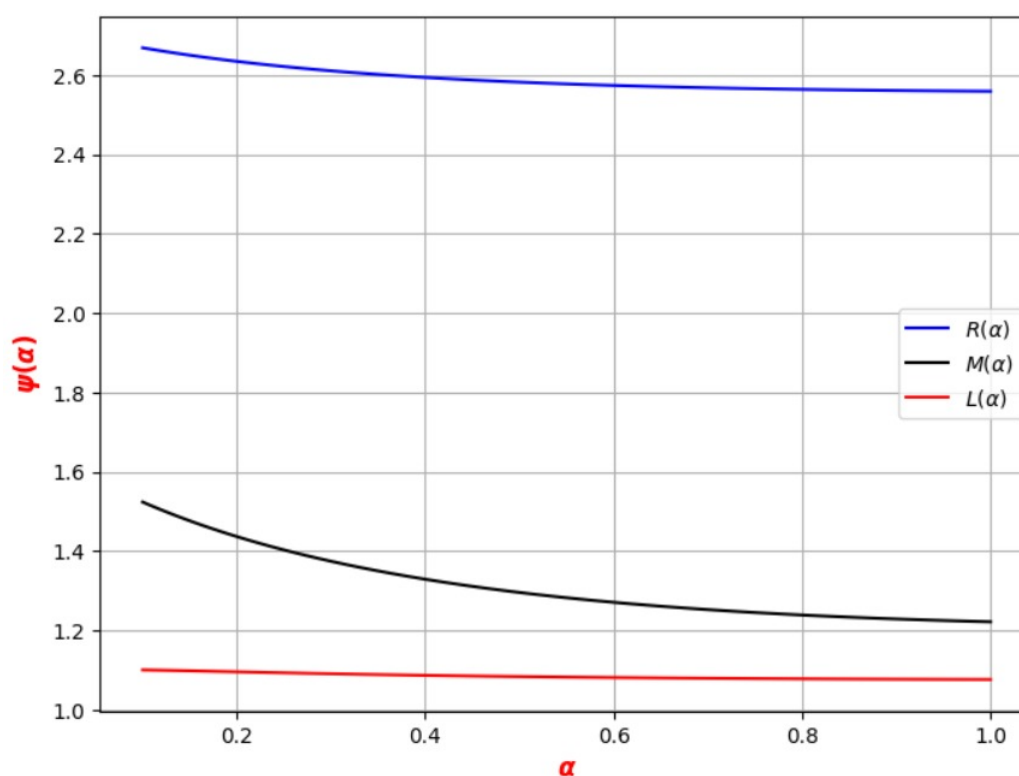


Figure 3. Graphical depiction of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\rho = 0.9$, and $\alpha \in (0, 1)$.

Next, we consider the 2D graphs, where the values for α are fixed while the values for ρ vary over the interval $(0, 1)$.

Example 3.9. Taking the same function as defined in Example 3.6, we have Table 4 and Figure 4, illustrating Theorem 3.1 for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.2$, and $\rho \in (0, 1)$. Their left-, middle- and right- terms are denoted by the symbols L , M , and R , respectively.

Table 4. Numerical demonstration of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.2$, and $\rho \in (0, 1)$.

ρ	R	M	L
0.1	2.7063	1.4362	1.2347
0.2	2.6963	1.4362	1.1224
0.3	2.6876	1.4362	1.0759
0.4	2.6793	1.4362	1.0475
0.5	2.6708	1.4362	1.0274
0.6	2.6621	1.4362	1.0122
0.7	2.6530	1.4362	1.0009
0.8	2.6437	1.4362	0.9929
0.9	2.6341	1.4362	0.9888

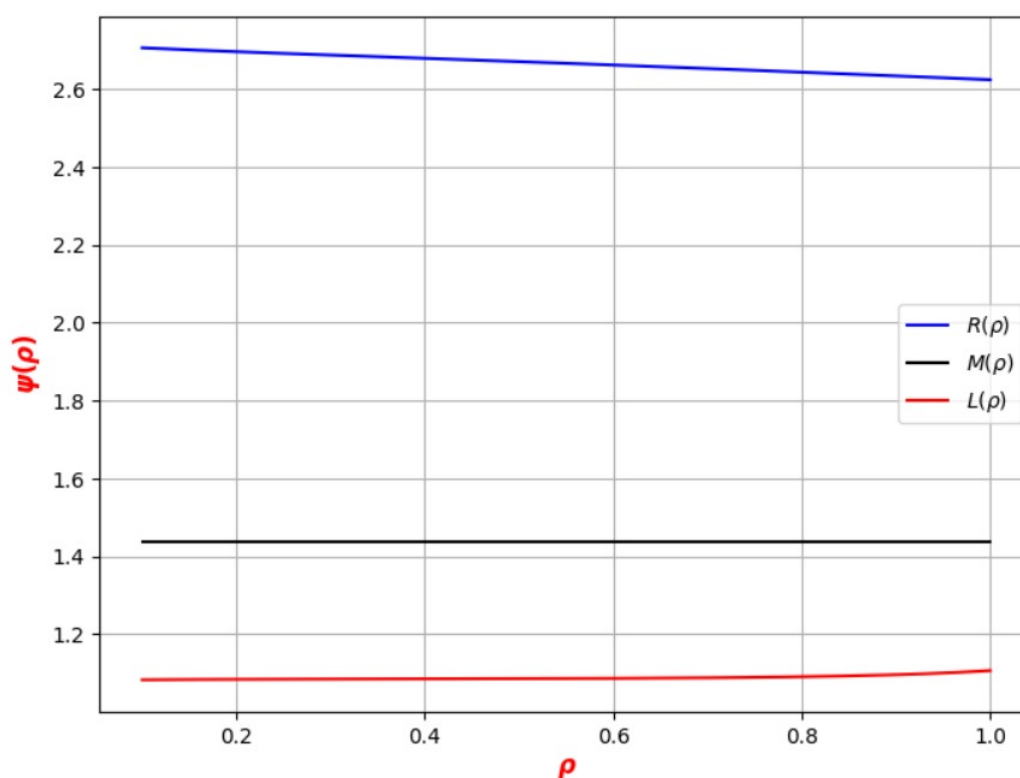


Figure 4. Graphical depiction of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.2$, and $\rho \in (0, 1)$.

Example 3.10. Taking the same function as defined in Example 3.6, we have Table 5 and Figure 5, illustrating Theorem 3.1 for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.6$, and $\rho \in (0, 1)$. Their left-, middle- and right-terms are denoted by the symbols L , M , and R , respectively.

Table 5. Numerical demonstration of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.6$, and $\rho \in (0, 1)$.

ρ	R	M	L
0.1	2.7037	1.2705	1.1080
0.2	2.6862	1.2705	1.0741
0.3	2.6696	1.2705	1.0565
0.4	2.6533	1.2705	1.0453
0.5	2.6371	1.2705	1.0377
0.6	2.6208	1.2705	1.0326
0.7	2.6047	1.2705	1.0295
0.8	2.5889	1.2705	1.0281
0.9	2.5584	1.2705	1.0281

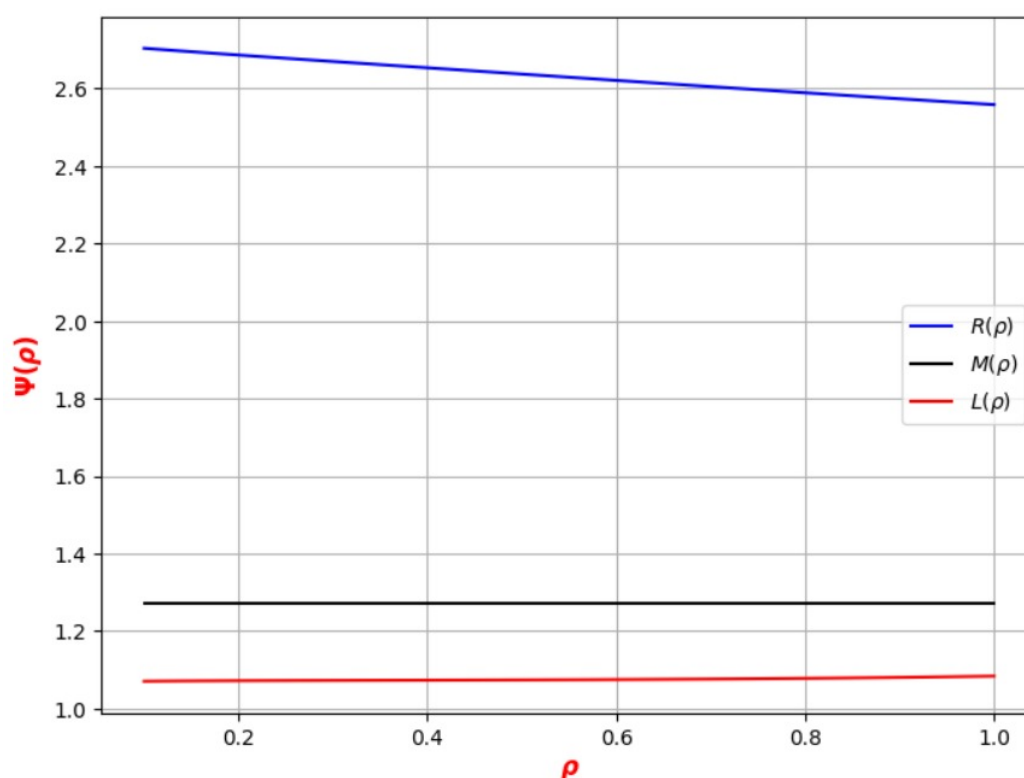


Figure 5. Graphical depiction of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.6$, and $\rho \in (0, 1)$.

Example 3.11. Taking the same function as defined in 3.6, we have the Table 6 and Figure 6, illustrate the Theorem 3.1 for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.9$, and $\rho \in (0, 1)$. Its left, middle, and right terms are denoted by the symbols L , M , and R , respectively.

Table 6. Numerical demonstration of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.9$, and $\rho \in (0, 1)$.

ρ	R	M	L
0.1	2.7048	1.2285	1.0752
0.2	2.6859	1.2285	1.0605
0.3	2.6672	1.2285	1.0528
0.4	2.6486	1.2285	1.0482
0.5	2.6301	1.2285	1.0455
0.6	2.6120	1.2285	1.0442
0.7	2.5942	1.2285	1.0438
0.8	2.5771	1.2285	1.0442
0.9	2.5606	1.2285	1.0468

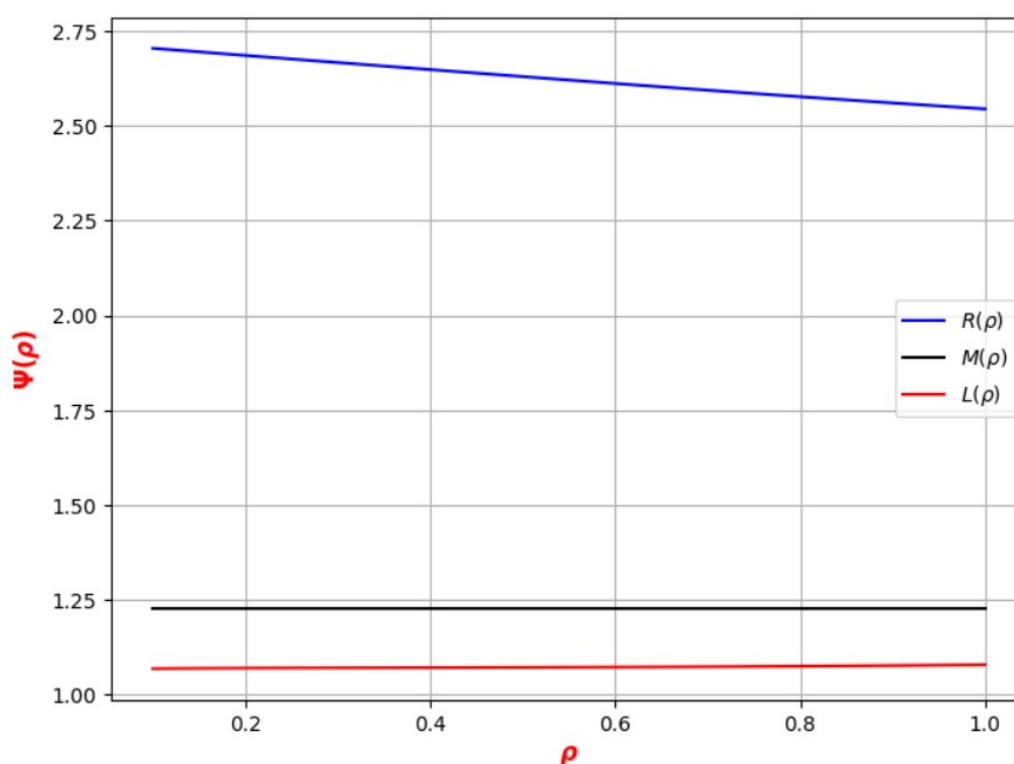


Figure 6. Graphical depiction of Theorem 3.1 via $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ for $\vartheta = 0$, $\nu = 1$, $\alpha = 0.9$, and $\rho \in (0, 1)$.

We now consider the 3D graphs, where the values for α and ρ both vary over the interval $(0, 1)$.

Example 3.12. Figure 7 confirms the authenticity of Theorem 3.1 for $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$.

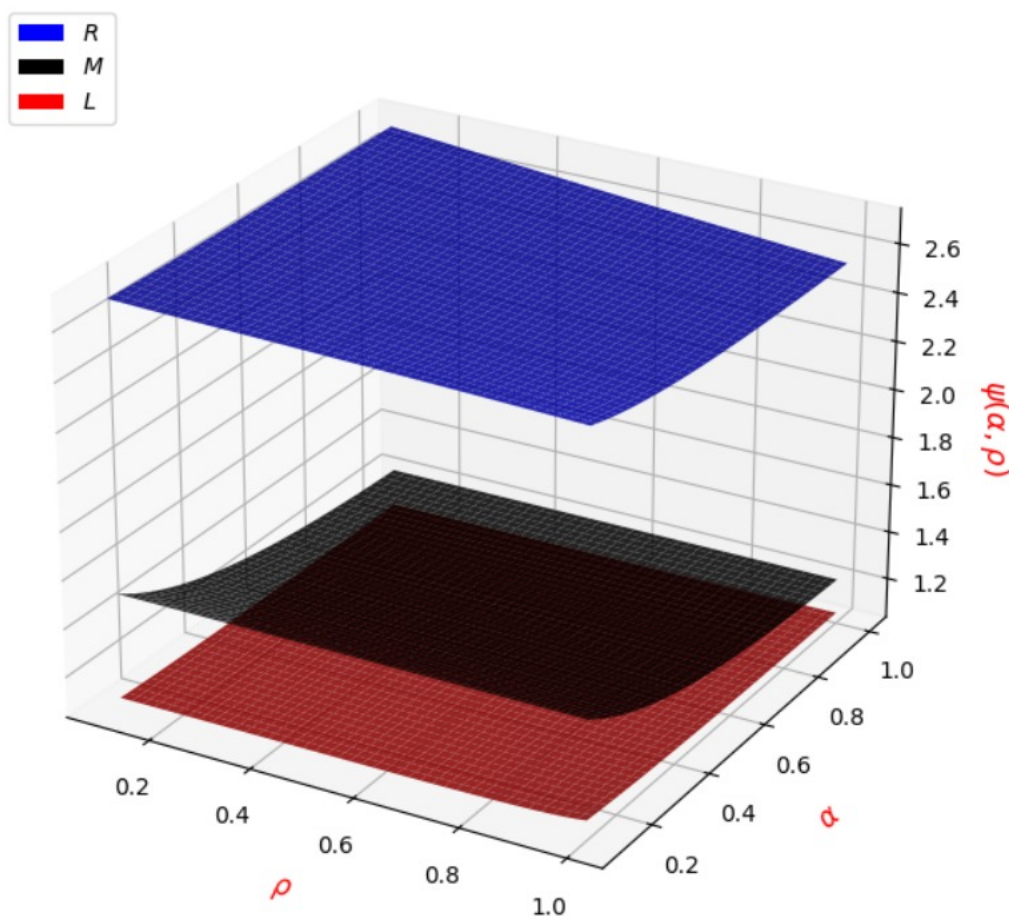


Figure 7. Graphical depiction of Theorem 3.1 for $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$, $\vartheta = 0$, $\nu = 1$, $\alpha \in (0, 1)$, and $\rho \in (0, 1)$.

Theorem 3.13. Let $\Psi, \varphi : [\vartheta^\rho, \nu^\rho] \rightarrow \mathfrak{R}$ be an $*$ -integrable function with $0 \leq \vartheta < \nu$. If Ψ and φ are positive and multiplicatively superquadratic functions on $[\vartheta^\rho, \nu^\rho]$, then

$$\begin{aligned}
 & \Psi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \varphi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \left(\int_{\vartheta}^{\nu} \left(\Psi \varphi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)^{\mathcal{H}} \right)^{d\mathcal{W}} \right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\
 & \leq \left[{}^\rho K_{\vartheta^+}^\alpha \Psi \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi \varphi(\vartheta^\rho) \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\
 & \leq \frac{G(\Psi(\vartheta^\rho) \Psi(\vartheta^\rho), \Psi(\nu^\rho) \Psi(\nu^\rho))}{\left[\int_{\vartheta}^{\nu} \left(\left((\Psi \Psi(|\nu^\rho - \mathcal{W}^\rho|))^{\mathcal{H}} (\Psi \Psi(|\mathcal{W}^\rho - \vartheta^\rho|))^{\mathcal{H}} \right)^{d\mathcal{W}} \right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha + 1}} \right]} \quad (3.4)
 \end{aligned}$$

holds $\forall \alpha > 0$ and $\rho > 0$.

Proof. Let functions Ψ and φ be positive and multiplicatively superquadratic and then

$$\ln \left(\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right)$$

$$\begin{aligned}
&= (\ln \circ \Psi) \left(\frac{\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho + \beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho}{2} \right) \\
&+ (\ln \circ \varphi) \left(\frac{\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho + \beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho}{2} \right) \\
&\leq \frac{1}{2} (\ln \circ \Psi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) + \frac{1}{2} (\ln \circ \Psi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) \\
&- \frac{1}{2} (\ln \circ \Psi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) - \frac{1}{2} (\ln \circ \Psi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) \\
&+ \frac{1}{2} (\ln \circ \varphi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) + \frac{1}{2} (\ln \circ \varphi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) \\
&- \frac{1}{2} (\ln \circ \varphi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) - \frac{1}{2} (\ln \circ \varphi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right).
\end{aligned}$$

Multiplying through by $\beta^{\alpha\rho-1}$ and integrating the inequality in β over $[0, 1]$, we arrive at

$$\begin{aligned}
&\int_0^1 \left(\beta^{\alpha\rho-1} (\ln \circ \Psi) \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) + \beta^{\alpha\rho-1} (\ln \circ \varphi) \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right) d\beta \\
&\leq \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) d\beta + \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \\
&- \frac{1}{2} \left(\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) \left(\frac{|(2\beta^\rho - 1)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta \right) \\
&+ \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) d\beta + \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \\
&- \frac{1}{2} \left(\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) \left(\frac{|(2\beta^\rho - 1)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta \right).
\end{aligned}$$

After performing simple computations and applying a change of variables, we arrive at

$$\begin{aligned}
&\ln \left(\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right) \\
&\leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}^\rho K_{\vartheta^+}^\alpha (\ln \circ (\Psi\varphi))(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ (\Psi\varphi))(\vartheta^\rho) \right] \\
&- \frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left(\int_{\vartheta}^{\nu} (\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} (\ln \circ (\Psi\varphi)) \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right) d\mathcal{W} \right. \\
&\left. + \int_{\vartheta}^{\nu} (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1} (\ln \circ (\Psi\varphi)) \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right) d\mathcal{W} \right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\exp \left\{ \ln \left(\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right) \right\} \\
&\leq \exp \left\{ \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}^\rho K_{\vartheta^+}^\alpha (\ln \circ (\Psi\varphi))(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ (\Psi\varphi))(\vartheta^\rho) \right] \right. \\
&\left. - \frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha} \int_{\vartheta}^{\nu} \left[(\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1} \right] (\ln \circ (\Psi\varphi)) \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right) d\mathcal{W} \right\}.
\end{aligned}$$

This implies that

$$\begin{aligned} & \Psi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \varphi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \\ & \leq \frac{[{}^\rho K_{\vartheta^+}^\alpha \Psi\varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi\varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}}}{\left(\int_{\vartheta}^{\nu} (\Psi\varphi(|\frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho|))^{(\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}} d\mathcal{W}\right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}}}, \end{aligned}$$

or

$$\begin{aligned} & \Psi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \varphi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right) \left(\int_{\vartheta}^{\nu} \left(\Psi\varphi\left(\left|\frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho\right|\right)^{\mathcal{H}}\right)^{d\mathcal{W}}\right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq [{}^\rho K_{\vartheta^+}^\alpha \Psi\varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi\varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}}. \end{aligned} \quad (3.5)$$

We have thereby proved the first part of inequality (3.4). For the second part, we analyze the right-hand side of inequality (3.5). Employing the definition of multiplicative superquadraticity, we arrive at:

$$\begin{aligned} & [{}^\rho K_{\vartheta^+}^\alpha \Psi\varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi\varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & = \exp\left\{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} [{}^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi\varphi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \Psi\varphi)(\vartheta^\rho)]\right\} \\ & = \exp\left\{\frac{\alpha\rho}{2} \left[\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho) \nu^\rho) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho) \nu^\rho) d\beta \right. \right. \\ & \quad \left. \left. + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \nu^\rho + (1-\beta^\rho) \vartheta^\rho) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \nu^\rho + (1-\beta^\rho) \vartheta^\rho) d\beta\right]\right\} \\ & \leq \exp\left\{\frac{\alpha\rho}{2} \left[\frac{\ln(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho)\Psi(\nu^\rho)\varphi(\nu^\rho))}{\alpha\rho} - \left[\int_0^1 \beta^{\rho(\alpha-1)} \left[\beta^\rho (\ln \circ \Psi\varphi)((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|) \right. \right. \right. \right. \\ & \quad \left. \left. \left. + (1-\beta^\rho)(\ln \circ \Psi\varphi)(\beta^\rho |\nu^\rho - \vartheta^\rho|)\right] \beta^{\rho-1} d\beta + \int_0^1 \beta^{\rho(\alpha-1)} \left[\beta^\rho (\ln \circ \Psi\varphi)((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|) \right. \right. \right. \\ & \quad \left. \left. \left. + (1-\beta^\rho)(\ln \circ \Psi\varphi)(\beta^\rho |\nu^\rho - \vartheta^\rho|)\right] \beta^{\rho-1} d\beta\right]\right]\right\}. \end{aligned} \quad (3.6)$$

Setting $\mathcal{W}^\rho = \beta^\rho \vartheta^\rho + (1-\beta^\rho) \nu^\rho$ in the first integral term, and $\mathcal{W}^\rho = \beta^\rho \nu^\rho + (1-\beta^\rho) \vartheta^\rho$ in the second integral term and after simple calculation, we attain:

$$\begin{aligned} & [{}^\rho K_{\vartheta^+}^\alpha \Psi\varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi\varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq G(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho), \Psi(\nu^\rho)\varphi(\nu^\rho)) \times \exp\left\{\frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}} \int_{\vartheta}^{\nu} ((\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}) \right. \\ & \quad \left. \times ((\mathcal{W}^\rho - \vartheta^\rho)(\ln \circ \Psi\varphi)(\nu^\rho - \mathcal{W}^\rho) + (\nu^\rho - \mathcal{W}^\rho)(\ln \circ \Psi\varphi)(\mathcal{W}^\rho - \vartheta^\rho)) d\mathcal{W}\right\} \\ & = \frac{G(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho), \Psi(\nu^\rho)\varphi(\nu^\rho))}{\left[\int_{\vartheta}^{\nu} \left((\Psi\varphi(|\nu^\rho - \mathcal{W}^\rho|))^{(\mathcal{W}^\rho - \vartheta^\rho)} (\Psi\varphi(|\mathcal{W}^\rho - \vartheta^\rho|))^{(\nu^\rho - \mathcal{W}^\rho)}\right)^{\mathcal{H}} d\mathcal{W}\right]^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}}}. \end{aligned} \quad (3.7)$$

Merging (3.5) and (3.7), we reach the required conclusion. $\square$

Remark 3.14. If $\rho = 1$ is put in Theorem 3.13, we attain the result involving RL operators.

$$\begin{aligned} & \Psi\left(\frac{\vartheta + \nu}{2}\right) \varphi\left(\frac{\vartheta + \nu}{2}\right) \left(\int_{\vartheta}^{\nu} \left(\Psi \varphi \left(\left| \frac{\vartheta + \nu}{2} - \mathcal{W} \right| \right)^{((\mathcal{W}-\vartheta)^{\alpha-1} + (\nu-\mathcal{W})^{\alpha-1})} \right)^{d\mathcal{W}} \right)^{\frac{\alpha}{2(\nu-\vartheta)^{\alpha}}} \\ & \leq [{}_*I_{\vartheta+}^{\alpha} \Psi \varphi(\nu) \cdot {}_*I_{\nu-}^{\alpha} \Psi \varphi(\vartheta)]^{\frac{\Gamma(\alpha+1)}{2(\nu-\vartheta)^{\alpha}}} \\ & \leq \frac{G(\Psi(\vartheta)\varphi(\vartheta), \Psi(\nu)\varphi(\nu))}{\left[\int_{\vartheta}^{\nu} \left((\Psi \varphi(|\nu - \mathcal{W}|))^{\mathcal{W}-\vartheta} (\Psi \varphi(|\mathcal{W} - \vartheta|))^{\nu-\mathcal{W}} \right)^{((\nu-\mathcal{W})^{\alpha-1} + (\mathcal{W}-\vartheta)^{\alpha-1})} d\mathcal{W} \right]^{\frac{\alpha}{2(\nu-\vartheta)^{\alpha+1}}}}. \end{aligned}$$

Remark 3.15. If $\alpha = 1$ is put in Theorem 3.13, we achieve

$$\begin{aligned} & \Psi\left(\frac{\vartheta^{\rho} + \nu^{\rho}}{2}\right) \varphi\left(\frac{\vartheta^{\rho} + \nu^{\rho}}{2}\right) \left(\int_{\vartheta}^{\nu} \left(\Psi \varphi \left(\left| \frac{\vartheta^{\rho} + \nu^{\rho}}{2} - \mathcal{W}^{\rho} \right| \right) \right)^{d\mathcal{W}} \right)^{\frac{\rho}{(\nu^{\rho}-\vartheta^{\rho})}} \\ & \leq \left(\int_{\vartheta}^{\nu} (\Psi \varphi(\mathcal{W}^{\rho}))^{d\mathcal{W}} \right)^{\frac{\rho}{\nu^{\rho}-\vartheta^{\rho}}} \\ & \leq \frac{G(\Psi(\vartheta^{\rho})\varphi(\vartheta^{\rho}), \Psi(\nu^{\rho})\varphi(\nu^{\rho}))}{\left[\int_{\vartheta}^{\nu} \left((\Psi \varphi(|\nu^{\rho} - \mathcal{W}^{\rho}|))^{\mathcal{W}^{\rho}-\vartheta^{\rho}} \cdot (\Psi \varphi(|\mathcal{W}^{\rho} - \vartheta^{\rho}|))^{\nu^{\rho}-\mathcal{W}^{\rho}} \right)^{d\mathcal{W}} \right]^{\frac{\rho}{(\nu^{\rho}-\vartheta^{\rho})^{\rho+1}}}}. \end{aligned}$$

Remark 3.16. If $\rho = 1$ and $\alpha = 1$ are put in Theorem 3.13, we achieve

$$\begin{aligned} & \Psi\left(\frac{\vartheta + \nu}{2}\right) \varphi\left(\frac{\vartheta + \nu}{2}\right) \left(\int_{\vartheta}^{\nu} \left(\Psi \varphi \left(\left| \frac{\vartheta + \nu}{2} - \mathcal{W} \right| \right) \right)^{d\mathcal{W}} \right)^{\frac{1}{(\nu-\vartheta)}} \\ & \leq \left(\int_{\vartheta}^{\nu} (\Psi \varphi(\mathcal{W}))^{d\mathcal{W}} \right)^{\frac{1}{\nu-\vartheta}} \\ & \leq \frac{G(\Psi(\vartheta)\varphi(\vartheta), \Psi(\nu)\varphi(\nu))}{\left[\int_{\vartheta}^{\nu} \left((\Psi \varphi(|\nu - \mathcal{W}|))^{\mathcal{W}-\vartheta} \cdot (\Psi \varphi(|\mathcal{W} - \vartheta|))^{\nu-\mathcal{W}} \right)^{d\mathcal{W}} \right]^{\frac{1}{(\nu-\vartheta)^2}}}. \end{aligned}$$

Example 3.17. Figure 8 confirms the authenticity of Theorem 3.13 for $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$ and $\varphi(\mathcal{W}) = \exp\{\mathcal{W}^6\}$.

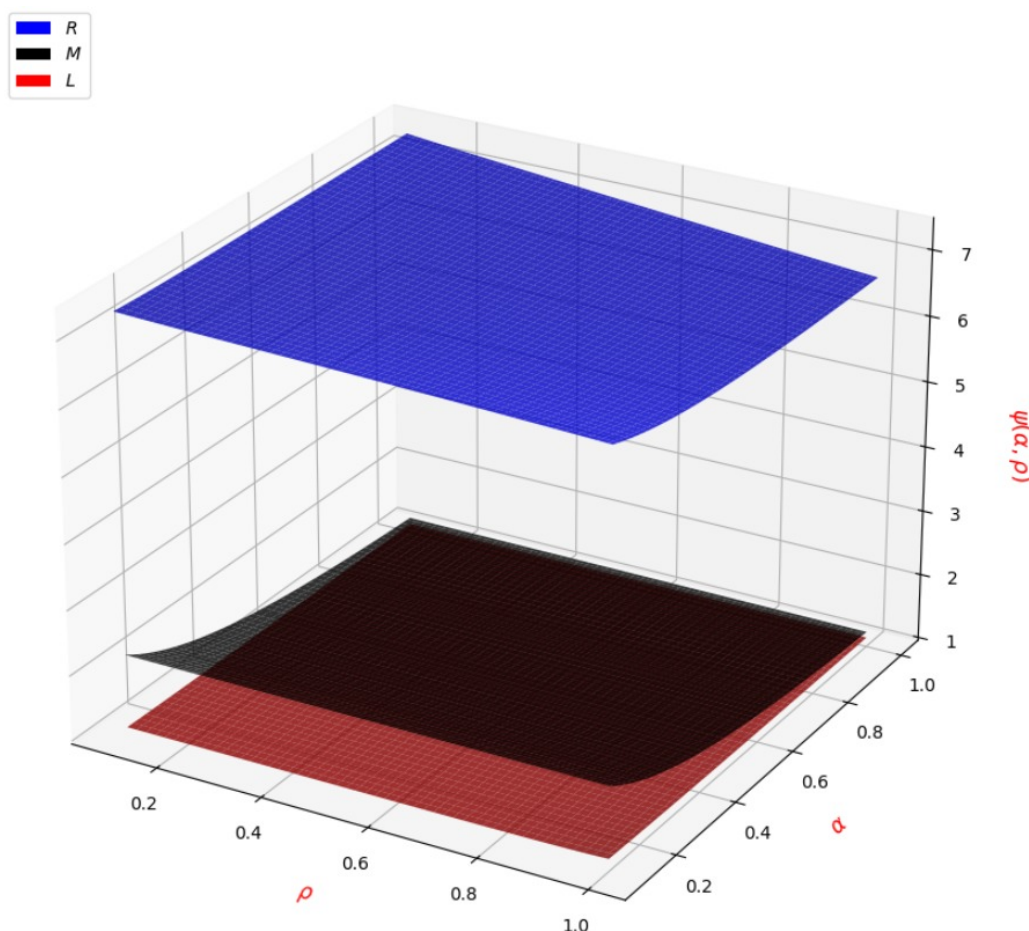


Figure 8. Graphical description of Theorem 3.13 for $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^4\}$, $\varphi(\mathcal{W}) = \exp\{\mathcal{W}^6\}$, $\vartheta = 0$, $\nu = 1$, $\alpha \in (0, 1)$, and $\rho \in (0, 1)$.

Theorem 3.18. Let $\Psi, \varphi : [\vartheta^\rho, \nu^\rho] \rightarrow \mathfrak{R}$ be a $*$ integrable function with $0 \leq \vartheta < \nu$. Let Ψ and φ be positive and multiplicatively superquadratic and subquadratic, respectively, on $[\vartheta^\rho, \nu^\rho]$. Then

$$\begin{aligned}
 & \left(\frac{\Psi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right)}{\varphi\left(\frac{\vartheta^\rho + \nu^\rho}{2}\right)} \right) \left(\int_{\vartheta}^{\nu} \left(\left(\frac{\Psi\left(|\frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho|\right)}{\varphi\left(|\frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho|\right)} \right)^{\mathcal{H}} \right)^{d\mathcal{W}} \right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\
 & \leq \left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\
 & \leq \frac{\frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))}}{\left[\int_{\vartheta}^{\nu} \left(\left(\frac{\Psi(\nu^\rho - \mathcal{W}^\rho)}{\varphi(\nu^\rho - \mathcal{W}^\rho)} \right)^{(\mathcal{W}^\rho - \vartheta^\rho)} \left(\frac{\Psi(\mathcal{W}^\rho - \vartheta^\rho)}{\varphi(\mathcal{W}^\rho - \vartheta^\rho)} \right)^{(\nu^\rho - \mathcal{W}^\rho)} \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha + 1}}}, \tag{3.8}
 \end{aligned}$$

holds $\forall \alpha > 0$ and $\rho > 0$, where $\varphi \neq 0$.

Proof. Let Ψ and φ be positive multiplicatively superquadratic and multiplicatively subquadratic,

respectively. Then

$$\begin{aligned}
 & \ln \left(\frac{\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)}{\varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)} \right) \\
 &= (\ln \circ \Psi) \left(\frac{\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho + \beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho}{2} \right) \\
 &\quad - (\ln \circ \varphi) \left(\frac{\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho + \beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho}{2} \right) \\
 &\leq \frac{1}{2} (\ln \circ \Psi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) + \frac{1}{2} (\ln \circ \Psi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) \\
 &\quad - \frac{1}{2} (\ln \circ \Psi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) - \frac{1}{2} (\ln \circ \Psi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) \\
 &\quad - \frac{1}{2} (\ln \circ \varphi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) - \frac{1}{2} (\ln \circ \varphi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) \\
 &\quad + \frac{1}{2} (\ln \circ \varphi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) + \frac{1}{2} (\ln \circ \varphi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right).
 \end{aligned}$$

Multiplying through by $\beta^{\alpha\rho-1}$ and integrating the inequality in β over $[0, 1]$, we arrive at

$$\begin{aligned}
 & \int_0^1 \left(\beta^{\alpha\rho-1} (\ln \circ \Psi) \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) - \beta^{\alpha\rho-1} (\ln \circ \varphi) \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right) d\beta \\
 &\leq \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) d\beta + \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \\
 &\quad - \frac{1}{2} \left(\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi) \left(\frac{|(2\beta^\rho - 1)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta \right) \\
 &\quad - \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) (\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) d\beta - \frac{1}{2} \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) (\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \\
 &\quad + \frac{1}{2} \left(\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) \left(\frac{|(1 - 2\beta^\rho)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi) \left(\frac{|(2\beta^\rho - 1)(\vartheta^\rho - \nu^\rho)|}{2} \right) d\beta \right).
 \end{aligned}$$

After straightforward calculations and an appropriate change of variables, we arrive at

$$\begin{aligned}
 & \ln \left(\frac{\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)}{\varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)} \right) \\
 &\leq \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) - ({}^\rho K_{\vartheta^+}^\alpha (\ln \circ \varphi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \varphi)(\vartheta^\rho)) \right] \\
 &\quad - \frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left(\int_{\vartheta}^{\nu} (\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} \ln \left(\frac{\Psi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)}{\varphi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)} \right) d\mathcal{W} \right. \\
 &\quad \left. + \int_{\vartheta}^{\nu} (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1} \ln \left(\frac{\Psi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)}{\varphi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)} \right) d\mathcal{W} \right).
 \end{aligned}$$

Thus, we have

$$\begin{aligned} & \exp \left\{ \ln \left(\frac{\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)}{\varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)} \right) \right\} \\ & \leq \exp \left\{ \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) - ({}^\rho K_{\vartheta^+}^\alpha (\ln \circ \varphi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \varphi)(\vartheta^\rho)) \right] \right. \\ & \quad \left. - \frac{\alpha \rho}{2(\nu^\rho - \vartheta^\rho)^\alpha} \int_{\vartheta}^{\nu} [(\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}] \ln \left(\frac{\Psi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)}{\varphi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)} \right) d\mathcal{W} \right\}. \end{aligned}$$

This implies that

$$\frac{\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)}{\varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)} \leq \frac{\left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}}}{\left(\int_{\vartheta}^{\nu} \left(\frac{\Psi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)}{\varphi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)} \right)^{(\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}} d\mathcal{W} \right)^{\frac{\alpha \rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}},$$

or

$$\begin{aligned} & \left(\frac{\Psi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)}{\varphi \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right)} \right) \left(\int_{\vartheta}^{\nu} \left(\frac{\Psi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)}{\varphi \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right)} \right)^{\mathcal{H}} d\mathcal{W} \right)^{\frac{\alpha \rho}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq \left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}}. \end{aligned} \quad (3.9)$$

We have thereby proved the first part of inequality (3.8). For the second part, we analyze the right-hand side of inequality (3.9). Employing the definition of multiplicative superquadraticity, we arrive at:

$$\begin{aligned} & \left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & = \exp \left\{ \frac{\rho^\alpha \Gamma(\alpha + 1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) \right. \right. \\ & \quad \left. \left. - ({}^\rho K_{\vartheta^+}^\alpha (\ln \circ \varphi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \varphi)(\vartheta^\rho)) \right] \right\} \\ & = \exp \left\{ \frac{\alpha \rho}{2} \left[\int_0^1 \beta^{\alpha \rho - 1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) d\beta \right. \right. \\ & \quad \left. \left. - \int_0^1 \beta^{\alpha \rho - 1} (\ln \circ \varphi)(\beta^\rho \vartheta^\rho + (1 - \beta^\rho) \nu^\rho) d\beta \right. \right. \\ & \quad \left. \left. + \int_0^1 \beta^{\alpha \rho - 1} (\ln \circ \Psi)(\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \right. \right. \\ & \quad \left. \left. - \int_0^1 \beta^{\alpha \rho - 1} (\ln \circ \varphi)(\beta^\rho \nu^\rho + (1 - \beta^\rho) \vartheta^\rho) d\beta \right] \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \exp \left\{ \frac{\alpha\rho}{2} \left[\frac{1}{\alpha\rho} \ln \left(\frac{\Psi(\vartheta^\rho)\Psi(\nu^\rho)}{\varphi(\vartheta^\rho)\Psi(\nu^\rho)} \right) \right. \right. \\
&\quad - \left[\int_0^1 \beta^{\rho(\alpha-1)} \left[\beta^\rho \ln \left(\frac{\Psi((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|)}{\varphi((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|)} \right) \right. \right. \\
&\quad \left. \left. + (1-\beta^\rho) \ln \left(\frac{\Psi(\beta^\rho|\nu^\rho - \vartheta^\rho|)}{\varphi(\beta^\rho|\nu^\rho - \vartheta^\rho|)} \right) \right] \beta^{\rho-1} d\beta \right. \\
&\quad \left. + \int_0^1 \beta^{\rho(\alpha-1)} \left[\beta^\rho \ln \left(\frac{\Psi((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|)}{\varphi((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|)} \right) \right. \right. \\
&\quad \left. \left. + (1-\beta^\rho) \ln \left(\frac{\Psi(\beta^\rho|\nu^\rho - \vartheta^\rho|)}{\varphi(\beta^\rho|\nu^\rho - \vartheta^\rho|)} \right) \right] \beta^{\rho-1} d\beta \right] \right\}.
\end{aligned}$$

Setting $\mathcal{W}^\rho = \beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho$ in the first integral term and $\mathcal{W}^\rho = \beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho$ in the second integral term, and after simple calculation, we attain:

$$\begin{aligned}
&\left[\frac{{}_\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}_\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}_\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}_\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\
&\leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \times \exp \left\{ \frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}} \int_{\vartheta}^{\nu} \left((\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1} \right) \right. \\
&\quad \times \left((\mathcal{W}^\rho - \vartheta^\rho) \ln \left(\frac{\Psi(\nu^\rho - \mathcal{W}^\rho)}{\varphi(\nu^\rho - \mathcal{W}^\rho)} \right) + (\nu^\rho - \mathcal{W}^\rho) \ln \left(\frac{\Psi(\mathcal{W}^\rho - \vartheta^\rho)}{\varphi(\mathcal{W}^\rho - \vartheta^\rho)} \right) \right) d\mathcal{W} \Big\} \\
&= \frac{\frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))}}{\left[\int_{\vartheta}^{\nu} \left(\left(\frac{\Psi(\nu^\rho - \mathcal{W}^\rho)}{\varphi(\nu^\rho - \mathcal{W}^\rho)} \right)^{(\mathcal{W}^\rho - \vartheta^\rho)} \left(\frac{\Psi(\mathcal{W}^\rho - \vartheta^\rho)}{\varphi(\mathcal{W}^\rho - \vartheta^\rho)} \right)^{(\nu^\rho - \mathcal{W}^\rho)} \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}}}. \tag{3.10}
\end{aligned}$$

Merging (3.9) and (3.10), we achieve the desired result. $\square$

Remark 3.19. If $\rho = 1$ is put in Theorem 3.18, we achieve the result involving $\mathbb{RL}$ integrals.

$$\begin{aligned}
&\left(\frac{\Psi\left(\frac{\vartheta+\nu}{2}\right)}{\varphi\left(\frac{\vartheta+\nu}{2}\right)} \right) \left(\int_{\vartheta}^{\nu} \left(\frac{\Psi(|\frac{\vartheta+\nu}{2} - \mathcal{W}|)}{\varphi(|\frac{\vartheta+\nu}{2} - \mathcal{W}|)} \right)^{((\mathcal{W}-\vartheta)^{\alpha-1} + (\nu-\mathcal{W})^{\alpha-1})} d\mathcal{W} \right)^{\frac{\alpha}{2(\nu-\vartheta)^\alpha}} \\
&\leq \left[\frac{{}_*I_{\vartheta^+}^\alpha \Psi(\nu) \cdot {}_*I_{\nu^-}^\alpha \Psi(\vartheta)}{{}_*I_{\vartheta^+}^\alpha \varphi(\nu) \cdot {}_*I_{\nu^-}^\alpha \varphi(\vartheta)} \right]^{\frac{\Gamma(\alpha+1)}{2(\nu-\vartheta)^\alpha}} \\
&\leq \frac{\frac{G(\Psi(\vartheta), \Psi(\nu))}{G(\varphi(\vartheta), \varphi(\nu))}}{\left[\int_{\vartheta}^{\nu} \left(\left(\frac{\Psi(\nu-\mathcal{W})}{\varphi(\nu-\mathcal{W})} \right)^{(\mathcal{W}-\vartheta)} \left(\frac{\Psi(\mathcal{W}-\vartheta)}{\varphi(\mathcal{W}-\vartheta)} \right)^{(\nu-\mathcal{W})} \right)^{((\nu-\mathcal{W})^{\alpha-1} + (\mathcal{W}-\vartheta)^{\alpha-1})} d\mathcal{W} \right]^{\frac{\alpha}{2(\nu-\vartheta)^{\alpha+1}}}}.
\end{aligned}$$

Remark 3.20. If $\alpha = 1$ is put in Theorem 3.18, we achieve

$$\left(\frac{\Psi\left(\frac{\vartheta^\rho+\nu^\rho}{2}\right)}{\varphi\left(\frac{\vartheta^\rho+\nu^\rho}{2}\right)} \right) \left(\int_{\vartheta}^{\nu} \left(\frac{\Psi\left(|\frac{\vartheta^\rho+\nu^\rho}{2} - \mathcal{W}^\rho\right|)}{\varphi\left(|\frac{\vartheta^\rho+\nu^\rho}{2} - \mathcal{W}^\rho\right|}\right) d\mathcal{W} \right)^{\frac{\rho}{(\nu^\rho - \vartheta^\rho)}} \leq \left(\int_{\vartheta}^{\nu} \left(\frac{\Psi(\mathcal{W}^\rho)}{\varphi(\mathcal{W}^\rho)} \right) d\mathcal{W} \right)^{\frac{\rho}{\nu^\rho - \vartheta^\rho}}$$

$$\leq \frac{\frac{G(\Psi(\vartheta^\rho), \Psi(v^\rho))}{G(\varphi(\vartheta^\rho), \varphi(v^\rho))}}{\left[\int_{\vartheta}^v \left(\left(\frac{\Psi(v^\rho - \mathcal{W}^\rho)}{\varphi(v^\rho - \mathcal{W}^\rho)} \right)^{(\mathcal{W}^\rho - \vartheta^\rho)} \left(\frac{\Psi(\mathcal{W}^\rho - \vartheta^\rho)}{\varphi(\mathcal{W}^\rho - \vartheta^\rho)} \right)^{(v^\rho - \mathcal{W}^\rho)} d\mathcal{W} \right)^{\frac{\rho}{(v^\rho - \vartheta^\rho)^{\rho+1}}} \right]}.$$

Remark 3.21. If $\rho = 1$ and $\alpha = 1$ are put in Theorem 3.18, we achieve

$$\begin{aligned} & \left(\frac{\Psi\left(\frac{\vartheta+v}{2}\right)}{\varphi\left(\frac{\vartheta+v}{2}\right)} \right) \left(\int_{\vartheta}^v \left(\frac{\Psi\left(\left|\frac{\vartheta+v}{2} - \mathcal{W}\right|\right)}{\varphi\left(\left|\frac{\vartheta+v}{2} - \mathcal{W}\right|\right)} \right)^{d\mathcal{W}} \right)^{\frac{1}{(v-\vartheta)}} \leq \left(\int_{\vartheta}^v \left(\frac{\Psi(\mathcal{W})}{\varphi(\mathcal{W})} \right)^{d\mathcal{W}} \right)^{\frac{1}{v-\vartheta}} \\ & \leq \frac{\frac{G(\Psi(\vartheta), \Psi(v))}{G(\varphi(\vartheta), \varphi(v))}}{\left[\int_{\vartheta}^v \left(\left(\frac{\Psi(v-\mathcal{W})}{\varphi(v-\mathcal{W})} \right)^{(\mathcal{W}-\vartheta)} \left(\frac{\Psi(\mathcal{W}-\vartheta)}{\varphi(\mathcal{W}-\vartheta)} \right)^{(v-\mathcal{W})} d\mathcal{W} \right)^{\frac{1}{(v-\vartheta)^2}} \right]}. \end{aligned}$$

Example 3.22. Figure 9 confirms the authenticity of Theorem 3.18 for $\varphi(\mathcal{W}) = \exp\{-\mathcal{W}^4\}$ and $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^6\}$.

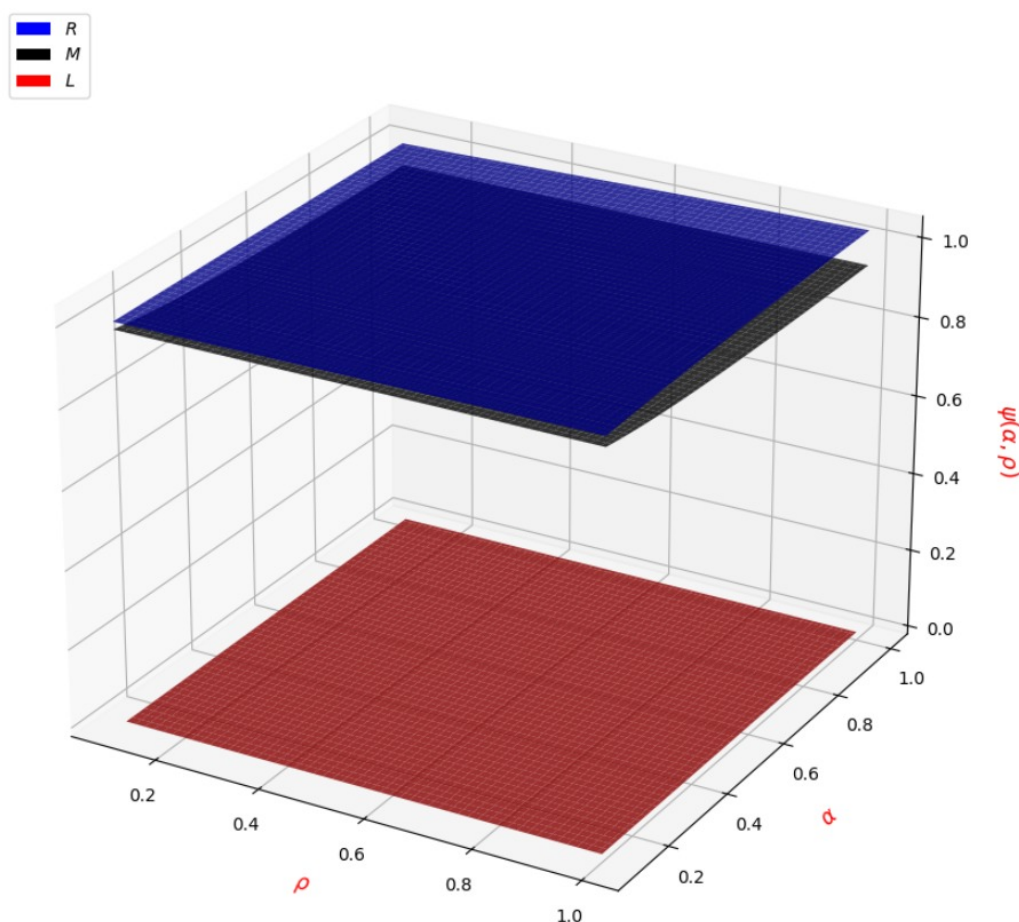


Figure 9. Graphical description of Theorem 3.18 for $\varphi(\mathcal{W}) = \exp\{-\mathcal{W}^4\}$, $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^6\}$, $\vartheta = 0$, $v = 1$, $\alpha \in (0, 1)$, and $\rho \in (0, 1)$.

Theorem 3.23. If Ψ is positively multiplicatively convex, and φ is a positive and multiplicatively subquadratic function on $[\vartheta^\rho, \nu^\rho]$, then

$$\begin{aligned} & \left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \\ & \times \left[\int_{\vartheta}^{\nu} \left(\left(\frac{(\Psi(\nu^\rho - \mathcal{W}^\rho))^{\mathcal{H}}}{(\varphi(\mathcal{W}^\rho - \vartheta^\rho))^{\mathcal{H}}} \right)^{d\mathcal{W}} \right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}} \right], \end{aligned} \quad (3.11)$$

holds $\forall \alpha > 0$ and $\rho > 0$.

Proof. Let us consider the left-hand side of (3.11):

$$\begin{aligned} & \left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & = \exp \left\{ \frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} [{}^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) - ({}^\rho K_{\vartheta^+}^\alpha (\ln \circ \varphi)(\nu^\rho) + {}^\rho K_{\nu^-}^\alpha (\ln \circ \varphi)(\vartheta^\rho))] \right\} \\ & = \exp \left\{ \frac{\alpha\rho}{2} \left[\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho) \nu^\rho) d\beta - \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho) \nu^\rho) d\beta \right. \right. \\ & \quad \left. \left. + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \nu^\rho + (1-\beta^\rho) \vartheta^\rho) d\beta - \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \nu^\rho + (1-\beta^\rho) \vartheta^\rho) d\beta \right] \right\} \\ & \leq \exp \left\{ \frac{\alpha\rho}{2} \left[\frac{1}{\alpha\rho} \ln \left(\frac{\Psi(\vartheta^\rho) \Psi(\nu^\rho)}{\varphi(\vartheta^\rho) \varphi(\nu^\rho)} \right) \right. \right. \\ & \quad \left. \left. - \left[\int_0^1 \beta^{\rho(\alpha-1)} [\beta^\rho (\ln \circ \Psi)((1-\beta^\rho) |\nu^\rho - \vartheta^\rho|) + (1-\beta^\rho) (\ln \circ \Psi)(\beta^\rho |\nu^\rho - \vartheta^\rho|)] \beta^{\rho-1} d\beta \right. \right. \right. \\ & \quad \left. \left. - \int_0^1 \beta^{\rho(\alpha-1)} [\beta^\rho (\ln \circ \varphi)((1-\beta^\rho) |\nu^\rho - \vartheta^\rho|) + (1-\beta^\rho) (\ln \circ \varphi)(\beta^\rho |\nu^\rho - \vartheta^\rho|)] \beta^{\rho-1} d\beta \right] \right\}. \end{aligned}$$

Setting $\mathcal{W}^\rho = \beta^\rho \vartheta^\rho + (1-\beta^\rho) \nu^\rho$ in the first integral term and $\mathcal{W}^\rho = \beta^\rho \nu^\rho + (1-\beta^\rho) \vartheta^\rho$ in the second integral term, and after simple calculation, we attain:

$$\begin{aligned} & \left[\frac{{}^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \times \exp \left\{ \frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}} \int_{\vartheta}^{\nu} ((\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}) \right. \\ & \quad \left. \times ((\mathcal{W}^\rho - \vartheta^\rho)(\ln \circ \Psi)(\nu^\rho - \mathcal{W}^\rho) - (\nu^\rho - \mathcal{W}^\rho)(\ln \circ \varphi)(\mathcal{W}^\rho - \vartheta^\rho)) d\mathcal{W} \right\} \\ & = \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \\ & \quad \times \left[\int_{\vartheta}^{\nu} \left(\left(\frac{(\Psi(\nu^\rho - \mathcal{W}^\rho))^{\mathcal{H}}}{(\varphi(\mathcal{W}^\rho - \vartheta^\rho))^{\mathcal{H}}} \right)^{d\mathcal{W}} \right)^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}} \right]. \end{aligned}$$

Hence the proof is complete. $\square$

Remark 3.24. If $\rho = 1$ is put in Theorem 3.23, we achieve the result involving $\mathbb{RL}$ operators.

$$\left[\frac{{}_*K_{\vartheta^+}^\alpha \Psi(\nu) \cdot {}_*K_{\nu^-}^\alpha \Psi(\vartheta)}{{}_*K_{\vartheta^+}^\alpha \varphi(\nu) \cdot {}_*K_{\nu^-}^\alpha \varphi(\vartheta)} \right]^{\frac{\Gamma(\alpha+1)}{2(\nu-\vartheta)^\alpha}} \leq \frac{G(\Psi(\vartheta), \Psi(\nu))}{G(\varphi(\vartheta), \varphi(\nu))} \\ \times \left[\int_{\vartheta}^{\nu} \left(\frac{(\Psi(\nu - \mathcal{W}))^{(\mathcal{W}-\vartheta)}}{(\varphi(\mathcal{W} - \vartheta))^{(\nu-\mathcal{W})}} \right)^{((\nu-\mathcal{W})^{\alpha-1} + (\mathcal{W}-\vartheta)^{\alpha-1})} d\mathcal{W} \right]^{\frac{\alpha}{2(\nu-\vartheta)^{\alpha+1}}}.$$

Remark 3.25. If $\alpha = 1$ is put in Theorem 3.23, we achieve

$$\left(\int_{\vartheta}^{\nu} \left(\frac{\Psi(\mathcal{W}^\rho)}{\varphi(\mathcal{W}^\rho)} \right)^{d\mathcal{W}} \right)^{\frac{\rho}{\nu^\rho - \vartheta^\rho}} \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \\ \times \left[\int_{\vartheta}^{\nu} \left(\frac{(\Psi(\nu^\rho - \mathcal{W}^\rho))^{(\mathcal{W}^\rho - \vartheta^\rho)}}{(\varphi(\mathcal{W}^\rho - \vartheta^\rho))^{(\nu^\rho - \mathcal{W}^\rho)}} \right)^{d\mathcal{W}} \right]^{\frac{\rho}{(\nu^\rho - \vartheta^\rho)^{\rho+1}}}.$$

Remark 3.26. If $\rho = 1$ and $\alpha = 1$ are put in Theorem 3.23, we achieve

$$\left(\int_{\vartheta}^{\nu} \left(\frac{\Psi(\mathcal{W})}{\varphi(\mathcal{W})} \right)^{d\mathcal{W}} \right)^{\frac{1}{\nu-\vartheta}} \leq \frac{G(\Psi(\vartheta), \Psi(\nu))}{G(\varphi(\vartheta), \varphi(\nu))} \\ \times \left[\int_{\vartheta}^{\nu} \left(\frac{(\Psi(\nu - \mathcal{W}))^{(\mathcal{W}-\vartheta)}}{(\varphi(\mathcal{W} - \vartheta))^{(\nu-\mathcal{W})}} \right)^{d\mathcal{W}} \right]^{\frac{1}{(\nu-\vartheta)^2}}.$$

Theorem 3.27. If Ψ is positive and multiplicatively superquadratic, and φ is a positive and multiplicatively concave function on $[\vartheta^\rho, \nu^\rho]$, then

$$\left[\frac{{}_*K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}_*K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}_*K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}_*K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \\ \times \left[\int_{\vartheta}^{\nu} \left(\frac{(\Psi(\nu^\rho - \mathcal{W}^\rho))^{(\mathcal{W}^\rho - \vartheta^\rho)}}{(\varphi(\mathcal{W}^\rho - \vartheta^\rho))^{(\nu^\rho - \mathcal{W}^\rho)}} \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}} \quad (3.12)$$

holds $\forall \alpha > 0$ and $\rho > 0$.

Proof. Let us consider the left-hand side of (3.12):

$$\left[\frac{{}_*K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}_*K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}_*K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}_*K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ = \exp \left\{ \frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} \left[{}_*K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + {}_*K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) - ({}_*K_{\vartheta^+}^\alpha (\ln \circ \varphi)(\nu^\rho) + {}_*K_{\nu^-}^\alpha (\ln \circ \varphi)(\vartheta^\rho)) \right] \right\} \\ = \exp \left\{ \frac{\alpha\rho}{2} \left[\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho) d\beta - \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho) d\beta \right. \right. \\ \left. \left. + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho) d\beta - \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho) d\beta \right] \right\}$$

$$\leq \exp \left\{ \frac{\alpha\rho}{2} \left[\frac{1}{\alpha\rho} \ln \left(\frac{\Psi(\vartheta^\rho)\Psi(\nu^\rho)}{\varphi(\vartheta^\rho)\varphi(\nu^\rho)} \right) - \left[\int_0^1 \beta^{\rho(\alpha-1)} [\beta^\rho (\ln \circ \Psi)((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|) + (1-\beta^\rho)(\ln \circ \Psi)(\beta^\rho|\nu^\rho - \vartheta^\rho|)] \beta^{\rho-1} d\beta - \int_0^1 \beta^{\rho(\alpha-1)} [\beta^\rho (\ln \circ \varphi)((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|) + (1-\beta^\rho)(\ln \circ \varphi)(\beta^\rho|\nu^\rho - \vartheta^\rho|)] \beta^{\rho-1} d\beta \right] \right] \right\}.$$

Setting $\mathcal{W}^\rho = \beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho$ in the first integral term and $\mathcal{W}^\rho = \beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho$ in the second integral term, and after simple calculation, we attain:

$$\begin{aligned} & \left[\frac{{}_\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot {}_\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho)}{{}_\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot {}_\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \exp \left\{ \frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}} \int_{\vartheta}^{\nu} ((\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}) \right. \\ & \quad \times ((\mathcal{W}^\rho - \vartheta^\rho)(\ln \circ \Psi)(\nu^\rho - \mathcal{W}^\rho) - (\nu^\rho - \mathcal{W}^\rho)(\ln \circ \varphi)(\mathcal{W}^\rho - \vartheta^\rho)) d\mathcal{W} \left. \right\} \\ & = \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \left[\int_{\vartheta}^{\nu} \left(\left(\frac{(\Psi(\nu^\rho - \mathcal{W}^\rho))^{(\mathcal{W}^\rho - \vartheta^\rho)}}{(\varphi(\mathcal{W}^\rho - \vartheta^\rho))^{(\nu^\rho - \mathcal{W}^\rho)}} \right)^{\mathcal{H}} \right)^{d\mathcal{W}} \right]^{\frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}}. \end{aligned}$$

Hence the proof is complete. $\square$

Remark 3.28. If $\rho = 1$ is put in Theorem 3.27, we achieve the result involving $\mathbb{RL}$ operators.

$$\begin{aligned} & \left[\frac{{}_* K_{\vartheta^+}^\alpha \Psi(\nu) \cdot {}_* K_{\nu^-}^\alpha \Psi(\vartheta)}{{}_* K_{\vartheta^+}^\alpha \varphi(\nu) \cdot {}_* K_{\nu^-}^\alpha \varphi(\vartheta)} \right]^{\frac{\Gamma(\alpha+1)}{2(\nu - \vartheta)^\alpha}} \\ & \leq \frac{\frac{G(\Psi(\vartheta), \Psi(\nu))}{G(\varphi(\vartheta), \varphi(\nu))}}{\left[\int_{\vartheta}^{\nu} \left(\left(\frac{(\Psi(\nu - \mathcal{W}))^{(\mathcal{W} - \vartheta)}}{(\varphi(\mathcal{W} - \vartheta))^{(\nu - \mathcal{W})}} \right)^{((\nu - \mathcal{W})^{\alpha-1} + (\mathcal{W} - \vartheta)^{\alpha-1})} \right)^{d\mathcal{W}} \right]^{\frac{\alpha}{2(\nu - \vartheta)^{\alpha+1}}}}. \end{aligned}$$

Remark 3.29. As a particular case, setting $\alpha = 1$ in Theorem 3.27 yields

$$\begin{aligned} & \left(\int_{\vartheta}^{\nu} \left(\frac{\Psi(\mathcal{W}^\rho)}{\varphi(\mathcal{W}^\rho)} \right)^{d\mathcal{W}} \right)^{\frac{\rho}{\nu^\rho - \vartheta^\rho}} \leq \frac{G(\Psi(\vartheta^\rho), \Psi(\nu^\rho))}{G(\varphi(\vartheta^\rho), \varphi(\nu^\rho))} \\ & \quad \times \left[\int_{\vartheta}^{\nu} \left(\frac{(\Psi(\nu^\rho - \mathcal{W}^\rho))^{(\mathcal{W}^\rho - \vartheta^\rho)}}{(\varphi(\mathcal{W}^\rho - \vartheta^\rho))^{(\nu^\rho - \mathcal{W}^\rho)}} \right)^{d\mathcal{W}} \right]^{\frac{-\rho}{(\nu^\rho - \vartheta^\rho)^{\rho+1}}}. \end{aligned}$$

Remark 3.30. If $\rho = 1$ and $\alpha = 1$ are put in Theorem 3.27, we achieve

$$\begin{aligned} & \left(\int_{\vartheta}^{\nu} \left(\frac{\Psi(\mathcal{W})}{\varphi(\mathcal{W})} \right)^{d\mathcal{W}} \right)^{\frac{1}{\nu - \vartheta}} \leq \frac{G(\Psi(\vartheta), \Psi(\nu))}{G(\varphi(\vartheta), \varphi(\nu))} \\ & \quad \times \left[\int_{\vartheta}^{\nu} \left(\frac{(\Psi(\nu - \mathcal{W}))^{(\mathcal{W} - \vartheta)}}{(\varphi(\mathcal{W} - \vartheta))^{(\nu - \mathcal{W})}} \right)^{d\mathcal{W}} \right]^{\frac{-1}{(\nu - \vartheta)^2}}. \end{aligned}$$

Theorem 3.31. If Ψ is positive and multiplicatively convex, and φ is a positive and multiplicatively superquadratic function on $[\vartheta^\rho, \nu^\rho]$, then

$$\begin{aligned} & [\cdot^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho) \cdot^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq \frac{G(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho), \Psi(\nu^\rho)\varphi(\nu^\rho))}{\left[\int_{\vartheta}^{\nu} \left(\left(\frac{(\Psi(|\nu^\rho - \mathcal{W}^\rho|))(\mathcal{W}^\rho - \vartheta^\rho)}{(\varphi(|\mathcal{W}^\rho - \vartheta^\rho|)) - (\nu^\rho - \mathcal{W}^\rho)} \right)^{\mathcal{H}} \right)^{d\mathcal{W}} \right]^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}}} \end{aligned} \quad (3.13)$$

holds $\forall \alpha > 0$ and $\rho > 0$.

Proof. Let us consider the left-hand side of (3.13):

$$\begin{aligned} & [\cdot^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho) \cdot^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & = \exp \left\{ \frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha} [\cdot^\rho K_{\vartheta^+}^\alpha (\ln \circ \Psi)(\nu^\rho) + \cdot^\rho K_{\nu^-}^\alpha (\ln \circ \Psi)(\vartheta^\rho) + \cdot^\rho K_{\vartheta^+}^\alpha (\ln \circ \varphi)(\nu^\rho) + \cdot^\rho K_{\nu^-}^\alpha (\ln \circ \varphi)(\vartheta^\rho)] \right\} \\ & = \exp \left\{ \frac{\alpha\rho}{2} \left[\int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \Psi)(\beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho) d\beta \right. \right. \\ & \quad \left. \left. + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho) d\beta + \int_0^1 \beta^{\alpha\rho-1} (\ln \circ \varphi)(\beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho) d\beta \right] \right\} \\ & \leq \exp \left\{ \frac{\alpha\rho}{2} \left[\frac{\ln(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho)\Psi(\nu^\rho)\varphi(\nu^\rho))}{\alpha\rho} - \left[\int_0^1 \beta^{\rho(\alpha-1)} \left[\beta^\rho (\ln \circ \Psi)((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|) \right. \right. \right. \right. \\ & \quad \left. \left. \left. + (1-\beta^\rho)(\ln \circ \Psi)(\beta^\rho |\nu^\rho - \vartheta^\rho|) \right] \beta^{\rho-1} d\beta + \int_0^1 \beta^{\rho(\alpha-1)} \left[\beta^\rho (\ln \circ \varphi)((1-\beta^\rho)|\nu^\rho - \vartheta^\rho|) \right. \right. \right. \\ & \quad \left. \left. \left. + (1-\beta^\rho)(\ln \circ \varphi)(\beta^\rho |\nu^\rho - \vartheta^\rho|) \right] \beta^{\rho-1} d\beta \right] \right] \right\}. \end{aligned}$$

Setting $\mathcal{W}^\rho = \beta^\rho \vartheta^\rho + (1-\beta^\rho)\nu^\rho$ in the first integral term and $\mathcal{W}^\rho = \beta^\rho \nu^\rho + (1-\beta^\rho)\vartheta^\rho$ in the second integral term, and after simple calculation, we attain:

$$\begin{aligned} & [\cdot^\rho K_{\vartheta^+}^\alpha \Psi(\nu^\rho) \cdot^\rho K_{\nu^-}^\alpha \Psi(\vartheta^\rho) \cdot^\rho K_{\vartheta^+}^\alpha \varphi(\nu^\rho) \cdot^\rho K_{\nu^-}^\alpha \varphi(\vartheta^\rho)]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^\alpha}} \\ & \leq G(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho), \Psi(\nu^\rho)\varphi(\nu^\rho)) \exp \left\{ \frac{-\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}} \int_{\vartheta}^{\nu} ((\mathcal{W}^\rho - \vartheta^\rho)^{\alpha-1} + (\nu^\rho - \mathcal{W}^\rho)^{\alpha-1}) \right. \\ & \quad \left. \times ((\mathcal{W}^\rho - \vartheta^\rho)(\ln \circ \Psi)(\nu^\rho - \mathcal{W}^\rho) + (\nu^\rho - \mathcal{W}^\rho)(\ln \circ \varphi)(\mathcal{W}^\rho - \vartheta^\rho)) d\mathcal{W} \right\} \\ & = \frac{G(\Psi(\vartheta^\rho)\varphi(\vartheta^\rho), \Psi(\nu^\rho)\varphi(\nu^\rho))}{\left[\int_{\vartheta}^{\nu} \left(\left(\frac{(\Psi(|\nu^\rho - \mathcal{W}^\rho|))(\mathcal{W}^\rho - \vartheta^\rho)}{(\varphi(|\mathcal{W}^\rho - \vartheta^\rho|)) - (\nu^\rho - \mathcal{W}^\rho)} \right)^{\mathcal{H}} \right)^{d\mathcal{W}} \right]^{\frac{\alpha\rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha\rho+1}}}}. \end{aligned}$$

Hence the proof is complete. $\square$

Remark 3.32. If $\rho = 1$ is put in Theorem 3.31, we attain the result involving $\mathbb{RL}$ operators.

$$[\cdot^\alpha I_{\vartheta^+}^\alpha \Psi(\nu) \cdot^\alpha I_{\nu^-}^\alpha \Psi(\vartheta) \cdot^\alpha I_{\vartheta^+}^\alpha \varphi(\nu) \cdot^\alpha I_{\nu^-}^\alpha \varphi(\vartheta)]^{\frac{\Gamma(\alpha+1)}{2(\nu - \vartheta)^\alpha}}$$

$$\leq \frac{G(\Psi(\vartheta)\varphi(\vartheta), \Psi(\nu)\varphi(\nu))}{\left[\int_{\vartheta}^{\nu} \left((\Psi(|\nu - \mathcal{W}|))^{\mathcal{W}-\vartheta} (\varphi(|\mathcal{W} - \vartheta|))^{\nu-\mathcal{W}} \right)^{((\nu-\mathcal{W})^{\alpha-1} + (\mathcal{W}-\vartheta)^{\alpha-1})} d\mathcal{W} \right]^{\frac{\alpha}{2(\nu-\vartheta)^{\alpha+1}}}.$$

Remark 3.33. As a particular case, setting $\alpha = 1$ in Theorem 3.31 yields

$$\begin{aligned} & \left(\int_{\vartheta}^{\nu} (\Psi(\mathcal{W}^{\rho})\varphi(\mathcal{W}^{\rho}))^{d\mathcal{W}} \right)^{\frac{\rho}{\nu^{\rho}-\vartheta^{\rho}}} \\ & \leq \frac{G(\Psi(\vartheta^{\rho})\varphi(\vartheta^{\rho}), \Psi(\nu^{\rho})\varphi(\nu^{\rho}))}{\left[\int_{\vartheta}^{\nu} \left((\Psi(|\nu^{\rho} - \mathcal{W}^{\rho}|))^{\mathcal{W}^{\rho}-\vartheta^{\rho}} \cdot (\varphi(|\mathcal{W}^{\rho} - \vartheta^{\rho}|))^{\nu^{\rho}-\mathcal{W}^{\rho}} \right)^{d\mathcal{W}} \right]^{\frac{\rho}{(\nu^{\rho}-\vartheta^{\rho})^{\rho+1}}}. \end{aligned}$$

Remark 3.34. If $\rho = 1$ and $\alpha = 1$ are put in Theorem 3.31, we achieve

$$\begin{aligned} & \left(\int_{\vartheta}^{\nu} (\Psi(\mathcal{W})\varphi(\mathcal{W}))^{d\mathcal{W}} \right)^{\frac{1}{\nu-\vartheta}} \\ & \leq \frac{G(\Psi(\vartheta)\varphi(\vartheta), \Psi(\nu)\varphi(\nu))}{\left[\int_{\vartheta}^{\nu} \left((\Psi(|\nu - \mathcal{W}|))^{\mathcal{W}-\vartheta} \cdot (\varphi(|\mathcal{W} - \vartheta|))^{\nu-\mathcal{W}} \right)^{d\mathcal{W}} \right]^{\frac{1}{(\nu-\vartheta)^2}}}. \end{aligned}$$

4. Applications

4.1. Special means

The arithmetic mean $A(\vartheta, \nu)$, the weighted arithmetic mean $\mathbb{W}(\vartheta, \nu, w_1, w_2)$, and the geometric mean $G(\vartheta, \nu)$ are introduced as follows.

$$\begin{aligned} A(\vartheta, \nu) &= \frac{\vartheta + \nu}{2}, \\ \mathbb{W}(\vartheta, \nu, w_1, w_2) &= \frac{w_1 \vartheta + w_2 \nu}{w_1 + w_2}, \\ G(\vartheta, \nu) &= \sqrt{\vartheta \nu}. \end{aligned}$$

Proposition 4.1. The inequality

$$\begin{aligned} & \exp \left\{ A^n(\vartheta^{\rho}, \nu^{\rho}) + \frac{G(\alpha, \rho)}{2(\nu^{\rho} - \vartheta^{\rho})^{\alpha}} \int_{\vartheta}^{\nu} G(2A((\mathcal{W}^{\rho} - \vartheta^{\rho})^{\alpha-1}, (\nu^{\rho} - \mathcal{W}^{\rho})^{\alpha-1}), (|A(\vartheta^{\rho}, \nu^{\rho}) - \mathcal{W}^{\rho}|)^n) d\mathcal{W} \right\} \\ & \leq [{}^{\rho}_* K_{\vartheta^+}^{\alpha} [\exp\{\nu^{n\rho}\}] \cdot {}^{\rho}_* K_{\nu^-}^{\alpha} [\exp\{\vartheta^{n\rho}\}]]^{\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(\nu^{\rho}-\vartheta^{\rho})^{\alpha}}} \\ & \leq \exp \left\{ 2A(\vartheta^{\rho n}, \nu^{\rho n}) - \frac{2G^2(\alpha, \rho)}{(\nu^{\rho} - \vartheta^{\rho})^{G^2(\alpha, \rho)+1}} \int_{\vartheta}^{\nu} G^2(A((\mathcal{W}^{\rho} - \vartheta^{\rho})^{\alpha-1}, (\nu^{\rho} - \mathcal{W}^{\rho})^{\alpha-1}), \right. \\ & \quad \left. A(G^2((\mathcal{W}^{\rho} - \vartheta^{\rho}), (\nu^{\rho} - \mathcal{W}^{\rho})^n), G^2((\mathcal{W}^{\rho} - \vartheta^{\rho}), (\nu^{\rho} - \mathcal{W}^{\rho})^n))) \right\} \end{aligned}$$

holds for every $[\vartheta, \nu] \subset [1, \infty)$.

Proof. Theorem 3.1 may be applied to establish the result for the multiplicatively superquadratic function $\Psi(\mathcal{W}) = \exp\{\mathcal{W}^n\}$, where $n \geq 3$. $\square$

4.2. Modified Bessel function

Making use of the modified Bessel function of first type $\mathfrak{I}_q$ [40],

$$\mathfrak{I}_q(\mathcal{W}) = \sum_{n=0}^{\infty} \frac{(\frac{\mathcal{W}}{2})^{q+2n}}{n! \Gamma(q+n+1)}, \quad \text{where } \mathcal{W} \in \mathfrak{R}.$$

We define, for all $q \geq 2$, the mapping $\Psi_q : [0, \infty) \rightarrow [0, \infty)$ as

$$\Psi_q(\mathcal{W}) = \mathcal{W}^q \mathfrak{I}_q(\mathcal{W}).$$

$$\Psi'_q(\mathcal{W}) = \mathcal{W}^q \mathfrak{I}_{q-1}(\mathcal{W}).$$

$$\Psi''_q(\mathcal{W}) = \mathcal{W}^{q-1} \mathfrak{I}_{q-1}(\mathcal{W}) + \mathcal{W}^q \mathfrak{I}_{q-2}(\mathcal{W}).$$

$$\Psi'''_q(\mathcal{W}) = 3\mathcal{W}^{q-1} \mathfrak{I}_{q-2}(\mathcal{W}) + \mathcal{W}^q \mathfrak{I}_{q-3}(\mathcal{W}).$$

$$\Psi^4_q(\mathcal{W}) = 3\mathcal{W}^{q-2} \mathfrak{I}_{q-2}(\mathcal{W}) + 6\mathcal{W}^{q-1} \mathfrak{I}_{q-3}(\mathcal{W}) + \mathcal{W}^q \mathfrak{I}_{q-4}(\mathcal{W}).$$

$$\Psi^5_q(\mathcal{W}) = 15\mathcal{W}^{q-2} \mathfrak{I}_{q-3}(\mathcal{W}) + 10\mathcal{W}^{q-1} \mathfrak{I}_{q-4}(\mathcal{W}) + \mathcal{W}^q \mathfrak{I}_{q-5}(\mathcal{W}).$$

Hence, by Lemma 1.6, if $\Psi'''_q(\mathcal{W})$ is convex and satisfies $\Psi'''_q(0) = \Psi''_q(0) = 0$, then $\Psi'_q(\mathcal{W})$ is superquadratic. Furthermore, if the logarithm of a function is superquadratic, then the function itself is referred to as multiplicatively superquadratic. It is worth noting that $\Psi'_q(\mathcal{W})$ can also be obtained by applying the logarithm to $\exp\{\Psi''_q(\mathcal{W})\}$.

Proposition 4.2. Suppose $q \geq 2$ and $\vartheta, \nu \in [1, \infty[$ such that $\vartheta < \nu$. In this case, the inequality is satisfied:

$$\begin{aligned} & \exp \left\{ \Psi''_q \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right\} \left(\int_{\vartheta}^{\nu} \left(\exp \left\{ \Psi''_q \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right) \right\} \right)^{\mathcal{H}} d\mathcal{W} \right)^{\frac{\alpha \rho}{2(\vartheta^\rho - \theta^\rho)^{\alpha}}} \\ & \leq [{}^\rho_* K_{\vartheta^+}^\alpha \exp\{\Psi''_q(\nu^\rho)\} \cdot {}^\rho_* K_{\nu^-}^\alpha \exp\{\Psi''_q(\vartheta^\rho)\}]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\vartheta^\rho - \theta^\rho)^{\alpha}}} \\ & \leq \frac{G(\exp\{\Psi''_q(\vartheta^\rho)\}, \exp\{\Psi''_q(\nu^\rho)\})}{\left[\int_{\vartheta}^{\nu} \left((\exp\{\Psi''_q(|\nu^\rho - \mathcal{W}^\rho|)\})^{(\mathcal{W}^\rho - \vartheta^\rho)} (\exp\{\Psi''_q(|\mathcal{W}^\rho - \vartheta^\rho|)\})^{(\nu^\rho - \mathcal{W}^\rho)} \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{\alpha \rho}{2(\vartheta^\rho - \theta^\rho)^{\alpha} \rho + 1}}}. \end{aligned} \quad (4.1)$$

Proof. The result (4.1) is established by applying Theorem 3.1 with the function $\exp\{\Psi''_q(\mathcal{W})\}$, where

$$\Psi''_q(\mathcal{W}) = \mathcal{W}^{q-1} \mathfrak{I}_{q-1}(\mathcal{W}) + \mathcal{W}^q \mathfrak{I}_{q-2}(\mathcal{W}).$$

□

4.3. Moment of random variables

While the distribution and density functions characterize the probability structure of a random variable, they do not always suffice to differentiate closely related distributions. Deeper insights are provided through measures such as mean, variance, skewness, kurtosis, and moments, which capture important aspects of the distribution's shape and variability. Additionally, moments provide valuable information for assessing the existence and limiting behavior of probability distributions under suitable conditions. Modern research has made use of mathematical inequalities for moment estimation, improving distributional analysis and comparison [41, 42].

In the context of probability theory, the k th moment of a random variable χ about an arbitrary location parameter $\mathcal{W}$ is written as $M_k(\mathcal{W})$. Its expression, for $k \geq 1$, is given by:

$$M_k(\mathcal{W}) = \int_{\vartheta}^{\nu} (\beta - \mathcal{W})^k \Upsilon(\beta) d\beta, \quad k = 1, 2, \dots \quad (4.2)$$

In (4.2), Υ represents the pdf, which is assumed to be uniform over the interval $[\vartheta, \nu]$. This assumption implies that every point within the interval is equally likely to represent the random variable. The uniform distribution serves as a natural choice when all possible outcomes in the specified range are treated as equally probable.

Based on (4.2), the following assertion holds.

Proposition 4.3. *Let $\Upsilon : I \subseteq [0, \infty) \rightarrow \mathfrak{R}^+$ be the uniform pdf of a random variable χ defined on $I = [\vartheta, \nu]$. Then, for $k \geq 1$, the following relation holds:*

$$\begin{aligned} & \exp \left\{ M_k \left(\frac{\vartheta^\rho + \nu^\rho}{2} \right) \right\} \left[\int_{\vartheta}^{\nu} \left(\exp \left\{ M_k \left(\left| \frac{\vartheta^\rho + \nu^\rho}{2} - \mathcal{W}^\rho \right| \right) \right\} \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{\alpha \rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha}}} \\ & \geq \left[{}^\rho K_{\vartheta^+}^\alpha \exp \{ M_k(\nu^\rho) \} \cdot {}^\rho K_{\nu^-}^\alpha \exp \{ M_k(\vartheta^\rho) \} \right]^{\frac{\rho^\alpha \Gamma(\alpha+1)}{2(\nu^\rho - \vartheta^\rho)^{\alpha}}} \\ & \geq \frac{G(\exp \{ M_k(\vartheta^\rho) \}, \exp \{ M_k(\nu^\rho) \})}{\left[\int_{\vartheta}^{\nu} \left((\exp \{ M_k(|\nu^\rho - \mathcal{W}^\rho|) \})^{(\mathcal{W}^\rho - \vartheta^\rho)} (\exp \{ M_k(|\mathcal{W}^\rho - \vartheta^\rho|) \})^{(\nu^\rho - \mathcal{W}^\rho)} \right)^{\mathcal{H}} d\mathcal{W} \right]^{\frac{\alpha \rho}{2(\nu^\rho - \vartheta^\rho)^{\alpha+1}}}}. \end{aligned} \quad (4.3)$$

Proof. It was established in [14] that the moment function $M_k(\mathcal{W})$ is subquadratic when the corresponding probability density function is uniform. Moreover, Moradi et al. [12] showed that if a function Ψ is subquadratic, then its exponential $\exp\{\Psi\}$ is multiplicatively subquadratic. Motivated by these results, we consider the function

$$\exp \{ M_k(\mathcal{W}) \},$$

which is multiplicatively subquadratic under the aforementioned assumptions. Consequently, by applying Remark 3.2 of Theorem 3.1 to the multiplicatively subquadratic function $\exp\{M_k(\mathcal{W})\}$, we obtain inequality (4.3), provided that Υ is the uniform probability density function defined on I . $\square$

5. Conclusions

In this paper, we derived $\mathcal{HHI}$ -type inequalities for multiplicatively superquadratic functions, as well as for the product and quotient of multiplicatively convex, concave, superquadratic, and subquadratic functions, by employing Katugampola fractional integral operators. Our results generalize known inequalities, since they reduce to $\mathcal{RL}$ cases when $\rho = 1$ and to Hadamard cases when $\rho \rightarrow 0^+$. The validity of the obtained results was further demonstrated through numerical and graphical illustrations, and their applicability was demonstrated through special means, functions, and the moments of random variables involving the uniform distribution. The following prospective studies emerge as direct extensions of this framework:

Problem 1: Inequalities of $\mathcal{HHI}$ -type for multiplicatively h -convex superquadratic functions via Katugampola fractional operators.

Problem 2: Higher-dimensional $\mathcal{HHI}$ -type inequalities for multiplicatively superquadratic functions.

Problem 3: $\mathcal{HHI}$ inequalities on time scales via Katugampola fractional integrals.

Problem 4: Applications of multiplicative fractional inequalities to numerical integration, optimization, and approximation theory.

Problem 5: Caputo fractional integral approach to multiplicatively superquadratic inequalities.

Problem 6: Unified non-singular fractional operator framework for product- and quotient- type multiplicative inequalities.

These directions collectively aim to broaden the theoretical foundation of multiplicative fractional inequalities, extend their structural generality, and deepen their applications in applied mathematical modeling, optimization theory, and stochastic analysis.

Author contributions

Conceptualization: Saad Ihsan Butt and Dawood Khan; Methodology: Saad Ihsan Butt and Dawood Khan; Formal analysis: Ghulam Jallani and Youngsoo Seol; Investigation: Dawood Khan and Ghulam Jallani; Writing—original draft preparation: Saad Ihsan Butt and Dawood Khan; Writing—review and editing: Ghulam Jallani and Youngsoo Seol; Supervision: Saad Ihsan Butt and Mohammed Alammar. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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