
*Research article***Complete convergence and complete integral convergence for weighted sums of arrays of rowwise m -AANA random variables under sublinear expectations****Zhiqiang Tang¹, Mengke Lv², and Wei Liu^{2,*}**¹ School of Science, Dalian Maritime University, Dalian 116026, China² School of Mathematics, Liaoning Normal University, Dalian, 116029, China*** Correspondence:** Email: wliu2210@lnnu.edu.cn.

Abstract: In the framework of sublinear expectations, we investigated weighted sums of arrays of rowwise m -asymptotically almost negatively associated (m -AANA) random variables. By applying truncation techniques for random variables, moment inequalities for m -AANA sequences, and properties of the Choquet integral, we established complete convergence and complete integral convergence theorems for such weighted sums under suitable moment conditions and tail-control conditions. The results applied to terminal sums of finite rows with nonnegative deterministic weights satisfying the stated maximal-weight summability condition. For rows of length n , this condition excluded weights whose sum was one. Within this scope, the results complemented existing convergence theorems for dependent arrays under sublinear expectations.

Keywords: sublinear expectation; arrays of rowwise m -AANA random variables; weighted sums; complete convergence; complete integral convergence

Mathematics Subject Classification: 60F15

1. Introduction Limit theorems are an important topic in probability theory and mathematical statistics. They play a fundamental role in the study of the law of large numbers, the central limit theorem, the law of the iterated logarithm, and the asymptotic properties of statistical estimation. In classical probability spaces, related results are usually established under a fixed probability measure and the independence assumption for random variables. However, in practical problems, such as financial risk analysis, actuarial insurance, engineering system modeling, and complex data analysis, random phenomena are often accompanied by model uncertainty, distributional uncertainty, and dependence among samples. In such cases, the traditional additive probability framework may not be fully applicable. To deal with this type of uncertainty, Peng [1,2] proposed the theory of sublinear

expectations, which laid the foundation for the study of limit theorems under a nonlinear probability framework. Subsequently, Hu and Zhou [3] investigated multidimensional central limit theorems and laws of large numbers under sublinear expectations. Zhang [4,5] established exponential inequalities under sublinear expectations and applied them to the study of the law of the iterated logarithm. Chen [6] discussed the strong law of large numbers under sublinear expectations, while Fang et al. [7] studied limit theorems with rates of convergence under sublinear expectations. These studies have promoted the development of limit theory in sublinear expectation spaces.

Complete convergence was introduced by Hsu and Robbins [8] in their study of the law of large numbers. Its core requirement is the convergence of the corresponding series of tail probabilities; therefore, it represents a stronger mode of convergence than convergence in probability. With the development of complete convergence theory, complete moment convergence, complete integral convergence, and other related convergence forms have been further studied. In sublinear expectation spaces, since the probability measure is no longer unique, the tail behavior of random variables usually cannot be described only by classical probabilities and linear expectations, but must instead be characterized by capacities and Choquet integrals. Therefore, complete integral convergence has become an important form for studying the convergence properties of weighted sums of random variable sequences or arrays under the framework of sublinear expectations. Regarding complete convergence and its related extensions, Chen and Wang [9] studied complete moment convergence for sequences of identically distributed φ -mixing random variables; Wu [10] investigated complete convergence for weighted sums of arrays of rowwise negatively dependent random variables; Wu, Peng and Hu [11] studied the limiting behavior of arrays of rowwise extended negatively dependent random variables; and Qiu and Chen [12] investigated the convergence of weighted sums of widely orthant dependent random variables. In this setting, Liang and Wu [13] studied complete convergence and complete integral convergence for extended negatively dependent random variables under sublinear expectations. Li and Wu [14] investigated complete integral convergence for arrays of rowwise extended independent random variables under sublinear expectations. Lu and Meng [15,16] established complete convergence and complete integral convergence for arrays of rowwise widely negative dependent (WND) random variables and their weighted sums, and Wang and Wu [17] further investigated complete convergence and complete integral convergence for weighted sums of arrays of rowwise WND random variables under sublinear expectations.

In many applications, the random variables involved are dependent rather than mutually independent and may exhibit different forms of negative dependence. Asymptotically almost negatively associated (AANA) random variables constitute an important extension of negatively associated random variables and have been extensively studied in connection with strong convergence and limit theorems. Wang et al. [18] investigated complete convergence and complete moment convergence for weighted sums of arrays of rowwise AANA random variables and established a Baum–Katz-type result. Moreover, Shen and Wang [19] derived exponential inequalities for arrays of rowwise AANA random variables and, on this basis, obtained complete convergence and complete moment convergence results under relatively weak conditions. The class of m -AANA random variables further incorporates a lag parameter m , which makes it possible to describe more complicated local dependence structures. In classical probability spaces, Nam et al. [20] studied maximal inequalities and the strong law of large numbers for m -AANA random variables, while Ko [21] extended related inequalities to random vectors in Hilbert spaces. Additionally, Xie et al. [22] further investigated the convergence properties of randomly weighted sums of arrays of rowwise m -AANA random variables and established complete convergence, complete moment convergence, and complete f -moment convergence results.

Convergence properties of rowwise AANA arrays have also been studied under sublinear expectations. In particular, Wang et al. [23, Theorems 3.1, 3.2 and Corollary 3.1] established L^q -convergence and complete q -th moment convergence for maximal partial sums of rowwise AANA arrays and obtained a deterministic-weight corollary. Nevertheless, these results do not cover the setting considered in this paper. First, the AANA condition used in [23] corresponds to the case $m=1$, whereas we treat rowwise m -AANA arrays for an arbitrary fixed positive integer m . Second, the main quantities investigated in [23] are maximal partial sums over an infinite row, while we consider terminal weighted sums of finite-row triangular arrays. Third, the convergence criteria, normalization conditions, and moment assumptions employed here differ from those in [23]. The results of Xie et al. [22] are closer to this work with respect to the m -AANA dependence structure, but they are established under a single additive probability measure and mainly concern randomly weighted sums. By contrast, we work under sublinear expectations, where upper and lower expectations, capacities, and Choquet integrals must be treated explicitly, and the weights are assumed to be nonnegative and deterministic. The largest-weight normalization and the summability condition (3.3) are also used in [17, Theorem 3.1]. That result concerns WND arrays and includes WND dominating factors in its moment and tail conditions, whereas our results entail the finite-row m -AANA assumption and the scaled variables Y_{ni} in the tail criterion. The terminal-sum results are not claimed to extend the maximal partial-sum results in [22, Theorems 3.1, 3.2] or [23, Theorems 3.1, 3.2 and Corollary 3.1].

The setting and assumptions considered here differ from those in [23] in the following respects. First, we establish complete convergence criteria for terminal weighted sums of finite-row arrays of rowwise m -AANA random variables under sublinear expectations. For a general mean-uncertain array, Theorem 3.1 gives separate upper-tail and lower-tail complete convergence conclusions. When the upper and lower expectations of every random variable coincide, these two one-sided conclusions can be combined to obtain the corresponding two-sided absolute-value result. Second, under additional integral summability conditions, Theorem 3.2 establishes the corresponding upper-tail and lower-tail complete integral convergence results in terms of Choquet integrals. The mean-certainty condition is required only when the two one-sided results are combined into a single absolute-value conclusion; it is not needed for either one-sided conclusion separately. Finally, the proofs combine truncation arguments, finite-row m -AANA moment estimates, capacity inequalities, and Choquet-integral techniques. These arguments address the non-additivity of capacities and the possible uncertainty of the means. The admissible weights must satisfy the maximal-weight summability condition (3.3). For rows of length n , nonnegative weights summing to one do not satisfy this condition; see Remark 3.2.

2. Preliminaries

Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathcal{H}$ be a linear space of real-valued $\mathcal{F}$ -measurable functions on Ω containing all constant functions. We assume that if $X_1, \dots, X_n \in \mathcal{H}$ and $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the class of locally Lipschitz functions satisfying

$$|\varphi(x) - \varphi(y)| \leq C_\varphi(1 + |x|^{q_\varphi} + |y|^{q_\varphi})|x - y|, \quad x, y \in \mathbb{R}^n,$$

for some constants $C_\varphi > 0$ and $q_\varphi \in \mathbb{N}$ depending only on φ . Elements of $\mathcal{H}$ are referred to as random variables.

(A1) For every event A used in the statements and proofs below, assume $I_A \in \mathcal{H}$, where I_A denotes the indicator function of A . This includes the level sets of the centered weighted sums, the truncated sums, and Y_{ni} , as well as all level sets used in the Choquet integrals.

Definition 2.1. [1] A functional $\mathbb{E}: \mathcal{H} \rightarrow \mathbb{R}$ is called a sublinear expectation if it satisfies the following properties:

- (1) Monotonicity: If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y], \forall X, Y \in \mathcal{H}$;
- (2) Constant preserving property: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$;
- (3) Subadditivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y], \forall X, Y \in \mathcal{H}$;
- (4) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

Then $\mathbb{E}$ is called a sublinear expectation, and the triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. The conjugate expectation of $\mathbb{E}$ is denoted by $\mathcal{E}$, and is defined by $\mathcal{E}[X] := -\mathbb{E}[-X], \forall X \in \mathcal{H}$.

Thus, for all $X \in \mathcal{H}$, one has $\mathcal{E}[X] \leq \mathbb{E}[X]$, therefore, $\mathcal{E}$ is also called the lower expectation, while $\mathbb{E}$ is called the upper expectation.

Remark 2.1. According to the above definition, the following properties of $\mathbb{E}$ can be obtained:

- (1) Translation invariance: $\mathbb{E}[X + c] = \mathbb{E}[X] + c, \forall c \in \mathbb{R}$.
- (2) Self-dominated property: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y], \forall X, Y \in \mathcal{H}$. Consequently, $|\mathbb{E}[X] - \mathbb{E}[Y]| \leq \mathbb{E}[|X - Y|]$.
- (3) Convexity: $\mathbb{E}[\alpha X + (1 - \alpha)Y] \leq \alpha \mathbb{E}[X] + (1 - \alpha)\mathbb{E}[Y], \alpha \in [0, 1]$.

Definition 2.2. [1] A set function $\mathbb{V}: \mathcal{F} \rightarrow [0, 1]$ is called a capacity if it satisfies the following conditions:

- (1) $\mathbb{V}(\emptyset) = 0, \mathbb{V}(\Omega) = 1$.
- (2) For any $A, B \in \mathcal{F}$ with $A \subseteq B$, one has $\mathbb{V}(A) \leq \mathbb{V}(B)$.

A capacity $\mathbb{V}$ is called subadditive if $\mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B) (\forall A, B \in \mathcal{F})$.

Under (A1), define the upper and lower capacities for the events considered below by $\mathbb{V}(A) := \mathbb{E}[I_A]$ and $\mathcal{V}(A) := \mathcal{E}[I_A]$. Then $\mathbb{V}$ is subadditive and $\mathbb{V}(A) + \mathcal{V}(A^c) = 1$. Here $A^c = \Omega \setminus A$.

For a random variable X , the Choquet integrals with respect to $\mathbb{V}$ and $\mathcal{V}$ are respectively defined by

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x) dx + \int_{-\infty}^0 [\mathbb{V}(X > x) - 1] dx,$$

and

$$C_{\mathcal{V}}(X) := \int_0^\infty \mathcal{V}(X > x) dx + \int_{-\infty}^0 [\mathcal{V}(X > x) - 1] dx,$$

whenever these integrals are well defined.

Remark 2.2. Under the standing assumption that $I_A \in \mathcal{H}$ for every event $A \in \mathcal{F}$ considered below, for every bounded nonnegative random variable $Z \in \mathcal{H}$, we have $\mathbb{E}[Z] \leq C_{\mathbb{V}}(Z)$. Indeed, suppose that $0 \leq Z \leq M$ for some $M > 0$. For each positive integer N , let $\Delta = \frac{M}{N}$. Then, pointwise, $Z \leq \Delta + \Delta \sum_{j=1}^{N-1} I_{\{Z > j\Delta\}}$. By the monotonicity, subadditivity, positive homogeneity, and constant-preserving property of $\mathbb{E}$, $\mathbb{E}[Z] \leq \Delta + \Delta \sum_{j=1}^{N-1} \mathbb{V}(Z > j\Delta)$. Since $t \mapsto \mathbb{V}(Z > t)$ is nonincreasing, letting $N \rightarrow \infty$ yields $\mathbb{E}[Z] \leq \int_0^M \mathbb{V}(Z > t) dt = C_{\mathbb{V}}(Z)$.

Definition 2.3 [16] (Complete convergence and complete integral convergence) Let $\{Z_n, n \geq 1\}$ be a sequence of random variables defined on the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. If for any $\varepsilon > 0, \sum_{n=1}^\infty \mathbb{V}(|Z_n| > \varepsilon) < \infty$, then Z_n is said to converge completely to zero under capacity $\mathbb{V}$.

Let $r > 0$. If for any

$$\varepsilon > 0, \sum_{n=1}^\infty C_{\mathbb{V}}((|Z_n| - \varepsilon)_+^r) < \infty,$$

then Z_n is said to converge completely integrally of order r to zero under the capacity $\mathbb{V}$, where $x_+ = \max\{x, 0\}$.

The classical notion of m -AANA random variables was introduced and studied by Nam et al. [20], and related maximal inequalities were extended to Hilbert-space-valued random vectors by Ko [21]. The following formulation under sublinear expectations is adopted from Sun [24]. For published work on the related AANA case $m = 1$ under sublinear expectations, see Wang et al. [23].

Definition 2.4. (m -AANA) Let $\{X_j, j \geq 1\}$ be a sequence of random variables defined on the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, and let $m \geq 1$ be an integer. If there exists a nonnegative sequence $\{\eta(j), j \geq 1\}$ with $\lim_{j \rightarrow \infty} \eta(j) = 0$ such that, for all $j \geq 1$ and $k \geq 1$,

$$\begin{aligned} & \mathbb{E} \left[f(X_j) g(X_{j+m}^{(k)}) \right] - \mathbb{E} [f(X_j)] \mathbb{E} [g(X_{j+m}^{(k)})] \\ & \leq \eta(j) \left\{ \mathbb{E} \left[\left(f(X_j) - \mathbb{E} f(X_j) \right)^2 \right] \right\}^{1/2} \left\{ \mathbb{E} \left[\left(g(X_{j+m}^{(k)}) - \mathbb{E} g(X_{j+m}^{(k)}) \right)^2 \right] \right\}^{1/2}, \end{aligned}$$

where $X_{j+m}^{(k)} = (X_{j+m}, X_{j+m+1}, \dots, X_{j+m+k-1})$, $m \geq 1$ is the lag parameter, $k \geq 1$ is the number (or dimension), of the subsequent consecutive variables, and f and g are continuous functions for which the above expectations exist and which are either nondecreasing or nonincreasing in each variable. Then $\{X_j, j \geq 1\}$ is called an m -AANA sequence of random variables, and $\{\eta(j), j \geq 1\}$ is called the sequence of mixing coefficients.

Since Definition 2.4 is formulated for an infinite sequence, we introduce the following finite-row formulation for the arrays considered here, using the rowwise terminology of [22].

Definition 2.5. (rowwise m -AANA arrays with finite rows) Let $\{X_{ni}: u_n \leq i \leq v_n, n \geq 1\} \subset \mathcal{H}$ be an array of random variables, and set $\ell_n = v_n - u_n + 1 < \infty$, $X_j^{(n)} = X_{n, u_n+j-1}$, $1 \leq j \leq \ell_n$. Fix an integer $m \geq 1$ and let $x_+ = \max\{x, 0\}$. The array is called rowwise m -AANA if, for every $n \geq 1$, there exist nonnegative coefficients $\{\eta_n(j): 1 \leq j \leq (\ell_n - m)_+\}$, such that, for all integers $j, k \geq 1$ satisfying $j + m + k - 1 \leq \ell_n$, we have

$$\begin{aligned} & \mathbb{E} \left[f(X_j^{(n)}) g(X_{j+m}^{(n,k)}) \right] - \mathbb{E} [f(X_j^{(n)})] \mathbb{E} [g(X_{j+m}^{(n,k)})] \\ & \leq \eta_n(j) \left\{ \mathbb{E} \left[\left(f(X_j^{(n)}) - \mathbb{E} f(X_j^{(n)}) \right)^2 \right] \right\}^{1/2} \left\{ \mathbb{E} \left[\left(g(X_{j+m}^{(n,k)}) - \mathbb{E} g(X_{j+m}^{(n,k)}) \right)^2 \right] \right\}^{1/2}, \end{aligned}$$

where $X_{j+m}^{(n,k)} = (X_{j+m}^{(n)}, X_{j+m+1}^{(n)}, \dots, X_{j+m+k-1}^{(n)})$. Here $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R}^k \rightarrow \mathbb{R}$ are continuous functions that are either nondecreasing (coordinatewise for g) or nonincreasing (coordinatewise for g), and for which all the displayed expectations and second moments are well defined and finite. If $\ell_n \leq m$, there is no admissible pair (j, k) , and the dependence condition is therefore vacuous. For notational convenience only, set $\eta_n(j) = 0$, $j > (\ell_n - m)_+$. This convention extends the coefficient sequence only, not the random row. In particular, no random variable $X_j^{(n)}$ with $j > \ell_n$ is introduced, and no dependence inequality is invoked for any unspecified infinite extension of the row. Consequently, for every fixed n , $\sum_{j=1}^{\infty} \eta_n(j)^2 = \sum_{j=1}^{(\ell_n - m)_+} \eta_n(j)^2 < \infty$, and $\lim_{j \rightarrow \infty} \eta_n(j) = 0$. The sequence $\{\eta_n(j), j \geq 1\}$ is called the mixing-coefficient sequence of the n -th row.

Remark 2.3. The finite-row dependence condition in Definition 2.5 is the finite-dimensional restriction of the dependence inequality in Definition 2.4. In particular, every finite segment of an infinite m -AANA sequence satisfying Definition 2.4 satisfies the finite-row dependence condition in

Definition 2.5 with the corresponding restricted mixing coefficients. Conversely, Definition 2.5 does not assert that a finite row admits an extension to an infinite m -AANA sequence. Notice that the zero-extension convention for the mixing coefficients does not by itself yield a uniform square-summability bound over the rows. Whenever such a uniform bound is required, it will be imposed explicitly in the corresponding result.

Under a classical linear expectation, independent and negatively associated finite rows satisfy Definition 2.5 with zero mixing coefficients, while $m=1$ gives the usual AANA case. Moreover, the uniform square-summability assumption is satisfied whenever $\eta_n(j) \leq c_j$, $n \geq 1$, $j \geq 1$, for some nonnegative sequence $\{c_j\}$ satisfying $\sum_{j=1}^{\infty} c_j^2 < \infty$. Typical examples include $c_j = C\rho^j$, $0 < \rho < 1$, and $c_j = Cj^{-\alpha}$, $\alpha > \frac{1}{2}$, where $C > 0$ is a constant.

Fix a constant $0 < \lambda < 1$. Let $G: [0, \infty) \rightarrow [0, 1]$ be a bounded, Lipschitz continuous, and nonincreasing function satisfying $G(t) = 1$ for $0 \leq t \leq \lambda$ and $G(t) = 0$ for $t \geq 1$. Then, for any $t \geq 0$, we have $I(t \geq 1) \leq 1 - G(t) \leq I(t > \lambda)$.

In Lemmas 2.1 and 2.3 below, we consider an arbitrary but fixed row n . For notational simplicity, the row index n is suppressed. Thus, we write $\ell = \ell_n$, $X_j = X_j^{(n)}$, $W_j = W_j^{(n)}$, $\eta(j) = \eta_n(j)$, whenever the corresponding notation is used. All the assertions in these two lemmas are understood rowwise. No new sequence of mixing coefficients is introduced.

Lemma 2.1. (closure of the finite-row m -AANA condition under monotone transformations) Let $\{X_j: 1 \leq j \leq \ell\}$ satisfy the finite-row m -AANA condition of Definition 2.5 with mixing coefficients $\{\eta(j): 1 \leq j \leq (\ell - m)_+\}$. Let $\varphi_j: \mathbb{R} \rightarrow \mathbb{R}$, $1 \leq j \leq \ell$, be continuous functions that are either nondecreasing or nonincreasing. Suppose that all the random variables and compositions appearing below belong to $\mathcal{H}$, and that the corresponding second moments are finite. Then $\{\varphi_j(X_j): 1 \leq j \leq \ell\}$ also satisfies the finite-row m -AANA condition with the same mixing coefficients $\{\eta(j)\}$.

Proof. Fix integers $j, k \geq 1$ satisfying $j + m + k - 1 \leq \ell$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R}^k \rightarrow \mathbb{R}$ be continuous functions that are either nondecreasing (coordinatewise for g) or nonincreasing (coordinatewise for g). Define $\tilde{f}(x) = f(\varphi_j(x))$ and $\tilde{g}(x_1, \dots, x_k) = g(\varphi_{j+m}(x_1), \dots, \varphi_{j+m+k-1}(x_k))$. If all the functions φ_i are nondecreasing, then $\tilde{f}$ and $\tilde{g}$ have the same monotonicity direction as f and g . If all the functions φ_i are nonincreasing, their monotonicity directions are reversed simultaneously, and hence $\tilde{f}$ and $\tilde{g}$ still have the same monotonicity direction.

Applying Definition 2.5 to $\tilde{f}$ and $\tilde{g}$, we obtain

$$\begin{aligned} & \mathbb{E} \left[f(\varphi_j(X_j)) g(\varphi_{j+m}(X_{j+m}), \dots, \varphi_{j+m+k-1}(X_{j+m+k-1})) \right] \\ & - \mathbb{E} \left[f(\varphi_j(X_j)) \right] \mathbb{E} \left[g(\varphi_{j+m}(X_{j+m}), \dots, \varphi_{j+m+k-1}(X_{j+m+k-1})) \right] \\ & \leq \eta(j) \left\{ \mathbb{E} \left[\left(f(\varphi_j(X_j)) - \mathbb{E} f(\varphi_j(X_j)) \right)^2 \right] \right\}^{1/2} \\ & \times \left\{ \mathbb{E} \left[\left(g(\varphi_{j+m}(X_{j+m}), \dots, \varphi_{j+m+k-1}(X_{j+m+k-1})) \right. \right. \right. \\ & \quad \left. \left. \left. - \mathbb{E} \left[g(\varphi_{j+m}(X_{j+m}), \dots, \varphi_{j+m+k-1}(X_{j+m+k-1})) \right] \right)^2 \right] \right\}^{1/2}, \end{aligned}$$

This is precisely the finite-row m -AANA dependence inequality for the transformed row. Therefore, $\{\varphi_j(X_j): 1 \leq j \leq \ell\}$ satisfies the finite-row m -AANA condition with the same mixing coefficients.

This completes the proof.

In particular, whenever the admissibility conditions stated above are satisfied, the conclusion applies to positive rescaling, deterministic translation, simultaneous sign reversal, and the Lipschitz truncation $\pi_t(x) = (-t) \vee (x \wedge t)$.

Lemma 2.2. Let X, Y be two random variables defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then we have

(1) Hölder inequality (see [25]). For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} (\mathbb{E}[|Y|^q])^{\frac{1}{q}}.$$

(2) Markov-Chebyshev inequality. Let $X \geq 0$, $p > 0$, and suppose that $X^p \in \mathcal{H}$ and $\mathbb{E}[X^p] < \infty$. Then, for every $x > 0$, $\mathbb{V}(X \geq x) \leq \frac{\mathbb{E}[X^p]}{x^p}$.

Indeed, the pointwise inequality $I_{\{X \geq x\}} \leq \frac{X^p}{x^p}$ and the monotonicity and positive homogeneity of $\mathbb{E}$ imply that

$$\mathbb{V}(X \geq x) = \mathbb{E}[I_{\{X \geq x\}}] \leq \frac{\mathbb{E}[X^p]}{x^p}.$$

Lemma 2.3. (finite-row second-moment estimate) Let $m \geq 1$ and $\ell \geq 1$ be integers. Suppose that the finite family $\{W_j; 1 \leq j \leq \ell\} \subset \mathcal{H}$ satisfies the finite-row m -AANA dependence condition in Definition 2.5, with mixing coefficients $\{\eta(j); 1 \leq j \leq (\ell - m)_+\}$. Assume that $\mathbb{E}[W_j] = 0$, $\mathbb{E}[W_j^2] < \infty$, $1 \leq j \leq \ell$. Put $D_\ell := \sum_{j=1}^{(\ell-m)_+} \eta(j)^2$. Then

$$\mathbb{E}\left[\left|\sum_{j=1}^{\ell} W_j\right|^2\right] \leq 2me^{4D_\ell} \sum_{j=1}^{\ell} \mathbb{E}[W_j^2].$$

Proof. Partition the index set $\{1, \dots, \ell\}$ into residue classes modulo m . For $1 \leq r \leq m$, let $I_r := \{r, r+m, r+2m, \dots\} \cap \{1, \dots, \ell\}$. Empty classes may be ignored. Fix a nonempty class I_r and write $I_r = \{i_1 < i_2 < \dots < i_q\}$. Thus, $i_{t+1} = i_t + m$, $1 \leq t \leq q-1$. For $1 \leq t \leq q$, define $S_t := \sum_{s=t}^q W_{i_s}$, $a_t^2 := \mathbb{E}[W_{i_t}^2]$, $V_t := \mathbb{E}[S_t^2]$. All the quantities, V_t are finite. Indeed, the elementary inequality

$$\left|\sum_{s=t}^q x_s\right|^2 \leq (q-t+1) \sum_{s=t}^q |x_s|^2,$$

together with the monotonicity, subadditivity, and positive homogeneity of $\mathbb{E}$ gives $V_t < \infty$.

Fix $1 \leq t \leq q-1$. Set $k_t := i_q - i_t - m + 1$ and define $g_t := \mathbb{R}^{k_t} \rightarrow \mathbb{R}$ by $g_t(x_1, \dots, x_{k_t}) := \sum_{h=0}^{q-t-1} x_{1+hm}$. Then g_t is continuous and coordinatewise nondecreasing, and $S_{t+1} = g_t(W_{i_t+m}, W_{i_t+m+1}, \dots, W_{i_q})$. Moreover, $i_t + m + k_t - 1 = i_q \leq \ell$. Consequently, the pair consisting of W_{i_t} and the above consecutive future block is admissible in Definition 2.5. Applying the finite-row m -AANA inequality with $f(x) = x$ and $g = g_t$, and using $\mathbb{E}[W_{i_t}] = 0$, we obtain

$$\mathbb{E}[W_{i_t} S_{t+1}] \leq \eta(i_t) a_t \{\mathbb{E}[(S_{t+1} - \mathbb{E}[S_{t+1}])^2]\}^{\frac{1}{2}}.$$

We next estimate the centered second moment of S_{t+1} . By subadditivity, $0 = \mathbb{E}[0] \leq \mathbb{E}[S_{t+1}] + \mathbb{E}[-S_{t+1}]$, and hence $-\mathbb{E}[S_{t+1}] \leq \mathbb{E}[-S_{t+1}] \leq \mathbb{E}[|S_{t+1}|]$. Since also $\mathbb{E}[S_{t+1}] \leq \mathbb{E}[|S_{t+1}|]$, it follows that $|\mathbb{E}[S_{t+1}]| \leq \mathbb{E}[|S_{t+1}|]$. By the Hölder inequality under sublinear expectations, $\mathbb{E}[|S_{t+1}|] \leq$

$(\mathbb{E}[S_{t+1}^2])^{\frac{1}{2}} = V_{t+1}$. Using the elementary inequality $(x - c)^2 \leq 2x^2 + 2c^2$, we therefore have

$$\mathbb{E}[(S_{t+1} - \mathbb{E}[S_{t+1}])^2] \leq 2\mathbb{E}[S_{t+1}^2] + 2|\mathbb{E}[S_{t+1}]|^2 \leq 4V_{t+1}.$$

It follows that

$$\mathbb{E}[W_{i_t} S_{t+1}] \leq 2\eta(i_t) a_t V_{t+1}^{\frac{1}{2}}.$$

Since $S_t = W_{i_t} + S_{t+1}$, the subadditivity and positive homogeneity of $\mathbb{E}$ yield

$$V_t = \mathbb{E}[(W_{i_t} + S_{t+1})^2] \leq a_t^2 + V_{t+1} + 2\mathbb{E}[W_{i_t} S_{t+1}] \leq a_t^2 + V_{t+1} + 4\eta(i_t) a_t V_{t+1}^{\frac{1}{2}}.$$

By $2uv \leq u^2 + v^2$, with $u = a_t$, $v = 2\eta(i_t) V_{t+1}^{\frac{1}{2}}$, we obtain $4\eta(i_t) a_t V_{t+1}^{\frac{1}{2}} \leq a_t^2 + 4\eta(i_t)^2 V_{t+1}$. Consequently, $V_t \leq 2a_t^2 + (1 + 4\eta(i_t)^2) V_{t+1}$, $1 \leq t \leq q - 1$.

Since $V_q = a_q^2$, iterating this inequality backward gives

$$V_1 \leq 2 \sum_{t=1}^{q-1} a_t^2 \prod_{s=1}^{t-1} (1 + 4\eta(i_s)^2) + a_q^2 \prod_{s=1}^{q-1} (1 + 4\eta(i_s)^2),$$

where an empty product is understood to be 1. Hence, $V_1 \leq 2 \sum_{t=1}^q a_t^2 \prod_{s=1}^{t-1} (1 + 4\eta(i_s)^2)$. Using $1 + x \leq e^x$ for $x \geq 0$, we have

$$\prod_{s=1}^{t-1} (1 + 4\eta(i_s)^2) \leq \exp\{4 \sum_{s=1}^{t-1} \eta(i_s)^2\} \leq e^{4D_\ell}.$$

The last inequality is valid because $i_s \leq \ell - m$ whenever $s \leq q - 1$. Therefore,

$$\mathbb{E}[|\sum_{j \in I_r} W_j|^2] \leq 2e^{4D_\ell} \sum_{j \in I_r} \mathbb{E}[W_j^2].$$

Finally, let $T_r := \sum_{j \in I_r} W_j$, with $T_r = 0$ when I_r is empty. Since $\sum_{j=1}^\ell W_j = \sum_{r=1}^m T_r$, the Cauchy–Schwarz inequality gives, pointwise, $|\sum_{r=1}^m T_r|^2 \leq m \sum_{r=1}^m |T_r|^2$. By monotonicity, subadditivity, and positive homogeneity,

$$\mathbb{E}[|\sum_{j=1}^\ell W_j|^2] \leq m \sum_{r=1}^m \mathbb{E}[|T_r|^2] \leq 2me^{4D_\ell} \sum_{r=1}^m \sum_{j \in I_r} \mathbb{E}[W_j^2] \leq 2me^{4D_\ell} \sum_{j=1}^\ell \mathbb{E}[W_j^2].$$

This completes the proof.

Corollary 2.1. Let $\{W_{ni} : u_n \leq i \leq v_n, n \geq 1\}$ be an array of rowwise m -AANA random variables satisfying Definition 2.5, where $m \geq 1$ is fixed. For each $n \geq 1$, let $\ell_n := v_n - u_n + 1$ and $\{\eta_n(j) : 1 \leq j \leq (\ell_n - m)_+\}$ be the mixing coefficients of the n -th row. Suppose that $\mathbb{E}[W_{ni}] = 0$, $\mathbb{E}[W_{ni}^2] < \infty$, $u_n \leq i \leq v_n$, $n \geq 1$, and that

$$D := \sup_{n \geq 1} \sum_{j=1}^{(\ell_n - m)_+} \eta_n(j)^2 < \infty.$$

Then for every $n \geq 1$,

$$\mathbb{E}\left[\left|\sum_{i=u_n}^{v_n} W_{ni}\right|^2\right] \leq 2me^{4D} \sum_{i=u_n}^{v_n} \mathbb{E}[W_{ni}^2].$$

Proof. Fix $n \geq 1$, and relabel the n -th row by $W_j := W_{n, u_n + j - 1}$, $1 \leq j \leq \ell_n$, and hence

$$D_\ell = \sum_{j=1}^{(\ell_n - m)_+} \eta_n(j)^2 \leq D.$$

Applying Lemma 2.3 to this fixed row gives

$$\mathbb{E} \left[\left| \sum_{i=u_n}^{v_n} W_{ni} \right|^2 \right] = \mathbb{E} \left[\left| \sum_{j=1}^{\ell_n} W_j \right|^2 \right] \leq 2me^{4D\ell} \sum_{j=1}^{\ell_n} \mathbb{E}[W_j^2] \leq 2me^{4D} \sum_{i=u_n}^{v_n} \mathbb{E}[W_{ni}^2].$$

This completes the proof.

Lemma 2.4. Let $Y \geq 0$ and $c > 0$. Then

$$C_{\mathbb{V}}(YI(Y > c)) = c\mathbb{V}(Y > c) + \int_c^\infty \mathbb{V}(Y > t)dt.$$

In particular, $\int_c^\infty \mathbb{V}(Y > t)dt \leq C_{\mathbb{V}}(YI(Y > c))$.

Proof. Since

$$\{YI(Y > c) > t\} = \begin{cases} \{Y > c\}, & 0 \leq t < c, \\ \{Y > t\}, & t \geq c, \end{cases}$$

the asserted identity follows directly from the definition of the Choquet integral. The second inequality is immediate.

Lemma 2.5. Let $X \in \mathcal{H}$ and $\mathbb{E}[|X|^2] < \infty$. Then

$$\mathbb{E}[|X - \mathbb{E}[X]|^2] \leq 4\mathbb{E}[|X|^2].$$

Similarly, we further have

$$\mathbb{E}[|X - \mathcal{E}[X]|^2] \leq 4\mathbb{E}[|X|^2].$$

Proof. Since $X \leq |X|$, by monotonicity, we have $\mathbb{E}[X] \leq \mathbb{E}[|X|]$.

On the other hand, by subadditivity, $0 = \mathbb{E}[X + (-X)] \leq \mathbb{E}[X] + \mathbb{E}[-X]$. Therefore, $-\mathbb{E}[X] \leq \mathbb{E}[-X]$.

Since $-X \leq |X|$, we have $\mathbb{E}[-X] \leq \mathbb{E}[|X|]$. Hence, $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$. For any $\alpha > 0$, we have $|X| \leq \frac{|X|^2}{2\alpha} + \frac{\alpha}{2}$. By monotonicity, positive homogeneity and translation invariance, we obtain

$$\mathbb{E}[|X|] \leq \frac{1}{2\alpha} \mathbb{E}[|X|^2] + \frac{\alpha}{2}.$$

If $\mathbb{E}[X^2] = 0$, the conclusion is immediate. Otherwise, taking $\alpha = (\mathbb{E}[|X|^2])^{\frac{1}{2}}$, we obtain $\mathbb{E}[|X|] \leq (\mathbb{E}[|X|^2])^{\frac{1}{2}}$. Therefore, $(\mathbb{E}[|X|])^2 \leq \mathbb{E}[|X|^2]$.

By the elementary inequality $|x - c|^2 \leq 2|x|^2 + 2|c|^2$, taking $c = \mathbb{E}[X]$, we have $|X - \mathbb{E}[X]|^2 \leq 2|X|^2 + 2|\mathbb{E}[X]|^2$.

Taking the upper expectation $\mathbb{E}$ on both sides gives

$$\mathbb{E}[|X - \mathbb{E}[X]|^2] \leq 2\mathbb{E}[|X|^2] + 2|\mathbb{E}[X]|^2.$$

Since $|\mathbb{E}[X]|^2 \leq (\mathbb{E}[|X|])^2 \leq \mathbb{E}[|X|^2]$, we obtain $\mathbb{E}[|X - \mathbb{E}[X]|^2] \leq 4\mathbb{E}[|X|^2]$.

Similarly, using $\mathcal{E}[X] = -\mathbb{E}[-X]$ and $|\mathcal{E}[X]| \leq \mathbb{E}[|X|]$, we obtain $\mathbb{E}[|X - \mathcal{E}[X]|^2] \leq 4\mathbb{E}[|X|^2]$.

Remark 2.4. In the subsequent proofs, Lemma 2.5 is used only for truncated variables, such as $\tilde{X}_{ni}$ and $U_{ni}(u)$. Since these variables are bounded, their second-order expectations are finite, and the assumptions of Lemma 2.5 are satisfied. This paper does not require the original random variables X_{ni} to have finite second moments.

Lemma 2.6. Let $1 \leq r < 2$, and let G be the truncation function fixed above. Let $Y \in \mathcal{H}$ be a nonnegative random variable and let $h, k > 0$. Define $\tilde{Y} := Y \left[1 - G\left(\frac{Y}{h}\right) \right]$. If $\mathbb{E}[Y] < \infty$ and

$C_{\mathbb{V}}(\tilde{Y}) < \infty$, then

$$\int_k^\infty u^{\frac{-2}{r}} \mathbb{E} \left[Y^{\frac{2}{r}} G \left(\frac{Y}{u} \right) \right] du \leq C_r \left(\frac{h}{k} \right)^{\frac{(2-r)}{r}} \mathbb{E}[Y] + C_r C_{\mathbb{V}}(\tilde{Y}),$$

where $C_r > 0$ depends only on r .

Proof. Put $p = \frac{2}{r} > 1$ and $W = (Y - h)^+$. By the elementary inequality

$$y^p \leq 2^{p-1} [h^{p-1}y + (y - h)_+^p], y \geq 0,$$

and the fact that $G \left(\frac{Y}{u} \right) = 0$ whenever $y \geq u$, we have

$$y^p G \left(\frac{Y}{u} \right) \leq 2^{p-1} h^{p-1} y + 2^{p-1} ((y - h)^+ \wedge u)^p.$$

Therefore,

$$\mathbb{E} \left[Y^p G \left(\frac{Y}{u} \right) \right] \leq C_p h^{p-1} \mathbb{E}[Y] + C_p \mathbb{E}[(W \wedge u)^p]. \quad (2.1)$$

Since $(W \wedge u)^p$ is bounded and nonnegative, Remark 2.2 yields

$$\mathbb{E}[(W \wedge u)^p] \leq C_{\mathbb{V}}((W \wedge u)^p) = p \int_0^u s^{p-1} \mathbb{V}(W > s) ds. \quad (2.2)$$

Combining (2.1) and (2.2), multiplying the resulting inequality by u^{-p} , and integrating over k, ∞), we obtain

$$\int_k^\infty u^{-p} \mathbb{E} \left[Y^p G \left(\frac{Y}{u} \right) \right] du \leq C_p \left(\frac{h}{k} \right)^{p-1} \mathbb{E}[Y] + C_p \int_k^\infty u^{-p} \int_0^u s^{p-1} \mathbb{V}(W > s) ds du. \quad (2.3)$$

Since the function $s \mapsto \mathbb{V}(W > s)$ is nonincreasing and hence Borel measurable, the integrand $(u, s) \mapsto u^{-p} s^{p-1} \mathbb{V}(W > s) I_{\{0 < s < u\}}$ is nonnegative and measurable. Therefore, Tonelli's theorem enables us to change the order of integration. Thus,

$$\begin{aligned} \int_k^\infty u^{-p} \int_0^u s^{p-1} \mathbb{V}(W > s) ds du &= \int_0^\infty s^{p-1} \mathbb{V}(W > s) \int_{\max\{k, s\}}^\infty u^{-p} du ds \\ &= \frac{1}{p-1} \left[k^{1-p} \int_0^k s^{p-1} \mathbb{V}(W > s) ds + \int_k^\infty \mathbb{V}(W > s) ds \right]. \end{aligned} \quad (2.4)$$

Since $k^{1-p} s^{p-1} = \left(\frac{s}{k} \right)^{p-1} \leq 1$, $0 \leq s \leq k$, it follows that

$$\int_k^\infty u^{-p} \int_0^u s^{p-1} \mathbb{V}(W > s) ds du \leq \frac{1}{p-1} \int_0^\infty \mathbb{V}(W > s) ds = \frac{1}{p-1} C_{\mathbb{V}}(W). \quad (2.5)$$

Substituting (2.5) into (2.3), we obtain

$$\int_k^\infty u^{-p} \mathbb{E} \left[Y^p G \left(\frac{Y}{u} \right) \right] du \leq C_p \left(\frac{h}{k} \right)^{p-1} \mathbb{E}[Y] + C_p C_{\mathbb{V}}(W).$$

Finally,

$$W = (Y - h)^+ \leq Y \left[1 - G \left(\frac{Y}{h} \right) \right] = \tilde{Y}.$$

Indeed, the inequality is immediate when $Y \leq h$, while, when $Y > h$, we have $G \left(\frac{Y}{h} \right) = 0$, and hence $\tilde{Y} = Y \geq Y - h = W$. By the monotonicity of the Choquet integral,

$$C_{\mathbb{V}}(W) \leq C_{\mathbb{V}}(\tilde{Y}).$$

Therefore,

$$\int_k^\infty u^{-p} \mathbb{E} \left[Y^p G \left(\frac{Y}{u} \right) \right] du \leq C_p \left(\frac{h}{k} \right)^{p-1} \mathbb{E}[Y] + C_p C_{\mathbb{V}}(\tilde{Y}).$$

Recalling that $p = \frac{2}{r}$ and $p - 1 = \frac{2-r}{r}$, the constant C_p depends only on r . Hence, by setting $C_r := \frac{C_2}{r}$, we obtain

$$\int_k^\infty u^{-\frac{2}{r}} \mathbb{E} \left[Y^{\frac{2}{r}} G \left(\frac{Y}{u} \right) \right] du \leq C_r \left(\frac{h}{k} \right)^{\frac{(2-r)}{r}} \mathbb{E}[Y] + C_r C_{\mathbb{V}}(\tilde{Y}).$$

3. Main results

Theorem 3.1. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of rowwise m -AANA random variables satisfying the finite-row formulation in Definition 2.5 on the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, where $m \geq 1$ is fixed. Assume that (A1) holds. For each $n \geq 1$, put $\ell_n := v_n - u_n + 1$, and let $\{\eta_n(j): 1 \leq j \leq (\ell_n - m)_+\}$ be the mixing coefficients of the n -th row. Assume that $D := \sup_{n \geq 1} \sum_{j=1}^{(\ell_n - m)_+} \eta_n(j)^2 < \infty$. Let $\{a_{ni}: u_n \leq i \leq v_n, n \geq 1\}$ be an array of nonnegative deterministic weights such that $a_n^* := \max_{u_n \leq i \leq v_n} a_{ni} > 0$, $n \geq 1$. Let $\{h_n, n \geq 1\}$ be a sequence of increasing positive constants satisfying $h_n \rightarrow \infty (n \rightarrow \infty)$. Assume that $1 \leq r < 2$. Define

$$k_n := (a_n^*)^{-r} = \left(\max_{u_n \leq i \leq v_n} a_{ni} \right)^{-r}, \quad n \geq 1,$$

$$Y_{ni} := k_n a_{ni}^r |X_{ni}|^r = \left(\frac{a_{ni}}{a_n^*} \right)^r |X_{ni}|^r, \quad u_n \leq i \leq v_n, \quad n \geq 1.$$

Furthermore, define

$$T_n := \sum_{i=u_n}^{v_n} a_{ni} (X_{ni} - \mathbb{E}[X_{ni}]), \quad \tilde{T}_n := \sum_{i=u_n}^{v_n} a_{ni} (X_{ni} - \mathcal{E}[X_{ni}]).$$

If the following conditions hold:

$$\sup_{n \geq 1} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r] < \infty, \quad (3.1)$$

$$\sum_{n=1}^\infty \frac{k_n}{h_n} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E} \left[|X_{ni}|^r \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right) \right] < \infty, \quad (3.2)$$

$$\sum_{n=1}^\infty \left(\frac{h_n}{k_n} \right)^{\frac{2-r}{r}} < \infty. \quad (3.3)$$

Then, for any $\varepsilon > 0$,

$$\sum_{n=1}^\infty \mathbb{V}(T_n > \varepsilon) < \infty, \quad (3.4)$$

and

$$\sum_{n=1}^\infty \mathbb{V}(\tilde{T}_n < -\varepsilon) < \infty. \quad (3.5)$$

In particular, if $\mathbb{E}[X_{ni}] = \mathcal{E}[X_{ni}]$, $u_n \leq i \leq v_n$, $n \geq 1$, then $T_n = \tilde{T}_n$. Hence, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V}(|T_n| > \varepsilon) < \infty. \quad (3.6)$$

Remark 3.1. The condition $\mathbb{E}[X_{ni}] = \mathcal{E}[X_{ni}]$ means that the mean-uncertainty interval $[\mathcal{E}[X_{ni}], \mathbb{E}[X_{ni}]]$ collapses to a single value. This condition does not require the sublinear expectation to be linear; its role here is only to ensure that $T_n = \tilde{T}_n$, so that the upper-tail and lower-tail complete-convergence conclusions in Theorem 3.1 can be combined into the corresponding two-sided conclusion. In particular, when the sublinear expectation $\mathbb{E}$ reduces to a classical linear expectation E , and the capacities $\mathbb{V}$ and $\mathbb{V}$ reduce to the same probability measure P , we have

$$T_n = \tilde{T}_n = \sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - E[X_{ni}]),$$

and hence, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|T_n| > \varepsilon) < \infty.$$

In the classical AANA case $m=1$, consider identically distributed, mean-zero variables whose finite rows are restrictions of infinite AANA rows with a common square-summable mixing bound. For $u_n = 1$, $v_n = n$, $a_{ni} = n^{-B}$, and $\frac{4}{3} \leq r < 2$, conditions (3.1)–(3.3) imply the hypotheses of [23, Corollary 3.1] with weights $b_{ni} = 1$ and normalization n^B . The conclusion obtained here agrees with its terminal-sum consequence.

Remark 3.2. By the definition of k_n , $0 \leq k_n^{\frac{1}{r}} a_{ni} = \frac{a_{ni}}{a_n^*} \leq 1$ and $\max_{u_n \leq i \leq v_n} k_n^{\frac{1}{r}} a_{ni} = 1$. Thus, k_n normalizes the largest weight in the n -th row. Moreover, condition (3.3) implies $\frac{h_n}{k_n} \rightarrow 0$. Since $h_n \rightarrow \infty$, it follows that $k_n \rightarrow \infty$, or equivalently, $a_n^* \rightarrow 0$. No assumption that the row length ℓ_n tends to infinity is imposed; rows of uniformly bounded length are also permitted, provided that the other conditions of Theorem 3.1 are satisfied.

In terms of the largest weight, condition (3.3) is equivalent to $\sum_{n=1}^{\infty} h_n^{\frac{(2-r)}{r}} (a_n^*)^{2-r} < \infty$. This restriction must be considered together with the moment and tail conditions (3.1) and (3.2).

Condition (3.3) does not cover the usual normalized weights when $u_n = 1$ and $v_n = n$. Indeed, if $a_{ni} \geq 0$ and $\sum_{i=1}^n a_{ni} = 1$, then $\max_{1 \leq i \leq n} a_{ni} \geq \frac{1}{n}$ and $k_n \leq n^r$. Consequently, for all sufficiently large n ,

$$\left(\frac{h_n}{k_n}\right)^{\frac{(2-r)}{r}} \geq \frac{1}{n^{2-r}},$$

so that (3.3) cannot hold because $0 < 2 - r \leq 1$. Thus, the ordinary weights $a_{ni} = \frac{1}{n}$ and the normalized power weights $a_{ni} = \frac{i^{-\delta}}{\sum_{j=1}^n j^{-\delta}}$ are outside the scope of Theorem 3.1.

Under the notation of Theorem 3.1, further define $\tilde{Y}_{ni} := Y_{ni} \left[1 - G\left(\frac{Y_{ni}}{h_n}\right)\right]$.

Theorem 3.2. Under the assumptions of Theorem 3.1, including (A1), suppose further that

$$\mathbb{E}[\tilde{Y}_{ni}] \leq C_{\mathbb{V}}(\tilde{Y}_{ni}), \quad u_n \leq i \leq v_n, \quad n \geq 1. \quad (3.7)$$

$$\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}) < \infty. \quad (3.8)$$

Then, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} C_{\mathbb{V}}((T_n - \varepsilon)_+^r) < \infty, \quad (3.9)$$

and

$$\sum_{n=1}^{\infty} C_{\mathbb{V}}\left((- \tilde{T}_n - \varepsilon)_+^r\right) < \infty. \quad (3.10)$$

If, in addition, $\mathbb{E}[X_{ni}] = \mathcal{E}[X_{ni}]$, $u_n \leq i \leq v_n$, $n \geq 1$, then $T_n = \tilde{T}_n$, and, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} C_{\mathbb{V}}((|T_n| - \varepsilon)_+^r) < \infty. \quad (3.11)$$

Remark 3.3. Since $\tilde{Y}_{ni}$ need not be bounded, condition (3.7) is imposed as an independent comparability assumption and does not follow directly from Remark 2.2. Moreover, for any $\delta > 0$, since $G(t) = 1$ for

$$0 \leq t \leq \lambda, \quad 0 \leq \tilde{Y}_{ni} \leq Y_{ni} I(Y_{ni} > \lambda h_n) \leq (\lambda h_n)^{\frac{-\delta}{r}} Y_{ni}^{\frac{1+\delta}{r}}.$$

By monotonicity and positive homogeneity of the Choquet integral,

$$\frac{1}{k_n} C_{\mathbb{V}}(\tilde{Y}_{ni}) \leq \lambda^{\frac{-\delta}{r}} \left(\frac{k_n}{h_n}\right)^{\frac{\delta}{r}} a_{ni}^{r+\delta} C_{\mathbb{V}}(|X_{ni}|^{r+\delta}).$$

Therefore, condition (3.8) follows if

$$\sum_{n=1}^{\infty} \left(\frac{k_n}{h_n}\right)^{\frac{\delta}{r}} \sum_{i=u_n}^{v_n} a_{ni}^{r+\delta} C_{\mathbb{V}}(|X_{ni}|^{r+\delta}) < \infty.$$

In particular, this condition holds if

$$\sup_{n,i} C_{\mathbb{V}}(|X_{ni}|^{r+\delta}) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{k_n}{h_n}\right)^{\frac{\delta}{r}} \sum_{i=u_n}^{v_n} a_{ni}^{r+\delta} < \infty.$$

These sufficient conditions do not replace the independent assumption (3.7).

When the sublinear expectation $\mathbb{E}$ reduces to a classical linear expectation E , and the capacities $\mathbb{V}$ reduce to the same probability measure P , the Choquet integral $C_{\mathbb{V}}$ reduces to the usual Lebesgue expectation. Hence, Theorem 3.2 yields

$$\sum_{n=1}^{\infty} E((|T_n| - \varepsilon)_+^r) < \infty, \quad \varepsilon > 0.$$

Corollary 3.1. Let $1 \leq r < 2$ and $\delta > 0$. Assume the dependence and space hypotheses of Theorem 3.1, with $u_n = 1$, $v_n = n$, and $\sup_{n,i} \mathbb{E}(|X_{ni}|^{r+\delta}) < \infty$. Let $a_{ni} = n^{-B}$ and $h_n = n^H$,

where

$$B > \frac{1}{2-r} + \frac{2}{r+\delta}, \quad \frac{2r}{r+\delta} < H < Br - \frac{r}{2-r}.$$

Then conditions (3.1)–(3.3) hold, and the conclusions of Theorem 3.1 follow. If condition (3.7) also holds and $\sup_{n,i} C_{\mathbb{V}}(|X_{ni}|^{r+\delta}) < \infty$, then condition (3.8) holds, and the conclusions of Theorem 3.2 follow.

Proof. For these weights, $k_n = n^{Br}$, $Y_{ni} = |X_{ni}|^r$. The row expression in (3.1) and the summands in (3.2) and (3.3) are bounded, respectively, by constant multiples of n^{1-Br} , $n^{1-H(1+\frac{\delta}{r})}$, and $n^{\frac{H(2-r)}{r-B(2-r)}}$. The stated parameter range ensures boundedness of the first expression and summability of the latter

two. Under the additional Choquet moment bound,

$$\frac{1}{k_n} \sum_{i=1}^n C_{\mathbb{V}}(\tilde{Y}_{ni}) = O\left(n^{\frac{1-Br-H\delta}{r}}\right),$$

which is summable because $\frac{Br+H\delta}{r} > 2$.

At $H = \frac{Br-r}{(2-r)}$, the series in (3.3) is harmonic and diverges. At $H = \frac{2r}{(r+\delta)}$, the displayed estimate for (3.2) is only of harmonic order and does not guarantee convergence.

Example 3.1. (i) A bounded case. Under the expectation-space and dependence assumptions of Theorem 3.1, let $u_n = 1$, $v_n = n$, and $a_{ni} = n^{-B}c_{ni}$, $0 \leq c_{ni} \leq 1$, $\max_{1 \leq i \leq n} c_{ni} = 1$. Suppose that $|X_{ni}| \leq M$ for a constant $M > 0$. For $1 \leq r < 2$, choose $h_n = n^H$, $B > \frac{1}{2-r}$, $0 < H < Br - \frac{r}{2-r}$. Then conditions (3.1)–(3.3) hold. The choices $c_{ni} = 1$ and $c_{ni} = \frac{i}{n}$ give equal and nonuniform weights, respectively.

Moreover, $k_n = n^{Br}$, and

$$0 \leq \tilde{Y}_{ni} = Y_{ni} \left[1 - G\left(\frac{Y_{ni}}{h_n}\right) \right] \leq Y_{ni} = c_{ni}^r |X_{ni}|^r \leq M^r.$$

By Remark 2.2, $\mathbb{E}[\tilde{Y}_{ni}] \leq C_{\mathbb{V}}(\tilde{Y}_{ni}) \leq M^r$, so condition (3.7) holds. Since $h_n \rightarrow \infty$ and $G(t) = 1$ for $0 \leq t \leq \lambda$, we have $\tilde{Y}_{ni} = 0$ for all sufficiently large n , uniformly in $1 \leq i \leq n$. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=1}^n C_{\mathbb{V}}(\tilde{Y}_{ni}) < \infty,$$

because only finitely many nonzero, finite terms remain. Thus, condition (3.8) also holds.

(ii) An unbounded case. Let $\mathbb{E} = E_P$, $\mathcal{H} = \bigcap_{p \geq 1} L^p(P)$ on a probability space $(\Omega, \mathcal{F}, P)$, and let X_{ni} , $1 \leq i \leq n$, be independent standard normal random variables within each row. Every $\mathcal{F}$ -measurable event indicator belongs to $\mathcal{H}$. Rowwise independence gives the finite-row m -AANA property with zero mixing coefficients, so $D=0$. Choose $r = \frac{3}{2}$, $\delta = \frac{3}{2}$, $a_{ni} = n^{-4}$, $h_n = n^2$. Then $k_n = n^6$, $Y_{ni} = |X_{ni}|^{\frac{3}{2}}$. For a standard normal random variable Z , put $K_{\lambda} = \lambda^{-1} E_P[|Z|^3] < \infty$. Since

$$0 \leq \tilde{Y}_{ni} \leq Y_{ni} I(Y_{ni} > \lambda n^2) \leq \frac{|X_{ni}|^3}{\lambda n^2},$$

we obtain

$$\begin{aligned} \sum_{i=1}^n a_{ni}^r E_P[|X_{ni}|^r] &= n^{-5} E_P[|Z|^{\frac{3}{2}}], \\ \frac{k_n}{h_n} \sum_{i=1}^n a_{ni}^r E_P \left[|X_{ni}|^r \left(1 - G\left(\frac{Y_{ni}}{h_n}\right) \right) \right] &= \frac{1}{h_n} \sum_{i=1}^n E_P[\tilde{Y}_{ni}] \leq K_{\lambda} n^{-3}, \end{aligned}$$

and

$$\left(\frac{h_n}{k_n}\right)^{\frac{(2-r)}{r}} = n^{\frac{-4}{3}}.$$

Thus, conditions (3.1)–(3.3) hold.

In this model, $\mathbb{V} = P$, and $\mathbb{E}[\tilde{Y}_{ni}] = C_{\mathbb{V}}(\tilde{Y}_{ni})$, which verifies (3.7). Moreover,

$$\frac{1}{k_n} \sum_{i=1}^n C_{\mathbb{V}}(\tilde{Y}_{ni}) = n^{-6} \sum_{i=1}^n E_P[\tilde{Y}_{ni}] \leq K_{\lambda} n^{-7}.$$

Consequently, condition (3.8) holds, and Theorem 3.2 applies.

Remark 3.4. The nonnegativity of the weights ensures that the mappings $x \mapsto a_{ni}x$ are nondecreasing and enables us to use the positive homogeneity of the sublinear expectation. If all weights in a row are nonpositive, one may set $b_{ni} = |a_{ni}|$ and apply the corresponding argument to $-X_{ni}$. Under mean uncertainty, however, this transformation interchanges the upper- and lower-expectation centers. Consequently, the common-sign case requires a corresponding reformulation of the centered sums and is not literally included in Theorems 3.1–3.2 as stated. Arbitrarily mixed signed weights within the same row are not covered by the argument, because the corresponding transformations do not have a common monotonicity direction, and the transformed array is not guaranteed to remain rowwise m -AANA. The fixed lag parameter m is used to obtain the uniform second-moment estimate in Lemma 2.3. Uniformly bounded row-dependent lag parameters m_n can be treated by taking $M = \sup_{n \geq 1} m_n < \infty$, whereas unbounded m_n would require additional uniform or summability conditions and is not considered in this paper.

4. Proofs

Throughout this section, $C > 0$ denotes a generic constant independent of n and, where applicable, of the integration variable u . Its value may vary from line to line and may depend on the fixed parameters appearing in the theorem under consideration.

Proof of Theorem 3.1.

Step 1. Truncation and event decomposition.

By the assumption $a_n^* > 0$, the quantity $k_n = (a_n^*)^{-r}$ is finite and strictly positive for every $n \geq 1$. Moreover, since $0 \leq a_{ni} \leq a_n^*$, we have $0 \leq k_n a_{ni}^r = \left(\frac{a_{ni}}{a_n^*}\right)^r \leq 1$, $u_n \leq i \leq v_n$. Consequently, $0 \leq Y_{ni} \leq |X_{ni}|^r$.

From condition (3.3), we know that the series $\sum_{n=1}^{\infty} \left(\frac{h_n}{k_n}\right)^{\frac{(2-r)}{r}}$ is convergent. Since all its terms are nonnegative, its general term tends to zero, that is, $\left(\frac{h_n}{k_n}\right)^{\frac{(2-r)}{r}} \rightarrow 0 (n \rightarrow \infty)$. Since $1 \leq r < 2$, we have $\frac{2-r}{r} > 0$. Therefore, $\frac{h_n}{k_n} \rightarrow 0 (n \rightarrow \infty)$.

Since $\frac{h_n}{k_n} \rightarrow 0$, there exists a positive integer N such that $h_n \leq k_n$ for all $n \geq N$.

For $c > 0$, define the truncation functions $\pi_c(x) := \max\{-c, \min\{x, c\}\}$, $\rho_c(x) := x - \pi_c(x)$, $c > 0$.

Thus, both π_c and ρ_c are continuous nondecreasing functions. Let

$$\bar{X}_{ni} := \pi_{\frac{1}{h_n^r}} \left(k_n^r a_{ni} X_{ni} \right), \quad (4.1)$$

$$\bar{Z}_{ni} := \rho_{\frac{1}{h_n^r}} \left(k_n^r a_{ni} X_{ni} \right) = k_n^r a_{ni} X_{ni} - \bar{X}_{ni}. \quad (4.2)$$

Here, X_{ni} is the original variable, $\bar{X}_{ni}$ is the truncation of $k_n^r a_{ni} X_{ni}$, and $\bar{Z}_{ni} = k_n^r a_{ni} X_{ni} - \bar{X}_{ni}$ is the corresponding remainder.

Notice that, if $a_{ni} = 0$, then $Y_{ni} = \bar{X}_{ni} = \bar{Z}_{ni} = 0$, and the corresponding terms in T_n and $\tilde{T}_n$

also vanish. For every $u_n \leq i \leq v_n$, define

$$\phi_{ni}(x) := \pi_{\frac{1}{h_n^r}} \left(k_n^{\frac{1}{r}} a_{ni} x \right).$$

Since $a_{ni} \geq 0$, the mapping ϕ_{ni} is continuous and nondecreasing; when $a_{ni} = 0$, it is simply the constant-zero mapping. Moreover,

$$|\bar{X}_{ni}| = \left| \pi_{\frac{1}{h_n^r}} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} \right) \right| \leq h_n^{\frac{1}{r}},$$

and hence $\mathbb{E}[|\bar{X}_{ni}|^2] \leq h_n^{\frac{2}{r}} < \infty$.

Therefore, Lemma 2.1 implies that the truncated finite row $\{\bar{X}_{ni}; u_n \leq i \leq v_n\}$ satisfies the finite-row m -AANA dependence condition with the same mixing coefficients. The remainder variables $\bar{Z}_{ni}$ are used below only through the pointwise tail bound in (4.4) and the resulting first-moment estimates; no m -AANA property of the remainder row is required.

By the definition of the truncation functions, for any $u_n \leq i \leq v_n$, we have

$$|\bar{X}_{ni}| \leq \min \left\{ Y_{ni}^{\frac{1}{r}}, h_n^{\frac{1}{r}} \right\}, \quad (4.3)$$

$$|\bar{Z}_{ni}| \leq k_n^{\frac{1}{r}} a_{ni} |X_{ni}| I(Y_{ni} > h_n) = Y_{ni}^{\frac{1}{r}} I(Y_{ni} > h_n). \quad (4.4)$$

Moreover, since

$$T_n = k_n^{\frac{-1}{r}} \sum_{i=u_n}^{v_n} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right),$$

if $T_n > \varepsilon$, then

$$\sum_{i=u_n}^{v_n} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) > k_n^{\frac{1}{r}} \varepsilon.$$

Since $k_n^{\frac{1}{r}} a_{ni} X_{ni} = \bar{X}_{ni} + \bar{Z}_{ni}$, we obtain

$$\begin{aligned} \sum_{i=u_n}^{v_n} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) &= \sum_{i=u_n}^{v_n} \left(\bar{X}_{ni} + \bar{Z}_{ni} - \mathbb{E}[\bar{X}_{ni}] + \mathbb{E}[\bar{X}_{ni}] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) \\ &= \sum_{i=u_n}^{v_n} \bar{Z}_{ni} + \sum_{i=u_n}^{v_n} (\bar{X}_{ni} - \mathbb{E}[\bar{X}_{ni}]) + \sum_{i=u_n}^{v_n} \left(\mathbb{E}[\bar{X}_{ni}] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right). \end{aligned}$$

If all three terms on the right-hand side are not greater than $\frac{1}{3} k_n^{\frac{1}{r}} \varepsilon$, then their sum is not greater than $\frac{1}{3} k_n^{\frac{1}{r}} \varepsilon$, which is a contradiction. Therefore, at least one of the three terms must be greater than $\frac{1}{3} k_n^{\frac{1}{r}} \varepsilon$. Write

$$A_{n1} := \left\{ \sum_{i=u_n}^{v_n} \bar{Z}_{ni} > \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon \right\},$$

$$A_{n2} := \left\{ \sum_{i=u_n}^{v_n} (\bar{X}_{ni} - \mathbb{E}[\bar{X}_{ni}]) > \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon \right\},$$

$$A_{n3} := \left\{ \sum_{i=u_n}^{v_n} \left(\mathbb{E}[\bar{X}_{ni}] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) > \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon \right\}.$$

Then $\{T_n > \varepsilon\} \subset A_{n1} \cup A_{n2} \cup A_{n3}$. Hence,

$$\mathbb{V}(T_n > \varepsilon) \leq \mathbb{V}(A_{n1}) + \mathbb{V}(A_{n2}) + \mathbb{V}(A_{n3}). \quad (4.5)$$

Step 2. Estimates of the decomposed events.

(i) Estimate of $\mathbb{V}(A_{n1})$.

If $Y_{ni} \leq h_n$ for all $u_n \leq i \leq v_n$, then, by the definition

$$Y_{ni} = \left(k_n^{\frac{1}{r}} a_{ni} |X_{ni}| \right)^r,$$

we obtain $k_n^{\frac{1}{r}} a_{ni} |X_{ni}| \leq h_n^{\frac{1}{r}}$. Therefore,

$$\pi_{\frac{1}{h_n^{\frac{1}{r}}}} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} \right) = k_n^{\frac{1}{r}} a_{ni} X_{ni},$$

From (4.2), we obtain

$$\bar{Z}_{ni} = k_n^{\frac{1}{r}} a_{ni} X_{ni} - \bar{X}_{ni} = k_n^{\frac{1}{r}} a_{ni} X_{ni} - \pi_{\frac{1}{h_n^{\frac{1}{r}}}} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} \right) = 0.$$

Thus, $\sum_{i=u_n}^{v_n} \bar{Z}_{ni} = 0$. Hence, the event A_{n1} does not occur. Therefore, $\cap_{i=u_n}^{v_n} \{Y_{ni} \leq h_n\} \subset A_{n1}^c$. Taking complements gives $A_{n1} \subset \cup_{i=u_n}^{v_n} \{Y_{ni} > h_n\}$. Thus, by the subadditivity of $\mathbb{V}$,

$$\mathbb{V}(A_{n1}) \leq \sum_{i=u_n}^{v_n} \mathbb{V}\{Y_{ni} > h_n\}. \quad (4.6)$$

For any $x \geq 0$, we have the pointwise inequality

$$I(x > h_n) \leq \frac{x}{h_n} \left(1 - G\left(\frac{x}{h_n}\right) \right).$$

Applying this inequality to the random variable Y_{ni} , we obtain

$$I(Y_{ni} > h_n) \leq \frac{Y_{ni}}{h_n} \left(1 - G\left(\frac{Y_{ni}}{h_n}\right) \right).$$

Taking upper expectations on both sides and using the monotonicity and positive homogeneity of $\mathbb{E}$, we obtain,

$$\mathbb{V}(Y_{ni} > h_n) \leq \mathbb{E} \left[\frac{Y_{ni}}{h_n} \left(1 - G\left(\frac{Y_{ni}}{h_n}\right) \right) \right].$$

Since $Y_{ni} = k_n a_{ni}^r |X_{ni}|^r$, we have

$$\mathbb{V}(Y_{ni} > h_n) \leq \frac{k_n}{h_n} \mathbb{E} \left[|X_{ni}|^r \left(1 - G\left(\frac{Y_{ni}}{h_n}\right) \right) \right]. \quad (4.7)$$

Substituting (4.7) into (4.6), we get

$$\mathbb{V}(A_{n1}) \leq \frac{k_n}{h_n} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E} \left[|X_{ni}|^r \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right) \right].$$

Then, by condition (3.2), we obtain

$$\sum_{n=1}^{\infty} \mathbb{V}(A_{n1}) < \infty. \quad (4.8)$$

(ii) Estimate of $\mathbb{V}(A_{n2})$.

Let $S_n = \sum_{i=u_n}^{v_n} (\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni})$. Since

$$\left\{ S > \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon \right\} \subset \left\{ |S| > \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon \right\},$$

we have

$$\mathbb{V}(A_{n2}) \leq \mathbb{V} \left(|S_n| > \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon \right).$$

Applying the Chebyshev inequality in Lemma 2.2 to the nonnegative random variable $|S_n|$ with $f(x) = x^2$, we obtain

$$\mathbb{V}(A_{n2}) \leq C k_n^{\frac{-2}{r}} \mathbb{E} \left[\left| \sum_{i=u_n}^{v_n} (\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni}) \right|^2 \right]. \quad (4.9)$$

For each fixed n and i , the mapping $x \mapsto x - \mathbb{E}\bar{X}_{ni}$ is continuous nondecreasing function. Hence, by Lemma 2.1, for each fixed n , the centered finite row $\{\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni}, u_n \leq i \leq v_n\}$ still satisfies the finite-row m -AANA dependence condition. Moreover, by the translation invariance of $\mathbb{E}$, $\mathbb{E}[\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni}] = 0$. Therefore, applying Corollary 2.1 to the centered finite row, we obtain

$$\mathbb{E} \left[\left| \sum_{i=u_n}^{v_n} (\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni}) \right|^2 \right] \leq 2me^{4D} \sum_{i=u_n}^{v_n} \mathbb{E}[|\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni}|^2]. \quad (4.10)$$

By Lemma 2.5, we further have

$$\mathbb{E}[|\bar{X}_{ni} - \mathbb{E}\bar{X}_{ni}|^2] \leq 4\mathbb{E}[|\bar{X}_{ni}|^2].$$

Therefore, $\mathbb{V}(A_{n2}) \leq C k_n^{\frac{-2}{r}} \sum_{i=u_n}^{v_n} \mathbb{E}[|\bar{X}_{ni}|^2]$. Furthermore, from (4.3), we know that

$$|\bar{X}_{ni}|^2 = \min \left\{ Y_{ni}^{\frac{2}{r}}, h_{ni}^{\frac{2}{r}} \right\} \leq h_n^{\frac{(2-r)}{r}} Y_{ni}.$$

Since $Y_{ni} = k_n a_{ni}^r |X_{ni}|^r$, we obtain

$$|\bar{X}_{ni}|^2 \leq k_n h_n^{\frac{(2-r)}{r}} a_{ni}^r |X_{ni}|^r.$$

Taking the upper expectation $\mathbb{E}$ on both sides gives

$$\mathbb{E}[|\bar{X}_{ni}|^2] \leq k_n h_n^{\frac{(2-r)}{r}} a_{ni}^r \mathbb{E}[|X_{ni}|^r]. \quad (4.11)$$

Substituting (4.10) and (4.11) into (4.9), we obtain

$$\mathbb{V}(A_{n2}) \leq C \left(\frac{h_n}{k_n} \right)^{\frac{2-r}{r}} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r].$$

By conditions (3.1) and (3.3), it follows that

$$\sum_{n=1}^{\infty} \mathbb{V}(A_{n2}) < \infty. \quad (4.12)$$

(iii) Estimate of $\mathbb{V}(A_{n3})$.

By the self-dominated property in Remark 2.1, we have

$$\left| \mathbb{E}[\bar{X}_{ni}] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq \mathbb{E}[|\bar{Z}_{ni}|].$$

Therefore,

$$\sum_{i=u_n}^{v_n} \left(\mathbb{E}[\bar{X}_{ni}] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) \leq \sum_{i=u_n}^{v_n} \mathbb{E}[|\bar{Z}_{ni}|].$$

From (4.4), we know that

$$|\bar{Z}_{ni}| \leq k_n^{\frac{1}{r}} a_{ni} |X_{ni}| I(Y_{ni} > h_n) = Y_{ni}^{\frac{1}{r}} I(Y_{ni} > h_n).$$

On the event $Y_{ni} > h_n$, we have

$$Y_{ni}^{\frac{1}{r}} \leq h_n^{\frac{-(r-1)}{r}} Y_{ni}.$$

Hence,

$$|\bar{Z}_{ni}| \leq Y_{ni}^{\frac{1}{r}} I(Y_{ni} > h_n) \leq h_n^{\frac{-(r-1)}{r}} Y_{ni} I(Y_{ni} > h_n) \leq h_n^{\frac{-(r-1)}{r}} Y_{ni} \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right).$$

Thus,

$$\sum_{i=u_n}^{v_n} \mathbb{E}[|\bar{Z}_{ni}|] \leq k_n h_n^{\frac{-(r-1)}{r}} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E} \left[|X_{ni}|^r \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right) \right].$$

Dividing both sides by $k_n^{\frac{1}{r}}$, we obtain

$$k_n^{\frac{-1}{r}} \sum_{i=u_n}^{v_n} \mathbb{E}[|\bar{Z}_{ni}|] \leq \left[\frac{k_n}{h_n} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E} \left[|X_{ni}|^r \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right) \right] \right] \left(\frac{h_n}{k_n} \right)^{\frac{1}{r}}.$$

By condition (3.2), the corresponding series is convergent, and hence its general term tends to zero. That is,

$$\frac{k_n}{h_n} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E} \left[|X_{ni}|^r \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right) \right] \rightarrow 0.$$

Moreover, by condition (3.3), we have $\frac{h_n}{k_n} \rightarrow 0$, and hence $\left(\frac{h_n}{k_n} \right)^{\frac{1}{r}} \rightarrow 0$. Therefore,

$$k_n^{\frac{-1}{r}} \sum_{i=u_n}^{v_n} \mathbb{E}[|\bar{Z}_{ni}|] \rightarrow 0.$$

Thus, there exists a positive integer N such that, for all $n \geq N$, we have

$$\sum_{i=u_n}^{v_n} \left| \mathbb{E} \bar{X}_{ni} - \mathbb{E} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} \right) \right| \leq \frac{1}{3} k_n^{\frac{1}{r}} \varepsilon.$$

Therefore, the event A_{n3} cannot occur. Hence,

$$\sum_{n=1}^{\infty} \mathbb{V}(A_{n3}) = \sum_{n=1}^{N-1} \mathbb{V}(A_{n3}) < \infty. \quad (4.13)$$

Summing both sides of (4.5) over $n \geq 1$, we obtain

$$\sum_{n=1}^{\infty} \mathbb{V}(T_n > \varepsilon) \leq \sum_{n=1}^{\infty} \mathbb{V}(A_{n1}) + \sum_{n=1}^{\infty} \mathbb{V}(A_{n2}) + \sum_{n=1}^{\infty} \mathbb{V}(A_{n3}).$$

Step 3. Completion of the one-sided complete convergence proof.

By (4.8), (4.12), and (4.13), the right-hand side is finite. Therefore, $\sum_{n=1}^{\infty} \mathbb{V}(T_n > \varepsilon) < \infty$. Thus, (3.4) is proved.

Step 4. The lower-tail estimate.

Since the function $x \mapsto -x$ is continuous and nonincreasing, Lemma 2.1 implies that the array $\{-X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is also rowwise m -AANA with the same mixing coefficient sequences. Moreover, $\mathbb{E}[-X_{ni}] = -\mathbb{E}[X_{ni}]$ and $|-X_{ni}| = |X_{ni}|$. Hence, conditions (3.1)–(3.3) remain valid for the array $\{-X_{ni}\}$. Applying the preceding upper-tail argument to this array, we obtain, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V} \left(\sum_{i=u_n}^{v_n} a_{ni} (-X_{ni} - \mathbb{E}[-X_{ni}]) > \varepsilon \right) < \infty.$$

Since

$$-X_{ni} - \mathbb{E}[-X_{ni}] = -X_{ni} + \mathbb{E}[X_{ni}] = -(X_{ni} - \mathbb{E}[X_{ni}]),$$

the above inequality is equivalent to

$$\sum_{n=1}^{\infty} \mathbb{V} \left(\sum_{i=u_n}^{v_n} a_{ni} (X_{ni} - \mathbb{E}[X_{ni}]) < -\varepsilon \right) < \infty.$$

Thus, (3.5) is proved.

Step 5. Two-sided complete convergence.

Furthermore, if $\mathbb{E}[X_{ni}] = \mathbb{E}[X_{ni}]$, then by (3.4) and (3.5), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{V}(|T_n| > \varepsilon) &\leq \sum_{n=1}^{\infty} \mathbb{V}(T_n > \varepsilon) + \sum_{n=1}^{\infty} \mathbb{V}(-T_n > \varepsilon) \\ &= \sum_{n=1}^{\infty} \mathbb{V}(T_n > \varepsilon) + \sum_{n=1}^{\infty} \mathbb{V}(T_n < -\varepsilon) < \infty. \end{aligned}$$

Therefore, (3.6) holds. The proof of Theorem 3.1 is completed.

Proof of Theorem 3.2. All integrals below are ordinary Lebesgue integrals with respect to the integration variable. Since the tail-capacity functions involved are nonnegative and nonincreasing in the integration variable, they are Borel measurable. Therefore, every interchange of summation and integration used below is justified by Tonelli's theorem.

Step 1. Decomposition of the upper-tail Choquet integral.

Under the notation of Theorem 3.1, let $T_n^* := k_n^{\frac{1}{r}} T_n$. Then $T_n = k_n^{\frac{-1}{r}} T_n^*$. By the definition of the Choquet integral,

$$C_{\mathbb{V}}((T_n - \varepsilon)_+^r) = \int_0^\infty \mathbb{V}\left(T_n > \varepsilon + t^{\frac{1}{r}}\right) dt.$$

Using the change of variable $u = k_n t$, so that, $dt = \frac{1}{k_n} du$, we obtain

$$C_{\mathbb{V}}((T_n - \varepsilon)_+^r) = \frac{1}{k_n} \int_0^\infty \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) du.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^\infty C_{\mathbb{V}}((T_n - \varepsilon)_+^r) &= \sum_{n=1}^\infty \frac{1}{k_n} \int_0^\infty \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) du \\ &= \sum_{n=1}^\infty \frac{1}{k_n} \int_0^{k_n} \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) du + \sum_{n=1}^\infty \frac{1}{k_n} \int_{k_n}^\infty \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) du \\ &= \sum_{n=1}^\infty P_n + \sum_{n=1}^\infty Q_n. \end{aligned}$$

where

$$P_n = \frac{1}{k_n} \int_0^{k_n} \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) du, \quad (4.14)$$

and

$$Q_n = \frac{1}{k_n} \int_{k_n}^\infty \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) du. \quad (4.15)$$

Thus, $C_{\mathbb{V}}((T_n - \varepsilon)_+^r) = P_n + Q_n$.

Step 2. Estimate on the small-parameter interval $[0, k_n]$.

For every $u \in [0, k_n]$, since $u^{\frac{1}{r}} \geq 0$, we have $k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}} \geq k_n^{\frac{1}{r}} \varepsilon$. Therefore, if $T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}$, then necessarily $T_n^* > k_n^{\frac{1}{r}} \varepsilon$. Hence, for each $u \in [0, k_n]$, we have the event inclusion

$$\left\{T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right\} \subset \left\{T_n^* > k_n^{\frac{1}{r}} \varepsilon\right\}.$$

By the monotonicity of the capacity $\mathbb{V}$, it follows that

$$\mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}\right) \leq \mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon\right). \quad (4.16)$$

Since $T_n^* = k_n^{\frac{1}{r}} T_n$ and $k_n^{\frac{1}{r}} > 0$, we have $T_n^* > k_n^{\frac{1}{r}} \varepsilon \Leftrightarrow k_n^{\frac{1}{r}} T_n > k_n^{\frac{1}{r}} \varepsilon \Leftrightarrow T_n > \varepsilon$. Therefore,

$$\mathbb{V}\left(T_n^* > k_n^{\frac{1}{r}} \varepsilon\right) = \mathbb{V}(T_n > \varepsilon). \quad (4.17)$$

Combining (4.14), (4.16) and (4.17), we obtain

$$\begin{aligned} P_n &= \frac{1}{k_n} \int_0^{k_n} \mathbb{V} \left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}} \right) du \leq \frac{1}{k_n} \int_0^{k_n} \mathbb{V}(T_n > \varepsilon) du \\ &= \frac{1}{k_n} \cdot k_n \mathbb{V}(T_n > \varepsilon) = \mathbb{V}(T_n > \varepsilon). \end{aligned} \quad (4.18)$$

By the upper-tail complete convergence conclusion (3.4) of Theorem 3.1, $\sum_{n=1}^{\infty} \mathbb{V}(T_n > \varepsilon) < \infty$. Consequently,

$$\sum_{n=1}^{\infty} P_n < \infty. \quad (4.19)$$

Step 3. Estimate on the large-parameter interval k_n, ∞ .

For fixed n and $u \geq k_n$, define the truncation functions by $\pi_c(x) = \max\{-c, \min(x, c)\}$ and $\rho_c(x) = x - \pi_c(x)$, $c > 0$. Thus, π_c and ρ_c are continuous nondecreasing functions. Define

$$U_{ni}(u) := \pi_{\frac{1}{u^{\frac{1}{r}}}} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} \right),$$

and

$$V_{ni}(u) := \rho_{\frac{1}{u^{\frac{1}{r}}}} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} \right) = k_n^{\frac{1}{r}} a_{ni} X_{ni} - U_{ni}(u).$$

By the definitions, we immediately obtain $k_n^{\frac{1}{r}} a_{ni} X_{ni} = U_{ni}(u) + V_{ni}(u)$, $u_n \leq i \leq v_n$. Moreover,

$$|U_{ni}(u)| \leq \min \left\{ Y_{ni}^{\frac{1}{r}}, u^{\frac{1}{r}} \right\}, \quad (4.20)$$

$$|V_{ni}(u)| \leq Y_{ni}^{\frac{1}{r}} I(Y_{ni} > u). \quad (4.21)$$

Therefore,

$$\begin{aligned} T_n^* &= k_n^{\frac{1}{r}} T_n = \sum_{i=u_n}^{v_n} \left(k_n^{\frac{1}{r}} a_{ni} X_{ni} - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) \\ &= \sum_{i=u_n}^{v_n} \left(U_{ni}(u) + V_{ni}(u) - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right) \\ &= \sum_{i=u_n}^{v_n} (U_{ni}(u) - \mathbb{E}[U_{ni}(u)]) + \sum_{i=u_n}^{v_n} V_{ni}(u) + \sum_{i=u_n}^{v_n} \left(\mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right). \end{aligned}$$

If $T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}}$, then, since $k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}} > 0$, $|T_n^*| = T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}} > u^{\frac{1}{r}}$. It follows from the preceding decomposition of T_n^* that at least one of the following three events occurs:

$$J_{n1}(u) := \left\{ \sum_{i=u_n}^{v_n} |V_{ni}(u)| > \frac{1}{3} u^{\frac{1}{r}} \right\},$$

$$J_{n2}(u) := \left\{ \left| \sum_{i=u_n}^{v_n} (U_{ni}(u) - \mathbb{E}[U_{ni}(u)]) \right| > \frac{1}{3} u^{\frac{1}{r}} \right\},$$

or

$$J_{n3}(u) := \left\{ \sum_{i=u_n}^{v_n} \left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| > \frac{1}{3} u^{\frac{1}{r}} \right\}.$$

Therefore,

$$\left\{ T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}} \right\} \subseteq J_{n1}(u) \cup J_{n2}(u) \cup J_{n3}(u).$$

By the subadditivity of the capacity $\mathbb{V}$, we have,

$$\mathbb{V} \left(T_n^* > k_n^{\frac{1}{r}} \varepsilon + u^{\frac{1}{r}} \right) \leq \mathbb{V}(J_{n1}(u)) + \mathbb{V}(J_{n2}(u)) + \mathbb{V}(J_{n3}(u)).$$

Define

$$Q_{nj} := \frac{1}{k_n} \int_{k_n}^{\infty} \mathbb{V}(J_{nj}(u)) du, \quad j = 1, 2, 3.$$

Then, $Q_n \leq Q_{n1} + Q_{n2} + Q_{n3}$.

(i) Estimate of Q_{n1} .

If $Y_{ni} \leq u$ for all $u_n \leq i \leq v_n$, then $\mathbb{V}_{ni}(u) = 0$. Hence, the event $J_{n1}(u)$ cannot occur. Therefore, $J_{n1}(u) \subset \bigcup_{i=u_n}^{v_n} \{Y_{ni} > u\}$. Thus, $\mathbb{V}(J_{n1}(u)) \leq \sum_{i=u_n}^{v_n} \mathbb{V}(Y_{ni} > u)$. Consequently,

$$Q_{n1} \leq \frac{1}{k_n} \sum_{i=u_n}^{v_n} \int_{k_n}^{\infty} \mathbb{V}(Y_{ni} > u) du.$$

By Lemma 2.4, we have

$$\int_{k_n}^{\infty} \mathbb{V}(Y_{ni} > u) du \leq C_{\mathbb{V}}(Y_{ni} I(Y_{ni} > k_n)).$$

Condition (3.3) implies that $\frac{h_n}{k_n} \rightarrow 0$. Hence, there exists a positive integer N_1 such that $h_n \leq k_n, n \geq N_1$. Thus, for all $n \geq N_1$, $\{Y_{ni} > k_n\} \subset \{Y_{ni} > h_n\}$. Moreover, since $G(t) = 0$ for $t \geq 1$, we have, on the event $\{Y_{ni} > h_n\}$, $1 - G\left(\frac{Y_{ni}}{h_n}\right) = 1$. Therefore,

$$Y_{ni} I(Y_{ni} > k_n) \leq Y_{ni} I(Y_{ni} > h_n) \leq Y_{ni} \left(1 - G\left(\frac{Y_{ni}}{h_n}\right) \right) = \tilde{Y}_{ni}.$$

Thus, for sufficiently large n , $Q_{n1} \leq \frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni})$.

On the other hand, for the finite number of n satisfying $k_n < h_n$, we decompose the integral

$$\text{as } \int_{k_n}^{\infty} \mathbb{V}(Y_{ni} > u) du = \int_{k_n}^{h_n} \mathbb{V}(Y_{ni} > u) du + \int_{h_n}^{\infty} \mathbb{V}(Y_{ni} > u) du,$$

Where

$$\int_{k_n}^{h_n} \mathbb{V}(Y_{ni} > u) du \leq h_n - k_n < \infty.$$

And

$$\int_{h_n}^{\infty} \mathbb{V}(Y_{ni} > u) du \leq C_{\mathbb{V}}(Y_{ni} I(Y_{ni} > h_n)) \leq C_{\mathbb{V}}(\tilde{Y}_{ni}) < \infty.$$

Since there are only finitely many n satisfying $k_n < h_n$, and since condition (3.8) implies that the corresponding $C_V(\cdot)$ terms are finite, the sum of these finite initial terms is finite. Hence,

$$\sum_{n=1}^{\infty} Q_{n1} < \infty. \quad (4.22)$$

(ii) Estimate of Q_{n2} .

By the definition of $J_{n2}(u)$ and the Chebyshev inequality in Lemma 2.2, we have

$$\mathbb{V}(J_{n2}(u)) \leq 9u^{\frac{-2}{r}} \mathbb{E} \left[\left| \sum_{i=u_n}^{v_n} (U_{ni}(u) - \mathbb{E}[U_{ni}(u)]) \right|^2 \right].$$

For each fixed $n \geq 1$, $u \geq k_n$, the mapping $x \mapsto x - \mathbb{E}[U_{ni}(u)]$ is continuous and nondecreasing. Hence, by Lemma 2.1, for each fixed n and $u \geq k_n$, the centered finite row $U_{ni}(u) - \mathbb{E}U_{ni}(u)$, $u_n \leq i \leq v_n$ still satisfies the finite-row m -AANA dependence condition. Moreover, by the translation invariance of $\mathbb{E}$, we have $\mathbb{E}[U_{ni}(u) - \mathbb{E}[U_{ni}(u)]] = 0$. Therefore, applying Corollary 2.1 to the centered finite row, we obtain

$$\mathbb{E} \left[\left| \sum_{i=u_n}^{v_n} (U_{ni}(u) - \mathbb{E}U_{ni}(u)) \right|^2 \right] \leq 2me^{4D} \sum_{i=u_n}^{v_n} \mathbb{E}[|U_{ni}(u) - \mathbb{E}[U_{ni}(u)]|^2].$$

By Lemma 2.5, we further have

$$\mathbb{E}[|U_{ni}(u) - \mathbb{E}[U_{ni}(u)]|^2] \leq 4\mathbb{E}[|U_{ni}(u)|^2].$$

Therefore,

$$\mathbb{V}(J_{n2}(u)) \leq cu^{\frac{-2}{r}} \sum_{i=u_n}^{v_n} \mathbb{E}[|U_{ni}(u)|^2].$$

Consequently,

$$Q_{n2} \leq \frac{c}{k_n} \sum_{i=u_n}^{v_n} \int_{k_n}^{\infty} u^{\frac{-2}{r}} \mathbb{E}[|U_{ni}(u)|^2] du.$$

Since $0 \leq G\left(\frac{Y_{ni}}{u}\right) \leq 1$, the quantity

$$Y_{ni}^{\frac{2}{r}} G\left(\frac{Y_{ni}}{u}\right) + u^{\frac{2}{r}} \left(1 - G\left(\frac{Y_{ni}}{u}\right)\right),$$

is a convex combination of $Y_{ni}^{\frac{2}{r}}$ and $u^{\frac{2}{r}}$. Hence, by (4.20), we have

$$|U_{ni}(u)|^2 \leq Y_{ni}^{\frac{2}{r}} G\left(\frac{Y_{ni}}{u}\right) + u^{\frac{2}{r}} \left(1 - G\left(\frac{Y_{ni}}{u}\right)\right).$$

Therefore, $Q_{n2} \leq Q_{n2}^{(1)} + Q_{n2}^{(2)}$, where

$$Q_{n2}^{(1)} := \frac{c}{k_n} \sum_{i=u_n}^{v_n} \int_{k_n}^{\infty} u^{\frac{-2}{r}} \mathbb{E} \left[Y_{ni}^{\frac{2}{r}} G\left(\frac{Y_{ni}}{u}\right) \right] du, \quad (4.23)$$

$$Q_{n2}^{(2)} := \frac{c}{k_n} \sum_{i=u_n}^{v_n} \int_{k_n}^{\infty} \mathbb{E} \left[1 - G\left(\frac{Y_{ni}}{u}\right) \right] du. \quad (4.24)$$

(a) Estimate of $Q_{n2}^{(1)}$.

By condition (3.1), $\mathbb{E}[Y_{ni}] < \infty$, while condition (3.8) implies $C_V(\tilde{Y}_{ni}) < \infty$. Hence, Lemma 2.6 is applicable. Therefore, applying Lemma 2.6 with $Y = Y_{ni}$, $h = h_n$, $k = k_n$, we obtain

$$\int_{k_n}^{\infty} u^{\frac{-2}{r}} \mathbb{E} \left[Y_{ni}^{\frac{2}{r}} G \left(\frac{Y_{ni}}{u} \right) \right] du \leq C_r \left(\frac{h_n}{k_n} \right)^{\frac{(2-r)}{r}} \mathbb{E}[Y_{ni}] + C_r C_{\mathbb{V}}[\tilde{Y}_{ni}], \quad (4.25)$$

where $C_r > 0$ depends only on r .

Substituting (4.25) into (4.23), we obtain

$$Q_{n2}^{(1)} \leq \frac{C}{k_n} \sum_{i=u_n}^{v_n} \left[\left(\frac{h_n}{k_n} \right)^{\frac{(2-r)}{r}} \mathbb{E}[Y_{ni}] + C_{\mathbb{V}}(\tilde{Y}_{ni}) \right].$$

Since $Y_{ni} = k_n a_{ni}^r |X_{ni}|^r$, we have

$$\frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbb{E}[Y_{ni}] = \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r].$$

Thus,

$$Q_{n2}^{(1)} \leq C \left(\frac{h_n}{k_n} \right)^{\frac{(2-r)}{r}} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r] + \frac{C}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}).$$

Summing over n , we get

$$\sum_{n=1}^{\infty} Q_{n2}^{(1)} \leq C \sum_{n=1}^{\infty} \left(\frac{h_n}{k_n} \right)^{\frac{(2-r)}{r}} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r] + C \sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}).$$

Since condition (3.1) gives $\sup_{n \geq 1} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r] < \infty$, there exists a constant $M > 0$ such that $\sup_{n \geq 1} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r] \leq M$. Thus, combined with condition (3.3), we obtain

$$\sum_{n=1}^{\infty} \left(\frac{h_n}{k_n} \right)^{\frac{(2-r)}{r}} \sum_{i=u_n}^{v_n} a_{ni}^r \mathbb{E}[|X_{ni}|^r] \leq M \sum_{n=1}^{\infty} \left(\frac{h_n}{k_n} \right)^{\frac{(2-r)}{r}} < \infty.$$

Furthermore, condition (3.8) gives $\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}) < \infty$. Hence, both nonnegative series on the right-hand side of the preceding estimate are finite. Consequently,

$$\sum_{n=1}^{\infty} Q_{n2}^{(1)} < \infty. \quad (4.26)$$

(b) Estimate of $Q_{n2}^{(2)}$.

If $1 - G\left(\frac{Y}{u}\right) > 0$, then $\frac{Y}{u} > \lambda$, that is $Y > \lambda u$. Therefore, pointwise, $1 - G\left(\frac{Y}{u}\right) \leq I(Y > \lambda u)$.

By the definition and monotonicity of the capacity, we obtain

$$\mathbb{E} \left[1 - G \left(\frac{Y}{u} \right) \right] \leq \mathbb{V}(Y > \lambda u).$$

Thus,

$$\int_{k_n}^{\infty} \mathbb{E} \left[1 - G \left(\frac{Y}{u} \right) \right] du \leq \int_{k_n}^{\infty} \mathbb{V}(Y > \lambda u) du = \frac{1}{\lambda} \int_{\lambda k_n}^{\infty} \mathbb{V}(Y > t) dt.$$

By Lemma 2.4, we have $\int_{\lambda k_n}^{\infty} \mathbb{V}(Y > t) dt \leq C_{\mathbb{V}}(Y I(Y > \lambda k_n))$. Therefore,

$$\int_{k_n}^{\infty} \mathbb{E} \left[1 - G \left(\frac{Y}{u} \right) \right] du \leq \frac{1}{\lambda} C_{\mathbb{V}}(Y I(Y > \lambda k_n)).$$

Applying the preceding estimate with $Y = Y_{ni}$, and then using Lemma 2.4, we obtain

$$\int_{k_n}^{\infty} \mathbb{E} \left[1 - G \left(\frac{Y_{ni}}{u} \right) \right] du \leq \frac{1}{\lambda} C_{\mathbb{V}}(Y_{ni} I(Y_{ni} > \lambda k_n)).$$

Condition (3.3) implies that $\frac{h_n}{k_n} \rightarrow 0$. Hence, for the fixed $\lambda > 0$, there exists a positive integer N_3 such that $h_n \leq \lambda k_n$ $n \geq N_3$. Thus, for all $n \geq N_3$, $\{Y_{ni} > \lambda k_n\} \subset \{Y_{ni} > h_n\}$. Consequently,

$$Y_{ni}(Y_{ni} > \lambda k_n) \leq Y_{ni} I(Y_{ni} > h_n) \leq Y_{ni} \left(1 - G \left(\frac{Y_{ni}}{h_n} \right) \right) = \tilde{Y}_{ni}.$$

Therefore, by the monotonicity of the Choquet integral, for all $n \geq N_3$,

$$\int_{k_n}^{\infty} \mathbb{E} \left[1 - G \left(\frac{Y_{ni}}{u} \right) \right] du \leq \frac{1}{\lambda} C_{\mathbb{V}}(\tilde{Y}_{ni}). \quad (4.27)$$

For the finitely many indices $n < N_3$, if $\lambda k_n \geq h_n$, the estimate in (4.27) remains valid. If $\lambda k_n < h_n$, then

$$\begin{aligned} \int_{\lambda k_n}^{\infty} \mathbb{V}(Y_{ni} > t) dt &= \int_{\lambda k_n}^{h_n} \mathbb{V}(Y_{ni} > t) dt + \int_{h_n}^{\infty} \mathbb{V}(Y_{ni} > t) dt \\ &\leq (h_n - \lambda k_n) + C_{\mathbb{V}}(Y_{ni} I(Y_{ni} > h_n)) \\ &\leq (h_n - \lambda k_n) + C_{\mathbb{V}}(\tilde{Y}_{ni}). \end{aligned}$$

Thus, each $Q_{n2}^{(2)}$ with $n < N_3$ is finite. Since there are only finitely many such indices, their total contribution is finite.

Substituting (4.27) into (4.24), we obtain

$$Q_{n2}^{(2)} \leq \frac{C}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}), \quad n \geq N_3.$$

Then, by condition (3.8), we get

$$\sum_{n=1}^{\infty} Q_{n2}^{(2)} < \infty. \quad (4.28)$$

By (4.26) and (4.28), we obtain

$$\sum_{n=1}^{\infty} Q_{n2} < \infty. \quad (4.29)$$

(iii) Estimate of Q_{n3} .

By the self-dominated property in Remark 2.1, we obtain

$$\left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq \mathbb{E}[|V_{ni}(u)|].$$

Therefore,

$$\sum_{i=u_n}^{v_n} \left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq \sum_{i=u_n}^{v_n} \mathbb{E}[|V_{ni}(u)|]. \quad (4.30)$$

When $u \geq k_n$, by (4.21), we know that

$$u^{\frac{-1}{r}} |V_{ni}(u)| \leq u^{\frac{-1}{r}} Y_{ni}^{\frac{1}{r}} I(Y_{ni} > u) \leq \left(\frac{Y_{ni}}{u} \right)^{\frac{1}{r}} I(Y_{ni} > u).$$

When $Y_{ni} > u$, we have $\frac{Y_{ni}}{u} \geq 1$. Since $\frac{1}{r} \leq 1$, it follows that $\left(\frac{Y_{ni}}{u} \right)^{\frac{1}{r}} \leq \frac{Y_{ni}}{u}$. Hence,

$$u^{\frac{-1}{r}} |V_{ni}(u)| \leq \left(\frac{Y_{ni}}{u}\right)^{\frac{1}{r}} I(Y_{ni} > u) \leq \frac{Y_{ni}}{u} I(Y_{ni} > u) \leq \frac{Y_{ni}}{k_n} I(Y_{ni} > u).$$

By condition (3.3), for sufficiently large n , we have $k_n \geq h_n$. Thus, $Y_{ni} > u \geq k_n \Rightarrow Y_{ni} > h_n$. Therefore,

$$u^{\frac{-1}{r}} |V_{ni}(u)| \leq \frac{Y_{ni}}{k_n} I(Y_{ni} > h_n) \leq \frac{\tilde{Y}_{ni}}{k_n}. \quad (4.31)$$

Taking the upper expectation $\mathbb{E}$ on both sides of (4.31) and summing over i , we obtain

$$u^{\frac{-1}{r}} \sum_{i=u_n}^{v_n} \mathbb{E} [|V_{ni}(u)|] \leq \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbb{E} [\tilde{Y}_{ni}].$$

Then, from (4.30), we have

$$u^{\frac{-1}{r}} \sum_{i=u_n}^{v_n} \left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq \frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbb{E} [\tilde{Y}_{ni}]. \quad (4.32)$$

By condition (3.7), we know that

$$\frac{1}{k_n} \sum_{i=u_n}^{v_n} \mathbb{E} [\tilde{Y}_{ni}] \leq \frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}). \quad (4.33)$$

Moreover, by condition (3.8), the nonnegative series $\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni})$ is convergent. Hence, its general term tends to zero, that is, $\frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}) \rightarrow 0$. Therefore, there exists a positive integer N_4 such that, for all $n \geq N_4$,

$$\frac{1}{k_n} \sum_{i=u_n}^{v_n} C_{\mathbb{V}}(\tilde{Y}_{ni}) \leq \frac{1}{3}. \quad (4.34)$$

By (4.32)–(4.34), when $n \geq N_4$ and $u \geq k_n$, we obtain

$$u^{\frac{-1}{r}} \sum_{i=u_n}^{v_n} \left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq \frac{1}{3}.$$

That is,

$$\sum_{i=u_n}^{v_n} \left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq \frac{1}{3} u^{\frac{1}{r}}.$$

Hence, $J_{n3}(u) = \emptyset$, $n \geq N_4$, $u \geq k_n$. Therefore, $Q_{n3} = 0$, $n \geq N_4$.

For the finite number of $n < N_4$, write $M_n := \sum_{i=u_n}^{v_n} \mathbb{E}[Y_{ni}]$. By condition (3.1), we know that $M_n < \infty$. Moreover,

$$\sum_{i=u_n}^{v_n} \left| \mathbb{E}[U_{ni}(u)] - \mathbb{E} \left[k_n^{\frac{1}{r}} a_{ni} X_{ni} \right] \right| \leq u^{\frac{-(r-1)}{r}} M_n.$$

Thus, when $u \geq 3M_n$, $u^{\frac{-(r-1)}{r}} M_n \leq \frac{1}{3} u^{\frac{1}{r}}$.

Consequently, $J_{n3}(u) = \emptyset$ for all $u \geq \max\{k_n, 3M_n\}$. Therefore,

$$Q_{n3} = \frac{1}{k_n} \int_{k_n}^{\infty} \mathbb{V}(J_{n3}(u)) du = \frac{1}{k_n} \int_{k_n}^{\max\{k_n, 3M_n\}} \mathbb{V}(J_{n3}(u)) du < \infty.$$

Since there are only finitely many such n , we have

$$\sum_{n=1}^{\infty} Q_{n3} < \infty. \quad (4.35)$$

Combining (4.22), (4.29) and (4.35), we obtain

$$\sum_{n=1}^{\infty} Q_n < \infty. \quad (4.36)$$

Step 4. Completion of the upper-tail proof.

Finally, combining (4.19) and (4.36), we obtain

$$\sum_{n=1}^{\infty} C_{\mathbb{V}}((T_n - \varepsilon)_+^r) < \infty.$$

Thus, the upper-tail conclusion (3.9) is proved.

Step 5. Lower-tail complete integral convergence.

Consider the array $X'_{ni} = -X_{ni}$, $u_n \leq i \leq v_n, n \geq 1$. Since the mapping $x \mapsto -x$ is continuous and nonincreasing, Lemma 2.1 implies that the array $\{X'_{ni}\}$ is also rowwise m -AANA with the same mixing-coefficient sequences. Moreover $|X'_{ni}| = |X_{ni}|$, and therefore $Y'_{ni} = Y_{ni}$, $\tilde{Y}'_{ni} = \tilde{Y}_{ni}$. Thus, all the assumptions of Theorem 3.1 and conditions (3.7) and (3.8) remain valid for the array $\{X'_{ni}\}$.

By the definition of the conjugate expectation, $\mathbb{E}[-X_{ni}] = -\mathcal{E}[X_{ni}]$. Hence, the upper-expectation-centered weighted sum associated with $\{X'_{ni}\}$ is

$$T'_n = \sum_{i=u_n}^{v_n} a_{ni}(-X_{ni} - \mathbb{E}[-X_{ni}]) = \sum_{i=u_n}^{v_n} a_{ni}(-X_{ni} + \mathcal{E}[X_{ni}]) = -\tilde{T}_n.$$

Applying the upper-tail conclusion (3.9), already proved in Step 4, to the array $\{X'_{ni}\}$, we obtain, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} C_{\mathbb{V}}\left((- \tilde{T}_n - \varepsilon)_+^r\right) < \infty.$$

Thus, the lower-tail conclusion (3.10) is proved.

Step 6. The mean-certain two-sided conclusion.

Suppose, in addition, that $\mathbb{E}[X_{ni}] = \mathcal{E}[X_{ni}]$, $u_n \leq i \leq v_n$, $n \geq 1$. Then $T_n = \tilde{T}_n$. For every $t \geq 0$,

$$\{(|T_n| - \varepsilon)_+^r > t\} = \{(T_n - \varepsilon)_+^r > t\} \cup \{(-T_n - \varepsilon)_+^r > t\}.$$

By the subadditivity of the capacity $\mathbb{V}$,

$$\mathbb{V}((|T_n| - \varepsilon)_+^r > t) \leq \mathbb{V}((T_n - \varepsilon)_+^r > t) + \mathbb{V}((-T_n - \varepsilon)_+^r > t).$$

Integrating both sides with respect to t , we obtain

$$C_{\mathbb{V}}((|T_n| - \varepsilon)_+^r) \leq C_{\mathbb{V}}((T_n - \varepsilon)_+^r) + C_{\mathbb{V}}((-T_n - \varepsilon)_+^r).$$

Since $T_n = \tilde{T}_n$, conclusions (3.9) and (3.10) imply that

$$\sum_{n=1}^{\infty} C_{\mathbb{V}}((|T_n| - \varepsilon)_+^r) < \infty.$$

Thus, (3.11) follows. The proof of Theorem 3.2 is complete.

5. Conclusions

In this paper, we establish complete convergence and complete integral convergence results for weighted sums of arrays of rowwise m -asymptotically almost negatively associated random variables

under sublinear expectations. By using truncation techniques, moment inequalities for m -AANA sequences, capacity estimates, and Choquet integral methods, we obtain one-sided complete convergence and complete integral convergence for T_n and $\tilde{T}_n$, together with the corresponding absolute-value conclusions under mean certainty. The results require the joint moment, tail, and weight conditions (3.1)–(3.3), with (3.7) and (3.8) additionally required for complete integral convergence. For rows $1 \leq i \leq n$, condition (3.3) excludes nonnegative weights satisfying $\sum_{i=1}^n a_{ni} = 1$.

The results are restricted to terminal sums of finite-row arrays with nonnegative deterministic weights and a fixed lag parameter m . Maximal partial sums, infinite-row arrays, mixed-sign or random weights, and unbounded row-dependent lag parameters require additional estimates and remain topics for future study.

Author contributions

Zhiqiang Tang: Conceptualization, Supervision, Funding acquisition; Mengke Lv: Methodology, Formal analysis, Writing-original draft, Writing-review and editing; Wei Liu: Conceptualization, Validation, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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