



*Research article***The wavelet characterization in Campanato spaces with variable exponents****Chenchen Niu¹, Hongbin Wang² and Zongguang Liu^{1,*}**¹ School of Science, China University of Mining and Technology (Beijing), Beijing 100083, China² School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255049, China*** Correspondence:** Email: liuzg@cumtb.edu.cn.

Abstract: This paper establishes the wavelet characterization of Campanato spaces with variable exponents by using compactly supported smooth wavelets with high vanishing moments. We apply the characterization to the boundedness of Riesz potentials.

Keywords: Campanato space; variable exponent; wavelet characterization; Riesz potential

Mathematics Subject Classification: 42B35, 46E30

1. Introduction

Campanato spaces, as generalizations of classical Lebesgue spaces and Hölder spaces, have been studied and applied in harmonic analysis and partial differential equations [16] since they were first introduced by Campanato in the 1960s [1]. As extensions of bounded mean oscillation (BMO) spaces and Lipschitz spaces, Campanato spaces serve as characterizations of function regularity. For related applications of function spaces and regularity theory to nonlinear partial differential equations, for example, [15, 20].

Function spaces with variable exponents have been studied in harmonic analysis, partial differential equations, and image processing [8, 14]. Since Kováčik and Rákosník generalized classical Lebesgue and Sobolev spaces to the variable exponent setting, a series of results has been established, including completeness, duality, and embedding theorems [12]. The boundedness of the Hardy-Littlewood maximal operator has been verified under the log-Hölder continuity condition [2, 3, 7]. Furthermore, Hardy spaces, Herz spaces, Morrey spaces, and Campanato spaces with variable exponents have been investigated [4, 11, 19]. In particular, the Campanato space with variable exponents is an extension of classical Campanato spaces and BMO spaces, inheriting the basic characteristics of constant exponent spaces while retaining the local adaptability of variable exponents [9, 21], and it provides a framework for studying the regularity of solutions to partial differential equations with variable exponents and the

boundedness of variable coefficient operators [24].

Unlike Fourier analysis, wavelet analysis and multiresolution analysis can characterize both spatial and frequency localizations and serve as tools for the characterization of function spaces [5, 13, 28]. Classical function spaces, including Besov spaces, Triebel-Lizorkin spaces, Herz spaces, and Hardy spaces, have been characterized by wavelets [13, 26, 29]. Wavelet characterizations of classical Hardy spaces were established by Meyer and other scholars [17, 18]. Subsequently, Fu established wavelet characterization of Hardy spaces with variable exponents by using discrete square functions and Peetre-type maximal functions [10]. Li and Sun applied multiresolution analysis to Besov-Q-type spaces and Triebel-Lizorkin-Q-type spaces and established corresponding wavelet characterizations [13]. In the framework of atomic and wavelet decompositions, Dechevsky [6] established general criteria for the boundedness of fractional integral and differential operators on a broad class of function spaces. His main result provides sharp estimates for the matrix coefficients of such operators with respect to localized bases, showing that they satisfy an almost-diagonal condition in terms of both scale and spatial separation. The Riesz potential is one of the operators in harmonic analysis and partial differential equations. The boundedness of variable order Riesz potentials provides a description for both critical and supercritical cases and generalizes the mapping properties of classical Riesz potentials to the variable-exponent setting [22]. Moreover, variable order Riesz potentials have been proved to be bounded from Morrey spaces with variable exponents to Campanato spaces with variable exponents [22, 23]. However, the wavelet characterization of Campanato spaces with variable exponents remains to be established.

Motivated by the above almost-diagonal approach, in the present paper we obtain matrix estimates for the Riesz potential with respect to the nonhomogeneous wavelet basis, adapted to the Campanato spaces with variable exponents. For such estimates to hold, the wavelet must possess sufficient smoothness, decay, and vanishing moments [5]; such properties are known to guarantee the decay of the Riesz transform [27], and the same regularity is sufficient for controlling the matrix coefficients of the Riesz potential. The wavelet characterization of Campanato spaces with variable exponents can transform continuous norm estimates into discrete wavelet coefficient estimates. This discrete form allows numerical implementation and allows operator boundedness to be established through coefficient estimates.

The paper is organized as follows: In Section 2, we recall the basic definitions and properties of variable exponent spaces and high-regularity orthogonal wavelet systems. In Section 3, we present the main wavelet characterization theorem for Campanato spaces with variable exponents. In Section 4, we apply this characterization to establish the boundedness of Riesz potentials on a bounded open set $\Omega \subset \mathbb{R}^n$.

Throughout this paper, we use the following notations:

- (1) $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. $\Omega \subset \mathbb{R}^n$ denotes an open set.
- (2) Given a measurable set $S \subset \mathbb{R}^n$, we denote the Lebesgue measure by $|S|$ and the characteristic function by χ_S .
- (3) Given a measurable set $S \subset \mathbb{R}^n$ such that $0 < |S| < \infty$ and a measurable function f on $\mathbb{R}^n$, we denote the mean value of f on S by f_S , namely

$$f_S := \frac{1}{|S|} \int_S f(x) dx.$$

- (4) A symbol C always stands for a positive constant independent of the main parameters.

(5) A cube $Q \subset \mathbb{R}^n$ is always assumed to be open and have sides parallel to the coordinate axes. Namely we can write

$$Q = \prod_{v=1}^n (x_v - r/2, x_v + r/2)$$

using a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and a constant $r > 0$. For any $x \in \mathbb{R}^n$ and $R \in (0, \infty)$, $Q(x, R)$ denotes the cube centered at x with side length R . For each cube $Q \subset \mathbb{R}^n$ and $k \in \mathbb{Z}$, we use c_Q to denote the center of Q , $\ell(Q)$ to denote the side length of Q , and $2^k Q$ to denote the cube concentric with Q having the side length $2^k \ell(Q)$.

(6) Let $\mathcal{D}$ denote the set of all dyadic cubes in $\mathbb{R}^n$. For each $j \in \mathbb{Z}$, let

$$\mathcal{D}_j := \{Q_{j,k} : k \in \mathbb{Z}^n\}$$

denote the dyadic cubes of side length 2^{-j} , where

$$Q_{j,k} = 2^{-j}([0, 1)^n + k).$$

(7) For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |y - x| \leq r\}$.

(8) We shall write $f \sim g$ if there exists a positive constant C , independent of f and g , such that $C^{-1}g \leq f \leq Cg$. Throughout this paper, C denotes a positive constant whose value may differ at each occurrence.

2. Preliminaries

In this section, we first recall the Lebesgue spaces with variable exponents $L^{p(\cdot)}$ with the Luxemburg-Nakano norm and log-Hölder continuity conditions [7, 9]. Then we present the definition of the Campanato spaces with variable exponents $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)$ introduced by Fan [9]. Finally, we recall Coiflet wavelets, which are compactly supported and smooth and have high vanishing moments. These properties are necessary for wavelet characterization [10, 13, 27].

2.1. Lebesgue spaces with variable exponents

Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : \Omega \rightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of all measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces are referred to as variable L^p spaces since they generalize the standard L^p spaces if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The space $L_{\text{loc}}^{p(\cdot)}(\Omega)$ is defined by

$$L_{\text{loc}}^{p(\cdot)}(\Omega) := \{f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset \Omega\}.$$

Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \longrightarrow [1, \infty)$ such that

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} \geq 1, \quad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty,$$

and $p'(x)$ denotes the conjugate exponent of $p(x)$, which satisfies

$$\frac{1}{p'(x)} + \frac{1}{p(x)} = 1.$$

Recall that the Hardy-Littlewood maximal operator M is defined by setting, for given $f \in L^1_{\operatorname{loc}}(\Omega)$ and $x \in \Omega$,

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q \cap \Omega} |f(y)| dy.$$

Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

Below we state several lemmas of the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ which will be used in the sequel.

Lemma 2.1. [22] *We say that $p \in \mathbb{P}^{\log}(\mathbb{R}^n)$ is log-Hölder continuous if $1 < p_- \leq p(x) \leq p_+ < \infty$ and for $x, y \in \mathbb{R}^n$ satisfying*

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad (2.1)$$

and

$$|p(x) - p(y)| \leq \frac{C}{\ln|x| + e}, \quad |y| \geq |x|, \quad (2.2)$$

where $p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$, $p_+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$, and C does not depend on x, y . In the case of a bounded Ω , the condition (2.1) may be equivalently written in the form

$$|p(x) - p(y)| \leq \frac{2\mathfrak{D}A}{\ln\left(\frac{2\mathfrak{D}}{|x-y|}\right)}, \quad x, y \in \Omega, \quad \mathfrak{D} = \operatorname{diam}(\Omega),$$

where A is a constant.

Lemma 2.2. [12, Theorem 2.1] (Generalized Hölder's inequality) *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p_- - 1/p_+.$$

Lemma 2.3. [7, Corollary 4.5.9] *Let $p(\cdot) \in \mathbb{P}^{\log}(\mathbb{R}^n)$. Then*

$$\| \chi_Q \|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}}, & |Q| \leq 2^n \text{ and } x \in Q; \\ |Q|^{\frac{1}{p(\infty)}}, & |Q| \geq 1, \end{cases}$$

for every cube (or ball) $Q \in \mathbb{R}^n$, where $p(\infty) = \lim_{x \rightarrow \infty} p(x)$.

Lemma 2.4. [11, Lemma 2.9] Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. There exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.5. [11, Corollary 2.6] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. There exist a constant $0 < \delta < 1$ and a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \frac{|B|}{|S|},$$

and

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|} \right)^\delta.$$

We next state some definitions of Campanato spaces.

2.2. Campanato spaces with variable exponents

First we give the classical definition of Campanato spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$.

$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \sup_Q \left(\frac{1}{|Q|^{\lambda/n}} \int_Q |f - f_Q|^p dx \right)^{1/p} < \infty.$$

The definition of Campanato spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$, which was introduced by Rafeiro and Samko [21], see [21, Definition 3.3], is defined as follows.

Definition 2.6. Let $p(\cdot) \in \mathbb{P}^{\log}(\mathbb{R}^n)$ and $\lambda(\cdot) \in L^\infty(\mathbb{R}^n)$ with $0 \leq \lambda(x) \leq \lambda_+ < \infty$ for all $x \in \mathbb{R}^n$. The Campanato spaces with variable exponents $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ are defined by

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) := \{f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n) : [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where the seminorm is

$$[f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} = \sup_{\substack{x_0 \in \mathbb{R}^n \\ \ell(Q) > 0}} (\ell(Q))^{-\frac{\lambda(x_0)}{p(x_0)}} \|(f - f_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

and χ_Q is the characteristic function of $Q = Q(x_0, \ell(Q))$. The norm of $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is given by

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}.$$

Remark 2.7. (i) The space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is a Banach space modulo additive constant: If $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ and $c \in \mathbb{R}$ is a constant, then $f + c \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ and $\|f + c\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}$.

(ii) Clearly, the seminorm $[\cdot]_{\mathcal{L}^{p,n}}$ is equivalent to the classical BMO norm whenever $p(\cdot) \equiv p$ ($1 \leq p < \infty$). In particular, $\mathcal{L}^{p,n}(\mathbb{R}^n)$ modulo constants coincides with $\text{BMO}(\mathbb{R}^n)$.

2.3. Wavelets

In the rest of this section, we introduce basic concepts and fundamental properties of wavelets. The results presented in this paper employ tensor-product real-valued orthogonal wavelets, specifically Coiflet wavelets. Recall that Coiflet wavelets are those that are sufficiently smooth and possess sufficiently high vanishing moments to meet the requirements for the study of the corresponding function spaces.

Let $\tilde{\phi}$ be an orthonormal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\tilde{\psi}$ be the corresponding orthonormal wavelet. Then the tensor product ansatz yields a scaling function ϕ and associated wavelets $\psi_1, \dots, \psi_{2^n-1}$, all defined now on $\mathbb{R}^n$; see, e.g., [28, Proposition 5.2]. To keep notation consistent, we set

$$\Phi^0 := \phi, \quad \Phi^\varepsilon := \psi_\varepsilon \quad (\varepsilon = 1, \dots, 2^n - 1).$$

We assume the existence of sufficiently large integers $m, M \in \mathbb{N}$ such that

- (i) $\Phi^0 \in C^m(\mathbb{R}^n)$ and $\text{supp } \Phi^0 \subset [-2^M, 2^M]^n$;
- (ii) $\Phi^\varepsilon \in C^m(\mathbb{R}^n)$ and $\text{supp } \Phi^\varepsilon \subset [-2^M, 2^M]^n$ for all $\varepsilon = 1, \dots, 2^n - 1$;
- (iii) $\int_{\mathbb{R}^n} x^\beta \Phi^\varepsilon(x) dx = 0$ for all multi-indices $|\beta| \leq m$ and all $\varepsilon = 1, \dots, 2^n - 1$ (see [28, Proposition 3.1]);
- (iv) $\int_{\mathbb{R}^n} x^\beta \Phi^0(x) dx = 0$ for all multi-indices $|\beta| \leq m$.

The inhomogeneous wavelet basis is now defined by

$$\begin{aligned} \Phi_{0,k}^0(x) &= \Phi^0(x - k), \quad k \in \mathbb{Z}^n, \\ \Phi_{j,k}^\varepsilon(x) &= 2^{jn/2} \Phi^\varepsilon(2^j x - k), \quad \varepsilon = 1, \dots, 2^n - 1, \quad j \in \mathbb{Z}_+, \quad k \in \mathbb{Z}^n. \end{aligned}$$

Lemma 2.8. *The system*

$$\{\Phi_{0,k}^0 : k \in \mathbb{Z}^n\} \cup \{\Phi_{j,k}^\varepsilon : \varepsilon = 1, \dots, 2^n - 1, \quad j \in \mathbb{Z}_+, \quad k \in \mathbb{Z}^n\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^n)$ (see [18, Section 3.9] or [25, Section 3.1]).

Correspondingly, for any $f \in L^2(\mathbb{R}^n)$, we define the projection onto the scaling function space V_0 by

$$P_0 f := \sum_{k \in \mathbb{Z}^n} \langle f, \Phi_{0,k}^0 \rangle \Phi_{0,k}^0,$$

and the projections onto the wavelet detail spaces W_j by

$$Q_j f := \sum_{\varepsilon=1}^{2^n-1} \sum_{k \in \mathbb{Z}^n} \langle f, \Phi_{j,k}^\varepsilon \rangle \Phi_{j,k}^\varepsilon, \quad j \in \mathbb{Z}_+.$$

Then, by Lemma 2.8, every $f \in L^2(\mathbb{R}^n)$ admits the orthogonal decomposition

$$f = P_0 f + \sum_{j=0}^{\infty} Q_j f \quad \text{in } L^2(\mathbb{R}^n).$$

3. Wavelet characterizations of function spaces

The main goal of this section is to derive a wavelet characterization for the Campanato spaces with variable exponents $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$. Using Coiflet wavelets, we show that the norm of $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is equivalent to a discrete wavelet coefficient norm. The argument is based on three key ingredients: the vanishing moment condition, local norm estimates, and appropriate inequalities for variable exponent spaces. The proof follows the wavelet characterization framework of [13] and employs the variable exponents techniques from [10].

Theorem 3.1. *Let $\lambda(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathbb{P}^{\log}(\mathbb{R}^n)$ with $0 < \lambda_- \leq \lambda_+ < \infty$, $p(\cdot) \in \mathbb{P}^{\log}(\mathbb{R}^n)$, and let the inhomogeneous wavelet basis be as in Lemma 2.8 with $m > \lambda_+$ and $\frac{\lambda(x)}{p(x)} > \frac{n}{2}$. Then any $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ admits the wavelet expansion*

$$f = \sum_{k \in \mathbb{Z}^n} a_k \Phi_{0,k}^0 + \sum_{\varepsilon=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} d_{j,k}^\varepsilon \Phi_{j,k}^\varepsilon \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \quad (3.1)$$

where

$$a_k = \langle f, \Phi_{0,k}^0 \rangle \quad (k \in \mathbb{Z}^n), \quad d_{j,k}^\varepsilon = \langle f, \Phi_{j,k}^\varepsilon \rangle \quad (\varepsilon = 1, \dots, 2^n - 1, j \in \mathbb{Z}_+, k \in \mathbb{Z}^n).$$

Moreover,

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp,$$

where

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp := \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \sup_{Q \in \mathcal{D}} 2^{j_Q \frac{\lambda(c_Q)}{p(c_Q)}} \left\| \left(\sum_{\varepsilon=1}^{2^n-1} \sum_{j=\max(0,j_Q)}^{\infty} 2^{jn} \sum_{\substack{k \in \mathbb{Z}^n \\ Q_{j,k} \subset Q}} |d_{j,k}^\varepsilon|^2 \chi_{Q_{j,k}}(x) \right)^{1/2} \right\|_{L^{p(\cdot)}(Q)}.$$

Conversely, if $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and its wavelet coefficients satisfy

$$\sup_{Q \in \mathcal{D}} 2^{j_Q \frac{\lambda(c_Q)}{p(c_Q)}} \left\| \left(\sum_{\varepsilon=1}^{2^n-1} \sum_{j=\max(0,j_Q)}^{\infty} 2^{jn} \sum_{\substack{k \in \mathbb{Z}^n \\ Q_{j,k} \subset Q}} |d_{j,k}^\varepsilon|^2 \chi_{Q_{j,k}}(\cdot) \right)^{1/2} \right\|_{L^{p(\cdot)}(Q)} < \infty,$$

then $f \in \mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

Proof. We need to prove $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp$, i.e., there exist positive constants C_1, C_2 such that $C_1 \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \leq C_2 \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp$.

First, we prove $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp$. Let Q be any dyadic cube with $\ell(Q) = 2^{-j_Q}$. For each $j \geq \max(0, j_Q)$, let $Q_{j,k} \subset Q$ be the dyadic subcubes of $\ell(Q_{j,k}) = 2^{-j}$. From the vanishing moments of the wavelets and the definition of the Campanato seminorm, we have

$$|d_{j,k}^\varepsilon| = |\langle f - f_{Q_{j,k}}, \Phi_{j,k}^\varepsilon \rangle| \leq \|(f - f_{Q_{j,k}}) \chi_{Q_{j,k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\Phi_{j,k}^\varepsilon \chi_{Q_{j,k}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

By Definition 2.6,

$$\|(f - f_{Q_{j,k}})\chi_{Q_{j,k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \left(\ell(Q_{j,k})\right)^{\frac{\lambda(c_{Q_{j,k}})}{p(c_{Q_{j,k}})}} = [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} 2^{-j\frac{\lambda(c_{Q_{j,k}})}{p(c_{Q_{j,k}})}}.$$

Using the support condition (ii) and the normalization of the wavelets,

$$\|\Phi_{j,k}^\varepsilon \chi_{Q_{j,k}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C 2^{jn/2} \|\chi_{Q_{j,k}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

We obtain

$$|d_{j,k}^\varepsilon| \leq C [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} 2^{-j\frac{\lambda(c_{Q_{j,k}})}{p(c_{Q_{j,k}})}} 2^{\frac{jn}{2}} \|\chi_{Q_{j,k}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Now fix Q and sum over all k with $Q_{j,k} \subset Q$. The number of such cubes is comparable to $2^{(j-j_Q)n}$. Consequently,

$$\sum_{k: Q_{j,k} \subset Q} \sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \leq C [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^2 2^{(j-j_Q)n} 2^{-2j\frac{\lambda(c_Q)}{p(c_Q)}} 2^{jn} \|\chi_{Q_{j,k}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}^2.$$

Since p and λ are log-Hölder continuous, so is λ/p , and there exist positive constants $C_1, C_2 > 0$ such that for any cube $Q_{j,k}$ of sufficiently small side length $\ell(Q_{j,k}) = 2^{-j}$ and with center satisfying $|c_{Q_{j,k}} - c_Q| \leq \sqrt{n}2^{-j_Q}$, we have

$$C_1 \ell(Q_{j,k})^{-\frac{\lambda(c_Q)}{p(c_Q)}} \leq \ell(Q_{j,k})^{-\frac{\lambda(c_{Q_{j,k}})}{p(c_{Q_{j,k}})}} \leq C_2 \ell(Q_{j,k})^{-\frac{\lambda(c_Q)}{p(c_Q)}}. \quad (3.2)$$

Summing over all $j \geq \max(0, j_Q)$, taking the square root, and by Lemma 2.4, we have

$$\begin{aligned} & \left\| \left(\sum_{j \geq \max(0, j_Q)} 2^{jn} \sum_{\varepsilon, k: Q_{j,k} \subset Q} |d_{j,k}^\varepsilon|^2 \chi_{Q_{j,k}} \right)^{1/2} \right\|_{L^{p(\cdot)}(Q)} \\ & \leq C \sum_{j \geq \max(0, j_Q)} [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} 2^{(j-j_Q)n/2} 2^{-j\frac{\lambda(c_Q)}{p(c_Q)}} 2^{jn} |Q_{j,k}| \frac{1}{|Q_{j,k}|} \|\chi_{Q_{j,k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{Q_{j,k}}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ & \leq C \sum_{j \geq \max(0, j_Q)} [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} 2^{-(j-j_Q)\left(\frac{\lambda(c_Q)}{p(c_Q)} - \frac{n}{2}\right)} 2^{-j_Q \frac{\lambda(c_Q)}{p(c_Q)}} \\ & \leq C [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} 2^{-j_Q \left(\frac{\lambda(c_Q)}{p(c_Q)}\right)}. \end{aligned}$$

Since $j - j_Q \geq 0$ and $\frac{\lambda(c_Q)}{p(c_Q)} > \frac{n}{2}$, the series $2^{-(j-j_Q)\left(\frac{\lambda(c_Q)}{p(c_Q)} - \frac{n}{2}\right)} < C$. If $j_Q < 0$, the series

$$\sum_{j=0}^{\infty} 2^{-(j-j_Q)\left(\frac{\lambda(c_Q)}{p(c_Q)} - \frac{n}{2}\right)} = 2^{j_Q \left(\frac{\lambda(c_Q)}{p(c_Q)} - \frac{n}{2}\right)} \sum_{j=0}^{\infty} 2^{-j \left(\frac{\lambda(c_Q)}{p(c_Q)} - \frac{n}{2}\right)}$$

converges since $\frac{\lambda(c_Q)}{p(c_Q)} > \frac{n}{2}$. Hence the conclusion still holds.

After moving the factor $2^{-j_Q \left(\frac{\lambda(c_Q)}{p(c_Q)}\right)}$ to the left-hand side of the inequality, take the supremum over all dyadic cubes Q . Since $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}$, together with the above estimate, we obtain

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\# \leq C \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}.$$

We now turn to the reverse inequality in the case $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \leq C_2 \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp$. Assume that $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp < \infty$. Let Q be any cube and choose $j_0 \in \mathbb{Z}$ such that $2^{-j_0-1} \leq \ell(Q) < 2^{-j_0}$. For $x \in Q$, we use the identity

$$f(x) - f_Q = \sum_{k \in \mathbb{Z}^n} a_k \left(\Phi_{0,k}^0(x) - (\Phi_{0,k}^0)_Q \right) + \sum_{\varepsilon=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} d_{j,k}^\varepsilon \left(\Phi_{j,k}^\varepsilon(x) - (\Phi_{j,k}^\varepsilon)_Q \right), \quad (3.3)$$

which follows from the unconditional convergence of the wavelet series in $L^1(Q)$.

For the scaling function part, we note that $P_0 f = \sum_k a_k \Phi_{0,k}^0$. If $j_0 \leq 0$, we simply use

$$\|(P_0 f - (P_0 f)_Q) \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2 \|P_0 f\|_{L^{p(\cdot)}(Q)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Multiplying by the factor $2^{j_0 \frac{\lambda(c_Q)}{p(c_Q)}} \leq 1$, we obtain the desired estimate controlled by $\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp$. If $j_0 > 0$, we exploit the C^m regularity of Φ^0 . By Taylor expansion,

$$|\Phi_{0,k}^0(x) - (\Phi_{0,k}^0)_Q| \leq C \ell(Q)^m = C 2^{-j_0 m},$$

hence

$$\|(P_0 f - (P_0 f)_Q) \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C 2^{-j_0 m} \sum_{k: \text{supp } \Phi_{0,k}^0 \cap Q \neq \emptyset} |a_k|.$$

Using $|a_k| \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and the finite overlap, we obtain

$$\|(P_0 f - (P_0 f)_Q) \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C 2^{-j_0 m} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Multiplying by $2^{j_0 \frac{\lambda(c_Q)}{p(c_Q)}}$, we obtain $C 2^{-j_0(m - \frac{\lambda(c_Q)}{p(c_Q)})} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}^\sharp$, since $m > \lambda_+ \geq \frac{\lambda(c_Q)}{p(c_Q)}$.

For the wavelet part, set $j_1 := \max(0, j_0)$. We split the sum into $j \geq j_1$ and $0 \leq j < j_1$ (the latter is empty when $j_0 \leq 0$). We begin with the case $j \geq j_1$. For such j , we have $\text{diam}(\text{supp } \Phi_{j,k}^\varepsilon) \leq C 2^{-j} \leq C \ell(Q)$. On the cube Q , we write the difference as

$$\Phi_{j,k}^\varepsilon(x) - (\Phi_{j,k}^\varepsilon)_Q = \Phi_{j,k}^\varepsilon(x) \chi_Q(x) - (\Phi_{j,k}^\varepsilon)_Q \chi_Q(x). \quad (3.4)$$

We proceed to estimate the two terms individually. We first consider the term $\Phi_{j,k}^\varepsilon(x) \chi_Q(x)$. By the support condition (ii) and the normalization of wavelets,

$$|\Phi_{j,k}^\varepsilon(x)| \leq C 2^{jn/2} \chi_{Q_{j,k}^*}(x),$$

where $Q_{j,k}^*$ is the unique dyadic cube of side length $2^{-j+2M+1}$ concentric with $Q_{j,k}$ such that $\text{supp } \Phi_{j,k}^\varepsilon \subset Q_{j,k}^*$. If $Q_{j,k}^* \cap Q = \emptyset$, then $\Phi_{j,k}^\varepsilon$ vanishes on Q , and this term contributes nothing. Because $2^{-j} \leq C \ell(Q)$ and both $Q_{j,k}^*$ and Q are cubes, if $Q_{j,k}^* \cap Q \neq \emptyset$, then we have the inclusions $Q_{j,k} \subset Q_{j,k}^* \subset 3Q$. Consequently,

$$\sum_{k \in \mathbb{Z}^n} |\Phi_{j,k}^\varepsilon(x)| \chi_Q(x) \leq C 2^{jn/2} \sum_{k: Q_{j,k} \subset 3Q} \chi_{Q_{j,k}^*}(x). \quad (3.5)$$

We now turn to $(\Phi_{j,k}^\varepsilon)_Q \chi_Q(x)$. The average over Q is controlled by

$$|(\Phi_{j,k}^\varepsilon)_Q| \leq \frac{1}{|Q|} \int_{Q \cap Q_{j,k}^*} |\Phi_{j,k}^\varepsilon(y)| dy \leq C 2^{jn/2} \frac{|Q_{j,k}^* \cap Q|}{|Q|} \leq C 2^{jn/2} \frac{|Q_{j,k} \cap 3Q|}{|Q|},$$

where we used that $Q_{j,k}^* \cap Q \neq \emptyset$ implies $Q_{j,k}^* \subset 3Q$ and $|Q_{j,k}^*| \lesssim |Q_{j,k}|$. Thus

$$|(\Phi_{j,k}^\varepsilon)_Q| \chi_Q(x) \leq C 2^{jn/2} \frac{|Q_{j,k} \cap 3Q|}{|Q|} \chi_Q(x). \quad (3.6)$$

Now insert (3.4)–(3.6) into the sum. For any $x \in Q$,

$$\begin{aligned} & \left| \sum_{j \geq j_1} \sum_{k, \varepsilon} d_{j,k}^\varepsilon (\Phi_{j,k}^\varepsilon(x) - (\Phi_{j,k}^\varepsilon)_Q) \right| \\ & \leq C \sum_{j \geq j_1} 2^{jn/2} \left(\sum_{k: Q_{j,k} \subset 3Q} \left(\sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \right)^{1/2} \chi_{Q_{j,k}^*}(x) + \sum_{k: Q_{j,k} \cap Q \neq \emptyset} \left(\sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \right)^{1/2} \frac{|Q_{j,k} \cap 3Q|}{|Q|} \chi_Q(x) \right) \\ & =: C \sum_{j \geq j_1} 2^{jn/2} (I_1 + I_2), \end{aligned}$$

where

$$\begin{cases} I_1 := \sum_{k: Q_{j,k} \subset 3Q} \left(\sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \right)^{1/2} \chi_{Q_{j,k}^*}(x); \\ I_2 := \sum_{k: Q_{j,k} \cap Q \neq \emptyset} \left(\sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \right)^{1/2} \frac{|Q_{j,k} \cap 3Q|}{|Q|} \chi_Q(x). \end{cases}$$

Here we used the Cauchy-Schwarz inequality $\sum_{\varepsilon=1}^{2^n-1} |d_{j,k}^\varepsilon| \leq C \left(\sum_{\varepsilon=1}^{2^n-1} |d_{j,k}^\varepsilon|^2 \right)^{1/2}$. We now estimate the $L^{p(\cdot)}(Q)$ norm of each term on the right-hand side.

We first handle the case I_1 . Fix $j \geq j_1$. Because the cubes $Q_{j,k}^*$ are obtained by enlarging the disjoint dyadic cubes $Q_{j,k}$ by a fixed factor, they have the bounded overlapping property:

$$\sum_{k: Q_{j,k} \subset 3Q} \chi_{Q_{j,k}^*}(x) \leq C_n.$$

Hence,

$$I_1 \leq \left(\sum_{k: Q_{j,k} \subset 3Q} \sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \right)^{1/2} \left(\sum_{k: Q_{j,k} \subset 3Q} \chi_{Q_{j,k}^*} \right)^{1/2} \leq C \left(\sum_{k: Q_{j,k} \subset 3Q} \sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \chi_{Q_{j,k}^*}(x) \right)^{1/2}. \quad (3.7)$$

We now consider the following estimate for I_2 .

$$\begin{aligned}
 & \sum_k \left(\sum_{\varepsilon} |d_{j,k}^{\varepsilon}|^2 \right)^{1/2} \frac{|Q_{j,k} \cap 3Q|}{|Q|} \chi_Q \\
 & \leq \left(\sum_k \sum_{\varepsilon} |d_{j,k}^{\varepsilon}|^2 \right)^{1/2} \left(\sum_k \left(\frac{|Q_{j,k} \cap 3Q|}{|Q|} \right)^2 \chi_Q \right)^{1/2} \\
 & \leq \left(\sum_k \sum_{\varepsilon} |d_{j,k}^{\varepsilon}|^2 \right)^{1/2} \left(N_Q \cdot \left(\frac{|3Q|}{|Q|} \right)^2 \cdot \chi_Q \right)^{1/2} \\
 & \leq C_n \left(\sum_{k: Q_{j,k} \subset 3Q} \sum_{\varepsilon} |d_{j,k}^{\varepsilon}|^2 \right)^{1/2} \chi_Q(x),
 \end{aligned} \tag{3.8}$$

where $C_n = 3^n \cdot \sup_Q \sqrt{N_Q}$, with N_Q denoting the number of indices k satisfying the given condition.

Combining (3.7) and (3.8), we deduce that for each $j \geq j_1$,

$$\left| \sum_{j \geq j_1} \sum_{k, \varepsilon} d_{j,k}^{\varepsilon} (\Phi_{j,k}^{\varepsilon}(x) - (\Phi_{j,k}^{\varepsilon})_Q) \right| \leq C \sum_{j \geq j_1} 2^{jn/2} \left(\sum_{k: Q_{j,k} \subset 3Q} \sum_{\varepsilon} |d_{j,k}^{\varepsilon}|^2 \chi_{Q_{j,k}^*}(x) \right)^{1/2}. \tag{3.9}$$

Integrating over Q and using the fact that $\bigcup_{j \geq j_1} \bigcup_{k: Q_{j,k} \subset 3Q} Q_{j,k}^* \subset 3Q$, we get

$$\left\| \sum_{j \geq j_1} \sum_{k, \varepsilon} d_{j,k}^{\varepsilon} (\Phi_{j,k}^{\varepsilon} - (\Phi_{j,k}^{\varepsilon})_Q) \right\|_{L^{p(\cdot)}(Q)} \leq C \left\| \left(\sum_{j \geq j_1} 2^{jn} \sum_{\varepsilon, k: Q_{j,k} \subset 3Q} |d_{j,k}^{\varepsilon}|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(3Q)}. \tag{3.10}$$

Since $\ell(3Q) \approx 2^{-j_0}$, by choosing $R = 3Q$ in the wavelet norm, we have

$$\left\| \left(\sum_{j \geq j_1} 2^{jn} \sum_{\varepsilon, k: Q_{j,k} \subset R} |d_{j,k}^{\varepsilon}|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(R)} \leq C 2^{-j_0} \frac{\lambda(c_R)}{p(c_R)} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^{\#}.$$

Substituting into (3.10) and using the log-Hölder continuity of λ/p as in (3.2) to replace the exponent at c_R by that at c_Q , we obtain

$$\left\| \sum_{j \geq j_1} \sum_{k, \varepsilon} d_{j,k}^{\varepsilon} (\Phi_{j,k}^{\varepsilon} - (\Phi_{j,k}^{\varepsilon})_Q) \right\|_{L^{p(\cdot)}(Q)} \leq C 2^{-j_0} \frac{\lambda(c_Q)}{p(c_Q)} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^{\#}. \tag{3.11}$$

Next we consider $0 \leq j < j_1$ when $j_0 > 0$. Since if $j_0 \leq 0$, there is no such term. Here the wavelet support is much larger than Q . By the C^m regularity and m vanishing moments of Φ^{ε} , for any $x, y \in Q$,

$$\Phi_{j,k}^{\varepsilon}(y) - \Phi_{j,k}^{\varepsilon}(x) = \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} D^{\alpha} \Phi_{j,k}^{\varepsilon}(y) (x-y)^{\alpha} + R_m(x, y).$$

Using the standard wavelet estimate

$$\|D^{\alpha} \Phi_{j,k}^{\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)} \leq C 2^{j(n/2+|\alpha|)},$$

we obtain, for $|\alpha| = m$, $|R_m(x, y)| \leq C2^{j(n/2+m)}|x - y|^m$. Since $x, y \in Q$, $|x - y| \leq C\ell(Q)$, we have

$$\begin{aligned}
 (\Phi_{j,k}^\varepsilon)_Q - \Phi_{j,k}^\varepsilon(x) &= \frac{1}{|Q|} \int_Q (\Phi_{j,k}^\varepsilon(y) - \Phi_{j,k}^\varepsilon(x)) dy \\
 &= \frac{1}{|Q|} \int_Q \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} D^\alpha \Phi_{j,k}^\varepsilon(y) (x - y)^\alpha dy + \frac{1}{|Q|} \int_Q R_m(x, y) dy \\
 &\leq \frac{1}{|Q|} \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \int_Q D^\alpha \Phi_{j,k}^\varepsilon(y) (x - y)^\alpha dy + \frac{1}{|Q|} \int_Q |R_m(x, y)| dy \\
 &\leq \frac{1}{|Q|} \sum_{1 \leq |\alpha| \leq m-1} \|D^\alpha \Phi_{j,k}^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \frac{1}{\alpha!} \int_Q |x - y|^\alpha dy + 2^{j(n/2+m)} |x - y|^m \\
 &\leq C \sum_{1 \leq |\alpha| \leq m} 2^{j(n/2+|\alpha|)} 2^{-j_0|\alpha|} \\
 &= C2^{j \cdot \frac{n}{2}} \sum_{1 \leq |\alpha| \leq m} 2^{(j-j_0)|\alpha|} \\
 &\leq C2^{j \cdot \frac{n}{2}} + C2^{j \cdot \frac{n}{2}} 2^{(j-j_0)m}.
 \end{aligned}$$

Moreover, since $\text{supp } \Phi_{j,k}^\varepsilon$ contains Q when $0 \leq j < j_1$, only those k for which $Q \subset 2^{M+1}Q_{j,k}$ can contribute. For each relevant (j, k) , we may choose a cube $\tilde{Q}_{j,k}$ of $\ell(\tilde{Q}_{j,k}) \sim 2^{-j}$ that contains both Q and $\text{supp } \Phi_{j,k}^\varepsilon$. Then the pointwise estimate above implies

$$|\Phi_{j,k}^\varepsilon(x) - (\Phi_{j,k}^\varepsilon)_Q| \leq C(2^{j \cdot \frac{n}{2}} + C2^{j \cdot \frac{n}{2}} 2^{(j-j_0)m}) \chi_{\tilde{Q}_{j,k}}(x).$$

The contribution of low scales is therefore

$$\begin{aligned}
 &\left| \sum_{0 \leq j < j_1} \sum_{k, \varepsilon} d_{j,k}^\varepsilon (\Phi_{j,k}^\varepsilon(x) - (\Phi_{j,k}^\varepsilon)_Q) \right| \\
 &\leq C \sum_{0 \leq j < j_1} 2^{jn/2} (2^{(j-j_0)m} + 1) \left(\sum_{\varepsilon, k: Q \subset \tilde{Q}_{j,k}} |d_{j,k}^\varepsilon|^2 \chi_{\tilde{Q}_{j,k}}(x) \right)^{1/2} \\
 &\leq C \sum_{0 \leq j < j_1} (2^{(j-j_0)m} + 1) \left(2^{jn} \sum_{\varepsilon, k: Q \subset \tilde{Q}_{j,k}} |d_{j,k}^\varepsilon|^2 \chi_{\tilde{Q}_{j,k}}(x) \right)^{1/2}.
 \end{aligned}$$

For each fixed $0 \leq j < j_1$, we choose a dyadic cube R with side length 2^{-j} such that $Q \subset R \subset C_j Q$ (with $C_j \sim 2^{j_0-j}$), which is possible because the cubes $\tilde{Q}_{j,k}$ that intersect Q have side length $\sim 2^{-j}$ and can be covered by a single dyadic cube of that scale. By the definition of the wavelet norm,

$$\begin{aligned}
 &\left\| \sum_{0 \leq j < j_1} \sum_{k, \varepsilon} d_{j,k}^\varepsilon (\Phi_{j,k}^\varepsilon - (\Phi_{j,k}^\varepsilon)_Q) \right\|_{L^{p(\cdot)}(Q)} \\
 &\leq C \sum_{0 \leq j < j_1} (2^{(j-j_0)m} + 1) \left\| \left(2^{jn} \sum_{k: Q \subset \tilde{Q}_{j,k}} \sum_{\varepsilon} |d_{j,k}^\varepsilon|^2 \chi_{\tilde{Q}_{j,k}} \right)^{1/2} \right\|_{L^{p(\cdot)}(Q)}
 \end{aligned}$$

$$\leq C \sum_{0 \leq j < j_1} (2^{(j-j_0)m} + 1) 2^{-j \frac{\lambda(c_R)}{p(c_R)}} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^\#.$$

By the log-Hölder continuity of λ/p as in (3.2), the factor $2^{-j \frac{\lambda(c_R)}{p(c_R)}}$ is comparable to $2^{-j \frac{\lambda(c_Q)}{p(c_Q)}}$. Consequently, inserting this estimate into the sum over $j < j_1$, we obtain

$$\left\| \sum_{0 \leq j < j_1} \sum_{k, \varepsilon} d_{j,k}^\varepsilon (\Phi_{j,k}^\varepsilon - (\Phi_{j,k}^\varepsilon)_Q) \right\|_{L^{p(\cdot)}(Q)} \leq C \sum_{0 \leq j < j_1} 2^{(j-j_0)(m - \frac{\lambda(c_Q)}{p(c_Q)})} 2^{-j_0 \frac{\lambda(c_Q)}{p(c_Q)}} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^\#.$$

Since $m > \lambda_+ \geq \frac{\lambda(c_Q)}{p(c_Q)}$, the series converges, and we get

$$\left\| \sum_{0 \leq j < j_1} \sum_{k, \varepsilon} d_{j,k}^\varepsilon (\Phi_{j,k}^\varepsilon - (\Phi_{j,k}^\varepsilon)_Q) \right\|_{L^{p(\cdot)}(Q)} \leq C 2^{-j_0 \frac{\lambda(c_Q)}{p(c_Q)}} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^\#. \quad (3.12)$$

Combining (3.11) and (3.12), we conclude that

$$\|(f - f_Q)\chi_Q\|_{L^{p(\cdot)}(Q)} \leq C 2^{-j_0 \frac{\lambda(c_Q)}{p(c_Q)}} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^\#.$$

Since $\ell(Q) \approx 2^{-j_0}$, taking the supremum over all cubes Q yields

$$\|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\mathbb{R}^n)}^\#.$$

This completes the proof. $\square$

4. Applications

In this section, we study the boundedness of Riesz potentials via the wavelet characterization established in Section 3. Let Ω be a bounded open set in $\mathbb{R}^n$ satisfying the property that there exists a constant $A > 0$ such that

$$|B(x_0, r) \cap \Omega| \geq A r^n$$

for every ball $B(x_0, r)$ centered at $x_0 \in \Omega$ with $r > 0$.

For $0 < \alpha < n$, the Riesz potential I_α can be defined either via the Fourier multiplier $(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha} \widehat{f}(\xi)$ or equivalently (up to a multiplicative constant) by the convolution integral

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

For this same domain Ω , the localized Riesz potential I_Ω^α is then given by

$$I_\Omega^\alpha f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \Omega.$$

In order to state the boundedness result, we shall require two families of function spaces, namely the Morrey spaces $M^{p(\cdot), \lambda(\cdot)}(\Omega)$ and the Campanato spaces $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$, introduced by Fan in [9]; see also Sections 3 and 4 of that paper. We now recall their definitions below. Let $\lambda : \Omega \rightarrow [0, \infty)$ be a

measurable function. The Morrey space with variable exponents $M^{p(\cdot), \lambda(\cdot)}(\Omega)$ consists of all measurable functions f for which $\|f\|_{M^{p(\cdot), \lambda(\cdot)}(\Omega)} < \infty$, with the norm

$$\|f\|_{M^{p(\cdot), \lambda(\cdot)}(\Omega)} := \sup_{\substack{x_0 \in \Omega \\ r > 0}} r^{-\frac{\lambda(x_0)}{p(x_0)}} \|f\|_{L^{p(\cdot)}(B \cap \Omega)}.$$

The Campanato space with variable exponents $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$ consists of all measurable functions f satisfying $\|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)} < \infty$. Its norm contains the $L^{p(\cdot)}$ -norm and is defined by

$$\|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \sup_{\substack{x_0 \in \Omega \\ r > 0}} r^{-\frac{\lambda(x_0)}{p(x_0)}} \|f - f_{B \cap \Omega}\|_{L^{p(\cdot)}(B \cap \Omega)}. \quad (4.1)$$

Since balls and cubes are interchangeable up to a constant factor, this norm is equivalent to the one given in Definition 2.6 in terms of cubes, and we shall use either formulation as convenient.

A special case of [22, Theorem 2] for constant order α states that if $p \in \mathbb{P}^{\log}(\Omega)$, λ satisfies the log-Hölder condition, and $0 \leq \lambda_- \leq \lambda(x) \leq \lambda_+ < n$, then

$$I_\Omega^\alpha : M^{p(\cdot), \lambda(\cdot)}(\Omega) \longrightarrow \mathcal{L}^{p(\cdot), \lambda(\cdot) + \alpha p(\cdot)}(\Omega)$$

is bounded.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p(\cdot) \in \mathbb{P}^{\log}(\Omega)$, $0 \leq \lambda_- \leq \lambda(x) \leq \lambda_+ < n$, and $\lambda(x)/p(x) \leq n\delta$ with δ in Lemma 2.5. Then there exists a constant $C > 0$ such that for all $f \in \mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$,*

$$\|f\|_{M^{p(\cdot), \lambda(\cdot)}(\Omega)} \leq C \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)},$$

i.e., $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega) \hookrightarrow M^{p(\cdot), \lambda(\cdot)}(\Omega)$.

Proof. Fix an arbitrary ball $B = B(x_0, r) \subset \Omega$. We shall prove that

$$\|f\|_{L^{p(\cdot)}(B)} \leq C r^{\frac{\lambda(x_0)}{p(x_0)}} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)}.$$

If r is large, say $r \geq r_0$ for some fixed $r_0 > 0$, then $r^{\frac{\lambda(x_0)}{p(x_0)}} \geq r_0^{\frac{\lambda(x_0)}{p(x_0)}} \geq c > 0$. Hence the trivial bound

$$\|f\|_{L^{p(\cdot)}(B)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq C r^{\frac{\lambda(x_0)}{p(x_0)}} \|f\|_{L^{p(\cdot)}(\Omega)} \quad (4.2)$$

with a constant $C = 1/c$. Thus we may assume without loss of generality that r is sufficiently small and $r < 1$.

Set $B_0 = B$, $B_j = 2^j B_0$ (same center x_0 , radius $2^j r$), $A_j = B_{j+1} \setminus B_j$, and $f_j := f_{A_j}$. Let J be the smallest positive integer such that B_J contains Ω , so that $2^{J+1} r \approx \text{diam}(\Omega)$.

For any $x \in B_0$,

$$|f(x)| \leq |f(x) - f_0| + |f_0|,$$

and

$$f_0 = f_J + \sum_{j=0}^{J-1} (f_j - f_{j+1}).$$

Consequently,

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(B_0)} &\leq \|f - f_0\|_{L^{p(\cdot)}(B_0)} + \|f_J\|_{L^{p(\cdot)}(\Omega)} \|\chi_{B_0}\|_{L^{p(\cdot)}(\Omega)} + \sum_{j=0}^{J-1} \|f_j - f_{j+1}\|_{L^{p(\cdot)}(\Omega)} \|\chi_{B_0}\|_{L^{p(\cdot)}(\Omega)} \\ &=: II_1 + II_2 + II_3. \end{aligned} \quad (4.3)$$

To see this, we begin by considering II_1 . By (4.1),

$$II_1 = \|f - f_0\|_{L^{p(\cdot)}(B_0)} \leq [f]_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)} r^{\frac{\lambda(x_0)}{p(x_0)}}.$$

Since $B_J \supset \Omega$, by Lemmas 2.2 and 2.3, and r is small enough and $r < 1$, we obtain

$$II_2 \leq C \frac{1}{|B_J|} \|f\|_{L^{p(\cdot)}(\Omega)} \|\chi_{B_J}\|_{L^{p'(\cdot)}(\Omega)} \|\chi_{B_0}\|_{L^{p(\cdot)}(\Omega)} \leq C \|f\|_{L^{p(\cdot)}(\Omega)} |B_0|^{1/p(x)} \leq C r^{\frac{\lambda(x_0)}{p(x_0)}} \|f\|_{L^{p(\cdot)}(\Omega)}.$$

For the above second inequality, we use the following result. We now estimate the constant factor

$$\frac{1}{|B_J|} \|\chi_{B_J}\|_{L^{p'(\cdot)}(\Omega)} \leq C_1.$$

Recall that B_J is the ball of radius $R_J \approx \text{diam}(\Omega)$ constructed in the dyadic annuli, so its volume $|B_J|$ is a fixed positive number independent of r . We distinguish two cases according to the size of $|B_J|$. For $|B_J| \leq 2^n$, by Lemma 2.3, we have

$$\frac{1}{|B_J|} \|\chi_{B_J}\|_{L^{p'(\cdot)}(\Omega)} \approx |B_J|^{\frac{1}{p'(\cdot)} - 1} = |B_J|^{-\frac{1}{p(\cdot)}} \leq \max \left\{ |B_J|^{-\frac{1}{p_-}}, |B_J|^{-\frac{1}{p_+}} \right\} \leq C_1.$$

Thus C_1 is a finite positive constant independent of r , depending only on Ω, n, p_- , and p_+ .

Next we consider the following estimate for II_3 . By Lemmas 2.2, 2.4, and 2.5, for each j , we have

$$\begin{aligned} II_3 &\leq C \sum_{j=0}^{J-1} \frac{1}{|B_j|} \|f - f_{j+1}\|_{L^{p(\cdot)}(B_{j+1})} \|\chi_{B_{j+1}}\|_{L^{p'(\cdot)}(\Omega)} \|\chi_{B_0}\|_{L^{p(\cdot)}(\Omega)} \\ &\leq C \sum_{j=0}^{J-1} \|f - f_{j+1}\|_{L^{p(\cdot)}(B_{j+1})} \frac{1}{|B_{j+1}|} \|\chi_{B_{j+1}}\|_{L^{p'(\cdot)}(\Omega)} \|\chi_{B_{j+1}}\|_{L^{p(\cdot)}(\Omega)} \frac{\|\chi_{B_0}\|_{L^{p(\cdot)}(\Omega)} |B_{j+1}|}{\|\chi_{B_{j+1}}\|_{L^{p(\cdot)}(\Omega)} |B_j|} \\ &\leq C \sum_{j=0}^{J-1} \|f - f_{j+1}\|_{L^{p(\cdot)}(B_{j+1})} \frac{\|\chi_{B_0}\|_{L^{p(\cdot)}(\Omega)} |B_{j+1}|}{\|\chi_{B_{j+1}}\|_{L^{p(\cdot)}(\Omega)} |B_j|} \\ &\leq C [f]_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)} \sum_{j=0}^{J-1} 2^{-(j+1)n\delta} (2^{j+1}r)^{\frac{\lambda(x_0)}{p(x_0)}} \\ &\leq C r^{\frac{\lambda(x_0)}{p(x_0)}} [f]_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)} \sum_{j=0}^{J-1} 2^{(j+1)\left(\frac{\lambda(x_0)}{p(x_0)} - n\delta\right)} \\ &\leq C r^{\frac{\lambda(x_0)}{p(x_0)}} [f]_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)}. \end{aligned}$$

By hypothesis, we have $\frac{\lambda(x_0)}{p(x_0)} - n\delta < 0$, hence the geometric series converges and the sum is bounded by a constant independent of r and J . Combining all the estimates back into (4.3), we complete the proof of

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq Cr^{\frac{\lambda(x_0)}{p(x_0)}} \left(\|f\|_{L^{p(\cdot)}(\Omega)} + [f]_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \right) = Cr^{\frac{\lambda(x_0)}{p(x_0)}} \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}.$$

Multiplying both sides by $r^{-\frac{\lambda(x_0)}{p(x_0)}}$ and taking the supremum over all cubes $Q \subset \Omega$ yields the desired inequality

$$\|f\|_{M^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C \|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}.$$

□

Remark 4.1. For a bounded open set $\Omega \subset \mathbb{R}^n$, one has the continuous embedding

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) \hookrightarrow M^{p(\cdot),\lambda(\cdot)}(\Omega).$$

Composing this with [22, Theorem 2] quoted above immediately yields

$$I_\Omega^\alpha : \mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) \longrightarrow \mathcal{L}^{p(\cdot),\lambda(\cdot)+\alpha p(\cdot)}(\Omega).$$

Thus, the boundedness of Riesz potentials between Campanato spaces with variable exponents is already a consequence of the known Campanato spaces with variable exponents to Campanato spaces with variable exponents estimate.

In order to apply the wavelet characterization, we introduce the matrix of I_α with respect to the nonhomogeneous orthonormal wavelet basis constructed from regular Coiflet wavelets. Let $\Phi^0 := \varphi$ be the scaling function and, for $\varepsilon \in \{1, \dots, 2^n - 1\}$, let Φ^ε be the corresponding wavelets with m vanishing moments and compact support. The nonhomogeneous basis consists of

$$\left\{ \Phi_{0,k}^0(x) \right\}_{k \in \mathbb{Z}^n} \cup \left\{ \Phi_{j,k}^\varepsilon(x) \right\}_{j \geq 0, k \in \mathbb{Z}^n, 1 \leq \varepsilon \leq 2^n - 1}.$$

For indices

$$(j, \varepsilon) \in (\{0\} \times \{0\}) \cup (\mathbb{Z}_{\geq 0} \times \{1, \dots, 2^n - 1\}), \quad k \in \mathbb{Z}^n,$$

and similarly for (j', ε') , we define the matrix elements

$$c_{j,k;j',k'}^{\varepsilon,\varepsilon'} = \left\langle I_\alpha \Phi_{j,k}^\varepsilon, \Phi_{j',k'}^{\varepsilon'} \right\rangle. \quad (4.4)$$

For uniformity, we keep the same symbol c for all blocks; the indices completely determine whether a scaling function or a wavelet is involved. The proof of the matrix decay estimates for the Riesz potential in Theorem 4.2 follows the almost diagonal approach in [6, Theorem 1], adapted here to the nonhomogeneous wavelet basis and to the Campanato spaces with variable exponents.

Theorem 4.2. Let $c_{j,k;j',k'}^{\varepsilon,\varepsilon'}$ be the matrix of I_α with respect to the nonhomogeneous wavelet basis of Section 3. Assume $m > \alpha + n/2$ and $m > \lambda_+$. Then for every integer $N \leq m$, there exists $C = C(\Phi, \alpha, n, N) > 0$ such that the following hold.

(1) If both entries are wavelets with $\varepsilon, \varepsilon' \geq 1$ and $j, j' \geq 0$, then

$$|c_{j,k;j',k'}^{\varepsilon,\varepsilon'}| \leq C 2^{-\alpha \min(j,j') - |j-j'| (m+\alpha+n/2)} \left(1 + 2^{\min(j,j')} |2^{-j}k - 2^{-j'}k'|\right)^{-n-N}, \quad (4.5)$$

and

$$\max \left(\sum_{k,\varepsilon} |c_{j,k;j',k'}^{\varepsilon,\varepsilon'}|^2, \sum_{k',\varepsilon'} |c_{j,k;j',k'}^{\varepsilon,\varepsilon'}|^2 \right)^{1/2} \leq C 2^{-\alpha \min(j,j') - |j-j'| (m+\alpha)}. \quad (4.6)$$

(2) If both entries are scaling functions, i.e., $\varepsilon = \varepsilon' = 0$ and $j = j' = 0$, then

$$|c_{0,k;0,k'}^{0,0}| \leq C (1 + |k - k'|)^{-n-\alpha}, \quad \left(\sum_k |c_{0,k;0,k'}^{0,0}|^2 \right)^{1/2} \leq C.$$

(3) If one entry is a wavelet and the other is the scaling function, i.e., either $\varepsilon \geq 1, \varepsilon' = 0, j \geq 0, j' = 0$, or the symmetric case, then

$$|c_{j,k;0,k'}^{\varepsilon,0}| \leq C 2^{-j(m+\alpha+n/2)} \left(1 + |2^{-j}k - k'|\right)^{-n-N}, \quad (4.7)$$

$$\max \left(\sum_{k,\varepsilon} |c_{j,k;0,k'}^{\varepsilon,0}|^2, \sum_{k'} |c_{j,k;0,k'}^{\varepsilon,0}|^2 \right)^{1/2} \leq C 2^{-j(m+\alpha)}. \quad (4.8)$$

The symmetric case $\varepsilon = 0, \varepsilon' \geq 1$ satisfies identical bounds.

Proof. Consider the coefficient mapping π defined by the nonhomogeneous wavelet basis of Section 3,

$$\pi f = (a_k(f), d_{j,k}^\varepsilon(f))_{k \in \mathbb{Z}^n, j \in \mathbb{Z}_{\geq 0}, 1 \leq \varepsilon \leq 2^n - 1},$$

where $a_k(f) = \langle f, \Phi_{0,k}^0 \rangle$ and $d_{j,k}^\varepsilon(f) = \langle f, \Phi_{j,k}^\varepsilon \rangle$. By the wavelet characterization Theorem 3.1, π is an isomorphism between $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$ and the sequence space C defined by the wavelet norm in that theorem. Its inverse is given by the expansion (3.1).

Identifying, as usual for a fixed wavelet basis, the matrix $(c_{j,k;j',k'}^{\varepsilon,\varepsilon'})$ with the linear operator represented by it, we obtain for the Riesz potential I^α

$$(c_{j,k;j',k'}^{\varepsilon,\varepsilon'}) = \pi I^\alpha \pi^{-1}.$$

We now prove the decay estimates for the three types of matrix entries. By symmetry it suffices to consider $j \geq j'$; the case $j < j'$ follows from the self-adjointness of I_α . Set $\ell = j - j' \geq 0$, so $\min(j, j') = j'$. Using the Fourier multiplier representation,

$$c_{j,k;j',k'}^{\varepsilon,\varepsilon'} = 2^{-(j+j')n/2} \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{-i(2^{-j}k - 2^{-j'}k') \cdot \xi} \widehat{\Phi^\varepsilon}(2^{-j}\xi) \overline{\widehat{\Phi^{\varepsilon'}}(2^{-j'}\xi)} d\xi.$$

Change variables $\eta = 2^{-j'}\xi$, so $\xi = 2^{j'}\eta$, $d\xi = 2^{j'n}d\eta$, and $2^{-j}\xi = 2^{-\ell}\eta$. The prefactor becomes $2^{-(j+j')n/2} \cdot 2^{j'n} = 2^{-\ell n/2}$. The phase rewrites as $e^{-i(2^{-\ell}k - k') \cdot \eta} = e^{i(k' - 2^{-\ell}k) \cdot \eta}$, and $|\xi|^{-\alpha} = 2^{-j'\alpha}|\eta|^{-\alpha}$. Consequently

$$\begin{aligned} c_{j,k;j',k'}^{\varepsilon,\varepsilon'} &= 2^{-j'\alpha - \ell n/2} \int_{\mathbb{R}^n} e^{it \cdot \eta} |\eta|^{-\alpha} \widehat{\Phi^\varepsilon}(2^{-\ell}\eta) \overline{\widehat{\Phi^{\varepsilon'}}(\eta)} d\eta, \quad t = k' - 2^{-\ell}k \\ &=: 2^{-j'\alpha - \ell n/2} J(t, \ell). \end{aligned} \quad (4.9)$$

To prove the decay estimates for the wavelet block (4.5) and (4.6), we argue directly via Fourier transform and vanishing moments. As a consistency check, note that by Remark 4.1, one also has $I_\Omega^\alpha \Phi^\varepsilon \in \mathcal{L}^{p(\cdot), \lambda(\cdot) + \alpha p(\cdot)}(\Omega)$, but the estimates below are independent of this fact.

Because $\Phi^{\varepsilon'}$ has m vanishing moments, its Fourier transform admits the representation

$$\widehat{\Phi^{\varepsilon'}}(\eta) = \sum_{|\mu|=m} \eta^\mu R_{\varepsilon', \mu}(\eta), \quad R_{\varepsilon', \mu} \in \mathcal{S}(\Omega).$$

Inserting this into the definition of $J(t, \ell)$ yields

$$J(t, \ell) = 2^{-\ell m} \sum_{\nu, \mu} \int_{\mathbb{R}^n} e^{it \cdot \eta} |\eta|^{-\alpha} \eta^{\nu+\mu} R_{\varepsilon, \nu}(2^{-\ell} \eta) \overline{R_{\varepsilon', \mu}(\eta)} d\eta.$$

Define $F_{\nu, \mu}(\eta) = |\eta|^{-\alpha} \eta^{\nu+\mu} R_{\varepsilon, \nu}(2^{-\ell} \eta) \overline{R_{\varepsilon', \mu}(\eta)}$, so that $J(t, \ell) = 2^{-\ell m} \sum_{\nu, \mu} \widehat{F_{\nu, \mu}}(-t)$. Using the standard Fourier-transform estimate, for any $N \leq m$,

$$|J(t, \ell)| \leq C 2^{-\ell m} (1 + |t|)^{-n-N}.$$

Returning to (4.9) and using $|t| \geq c 2^{\min(j, j')} |2^{-j}k - 2^{-j'}k'|$, we have

$$c_{j, k; j', k'}^{\varepsilon, \varepsilon'} = 2^{-j' \alpha - \ell n/2} J(t, \ell) \lesssim 2^{-j' \alpha} 2^{-\ell(m+n/2)} (1 + |t|)^{-n-N}.$$

The opposite case $j < j'$ is symmetric, so we deduce precisely (4.5).

Again assume $j \geq j'$ and set $\ell = j - j'$. The compact support of the wavelets implies that $c_{j, k; j', k'}^{\varepsilon, \varepsilon'} \neq 0$ only when $|2^{-j}k - 2^{-j'}k'| \leq C(2^{-j} + 2^{-j'}) \leq C2^{-j'}$, i.e., $|k' - 2^{-\ell}k| \leq C2^\ell$. Thus, for fixed j, j', k' , the number of relevant k is at most $C2^{n\ell}$. Using (4.5), we bound each term by $C 2^{-\alpha j' - \ell(m+\alpha+n/2)}$. Squaring and summing over k, ε gives

$$\sum_{k, \varepsilon} |c|^2 \leq C 2^{n\ell} \cdot 2^{-2\alpha j' - 2\ell(m+\alpha+n/2)} = C 2^{-2\alpha j' - 2\ell(m+\alpha)}.$$

Thus

$$\left(\sum_{k, \varepsilon} |c|^2 \right)^{1/2} \leq C 2^{-\alpha j' - \ell(m+\alpha)} = C 2^{-\alpha \min(j, j') - |j - j'| (m+\alpha)}.$$

The symmetric argument for the sum over k', ε' gives the same bound, proving (4.6).

For the second block $\varepsilon = \varepsilon' = 0$ and $j = j' = 0$, we have

$$c_{0, k; 0, k'}^{0, 0} = \langle I_\alpha \varphi(\cdot - k), \varphi(\cdot - k') \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(k' - k) \cdot \xi} |\xi|^{-\alpha} |\widehat{\varphi}(\xi)|^2 d\xi.$$

Since φ is sufficiently smooth, the inverse Fourier transform of $|\xi|^{-\alpha} |\widehat{\varphi}(\xi)|^2$ decays like $(1 + |x|)^{-n-\alpha}$. Hence $|c_{0, k; 0, k'}^{0, 0}| \leq C(1 + |k - k'|)^{-n-\alpha}$, from which the ℓ^2 bound follows immediately.

For the mixed block with $\varepsilon \geq 1, \varepsilon' = 0, j \geq 0$, and $j' = 0$, similar Fourier arguments yield

$$|c_{j, k; 0, k'}^{\varepsilon, 0}| \leq C 2^{-j(m+\alpha+n/2)} (1 + |2^{-j}k - k'|)^{-n-N},$$

and the ℓ^2 estimate follows by counting. The symmetric case $\varepsilon = 0, \varepsilon' \geq 1$ is identical. This completes the proof of Theorem 4.2. $\square$

Remark 4.3. In the context of Campanato spaces with variable exponents, applying the estimates of Theorem 4.2 to the wavelet expansion of $I_\Omega^\alpha f$ and using the compact support of the wavelets yields the following key local square-function inequality:

$$\left(\sum_{\varepsilon, k: Q_{j,k} \subset Q} |d_{j,k}^\varepsilon(I_\Omega^\alpha f)|^2 \right)^{1/2} \leq C 2^{-j\alpha} \left(\sum_{\varepsilon, k: Q_{j,k} \subset 3Q} |d_{j,k}^\varepsilon(f)|^2 \right)^{1/2} + C 2^{-j(m+\alpha)} \|f\|_{L^{p(\cdot)}(\Omega)}, \quad (4.10)$$

where C is independent of j and Q . The second term arises from the mixed blocks and can be absorbed into the first when m is large; it does not affect the subsequent Campanato estimates.

Theorem 4.4. Let $p(\cdot) \in \mathbb{P}^{\log}(\Omega)$, $\lambda(\cdot), \mu(\cdot) \in L^\infty(\Omega)$ satisfy the log-Hölder conditions, with $0 \leq \lambda(x), \mu(x) \leq n$. Assume that for a.e. $x \in \Omega$,

$$\mu(x) = \lambda(x) + \alpha p(x),$$

and that the wavelet regularity satisfies $m > \|\lambda\|_\infty$. Then the Riesz potential I_α is bounded from $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{p(\cdot), \mu(\cdot)}(\Omega)$, i.e.,

$$\|I_\Omega^\alpha f\|_{\mathcal{L}^{p(\cdot), \mu(\cdot)}(\Omega)} \leq C \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)},$$

where $C > 0$ is independent of f .

Proof. We work with the wavelet norm equivalence proved in Theorem 3.1. Fix a dyadic cube Q with side length $\ell(Q) = 2^{-j}$ and center c_Q . It suffices to estimate

$$S_Q(I_\Omega^\alpha f) := 2^{j\left(\frac{\mu(c_Q)}{p(c_Q)} + \frac{n}{2}\right)} \left\| \left(\sum_{\varepsilon, k: Q_{j,k} \subset Q} |d_{j,k}^\varepsilon(I_\Omega^\alpha f)|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(Q)}. \quad (4.11)$$

Expand f and $I_\Omega^\alpha f$ in the nonhomogeneous wavelet basis:

$$f = \sum_{k \in \mathbb{Z}^n} a_k(f) \Phi_{0,k}^0 + \sum_{\varepsilon=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} d_{j,k}^\varepsilon(f) \Phi_{j,k}^\varepsilon,$$

and similarly for $I_\Omega^\alpha f$ with coefficients $a_k(I_\Omega^\alpha f)$, $d_{j,k}^\varepsilon(I_\Omega^\alpha f)$. Using the matrix elements (4.4),

$$d_{j,k}^\varepsilon(I_\Omega^\alpha f) = \sum_{k'} a_{k'}(f) c_{j,k;0,k'}^{\varepsilon,0} + \sum_{\varepsilon'=1}^{2^n-1} \sum_{j'=0}^{\infty} \sum_{k'} d_{j',k'}^{\varepsilon'}(f) c_{j,k;j',k'}^{\varepsilon,\varepsilon'}.$$

The first sum involves the mixed block estimates and the second the wavelet-wavelet block. By the large vanishing moment condition $m > \lambda_+$, the mixed block terms carry an extra factor $2^{-j(m+\alpha)}$ (see (4.8)), which makes their contribution negligible for the Campanato seminorm. More precisely, applying the ℓ^2 bound of Theorem 4.2(3) and the fact that $\|a(f)\|_{\ell^2} \leq C \|f\|_{L^2} \leq C \|f\|_{L^{p(\cdot)}}$, we obtain

$$\left(\sum_{k, \varepsilon} \left| \sum_{k'} a_{k'}(f) c_{j,k;0,k'}^{\varepsilon,0} \right|^2 \right)^{1/2} \leq C 2^{-j(m+\alpha)} \|f\|_{L^{p(\cdot)}(\Omega)}.$$

Since $2^{j\frac{\mu}{p}} \approx 2^{j(\frac{\lambda}{p} + \alpha)}$, multiplying by this factor gives $2^{-j(m-\lambda_+/p)}$, which tends to 0 as $j \rightarrow \infty$ because $m > \lambda_+$. Thus this part is bounded by a constant multiple of $\|f\|_{L^{p(\cdot)}(\Omega)}$ and does not affect the supremum over Q .

For the main contribution, we follow the same pattern as in the homogeneous case. We split the sum over j' into $|j - j'| \leq L$ and $|j - j'| > L$. The distant scales are handled by the exponential decay in (4.5) exactly as before. For close scales, (4.5) and Schur's test yield the local square function estimate

$$\left(\sum_{\substack{\varepsilon, k: \\ Q_{j,k} \subset Q}} \left| \sum_{\substack{\varepsilon', j', k': \\ |j-j'| \leq L}} d_{j',k'}^{\varepsilon'} c_{j,k;j',k'}^{\varepsilon, \varepsilon'} \right|^2 \right)^{1/2} \leq C 2^{-j\alpha} \left(\sum_{j': |j-j'| \leq L} \sum_{\substack{\varepsilon', k': \\ Q_{j',k'} \subset 3Q}} |d_{j',k'}^{\varepsilon'}|^2 \right)^{1/2}.$$

Taking the $L^{p(\cdot)}(Q)$ norm, covering $3Q$ by dyadic cubes Q_i of the same size, and applying the wavelet norm definition to each Q_i , we obtain

$$\begin{aligned} S_Q(I_\Omega^\alpha f) &\leq C 2^{j(\frac{\mu(c_Q)}{p(c_Q)} + \frac{n}{2} - \alpha)} \sum_i 2^{-j(\frac{\lambda(c_{Q_i})}{p(c_{Q_i})} + \frac{n}{2})} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)}^\# \\ &\leq C 2^{j(\frac{\lambda(c_Q)}{p(c_Q)} + \frac{n}{2})} 2^{-j(\frac{\lambda(c_Q)}{p(c_Q)} + \frac{n}{2})} \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)}^\# = C \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)}^\#. \end{aligned}$$

Taking the supremum over all dyadic cubes Q and adding the $L^{p(\cdot)}$ norm (which is controlled by the known $L^{p(\cdot)}$ -boundedness of I_α or directly by the above estimate for $j = 0$), we conclude

$$\|I_\Omega^\alpha f\|_{\mathcal{L}^{p(\cdot), \mu(\cdot)}(\Omega)}^\# \leq C \|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)}^\#.$$

Finally, the norm equivalence of Theorem 3.1 transforms this discrete inequality into the desired continuous bound. This completes the proof. $\square$

Author contributions

Chenchen Niu: Conceptualization, Writing—original draft; Hongbin Wang: Supervision, Writing—review & editing; Zongguang Liu: Supervision, Writing—review & editing, Funding acquisition. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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