



Research article

Simons' inequality in warped product manifolds and some applications

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Abstract: Let

$$(\overline{M}^{m+1}, \bar{g}) = (I \times_f F^m, dt^2 + f(t)^2 g_F),$$

where F has constant sectional curvature, and let $\Sigma^m \subset \overline{M}^{m+1}$ be a two-sided minimal hypersurface. We derived an explicit Simons identity in terms of

$$\alpha = -\frac{f''}{f}, \quad \beta = \frac{k_F - (f')^2}{f^2}, \quad \Xi = (\alpha - \beta)' - 2\frac{f'}{f}(\alpha - \beta).$$

When $\Xi = 0$, we determined the optimal angle-free algebraic lower coefficient for the curvature term and obtained a compact pinching theorem, recovering the classical spherical gap constant. For stable minimal hypersurfaces, a power Caccioppoli estimate, local Sobolev inequality, and Moser iteration yield, for $3 \leq m \leq 5$,

$$|A|^2(x_0) \leq C \left(r^{-2} + \sup_{B_{2r}(x_0)} (|\alpha| + |\beta| + |\Xi|^{2/3}) \right).$$

Thus the classical Schoen–Simon–Yau (SSY) curvature-estimate mechanism extends to this warped-product setting with an explicit scale-correct curvature term.

Keywords: minimal hypersurface; Simons identity; warped product; second fundamental form; pinching; stability; curvature estimate

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1. Introduction

The Simons identity is a fundamental bridge between the extrinsic geometry of a minimal hypersurface and elliptic analysis. For a minimal hypersurface $\Sigma^m \subset \mathbb{R}^{m+1}$, Simons' formula gives

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4. \quad (1.1)$$

Together with the refined Kato inequality

$$|\nabla A|^2 \geq \left(1 + \frac{2}{m}\right)|\nabla|A||^2, \quad (1.2)$$

this yields

$$\Delta|A|^2 \geq -2|A|^4 + 2\left(1 + \frac{2}{m}\right)|\nabla|A||^2. \quad (1.3)$$

The combination of (1.3) with the stability inequality is one of the main analytic ingredients in the curvature estimates of Schoen–Simon–Yau [22]. Their argument produces a characteristic integrability range $p < 4 + \sqrt{8/m}$, from which the classical low-dimensional threshold $m \leq 5$ emerges when one seeks an exponent strictly larger than the critical dimension. Subsequent refinements and regularity arguments were developed by Simon and by Schoen–Simon [21, 23]; see also [9] for a modern partial differential equation (PDE)-oriented account. For stable minimal surfaces in three-dimensional ambient manifolds, Schoen also established curvature estimates [20].

The corresponding problem in a general ambient manifold is more delicate. The Codazzi equation contains an ambient curvature defect, and after commuting covariant derivatives of A , first covariant derivatives of the ambient curvature appear. In particular, a formula of the schematic form

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4 + \overline{\text{Rm}} * A * A$$

is incomplete unless the ambient curvature is parallel. Simons' general second-order system already exhibits a term depending on $\overline{\nabla}\text{Rm}$; see [24]. Recent studies have addressed rigidity phenomena for totally real submanifolds in complex space forms [2].

The classical theory was subsequently extended in several directions. Nomizu and Smyth derived a Simons-type formula for constant-mean-curvature hypersurfaces in space forms [17], while the spherical rigidity and pinching theory was developed further by Chern–do Carmo–Kobayashi and Lawson [8, 13]. Stability questions for constant-mean-curvature hypersurfaces were also studied by Barbosa and do Carmo [5]. In product geometries, Fetcu and Rosenberg obtained Simons-type formulas for submanifolds with parallel mean curvature [10]. Warped products themselves form a standard and flexible ambient class, beginning with the curvature formulas of Bishop and O'Neill [6]; rigidity and constant-mean-curvature questions in this setting were studied by Montiel, Alías–Dajczer, and Brendle [1, 7, 16]. Recent work has also investigated geometric inequalities and optimal estimates for warped-product submanifolds in various ambient geometries [3, 4]. In a related direction, Lobos–Melara–Santos derived a Simons-type formula for spacelike submanifolds in semi-Riemannian warped products [14]. The present paper concerns the Riemannian minimal-hypersurface problem and focuses on an explicit reduction of both the zeroth-order and derivative-curvature contractions in the model $I \times_f F^m(k_F)$.

The goal of this paper is to make this structure fully explicit in the important class

$$\overline{M}^{m+1} = I \times_f F^m.$$

When the fiber F is a space form, the ambient curvature is determined by two sectional curvatures,

$$\alpha = -\frac{f''}{f}, \quad \beta = \frac{k_F - (f')^2}{f^2}.$$

The difference $\gamma = \alpha - \beta$ measures the anisotropy between radial and fiber planes. We show that every zeroth-order ambient term in Simons' formula can be reduced to α, β, θ , and $A(T^\top)$, while all first-derivative curvature terms collapse to the scalar

$$\Xi = (\alpha - \beta)' - 2\frac{f'}{f}(\alpha - \beta).$$

Our first main theorem is the following exact identity.

Theorem 1.1 (Explicit warped-product Simons identity). *Let $m \geq 2$, let $F^m(k_F)$ be a space form of constant sectional curvature k_F , and let*

$$\overline{M}^{m+1} = I \times_f F^m(k_F)$$

with the metric $dt^2 + f^2 g_F$. Let $\Sigma^m \subset \overline{M}^{m+1}$ be a two-sided minimal hypersurface with unit normal ν , second fundamental form A , and

$$T = \partial_t, \quad \theta = \langle \nu, T \rangle, \quad T^\top = T - \theta \nu.$$

Set

$$\alpha = -\frac{f''}{f}, \quad \beta = \frac{k_F - (f')^2}{f^2}, \quad \gamma = \alpha - \beta, \quad \rho = \frac{f'}{f},$$

and

$$\Xi = \gamma' - 2\rho\gamma.$$

With the curvature convention fixed in Section 2, one has

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 - |A|^4 \\ &\quad + [m\beta + \gamma(1 - (m+1)\theta^2)]|A|^2 \\ &\quad + 2m\gamma|A(T^\top)|^2 - m\theta\Xi A(T^\top, T^\top). \end{aligned} \tag{1.4}$$

The formula immediately recovers the space-form identity because $\alpha = \beta = c$ and $\Xi = 0$:

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + mc|A|^2 - |A|^4.$$

The second main point concerns the optimal curvature coefficient available from (1.4) when the derivative term vanishes.

Theorem 1.2 (Optimal angle-free curvature coefficient). *Assume the hypotheses of Theorem 1.1 and suppose $\Xi = 0$. Define*

$$\kappa_{\text{wp}}(t) = \begin{cases} m(2\beta - \alpha), & \alpha \geq \beta, \\ (2m - 1)\alpha - (m - 1)\beta, & \alpha < \beta. \end{cases} \quad (1.5)$$

Then

$$\frac{1}{2}\Delta|A|^2 \geq |\nabla A|^2 + (\kappa_{\text{wp}} - |A|^2)|A|^2. \quad (1.6)$$

Moreover, at each fixed t , the coefficient $\kappa_{\text{wp}}(t)$ is the largest number depending only on $\alpha(t), \beta(t)$ for which the quadratic ambient curvature term in (1.4) is bounded below by $\kappa_{\text{wp}}(t)|A|^2$ for every angle $\theta \in [-1, 1]$ and every nonzero trace-free symmetric bilinear form A .

Here “optimal” is deliberately used in an algebraic sense. It does not assert that every equality configuration is realized by a global minimal hypersurface.

A direct maximum-principle consequence is:

Corollary 1.3 (Compact pinching). *Under the hypotheses of Theorem 1.2, assume Σ is compact and*

$$\kappa_0 := \inf_{\Sigma} \kappa_{\text{wp}} > 0.$$

If

$$\sup_{\Sigma} |A|^2 < \kappa_0,$$

then $A \equiv 0$. If $|A|^2 \leq \kappa_0$ and $\kappa_{\text{wp}} \equiv \kappa_0$ along Σ , then either $A \equiv 0$, or

$$|A|^2 \equiv \kappa_0, \quad \nabla A \equiv 0,$$

and equality holds in the algebraic estimate of Theorem 1.2 at every point.

For the unit sphere, represented as $(0, \pi) \times_{\sin t} \mathbb{S}^m$, one has $\alpha = \beta = 1$, hence $\kappa_{\text{wp}} = m$. Therefore the classical sharp threshold $|A|^2 < m$ is recovered.

A third main result converts the differential inequality into a genuine pointwise curvature estimate. For a domain $\Omega \subset \Sigma$, define the scale-correct warped curvature quantity

$$\Lambda_{\text{wp}}(\Omega) = \sup_{\Omega} (|\alpha| + |\beta| + |\Xi|^{2/3}). \quad (1.7)$$

Here and below, C_m denotes a positive constant depending only on m , while $C_{m,\varepsilon}$ depends only on m and ε . The supremum in $\Lambda_{\text{wp}}(\Omega)$ is taken over the indicated region $\Omega \subset \Sigma$; in (1.8), this region is $B_{2r}^{\Sigma}(x_0)$.

Theorem 1.4 (Warped Schoen–Simon–Yau curvature estimate). *Let $3 \leq m \leq 5$, and let $\Sigma^m \subset I \times_f F^m(k_F)$ be two-sided, minimal, and stable. Fix $x_0 \in \Sigma$ and assume $B_{2r}^{\Sigma}(x_0) \Subset \Sigma$. Suppose that on every concentric ball $B_s^{\Sigma}(x_0)$, $0 < s \leq 2r$, one has the scale-invariant Sobolev inequality*

$$\left(\int_{B_s} |\phi|^{2\chi} d\mu_{\Sigma} \right)^{1/\chi} \leq S_0 \int_{B_s} (|\nabla \phi|^2 + s^{-2} \phi^2) d\mu_{\Sigma}, \quad \chi = \frac{m}{m-2},$$

and the volume bound

$$|B_s^{\Sigma}(x_0)| \leq V_0 s^m.$$

Then

$$|A|^2(x_0) \leq C(m, S_0, V_0) \left[r^{-2} + \Lambda_{\text{wp}}(B_{2r}^{\Sigma}(x_0)) \right]. \quad (1.8)$$

The proof uses the scalar warped Simons inequality, stability, a power Caccioppoli estimate, a Sobolev gain, and a final Moser iteration. The restriction $m \leq 5$ is not inserted ad hoc: it is precisely the range in which the admissible Caccioppoli exponents include one giving $|A| \in L^p$ for some $p > m$, so that the nonlinear potential in the Simons inequality becomes supercritical for Moser iteration.

The quantity

$$\Lambda_{\text{wp}} = \sup(|\alpha| + |\beta| + |\Xi|^{2/3})$$

is the natural scale-correct combination of the zeroth- and first-order ambient curvature effects: α and β have the scale of curvature, while $|\Xi|^{2/3}$ has the same homogeneity. Consequently, the pointwise estimate records the dependence on the warping data f, f', f'', f''' without hiding these effects in a generic ambient curvature constant.

The paper is organized into five numbered sections. After this introduction, Section 2 collects the general ambient Simons identity and the warped-product curvature formulas. In Section 3, the ambient contractions are evaluated explicitly and combined with a perturbed refined Kato inequality. Section 4 contains the optimal algebraic coefficient, the compact pinching theorem, the space-form consistency checks, and the equality/model discussion. Finally, Section 5 develops the analytic applications: the non-parallel-curvature perturbation, stability, a power Caccioppoli inequality, the Sobolev gain, and the Schoen–Simon–Yau-type pointwise estimate (1.8).

2. Geometric preliminaries and the general Simons formula

Let $(\bar{M}^{m+1}, \bar{g})$ be a Riemannian manifold and let $\Sigma^m \subset \bar{M}^{m+1}$ be a two-sided immersed hypersurface with unit normal ν . We use the curvature convention

$$\bar{R}(X, Y)Z = \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_X \bar{\nabla}_Y Z + \bar{\nabla}_{[X, Y]} Z \quad (2.1)$$

and

$$\bar{R}_{ABCD} = \langle \bar{R}(E_A, E_B)E_C, E_D \rangle.$$

Our convention is chosen so that a space form of sectional curvature c satisfies

$$\bar{R}_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \quad (2.2)$$

Let $e_1, \dots, e_m$ be a local orthonormal tangent frame. We write

$$h_{ij} = \langle \bar{\nabla}_{e_i} e_j, \nu \rangle, \quad A = (h_{ij}), \quad H = \sum_i h_{ii}.$$

Throughout the main part of the paper,

$$H = 0.$$

The Codazzi equation reads

$$h_{ij;k} - h_{ik;j} = \bar{R}_{vikj}. \quad (2.3)$$

For a minimal hypersurface, differentiating $H = 0$ gives

$$\sum_i h_{ii;k} = 0. \quad (2.4)$$

We next state the index-level Simons identity in a form directly adapted to our later substitution.

Proposition 2.1 (General ambient identity). *Let $\Sigma^m \subset \overline{M}^{m+1}$ be minimal. At a point p , choose the tangent frame geodesic for the induced connection. Then*

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4 + Q_{\overline{R}}(A) + \mathcal{D}_{\overline{\nabla R}}(A), \quad (2.5)$$

where

$$\begin{aligned} Q_{\overline{R}}(A) = & - \sum_{i,j,k} h_{ij}^2 \overline{R}_{vkvk} \\ & + 2 \sum_{m,i,j,k} h_{mj} h_{ij} \overline{R}_{mkik} + 2 \sum_{m,i,j,k} h_{mk} h_{ij} \overline{R}_{mijk} \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \mathcal{D}_{\overline{\nabla R}}(A) = & \sum_{i,j,k} h_{ij} (\overline{\nabla}_{e_k} \overline{R})_{vikj} \\ & - \sum_{i,j,k} h_{ij} (\overline{\nabla}_{e_j} \overline{R})_{vkik}. \end{aligned} \quad (2.7)$$

In particular,

$$\frac{1}{2}\Delta|A|^2 \geq |\nabla A|^2 - |A|^4 - C_m |\overline{\text{Rm}}| |A|^2 - C_m |\overline{\nabla \text{Rm}}| |A|. \quad (2.8)$$

Proof. At p ,

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i,j} h_{ij} \Delta h_{ij}.$$

Starting from (2.3), commute the second derivatives of h_{ij} . The commutator terms acting on the two covariant indices of h are expressed through the intrinsic curvature of Σ , and the Gauss equation

$$R_{ijkl}^\Sigma = \overline{R}_{ijkl} + h_{ik} h_{jl} - h_{il} h_{jk}$$

splits them into ambient-curvature terms and cubic expressions in h . After contraction with h_{ij} , the cubic part equals $-|A|^4$ because $H = 0$. The differentiated Codazzi defect yields the two terms in (2.7). Collecting the remaining curvature contractions gives (2.6). This is the hypersurface specialization of the second-order system derived by Simons; compare also the index formula used by Schoen–Simon–Yau [22, 24].

Finally, every term in $Q_{\overline{R}}$ contains two factors of A and one factor of $\overline{\text{Rm}}$, while every term in $\mathcal{D}_{\overline{\nabla R}}$ contains one factor of A and one factor of $\overline{\nabla \text{Rm}}$. Since the number of contractions depends only on m , Cauchy–Schwarz gives (2.8). $\square$

Remark 2.2. *The derivative term $\mathcal{D}_{\overline{\nabla R}}$ vanishes in locally symmetric ambient spaces, in particular, in all space forms. It is generally nonzero in a warped product with variable radial and fiber sectional curvatures.*

2.1. Warped-product geometry

The formulas used below are the standard Bishop–O’Neill warped-product identities; see [6, 18] and, for background on curvature conventions [19].

Let

$$\overline{M} = I \times_f F^m, \quad \bar{g} = dt^2 + f(t)^2 g_F, \quad f > 0.$$

Write

$$T = \partial_t, \quad \rho = \frac{f'}{f}.$$

In the following connection formulas, horizontal and vertical vector fields are understood as their natural lifts to the warped product $I \times_f F^m$.

If U, V are vertical vector fields, then the Levi-Civita connection is

$$\bar{\nabla}_T T = 0, \quad \bar{\nabla}_U T = \bar{\nabla}_T U = \rho U, \quad (2.9)$$

and

$$\bar{\nabla}_U V = \nabla_U^F V - \rho \langle U, V \rangle T, \quad (2.10)$$

where the vertical term is interpreted after the standard lift and rescaling.

Equivalently, for an arbitrary ambient vector X ,

$$\bar{\nabla}_X T = \rho(X - \langle X, T \rangle T). \quad (2.11)$$

Assume from now on that F has constant sectional curvature k_F . Define

$$\alpha = -\frac{f''}{f}, \quad \beta = \frac{k_F - (f')^2}{f^2}, \quad \gamma = \alpha - \beta. \quad (2.12)$$

Thus α is the sectional curvature of every plane containing T , whereas β is the sectional curvature of every vertical two-plane.

Let

$$\eta = \bar{g}(T, \cdot).$$

For arbitrary ambient vectors, the full Riemann tensor can be written as

$$\bar{R} = \beta G + \gamma U, \quad (2.13)$$

where

$$G_{ABCD} = \bar{g}_{AC}\bar{g}_{BD} - \bar{g}_{AD}\bar{g}_{BC} \quad (2.14)$$

and

$$\begin{aligned} U_{ABCD} = & \eta_A \eta_C \bar{g}_{BD} + \eta_B \eta_D \bar{g}_{AC} \\ & - \eta_A \eta_D \bar{g}_{BC} - \eta_B \eta_C \bar{g}_{AD}. \end{aligned} \quad (2.15)$$

Indeed, for a unit vertical vector U ,

$$\bar{R}_{TUTU} = \beta + \gamma = \alpha,$$

while for an orthonormal vertical pair U, V ,

$$\bar{R}_{UVUV} = \beta.$$

The Ricci tensor is therefore

$$\overline{\text{Ric}}(T, T) = m\alpha, \quad (2.16)$$

and, for a unit vertical vector U ,

$$\overline{\text{Ric}}(U, U) = \alpha + (m-1)\beta. \quad (2.17)$$

The derivative of U is controlled by $\bar{\nabla}\eta$. From (2.11),

$$(\bar{\nabla}_X \eta)(Y) = \rho(\langle X, Y \rangle - \eta(X)\eta(Y)). \quad (2.18)$$

3. Explicit Simons identity and the refined Kato estimate

Let $\Sigma^m \subset I \times_f F^m$ be a two-sided minimal hypersurface. Set

$$\theta = \langle \nu, T \rangle, \quad X := T^\top = T - \theta \nu. \quad (3.1)$$

Then

$$|X|^2 = 1 - \theta^2. \quad (3.2)$$

At a point, choose an orthonormal tangent frame $e_1, \dots, e_m$, and write

$$x_i = \langle e_i, T \rangle.$$

Hence

$$X = \sum_i x_i e_i, \quad \sum_i x_i^2 = 1 - \theta^2.$$

3.1. The zeroth-order curvature contraction

Lemma 3.1. *For the curvature tensor (2.13),*

$$Q_{\bar{R}}(A) = [m\beta + \gamma(1 - (m+1)\theta^2)]|A|^2 + 2m\gamma|A(X)|^2. \quad (3.3)$$

Proof. The βG part is exactly the space-form contraction. Substituting (2.14) into (2.6) and using $H = 0$ gives

$$Q_{\beta G}(A) = m\beta|A|^2. \quad (3.4)$$

It remains to evaluate the U -part. We use repeatedly

$$\eta(\nu) = \theta, \quad \eta(e_i) = x_i.$$

For the first term of (2.6),

$$U_{\nu k \nu k} = \theta^2 + x_k^2.$$

Therefore

$$-\sum_{i,j,k} h_{ij}^2 U_{\nu k \nu k} = -(1 + (m-1)\theta^2)|A|^2. \quad (3.5)$$

Next,

$$U_{mkik} = x_m x_i + x_k^2 \delta_{mi} - x_m x_k \delta_{ki} - x_k x_i \delta_{mk}.$$

Hence

$$\begin{aligned} & 2 \sum_{m,i,j,k} h_{mj} h_{ij} U_{mkik} \\ &= 2 \left((m-2)|A(X)|^2 + (1-\theta^2)|A|^2 \right). \end{aligned} \quad (3.6)$$

Finally,

$$U_{mijk} = x_m x_j \delta_{ik} + x_i x_k \delta_{mj} - x_m x_k \delta_{ij} - x_i x_j \delta_{mk}.$$

Since $H = 0$,

$$2 \sum_{m,i,j,k} h_{mk} h_{ij} U_{mijk} = 4|A(X)|^2. \quad (3.7)$$

Adding (3.5)–(3.7) gives

$$Q_U(A) = (1 - (m+1)\theta^2)|A|^2 + 2m|A(X)|^2.$$

Combining with (3.4) proves (3.3). $\square$

3.2. The derivative-curvature contraction

The derivative term admits an unexpectedly simple reduction.

Lemma 3.2. *Define*

$$\Xi = \gamma' - 2\rho\gamma. \quad (3.8)$$

Then

$$\mathcal{D}_{\overline{\nabla}R}(A) = -m\theta \Xi A(X, X). \quad (3.9)$$

Proof. Since G is parallel,

$$\overline{\nabla}R = d\beta \otimes G + d\gamma \otimes U + \gamma \overline{\nabla}U.$$

Both contractions in (2.7) contain one normal and three tangential slots. Hence the $d\beta \otimes G$ contribution vanishes identically.

Because $d\gamma = \gamma'\eta$,

$$e_k(\gamma) = \gamma' x_k, \quad e_j(\gamma) = \gamma' x_j.$$

Furthermore

$$U_{vikj} = \theta(x_k \delta_{ij} - x_j \delta_{ik})$$

and

$$U_{vkik} = \theta(x_i - x_k \delta_{ki}).$$

Using $H = 0$, direct contraction gives

$$\begin{aligned} \sum_{i,j,k} h_{ij} x_k U_{vikj} &= \theta(|X|^2 H - A(X, X)) = -\theta A(X, X), \\ \sum_{i,j,k} h_{ij} x_j U_{vkik} &= (m-1)\theta A(X, X). \end{aligned} \quad (3.10)$$

Therefore, taking account of the subtraction in (2.7), the $d\gamma$ -contribution is

$$-m\theta\gamma' A(X, X).$$

It remains to differentiate U . From (2.18), at the point under consideration,

$$(\overline{\nabla}_{e_a} \eta)(v) = -\rho\theta x_a$$

and

$$(\overline{\nabla}_{e_a} \eta)(e_c) = \rho(\delta_{ac} - x_a x_c).$$

Differentiating the four terms in (2.15) gives

$$(\bar{\nabla}_{e_a} U)_{vbcd} = \rho\theta [(\delta_{ac} - 2x_a x_c)\delta_{bd} - (\delta_{ad} - 2x_a x_d)\delta_{bc}].$$

Consequently, the two contractions occurring in (2.7) are

$$\begin{aligned} \sum_{i,j,k} h_{ij}(\bar{\nabla}_{e_k} U)_{vikj} &= 2\rho\theta A(X, X), \\ \sum_{i,j,k} h_{ij}(\bar{\nabla}_{e_j} U)_{vikj} &= -2(m-1)\rho\theta A(X, X). \end{aligned} \quad (3.11)$$

Thus, again using the subtraction in (2.7), the $\gamma\bar{\nabla}U$ -contribution is

$$2m\rho\gamma\theta A(X, X).$$

Substituting (3.10) and (3.11) into (2.7) gives

$$\begin{aligned} \mathcal{D}_{\bar{\nabla}R}(A) &= -m\theta\gamma' A(X, X) + 2m\rho\gamma\theta A(X, X) \\ &= -m\theta(\gamma' - 2\rho\gamma)A(X, X) \\ &= -m\theta\Xi A(X, X), \end{aligned}$$

as claimed. $\square$

Proof of Theorem 1.1. Substitute Lemmas 3.1 and 3.2 into the general identity (2.5). $\square$

Remark 3.3. *The condition*

$$\Xi = 0$$

is equivalent to

$$\left(\frac{\gamma}{f^2}\right)' = 0.$$

Thus

$$\gamma = Cf^2$$

for some constant C on each connected interval. Every space form satisfies this condition because $\gamma = 0$, but $\Xi = 0$ is weaker than the constant sectional curvature.

3.3. Refined Kato inequality and the Codazzi defect

The Euclidean refined Kato inequality relies on complete symmetry of ∇A . This is the same structural feature underlying the classical Simons–Schoen–Simon–Yau estimates [21,22,24]. In the present warped product, the Codazzi defect can also be computed explicitly.

From (2.13),

$$\bar{R}_{vikj} = \gamma U_{vikj}$$

because $G_{vikj} = 0$. Thus

$$h_{ij;k} - h_{ik;j} = \gamma\theta(x_k\delta_{ij} - x_j\delta_{ik}), \quad (3.12)$$

up to the equivalent sign convention obtained by interchanging the final two curvature slots.

Consequently,

$$|\text{Alt}_{jk} \nabla A|^2 \leq C_m \gamma^2 \theta^2 (1 - \theta^2). \quad (3.13)$$

Proposition 3.4 (Perturbed refined Kato inequality). *For every $\varepsilon > 0$, there is $C_{m,\varepsilon} > 0$ such that*

$$|\nabla A|^2 \geq \left(1 + \frac{2}{m} - \varepsilon\right) |\nabla |A||^2 - C_{m,\varepsilon} \gamma^2 \theta^2 (1 - \theta^2). \quad (3.14)$$

If

$$\gamma \theta (1 - \theta^2)^{1/2} = 0,$$

then the sharp estimate

$$|\nabla A|^2 \geq \left(1 + \frac{2}{m}\right) |\nabla |A||^2 \quad (3.15)$$

holds.

Proof. At a point $p \in \Sigma$ with $|A|(p) > 0$, choose an orthonormal frame such that A is diagonal at p , and write

$$T_{ijk} = h_{ij;k}.$$

Since $h_{ij} = h_{ji}$, we have

$$T_{ijk} = T_{jik}.$$

Moreover, since $H = 0$,

$$\sum_i T_{iik} = \nabla_k H = 0 \quad \text{for every } k. \quad (3.16)$$

Let $\mathcal{V}_m$ denote the Euclidean space of tensors $T = (T_{ijk})$ satisfying

$$T_{ijk} = T_{jik}, \quad \sum_i T_{iik} = 0,$$

and let $\mathcal{S}_m \subset \mathcal{V}_m$ be the subspace of completely symmetric tensors. Because of (3.16), every tensor in $\mathcal{S}_m$ is completely symmetric and trace-free.

Define the linear map

$$\mathcal{A} : \mathcal{V}_m \longrightarrow \mathcal{W}_m, \quad (\mathcal{A}T)_{ijk} = T_{ijk} - T_{ikj}.$$

Then

$$\ker \mathcal{A} = \mathcal{S}_m.$$

Since $\mathcal{V}_m$ is finite-dimensional, the restriction of $\mathcal{A}$ to $\mathcal{S}_m^\perp$ is injective. Hence there is a constant $c_m > 0$, depending only on m , such that

$$\text{dist}(T, \mathcal{S}_m)^2 \leq c_m |\mathcal{A}T|^2. \quad (3.17)$$

Let

$$T = S + E, \quad S \in \mathcal{S}_m, \quad E \in \mathcal{S}_m^\perp$$

be the orthogonal decomposition. Thus

$$|T|^2 = |S|^2 + |E|^2, \quad |E|^2 \leq c_m |\mathcal{A}T|^2. \quad (3.18)$$

We now compare these tensors with $\nabla|A|$. Since A is diagonal at p , writing its principal curvatures as $\lambda_1, \dots, \lambda_m$, we have

$$|A| \nabla_k |A| = \sum_i \lambda_i T_{iik}.$$

For a three-tensor Q , set

$$(L_A Q)_k = \frac{1}{|A|} \sum_i \lambda_i Q_{iik}.$$

Then

$$\nabla|A| = L_A T.$$

By Cauchy–Schwarz,

$$|L_A Q| \leq |Q|. \quad (3.19)$$

For the completely symmetric trace-free tensor S , the classical finite-dimensional refined Kato estimate gives

$$|S|^2 \geq \left(1 + \frac{2}{m}\right) |L_A S|^2. \quad (3.20)$$

This is precisely the algebraic estimate underlying the classical refined Kato inequality for minimal hypersurfaces [21, 22, 24].

Set

$$c_m^* = 1 + \frac{2}{m}.$$

Since

$$L_A S = L_A T - L_A E = \nabla|A| - L_A E,$$

for every $0 < \delta < 1$,

$$|L_A S|^2 \geq (1 - \delta) |\nabla|A||^2 - \left(\frac{1}{\delta} - 1\right) |L_A E|^2.$$

Using (3.19),

$$|L_A S|^2 \geq (1 - \delta) |\nabla|A||^2 - \left(\frac{1}{\delta} - 1\right) |E|^2.$$

Therefore, by (3.18) and (3.20),

$$\begin{aligned} |\nabla A|^2 &= |T|^2 \geq |S|^2 \\ &\geq c_m^* (1 - \delta) |\nabla|A||^2 - c_m^* \left(\frac{1}{\delta} - 1\right) |E|^2. \end{aligned}$$

Given $\varepsilon > 0$, choose

$$\delta = \frac{\varepsilon}{c_m^*}$$

when $0 < \varepsilon < c_m^*$. Then

$$c_m^* (1 - \delta) = 1 + \frac{2}{m} - \varepsilon,$$

and hence

$$|\nabla A|^2 \geq \left(1 + \frac{2}{m} - \varepsilon\right) |\nabla|A||^2 - C_{m,\varepsilon} |\mathcal{A}T|^2. \quad (3.21)$$

For $\varepsilon \geq c_m^*$, the assertion is immediate after enlarging the constant.

Finally, by the Codazzi equation,

$$(\mathcal{A}T)_{ijk} = h_{ij;k} - h_{ik;j} = \gamma\theta(x_k\delta_{ij} - x_j\delta_{ik}),$$

and therefore, by (3.13),

$$|\mathcal{A}T|^2 \leq C_m\gamma^2\theta^2(1 - \theta^2).$$

Substituting this into (3.21) gives (3.14).

If

$$\gamma\theta(1 - \theta^2)^{1/2} = 0,$$

then the Codazzi defect vanishes, so $E = 0$ and $T = S$. Thus (3.20) yields directly

$$|\nabla A|^2 \geq \left(1 + \frac{2}{m}\right)|\nabla|A||^2,$$

which is (3.15). □

Combining Theorem 1.1 and proposition 3.4 yields a scalar Simons inequality suitable for iteration:

$$\begin{aligned} \Delta|A|^2 \geq & -2|A|^4 + 2\left(1 + \frac{2}{m} - \varepsilon\right)|\nabla|A||^2 \\ & + 2\kappa_{\text{wp}}|A|^2 - \frac{4m}{3\sqrt{3}}|\Xi||A| - C_{m,\varepsilon}\gamma^2\theta^2(1 - \theta^2). \end{aligned} \quad (3.22)$$

4. Sharp pinching, equality cases, and model geometries

We now optimize the quadratic curvature term in (1.4) over the angle and over all trace-free symmetric forms.

Fix a point and suppose $X \neq 0$. Rotate the tangent frame so that

$$X = \sqrt{s}e_1, \quad s = |X|^2 = 1 - \theta^2.$$

Then

$$|A(X)|^2 = s \sum_j h_{1j}^2.$$

The following elementary optimization is crucial.

Lemma 4.1 (Row-energy bound). *Let $A = (h_{ij})$ be a nonzero trace-free symmetric $m \times m$ matrix, where $m \geq 2$. Then*

$$0 \leq \frac{\sum_j h_{1j}^2}{|A|^2} \leq \frac{m-1}{m}. \quad (4.1)$$

For $m = 2$, the ratio is identically $1/2$. For $m \geq 3$, the lower value 0 is attainable, while the upper value $(m-1)/m$ is attained by a diagonal matrix proportional to

$$\text{diag}\left(1, -\frac{1}{m-1}, \dots, -\frac{1}{m-1}\right).$$

Proof. Write

$$a = h_{11}, \quad b_\mu = h_{1\mu}, \quad 2 \leq \mu \leq m.$$

Then

$$\sum_j h_{1j}^2 = a^2 + \sum_{\mu=2}^m b_\mu^2.$$

The off-diagonal entries b_μ contribute $2 \sum b_\mu^2$ to $|A|^2$.

For fixed a , the trace-free condition gives

$$\sum_{\mu=2}^m h_{\mu\mu} = -a.$$

By Cauchy–Schwarz,

$$\sum_{\mu=2}^m h_{\mu\mu}^2 \geq \frac{a^2}{m-1}.$$

Discarding all other nonnegative entries,

$$|A|^2 \geq a^2 + \frac{a^2}{m-1} + 2 \sum_{\mu=2}^m b_\mu^2.$$

Consequently,

$$\frac{a^2 + \sum b_\mu^2}{|A|^2} \leq \max \left\{ \frac{m-1}{m}, \frac{1}{2} \right\} = \frac{m-1}{m}.$$

Equality is realized by the stated diagonal form. For $m \geq 3$, a nonzero trace-free matrix supported entirely in the $(m-1)$ -dimensional orthogonal complement of e_1 gives the lower value zero. When $m = 2$, every trace-free symmetric matrix has the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

and the ratio equals $1/2$. □

Let

$$z = \theta^2 \in [0, 1], \quad s = 1 - z.$$

From Lemma 3.1,

$$\frac{Q_{\bar{R}}(A)}{|A|^2} = m\beta + \gamma(1 - (m+1)z) + 2m\gamma(1-z)r, \quad (4.2)$$

where

$$r = \frac{\sum_j h_{1j}^2}{|A|^2}.$$

Proposition 4.2 (Pointwise algebraic optimum). *For every $m \geq 2$,*

$$Q_{\bar{R}}(A) \geq \kappa_{\text{wp}}(t)|A|^2, \quad (4.3)$$

where κ_{wp} is given by (1.5). The coefficient is optimal among bounds depending only on α, β .

Proof. Suppose first that

$$\gamma = \alpha - \beta \geq 0.$$

For fixed z , the last term of (4.2) is nonnegative, so the smallest admissible value is obtained by minimizing r . For $m \geq 3$, this is $r = 0$; for $m = 2$, $r = 1/2$, but the final angle-free minimum is the same. Indeed, when $m = 2$, substituting $r = 1/2$ into (4.2) gives

$$\frac{Q_{\bar{R}}(A)}{|A|^2} = 2\beta + \gamma(3 - 5z).$$

If $\gamma \geq 0$, this is minimized at $z = 1$, giving $2\beta - 2\gamma = 2(2\beta - \alpha)$. The resulting expression is minimized at $z = 1$, and gives

$$m\beta - m\gamma = m(2\beta - \alpha).$$

Now suppose

$$\gamma < 0.$$

The last term in (4.2) is minimized by maximizing r . By Lemma 4.1,

$$r_{\max} = \frac{m-1}{m}.$$

Thus

$$\frac{Q_{\bar{R}}(A)}{|A|^2} \geq m\beta + \gamma(2m - 1 - (3m - 1)z).$$

For $m = 2$, the same substitution $r = 1/2$ gives $2\beta + \gamma(3 - 5z)$; since $\gamma < 0$, its minimum occurs at $z = 0$ and equals $2\beta + 3\gamma = 3\alpha - \beta$, exactly the $m = 2$ value of the coefficient below. Since $\gamma < 0$, this is minimized at $z = 0$, yielding

$$m\beta + (2m - 1)(\alpha - \beta) = (2m - 1)\alpha - (m - 1)\beta.$$

The equality configurations described in Lemma 4.1, together with $z = 1$ in the first case and $z = 0$ in the second, show that no larger coefficient depending only on α, β can hold for all trace-free A and all angles. $\square$

Proof of Theorem 1.2. If $\Xi = 0$, the final term in (1.4) vanishes. We apply Proposition 4.2 to the remaining ambient curvature term. $\square$

Theorem 4.3 (Compact warped-product gap theorem). *Assume $\Xi = 0$ along a compact minimal hypersurface $\Sigma^m \subset I \times_f F^m(k_F)$. If*

$$\kappa_0 := \inf_{\Sigma} \kappa_{\text{wp}} > 0$$

and

$$|A|^2 < \kappa_0 \quad \text{on } \Sigma,$$

then Σ is totally geodesic.

If $\kappa_{\text{wp}} \equiv \kappa_0 > 0$ along Σ and

$$|A|^2 \leq \kappa_0,$$

then either $A \equiv 0$ or

$$|A|^2 \equiv \kappa_0, \quad \nabla A \equiv 0,$$

and the equality configuration of Proposition 4.2 holds pointwise.

Proof. Set $u = |A|^2$. By (1.6),

$$\frac{1}{2}\Delta u \geq |\nabla A|^2 + u(\kappa_{\text{wp}} - u).$$

Suppose $u \not\equiv 0$ and let p be a maximum point. Under the strict pinching assumption,

$$0 < u(p) < \kappa_0 \leq \kappa_{\text{wp}}(p).$$

Hence

$$0 \geq \frac{1}{2}\Delta u(p) \geq |\nabla A|^2(p) + u(p)(\kappa_{\text{wp}}(p) - u(p)) > 0,$$

where is a contradiction.

For the weak pinching statement, we integrate over the compact hypersurface:

$$0 = \frac{1}{2} \int_{\Sigma} \Delta u \, d\mu_{\Sigma} \geq \int_{\Sigma} |\nabla A|^2 \, d\mu_{\Sigma} + \int_{\Sigma} u(\kappa_0 - u) \, d\mu_{\Sigma}.$$

Both integrands are nonnegative. Therefore

$$\nabla A \equiv 0, \quad u(\kappa_0 - u) \equiv 0.$$

On each connected component, u is constant, hence $u \equiv 0$ or $u \equiv \kappa_0$. Equality in the integrated estimate forces equality in Proposition 4.2 pointwise. $\square$

4.1. Model geometries and equality cases

A key consistency requirement is that the explicit warped-product formula recover the classical identities without loss.

Euclidean space. Use polar coordinates

$$\mathbb{R}^{m+1} \setminus \{0\} = (0, \infty) \times_r \mathbb{S}^m.$$

Here

$$f(r) = r, \quad k_F = 1,$$

so

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \Xi = 0.$$

Thus (1.4) reduces to

$$\frac{1}{2}\Delta |A|^2 = |\nabla A|^2 - |A|^4.$$

The unit sphere. On

$$\mathbb{S}^{m+1} = (0, \pi) \times_{\sin t} \mathbb{S}^m,$$

we have

$$\alpha = -\frac{(-\sin t)}{\sin t} = 1$$

and

$$\beta = \frac{1 - \cos^2 t}{\sin^2 t} = 1.$$

Therefore

$$\gamma = 0, \quad \Xi = 0,$$

and

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + m|A|^2 - |A|^4. \quad (4.4)$$

Consequently,

$$|A|^2 < m \implies A \equiv 0$$

for compact minimal hypersurfaces. The threshold m is exactly the classical Simons–Chern–do Carmo–Kobayashi pinching constant in codimension one [8, 24]; the surrounding rigidity theory is also closely related to Lawson’s work [13].

Hyperbolic space. On

$$\mathbb{H}^{m+1} = (0, \infty) \times_{\sinh t} \mathbb{S}^m,$$

one has

$$\alpha = -1, \quad \beta = -1,$$

and hence

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - m|A|^2 - |A|^4.$$

There is no positive compact pinching coefficient, reflecting the different role of ambient negative curvature.

The condition $\Xi = 0$ and model warping functions. The condition

$$\Xi = 0 \iff \gamma' = 2\rho\gamma$$

is equivalent to

$$\gamma = Cf^2$$

for some constant C . Since

$$\gamma = -\frac{f''}{f} - \frac{k_F - (f')^2}{f^2},$$

the condition $\Xi = 0$ is therefore equivalent to

$$(f')^2 - ff'' - k_F = Cf^4. \quad (4.5)$$

This equation admits a useful first integral. Indeed, define

$$D(t) = \frac{(f')^2 - k_F + Cf^4}{f^2}.$$

Using (4.5), we obtain

$$D' = \frac{2f'}{f^3} (ff'' - (f')^2 + k_F + Cf^4) = 0.$$

Hence D is constant on each connected component of I , and

$$(f')^2 = k_F + Df^2 - Cf^4. \quad (4.6)$$

Combining (4.5) and (4.6), we further obtain

$$f'' = Df - 2Cf^3. \quad (4.7)$$

Consequently,

$$\alpha = -D + 2Cf^2, \quad \beta = -D + Cf^2, \quad \gamma = Cf^2. \quad (4.8)$$

Space-form case. When $C = 0$, we have $\gamma = 0$, and hence $\alpha = \beta = -D$. Thus the ambient manifold has constant sectional curvature $-D$. In particular, the standard Euclidean, spherical, and hyperbolic models are recovered from the corresponding choices of the warping function.

Non-space-form case. Suppose now that $C \neq 0$. Then

$$\alpha - \beta = Cf^2,$$

so the radial and vertical sectional curvatures are different wherever $f > 0$. Thus these models are genuinely non-space-form warped products, while the derivative-curvature term in the Simons identity still vanishes identically.

For comparison, Zhan and Hou [25] considered the related condition

$$k_F - (f')^2 + ff'' = c_1 f^4, \quad c_1 > 0,$$

in their study of pinching phenomena for compact minimal submanifolds in warped products. In the notation of (4.5), their assumption corresponds to $C = -c_1 < 0$. We now consider the opposite sign $C > 0$, for which the optimal angle-free coefficient obtained above has a particularly simple form.

Corollary 4.4 (A non-space-form gap family). *Let*

$$\overline{M}^{m+1} = I \times_f F^m(k_F), \quad m \geq 2,$$

where $f > 0$ satisfies

$$f'' = Df - 2Cf^3 \quad (4.9)$$

for constants $C > 0$ and $D < 0$, and suppose that its first integral is

$$(f')^2 = k_F + Df^2 - Cf^4.$$

Then

$$\Xi \equiv 0, \quad \alpha - \beta = Cf^2 > 0,$$

and the optimal angle-free curvature coefficient in Theorem 1.2 is the positive constant

$$\kappa_{\text{wp}} \equiv -mD. \quad (4.10)$$

Consequently, if $\Sigma^m \subset \overline{M}^{m+1}$ is a compact two-sided minimal hypersurface satisfying

$$|A|^2 < -mD \quad \text{on } \Sigma, \quad (4.11)$$

then Σ is totally geodesic.

Proof. By (4.8),

$$\alpha = -D + 2Cf^2, \quad \beta = -D + Cf^2, \quad \gamma = Cf^2.$$

Therefore

$$\Xi = \gamma' - 2\frac{f'}{f}\gamma = 2Cff' - 2\frac{f'}{f}Cf^2 = 0.$$

Since $C > 0$ and $f > 0$, one has

$$\alpha > \beta.$$

Hence the first branch of (1.5) gives

$$\begin{aligned} \kappa_{\text{wp}} &= m(2\beta - \alpha) \\ &= m(2(-D + Cf^2) - (-D + 2Cf^2)) \\ &= -mD. \end{aligned}$$

Since $D < 0$, the coefficient is positive. The conclusion follows directly from Theorem 4.3. $\square$

The preceding family contains explicit anisotropic models for which the optimal coefficient nevertheless agrees with the classical spherical value.

Example 4.5 (A non-space-form model with the spherical algebraic threshold). *Let*

$$k_F = 1, \quad D = -1, \quad C = \lambda > 0,$$

and let f solve

$$f'' = -f - 2\lambda f^3, \quad f(0) = 0, \quad f'(0) = 1. \quad (4.12)$$

The corresponding first integral is

$$(f')^2 = 1 - f^2 - \lambda f^4. \quad (4.13)$$

Let $a_\lambda > 0$ be determined by

$$1 - a_\lambda^2 - \lambda a_\lambda^4 = 0, \quad a_\lambda^2 = \frac{\sqrt{1 + 4\lambda} - 1}{2\lambda}.$$

Then f increases from 0 to a_λ until its first maximum at some $T_\lambda > 0$, and $f > 0$ on $(0, 2T_\lambda)$. On the increasing branch,

$$t = \int_0^{f(t)} \frac{ds}{\sqrt{1 - s^2 - \lambda s^4}}.$$

For the warped product

$$(0, 2T_\lambda) \times_f \mathbb{S}^m,$$

formula (4.8) gives

$$\alpha = 1 + 2\lambda f^2, \quad \beta = 1 + \lambda f^2.$$

Thus

$$\alpha - \beta = \lambda f^2 > 0,$$

so the ambient metric does not have constant sectional curvature. Nevertheless,

$$\kappa_{\text{wp}} = m(2\beta - \alpha) = m.$$

Therefore every compact two-sided minimal hypersurface satisfying

$$|A|^2 < m$$

is totally geodesic. Thus the optimal angle-free coefficient coincides with the classical spherical value m , although the ambient sectional curvature is anisotropic.

Parallel second fundamental form. If $\Xi = 0$ and $\nabla A = 0$, then (1.4) gives the pointwise algebraic relation

$$|A|^4 = [m\beta + \gamma(1 - (m+1)\theta^2)]|A|^2 + 2m\gamma|A(X)|^2. \quad (4.14)$$

Thus a non-totally-geodesic minimal hypersurface with parallel A must saturate the warped curvature balance. In the spherical case, this reduces to

$$|A|^2 = m.$$

Local curvature gap. Suppose $\Xi = 0$ and

$$\kappa_{\text{wp}} \geq \kappa_0 > 0$$

on a connected open set $\Omega \subset \Sigma$. If

$$|A|^2 \leq \kappa_0 - \varepsilon$$

and $u = |A|^2$ attains a positive interior maximum in Ω , then (1.6) gives a contradiction. Thus either $A = 0$ at the maximum scale or the maximum is forced to the boundary. This is the local form of the compact gap principle and is useful in blow-up arguments.

Remark 4.6 (Sharpness and scope). *For clarity, we separate three notions.*

- (i) *The identity (1.4) is exact once the curvature convention is fixed.*
- (ii) *The coefficient κ_{wp} in (1.5) is optimal for the purely algebraic problem of minimizing the zeroth-order curvature contraction over all trace-free symmetric A and all angle functions.*
- (iii) *A global classification of the equality case is a separate geometric problem. In the sphere, the classical theory identifies Clifford hypersurfaces in the hypersurface equality case. In a general warped product, realizing every algebraic equality configuration by a complete or compact minimal hypersurface requires additional integrability and global arguments.*

This distinction prevents an algebraically optimal estimate from being mistaken for a complete geometric classification theorem.

5. Perturbative estimates, stability, and pointwise curvature bounds

The exact formula of Section 3 contains two types of lower-order effects that are absent in a space form: the anisotropy $\gamma = \alpha - \beta$ and the derivative-curvature coefficient $\Xi = \gamma' - 2\rho\gamma$. In this section, we organize these terms at their natural scale and combine the resulting differential inequality with stability. The outcome is a local Schoen–Simon–Yau-type pointwise estimate.

For a relatively compact region $\Omega \subset \Sigma$, set

$$\Lambda_{\text{wp}}(\Omega) := \sup_{\Omega} (|\alpha| + |\beta| + |\Xi|^{2/3}). \quad (5.1)$$

The exponent $2/3$ is forced by scaling: α, β have the dimension of curvature, whereas Ξ has the dimension of a first covariant derivative of curvature. Thus $|\Xi|^{2/3}$ has the same dimension as $|A|^2$.

5.1. A scale-correct scalar inequality

Let $v = |A|$. From (3.22), after choosing

$$0 < \varepsilon < \frac{2}{m}$$

and writing

$$a_{\varepsilon} := \frac{2}{m} - \varepsilon > 0,$$

we obtain the following useful form of the warped Simons inequality.

Proposition 5.1 (Scalar warped Simons inequality). *Let $\Omega \Subset \Sigma$ and set $\Lambda = \Lambda_{\text{wp}}(\Omega)$. Then on $\Omega \cap \{v > 0\}$,*

$$v\Delta v \geq -v^4 + a_{\varepsilon}|\nabla v|^2 - C_m\Lambda v^2 - C_{m,\varepsilon}\Lambda^2. \quad (5.2)$$

The same inequality holds weakly on all of Ω after the standard regularization $v_{\tau} = (v^2 + \tau^2)^{1/2}$ and passage $\tau \downarrow 0$.

Proof. Starting from (3.22) and using

$$\Delta v^2 = 2v\Delta v + 2|\nabla v|^2,$$

we obtain

$$\begin{aligned} v\Delta v \geq & -v^4 + \left(\frac{2}{m} - \varepsilon\right)|\nabla v|^2 + \kappa_{\text{wp}}v^2 \\ & - \frac{2m}{3\sqrt{3}}|\Xi|v - C_{m,\varepsilon}\gamma^2\theta^2(1 - \theta^2). \end{aligned}$$

Since κ_{wp} is a dimensional linear combination of α, β ,

$$|\kappa_{\text{wp}}| \leq C_m\Lambda.$$

Moreover

$$|\gamma| \leq |\alpha| + |\beta| \leq \Lambda, \quad |\Xi| \leq \Lambda^{3/2}.$$

Young's inequality in the scale-correct form

$$\Lambda^{3/2}v \leq \frac{1}{2}\Lambda v^2 + \frac{1}{2}\Lambda^2$$

then gives (5.2). $\square$

When $\Xi \neq 0$, this formulation is preferable to an estimate containing Ξ^2/δ because every lower-order term is expressed through one curvature scale Λ .

We now impose stability. With our Laplacian convention, a two-sided minimal hypersurface is stable when

$$\int_{\Sigma} (v^2 + \overline{\text{Ric}}(v, v)) \phi^2 d\mu_{\Sigma} \leq \int_{\Sigma} |\nabla \phi|^2 d\mu_{\Sigma} \quad (5.3)$$

for every compactly supported smooth ϕ . In the present warped product, the normal Ricci curvature is

$$\overline{\text{Ric}}(v, v) = \alpha(1 + (m-1)\theta^2) + (m-1)\beta(1 - \theta^2). \quad (5.4)$$

Hence

$$|\overline{\text{Ric}}(v, v)| \leq C_m \Lambda.$$

Consequently, (5.3) implies

$$\int_{\Sigma} v^2 \phi^2 d\mu_{\Sigma} \leq \int_{\Sigma} |\nabla \phi|^2 d\mu_{\Sigma} + C_m \Lambda \int_{\Sigma} \phi^2 d\mu_{\Sigma} \quad (5.5)$$

for every $\phi \in C_c^{\infty}(\Omega)$.

5.2. Power Caccioppoli estimates and the SSY exponent range

The next proposition is the analytic strengthening of the basic Caccioppoli estimate. It is the point at which the classical Schoen–Simon–Yau exponent range reappears.

Proposition 5.2 (Power Caccioppoli inequality). *Assume that Σ is stable in Ω , let $\Lambda = \Lambda_{\text{wp}}(\Omega)$, and fix $0 < \varepsilon < 2/m$. Let $q \geq 1$ satisfy*

$$\sigma_q := \frac{2}{m} - \varepsilon - (q-1)^2 > 0. \quad (5.6)$$

Then for every $\eta \in C_c^{\infty}(\Omega)$,

$$\begin{aligned} & \int_{\Sigma} \eta^2 v^{2q+2} d\mu_{\Sigma} + \sigma_q \int_{\Sigma} \eta^2 v^{2q-2} |\nabla v|^2 d\mu_{\Sigma} \\ & \leq C_{m,q,\varepsilon} \int_{\Sigma} (|\nabla \eta|^2 + \Lambda \eta^2) v^{2q} d\mu_{\Sigma} + C_{m,q,\varepsilon} \Lambda^{q+1} |\text{supp } \eta|. \end{aligned} \quad (5.7)$$

In particular, for $q = 1$,

$$\begin{aligned} \int_{\Sigma} \eta^2 (v^4 + |\nabla v|^2) d\mu_{\Sigma} & \leq C_{m,\varepsilon} \int_{\Sigma} (|\nabla \eta|^2 + \Lambda \eta^2) v^2 d\mu_{\Sigma} \\ & \quad + C_{m,\varepsilon} \Lambda^2 |\text{supp } \eta|. \end{aligned} \quad (5.8)$$

Proof. We give the calculation because it is precisely the step that controls the admissible exponent. We work first with $v_\tau = (v^2 + \tau^2)^{1/2}$; to avoid cumbersome notation, we suppress the subscript τ . Multiply (5.2) by $\eta^2 v^{2q-2}$ and integrate by parts. Set

$$G = \int \eta^2 v^{2q-2} |\nabla v|^2 d\mu_\Sigma, \quad J = \int \eta v^{2q-1} \langle \nabla \eta, \nabla v \rangle d\mu_\Sigma,$$

and

$$I = \int \eta^2 v^{2q+2} d\mu_\Sigma.$$

Since

$$\int \eta^2 v^{2q-1} \Delta v d\mu_\Sigma = -(2q-1)G - 2J,$$

we obtain

$$I \geq (2q-1 + a_\varepsilon)G + 2J - C_m \Lambda \int \eta^2 v^{2q} d\mu_\Sigma - C_{m,\varepsilon} \Lambda^2 \int \eta^2 v^{2q-2} d\mu_\Sigma. \quad (5.9)$$

For $q \geq 1$,

$$\Lambda^2 v^{2q-2} \leq C_q (\Lambda v^{2q} + \Lambda^{q+1}),$$

which follows after dividing by Λ^{q+1} and using $y^{q-1} \leq 1 + y^q$.

Next, we apply stability (5.5) with $\phi = \eta v^q$. Expanding the gradient gives

$$I \leq q^2 G + 2qJ + \int v^{2q} |\nabla \eta|^2 d\mu_\Sigma + C_m \Lambda \int \eta^2 v^{2q} d\mu_\Sigma. \quad (5.10)$$

Combining (5.9) and (5.10) yields

$$\sigma_q G \leq 2(q-1)J + C \int (|\nabla \eta|^2 + \Lambda \eta^2) v^{2q} d\mu_\Sigma + C \Lambda^{q+1} |\text{supp } \eta|.$$

By Cauchy–Young,

$$2|q-1|J \leq \frac{\sigma_q}{2} G + C_{q,\sigma_q} \int v^{2q} |\nabla \eta|^2 d\mu_\Sigma.$$

Thus G is controlled by the right-hand side of (5.7). Substituting this estimate back into (5.10), and estimating J once more by Cauchy–Young, gives the bound for I . Finally, let $\tau \downarrow 0$. $\square$

If $p = 2q + 2$, the restriction (5.6) becomes

$$p < 4 + 2\sqrt{\frac{2}{m} - \varepsilon}.$$

Letting $\varepsilon \downarrow 0$ recovers the classical upper endpoint

$$p < 4 + \sqrt{\frac{8}{m}} \quad (5.11)$$

appearing in the Schoen–Simon–Yau integral curvature estimates [22]. This agreement is a useful consistency check: the warped terms change the lower-order scale but not the algebraic source of the critical exponent.

5.3. Sobolev gain and Moser iteration

We record the analytic lemma used in the final step. Its proof is included to make the pointwise estimate self-contained once a local Sobolev inequality is available. The argument follows the classical elliptic Moser iteration framework; see, for example, [11]. Here the iteration is adapted to a potential in L^s with $s > m/2$.

Lemma 5.3 (Moser iteration with a supercritical potential). *Let $m \geq 3$, $s > m/2$, and let B_R be a geodesic ball satisfying*

$$\left(\int_{B_R} |\psi|^{2\chi} d\mu_\Sigma \right)^{1/\chi} \leq S_0 \int_{B_R} (|\nabla \psi|^2 + R^{-2} \psi^2) d\mu_\Sigma, \quad \chi = \frac{m}{m-2}, \quad (5.12)$$

for all $\psi \in C_c^1(B_R)$. Suppose $w \geq 0$ satisfies weakly

$$\Delta w \geq -hw$$

on B_R , where $h \in L^s(B_R)$. Then

$$\sup_{B_{R/2}} w \leq CR^{-m/2} \|w\|_{L^2(B_R)}, \quad (5.13)$$

where

$$C = C(m, S_0, s, R^{2-m/s} \|h\|_{L^s(B_R)}).$$

Proof. By scaling, it is enough to consider $R = 1$. Let η be a cut-off supported in B_ρ , equal to one on B_σ , with

$$|\nabla \eta| \leq \frac{2}{\rho - \sigma}.$$

For $p \geq 2$, test the weak inequality by $\eta^2 w^{p-1}$. The usual integration-by-parts calculation gives

$$\int |\nabla(\eta w^{p/2})|^2 d\mu_\Sigma \leq Cp^2 \left[(\rho - \sigma)^{-2} \int_{B_\rho} w^p d\mu_\Sigma + \int_{B_\rho} h \eta^2 w^p d\mu_\Sigma \right]. \quad (5.14)$$

By Hölder,

$$\int h(\eta w^{p/2})^2 \leq \|h\|_{L^s} \|\eta w^{p/2}\|_{L^{2s/(s-1)}}^2.$$

Because $s > m/2$,

$$2 < \frac{2s}{s-1} < 2\chi.$$

Interpolation between L^2 and $L^{2\chi}$, followed by Young's inequality, therefore gives for every $\delta > 0$,

$$\|\eta w^{p/2}\|_{L^{2s/(s-1)}}^2 \leq \delta \|\eta w^{p/2}\|_{L^{2\chi}}^2 + C_{\delta, m, s} \|\eta w^{p/2}\|_{L^2}^2.$$

Choose δ so that the first term can be absorbed after applying the Sobolev inequality (5.12) to (5.14). We obtain an iterative estimate of the form

$$\|w\|_{L^{p\chi}(B_\sigma)} \leq \left[\frac{Cp^\mu}{(\rho - \sigma)^2} \right]^{1/p} \|w\|_{L^p(B_\rho)},$$

where C, μ depend only on $m, S_0, s, \|h\|_{L^s(B_1)}$. Taking $p_j = 2\chi^j$ and a nested sequence of radii decreasing to $1/2$, the product of the factors converges. Letting $j \rightarrow \infty$ gives (5.13). $\square$

We can now prove the pointwise estimate announced in the Introduction.

Theorem 5.4 (Local pointwise curvature estimate). *Let $3 \leq m \leq 5$, and let*

$$\Sigma^m \subset I \times_f F^m(k_F)$$

be a two-sided stable minimal hypersurface. Fix $x_0 \in \Sigma$ and assume

$$B_{2r} := B_{2r}^\Sigma(x_0) \Subset \Sigma.$$

Suppose that for every $0 < s \leq 2r$, the following hold:

(i) *the local Sobolev inequality*

$$\left(\int_{B_s} |\phi|^{2\chi} d\mu_\Sigma \right)^{1/\chi} \leq S_0 \int_{B_s} (|\nabla \phi|^2 + s^{-2} \phi^2) d\mu_\Sigma, \quad \chi = \frac{m}{m-2}, \quad (5.15)$$

for every $\phi \in C_c^1(B_s)$;

(ii) *the volume-ratio bound*

$$|B_s| \leq V_0 s^m. \quad (5.16)$$

Then

$$|A|^2(x_0) \leq C(m, S_0, V_0) \left[r^{-2} + \Lambda_{\text{wp}}(B_{2r}) \right]. \quad (5.17)$$

Proof. Set

$$\Lambda = \Lambda_{\text{wp}}(B_{2r})$$

and

$$R = \min\{r, \Lambda^{-1/2}\},$$

with the convention $\Lambda^{-1/2} = +\infty$ when $\Lambda = 0$. After rescaling the ambient and induced metrics by R^{-2} , it is enough to prove a universal bound at x_0 under

$$R = 1, \quad \Lambda \leq 1.$$

The Sobolev constant and the volume-ratio constant are unchanged by this rescaling.

Choose cut-offs on the concentric balls below; all constants in their gradient bounds are universal. First use stability (5.5) with a cut-off equal to one on $B_{3/2}$ and supported in B_2 . Since $\Lambda \leq 1$ and $|B_2| \leq C_m V_0$,

$$\int_{B_{3/2}} v^2 d\mu_\Sigma \leq C(m, V_0). \quad (5.18)$$

Next apply the $q = 1$ case (5.8) with a cut-off supported in $B_{3/2}$ and equal to one on $B_{7/5}$. Using (5.18) gives

$$\int_{B_{7/5}} (v^4 + |\nabla v|^2) d\mu_\Sigma \leq C(m, V_0). \quad (5.19)$$

Applying the Sobolev inequality (5.15) to the cut-off times v , and using (5.18) and (5.19), we obtain

$$\|v\|_{L^{2\chi}(B_{4/3})} \leq C(m, S_0, V_0). \quad (5.20)$$

We now use the dimensional restriction. Choose $\varepsilon > 0$ small enough that there exists q satisfying

$$\max \left\{ 1, \frac{m}{2} - 1 \right\} < q < \min \left\{ \chi, 1 + \sqrt{\frac{2}{m} - \varepsilon} \right\}. \quad (5.21)$$

Such a q exists exactly for $3 \leq m \leq 5$. Indeed, the nontrivial condition is

$$\frac{m}{2} - 1 < 1 + \sqrt{\frac{2}{m}},$$

and equivalently,

$$m < 4 + 2\sqrt{\frac{2}{m}},$$

which holds for $m \leq 5$ and fails at $m = 6$.

Because $q < \chi$, (5.20) and the volume bound control $\|v\|_{L^{2q}(B_{4/3})}$. We apply Proposition 5.2 with this q and a cut-off equal to one on $B_{5/4}$. We obtain

$$\int_{B_{5/4}} v^p d\mu_\Sigma \leq C(m, S_0, V_0), \quad p := 2q + 2 > m. \quad (5.22)$$

Set

$$s = \frac{p}{2} = q + 1 > \frac{m}{2}, \quad w = v^2 + 1.$$

Then (5.22) gives

$$\|w\|_{L^s(B_{5/4})} \leq C(m, S_0, V_0). \quad (5.23)$$

Finally, (3.22), together with $\Lambda \leq 1$, implies weakly

$$\Delta w \geq -C_m w^2. \quad (5.24)$$

We can write this as

$$\Delta w \geq -hw, \quad h = C_m w.$$

By (5.23), $h \in L^s$ with $s > m/2$ and a bounded norm. Moreover, (5.19) and the volume bound imply

$$\|w\|_{L^2(B_{6/5})} \leq C(m, V_0).$$

Applying Lemma 5.3 on a ball contained in $B_{5/4}$ therefore yields

$$w(x_0) \leq C(m, S_0, V_0).$$

Thus, in the rescaled metric,

$$|A|^2(x_0) \leq C(m, S_0, V_0).$$

Returning to the original scale gives

$$|A|^2(x_0) \leq CR^{-2} \leq C(r^{-2} + \Lambda),$$

which is (5.17). □

Corollary 5.5 (Estimate written directly in terms of the warping function). *Under the hypotheses of Theorem 5.4, define*

$$\begin{aligned} \mathcal{K}_f := & \frac{|f''|}{f} + \frac{|k_F - (f')^2|}{f^2} \\ & + \left| \left(-\frac{f''}{f} - \frac{k_F - (f')^2}{f^2} \right)' - 2\frac{f'}{f} \left(-\frac{f''}{f} - \frac{k_F - (f')^2}{f^2} \right) \right|^{2/3}. \end{aligned} \quad (5.25)$$

Then

$$|A|(x_0) \leq C(m, S_0, V_0) \left[r^{-1} + \left(\sup_{B_{2r}} \mathcal{K}_f \right)^{1/2} \right]. \quad (5.26)$$

Proof. The first two terms of $\mathcal{K}_f$ are $|\alpha|$ and $|\beta|$, while the final term is $|\Xi|^{2/3}$. Hence

$$\sup_{B_{2r}} \mathcal{K}_f = \Lambda_{\text{wp}}(B_{2r}).$$

Take the square root of (5.17). □

Remark 5.6 (On the Sobolev hypothesis). *The Sobolev assumption (5.15) is written explicitly in the statement because it cleanly separates the warped Simons calculation from the ambient geometric input. In applications it can be supplied, on sufficiently small supports and under the usual ambient curvature and injectivity-radius hypotheses, by the Riemannian submanifold Sobolev inequalities of Hoffman–Spruck [12]. In Euclidean space, it reduces to the Michael–Simon framework [15]. The volume-ratio assumption is the analogue of the local area control used in classical curvature-estimate arguments [21–23].*

Remark 5.7 (Why the dimension $m \leq 5$ appears). *The proof shows exactly where the dimensional threshold enters. Stability and the refined Kato term allow powers up to*

$$p < 4 + \sqrt{\frac{8}{m}}.$$

To turn the nonlinear inequality $\Delta(v^2 + 1) \geq -C(v^2 + 1)^2$ into a linear Moser problem with an L^s -potential satisfying $s > m/2$, one needs $p > m$. These two requirements are simultaneously compatible for $m \leq 5$. This is the same mechanism behind the low-dimensional range in the original Schoen–Simon–Yau argument [22]. Higher dimensions require additional input or a different regularity mechanism, as already occurs in the classical theory [21, 23].

The pointwise estimate completes the analytic application of the explicit warped Simons identity. The remaining classification problem concerns equality configurations for non-space-form solutions of $\Xi = 0$ with positive κ_{wp} , which is separate from the curvature-estimate argument above.

6. Conclusions

We have derived an explicit Simons identity for two-sided minimal hypersurfaces in warped products with space-form fibers, expressing the ambient curvature contributions in terms of α , β , and the

derivative-curvature quantity Ξ . When $\Xi = 0$, we obtained an optimal angle-independent algebraic lower bound and corresponding pinching results for compact minimal hypersurfaces, recovering the classical spherical gap as a special case. For stable minimal hypersurfaces, the scalar Simons inequality, stability, and power Caccioppoli estimates lead to a local pointwise curvature bound in dimensions $3 \leq m \leq 5$, under the stated Sobolev and volume growth assumptions. The estimate incorporates the ambient geometry through the scale-correct quantity $|\alpha| + |\beta| + |\Xi|^{2/3}$. These results clarify the role of warped-product curvature in both rigidity and local curvature estimates. The classification of equality cases in non-space-form warped products remains a natural question for further investigation.

Author contributions

All authors have contributed equally to the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors formulated the research problem and developed the initial mathematical approach. ChatGPT-5.6 Sol (OpenAI) was subsequently used as an interactive aid in exploring possible mathematical approaches and in developing, checking, and refining selected algebraic estimates, curvature computations, and steps in the proofs. It was also used for English-language editing, improving the clarity of the mathematical exposition, LaTeX formatting, and checking citation consistency. All mathematical arguments and calculations in the final manuscript were independently verified by the authors, who also reviewed the references and take full responsibility for the correctness, originality, and integrity of the work.

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Conflict of interest

All authors declare they have no conflict of interest in this paper.

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