



*Research article***Event-triggered sparse anchor-assisted residual-cancellation neurodynamics for moving KKT-path tracking in time-varying second-order cone optimization****Xu Wu¹, Juhe Sun^{2,*}, Li Wang² and Yunlai Zhong¹**¹ College of Artificial Intelligence, Shenyang Aerospace University, Shenyang 110136, China² College of Science, Shenyang Aerospace University, Shenyang 110136, China*** Correspondence:** Email: juhesun070133@163.com.

Abstract: This work developed an event-triggered sparse anchor-assisted residual-cancellation neurodynamic method for tracking the moving Karush–Kuhn–Tucker (KKT) path of time-varying strongly convex optimization problems with product second-order cone (SOC) constraints. The resulting continuous-time architecture is a zeroing neural network (ZNN)-type computational-intelligence framework for online conic optimization in time-varying engineering environments. The model was motivated by three-finger contact-force allocation, where quasi-static wrench balance and exact Coulomb friction constraints yield a strongly convex SOC program. A smoothed natural-residual (NR) KKT system was constructed to regularize cone complementarity and define a differentiable moving primal–dual path. A feedforward residual-cancellation law compensates for explicit temporal variation. An anchor tracks the residual-zero path, while replicated solver units exchange delayed state information and receive event-triggered Top- p sparse corrections. The Top- p rule gives an explicit coverage ratio without an additional sparse-coverage assumption. Under uniform residual regularity on a forward-invariant domain, bounded communication delays and implementation disturbances, and a computable impulsive-delay condition, local exponential anchor tracking and disturbance-dependent synchronization were established. The complete estimate separates delay-induced mismatch, implementation disturbance, anchor residual error, and smoothing bias. Experiments on three-finger grasping and four-contact force allocation demonstrated accurate moving-path tracking, a lower aggregate squared correction magnitude relative to matched periodic updates, and finite-horizon recovery across multiple solver-fault amplitudes, directions, and disturbance scenarios.

Keywords: time-varying second-order cone optimization; moving KKT path; residual-cancellation neurodynamics; event-triggered Top- p correction; delayed synchronization; robotic contact-force allocation; computational intelligence; zeroing neural network

Mathematics Subject Classification: 34D06, 65K05, 90C25, 90C30

1. Introduction

Contact-force allocation is a basic computation in contact-rich robotic manipulation. For a multi-finger grasp, contact forces must balance the external wrench while satisfying Coulomb friction limits, as described in grasp-closure analysis and standard robotic manipulation models [3, 26]. Grasp planning and task-oriented force selection provide related optimization formulations [9, 19]. When the external wrench or desired force distribution varies with time, the solver must track a moving force allocation rather than converge only to a fixed equilibrium, and its numerical accuracy must be reflected in wrench balance, force tracking, and friction margins [30].

Retaining the circular Coulomb friction cones gives an exact second-order cone (SOC) representation, avoiding the direction-dependent approximation introduced by friction pyramids. Together with quasi-static wrench balance and a positive-definite quadratic force-selection criterion, this yields a time-varying strongly convex SOC program and an equivalent structured variational inequality [1, 23]. The scope is therefore narrower than that of a general second-order cone constrained variational inequality (SOCCVI) with an arbitrary nonpotential mapping. Although strong convexity guarantees uniqueness of the instantaneous primal optimizer, the KKT system still contains nonsmooth cone complementarity. Perturbation analysis provides regularity tools for second-order cone programs [4], while SOC functions and complementarity methods are treated in [5, 6]. Smooth reformulations of SOC complementarity have also been developed [7, 10].

Static convergence alone does not ensure accurate online tracking. Time variation moves the primal-dual solution, and prediction-correction methods reduce the tracking lag by exploiting derivatives of the problem data [27]. Neurodynamic optimization provides a continuous-time alternative in which optimizers are represented by equilibria or residual zeros. Hopfield and Tank [14] introduced an early neural-computation approach to optimization. Projection neural networks were subsequently developed for monotone variational inequalities [32] and nonlinear convex programs with linear constraints [34], while a broader overview is provided in [35]. For SOC-constrained problems, recurrent neural networks have been developed for second-order cone programs [15], smoothed natural-residual formulations of SOC-constrained convex programs [25], and SOC-constrained variational inequalities [28]. In particular, Sun et al. [29] related exponential stability to nonsingularity of the KKT Jacobian. These studies, however, mainly concern fixed KKT points rather than the lag induced by a moving conic KKT path.

Zhang and Ge [41] developed an early recurrent neural-network model for time-varying matrix inversion using derivative information to track time-varying solutions. Zeroing neural networks and residual-cancellation methods similarly incorporate explicit derivative information to compensate for temporal variation [11, 17, 18]. Existing time-varying models nevertheless focus mainly on matrix equations, smooth quadratic programs, equality-constrained problems, or nonconic distributed optimization. Exact product-cone complementarity, smoothing bias, and local regularity of a moving conic KKT path have not been treated jointly in a delayed residual-neurodynamic architecture.

Event-triggered computation offers a complementary mechanism for reducing unnecessary corrective action. Recent approaches address distributed time-varying optimization, discrete-time zeroing neural networks, and event-dependent robotic control [2, 13, 16]. Their settings differ from the replicated architecture considered here, where each solver evaluates the complete residual of the same conic problem. Delays and local numerical disturbances further motivate pinning and impulsive synchronization techniques. Impulsive network synchronization has been studied using general

synchronization criteria [24] and hybrid adaptive control under stochastic perturbations [37]. Event-triggered impulsive synchronization has been investigated for delayed memristive neural networks [21], while pinning impulsive synchronization of delayed neural networks has been analyzed in [22, 38]. Impulsive control with actuation delays is considered in [40]. However, sparse impulses applied to only part of a delayed network do not automatically contract the total synchronization error, and the reference trajectory in conventional synchronization is generally prescribed independently of an optimization-generated KKT path.

The proposed architecture is therefore not a conventional distributed optimizer. Variables and problem data are not partitioned; instead, replicated solver units evaluate the same complete KKT residual under delayed information exchange and implementation disturbances. An anchor supplies the moving residual-tracking reference, while event-triggered sparse corrections restore node-to-anchor consistency. The objective is computational redundancy and recovery rather than reducing the arithmetic complexity of a single centralized conic solve. Accordingly, the aggregate squared correction magnitude is reported separately from communication and runtime quantities.

This paper develops an event-triggered sparse anchor-assisted residual-cancellation neurodynamic method for moving KKT-path tracking in time-varying strongly convex optimization under product SOC constraints. The contact-force SOC program serves as the motivating engineering model, while the main methodological contribution is the integration of residual-cancellation tracking, delayed solver replication, and sparse event-triggered correction around an optimization-generated reference. The main contributions are: (i) An NR-smoothed KKT residual and its explicit temporal derivative define a residual-cancellation field for tracking the moving smoothed primal–dual path; (ii) the field is embedded in a delayed anchor–solver architecture for replicated optimization modules; (iii) an event-triggering rule determines when correction occurs, while a weighted Top- p rule selects the corrected units and gives the explicit coverage ratio p/N_s and a computable contraction factor without an additional sparse-coverage assumption; and (iv) under uniform residual regularity, bounded delays and disturbances, and an impulsive-delay condition, the analysis yields anchor-tracking and solver-synchronization estimates that separate residual error, delay mismatch, implementation disturbance, and smoothing bias.

The method is evaluated on three-finger grasping and four-contact force redistribution using frozen-time SOC program references, published neurodynamic baselines, ablations, event-trigger diagnostics, sensitivity tests, fault-recovery experiments, and conditioning studies. These experiments assess tracking accuracy, aggregate squared correction magnitude, redundancy, and numerical limitations.

2. Time-varying three-finger force allocation under SOC constraints

This section derives a time-varying SOC-constrained optimization model for three-finger contact-force allocation. The physical feasible set is determined by quasi-static wrench balance and exact Coulomb friction cones, while a strongly convex quadratic criterion selects one force distribution from this set. The resulting model is subsequently characterized as a strongly monotone variational inequality over a time-varying affine section of a product SOC.

Consider a rigid object grasped by three fingertips. For the i th contact, let

$$f_i = \begin{bmatrix} f_{it} \\ f_{in} \end{bmatrix} \in \mathbb{R}^3, \quad f_{it} = \begin{bmatrix} f_{ix} \\ f_{iy} \end{bmatrix} \in \mathbb{R}^2, \quad f_{in} = f_{iz} \in \mathbb{R}, \quad i = 1, 2, 3,$$

where f_{it} and f_{in} are the tangential and normal components in the local contact frame, respectively. The stacked contact-force vector is

$$f = (f_1^\top, f_2^\top, f_3^\top)^\top \in \mathbb{R}^9.$$

2.1. Physical admissibility and force selection

Let $p_i \in \mathbb{R}^3$ denote the vector from the object center of mass to the i th contact point, and let $R_i \in \mathbb{R}^{3 \times 3}$ map vectors from the local contact frame to the object frame. The force f_i generates the object-frame wrench

$$\begin{bmatrix} R_i f_i \\ p_i \times (R_i f_i) \end{bmatrix} = \begin{bmatrix} R_i \\ [p_i]_\times R_i \end{bmatrix} f_i,$$

where, for $a = (a_1, a_2, a_3)^\top$,

$$[a]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad [a]_\times b = a \times b.$$

Define

$$G_{\text{gr},i} = \begin{bmatrix} R_i \\ [p_i]_\times R_i \end{bmatrix} \in \mathbb{R}^{6 \times 3}$$

and

$$G_{\text{gr}} = \begin{bmatrix} G_{\text{gr},1} & G_{\text{gr},2} & G_{\text{gr},3} \end{bmatrix} \in \mathbb{R}^{6 \times 9}.$$

The vector $G_{\text{gr}} f$ is the total wrench generated by the three contact forces. Under the quasi-static rigid-body assumption, this wrench must balance the external wrench $\omega_{\text{ext}}(t) \in \mathbb{R}^6$, giving

$$G_{\text{gr}} f + \omega_{\text{ext}}(t) = 0. \quad (2.1)$$

This relation is the standard Newton–Euler equilibrium condition used in multi-finger grasp analysis [3, 19, 26].

At each contact, the point-contact-with-friction model requires a compressive normal force and restricts the tangential force to the Coulomb friction cone:

$$\|f_{it}\| \leq \theta_i f_{in}, \quad f_{in} \geq 0,$$

where $\theta_i > 0$ is the friction coefficient. These inequalities exclude tensile contact forces and preserve the exact circular geometry of the local friction cone [3, 9, 26].

The wrench-balance and friction constraints determine physical admissibility but generally do not specify a unique force distribution. Indeed, multi-finger grasps may possess internal-force degrees of freedom, so different contact-force vectors can generate the same object wrench while satisfying all contact constraints [19, 26]. A force-allocation criterion is therefore required. Quadratic criteria are commonly used to penalize contact effort, redistribute loads among the fingers, or track a prescribed force profile while retaining convexity [19, 33].

Accordingly, the instantaneous contact-force allocation problem is

$$\begin{aligned} \min_{f \in \mathbb{R}^9} \quad & J(f, t) = \frac{1}{2} f^\top H f + r(t)^\top f, \\ \text{s.t.} \quad & G_{\text{gr}} f + \omega_{\text{ext}}(t) = 0, \\ & \|f_{it}\| \leq \theta_i f_{in}, \quad f_{in} \geq 0, \quad i = 1, 2, 3, \end{aligned} \quad (\mathcal{G}_t)$$

where $H = H^\top > 0$ is the force-weighting matrix and $r(t) \in \mathbb{R}^9$ is a time-varying force-preference vector. The objective in $(\mathcal{G}_t)$ is a selection rule over the physical feasible set and does not modify the underlying contact mechanics.

If a reference force $f_{\text{ref}}(t)$ is prescribed, the weighted tracking objective

$$\frac{1}{2} \|f - f_{\text{ref}}(t)\|_H^2, \quad \|v\|_H^2 := v^\top H v$$

is equivalent, up to a term independent of f , to the objective in $(\mathcal{G}_t)$ with

$$r(t) = -H f_{\text{ref}}(t).$$

2.2. Exact SOC reformulation

For each contact, introduce the scaled variable

$$x_i = \begin{bmatrix} \theta_i f_{\text{in}} \\ f_{\text{it}} \end{bmatrix} \in \mathbb{R}^3.$$

Let

$$\mathcal{Q}_3 = \left\{ \begin{bmatrix} u_0 \\ \bar{u} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^2 \mid \|\bar{u}\| \leq u_0 \right\}$$

be the three-dimensional second-order cone. Then the Coulomb constraint at the i th contact is represented exactly by

$$x_i \in \mathcal{K}_i, \quad \mathcal{K}_i := \mathcal{Q}_3.$$

For all three contacts, define

$$x = (x_1^\top, x_2^\top, x_3^\top)^\top \in \mathbb{R}^9, \quad \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3.$$

Thus, $x \in \mathcal{K}$ is equivalent to satisfaction of the three Coulomb friction constraints. Since every second-order cone is self-dual, the product cone satisfies

$$\mathcal{K}^* = \mathcal{K}.$$

The physical force is recovered through

$$f_i = T_i x_i, \quad T_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \theta_i^{-1} & 0 & 0 \end{bmatrix}.$$

Let

$$T = \text{diag}(T_1, T_2, T_3).$$

Because $\theta_i > 0$, every T_i , and hence T , is nonsingular. Therefore,

$$f = T x$$

defines a bijection between the physical force variables and the scaled cone variables.

Substituting $f = Tx$ into (2.1) and the objective of $(\mathcal{G}_t)$, define

$$G := G_{\text{gr}}T, \quad Q := T^\top HT, \quad q(t) := T^\top r(t).$$

Since $H > 0$ and T is nonsingular,

$$Q = Q^\top > 0.$$

The physical problem $(\mathcal{G}_t)$ is therefore equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^9} \quad & \Phi(x, t) = \frac{1}{2}x^\top Qx + q(t)^\top x, \\ \text{s.t.} \quad & Gx + \omega_{\text{ext}}(t) = 0, \\ & x \in \mathcal{K}, \end{aligned} \tag{\mathcal{P}_t}$$

with the feasible set

$$\Omega(t) = \{x \in \mathcal{K} \mid Gx + \omega_{\text{ext}}(t) = 0\}. \tag{2.2}$$

The transformation is exact: it preserves feasibility, objective values, and the one-to-one correspondence between f and x . If a reference scaled force $x_{\text{ref}}(t)$ is prescribed, it can be incorporated by setting

$$q(t) = -Qx_{\text{ref}}(t).$$

2.3. Well-posedness and variational inequality characterization

Assumption 2.1. For every $t \geq 0$, the feasible set $\Omega(t)$ is nonempty. The matrix G has full row rank, and $q(t)$ and $\omega_{\text{ext}}(t)$ are continuously differentiable and bounded together with their first derivatives.

Because T is nonsingular, $\text{rank}(G) = \text{rank}(G_{\text{gr}})$. Hence, the full-row-rank condition is directly verifiable from the contact geometry and excludes redundant wrench-balance equations. It is weaker than the KKT-residual regularity required later for moving-path tracking.

Under Assumption 2.1, $\Omega(t)$ is nonempty, closed, and convex. Since $Q > 0$, problem $(\mathcal{P}_t)$ has a unique primal optimizer $x^*(t)$. The existence, local uniqueness, and regularity of the smoothed primal–dual path used in the neurodynamic model are specified separately in Assumption 3.1.

Define

$$F(x, t) = \nabla_x \Phi(x, t) = Qx + q(t). \tag{2.3}$$

The first-order optimality condition is equivalently expressed as:

$$\text{find } x^*(t) \in \Omega(t) \text{ such that } (y - x^*(t))^\top F(x^*(t), t) \geq 0, \quad \forall y \in \Omega(t), \tag{2.4}$$

by the standard variational-inequality characterization of differentiable convex programs [8, 12].

Because $Q > 0$, $F(\cdot, t)$ is strongly monotone. Consequently, (2.4) is a strongly monotone variational inequality over a time-varying affine section of a product SOC. The VI form is used to construct the KKT residual of the strongly convex program $(\mathcal{P}_t)$; no applicability to arbitrary nonpotential SOCCVIs is claimed.

3. Smoothed KKT residual and moving solution path

This section constructs a differentiable KKT residual for the time-varying SOC-constrained program derived in Section 2. The stationarity and equality-feasibility conditions are smooth, whereas the cone complementarity condition is nonsmooth on the boundary of the product cone. Following the natural-residual and smoothed metric-projector framework [7, 10, 29], we replace the cone projector by a smooth spectral approximation and characterize the resulting moving primal–dual path.

3.1. Nonsmooth KKT system and natural residual

To separate the residual construction from the dimension of the three-finger example, consider the general product-cone program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \Phi(x, t) = \frac{1}{2}x^\top Qx + q(t)^\top x, \\ \text{s.t.} \quad & Gx + \omega_{\text{ext}}(t) = 0, \\ & x \in \mathcal{K}, \end{aligned} \tag{P}_t^g$$

where $Q = Q^\top > 0$, $G \in \mathbb{R}^{\ell \times n}$, and $\mathcal{K} \subset \mathbb{R}^n$ is a Cartesian product of SOCs. The model in Section 2 corresponds to $n = 9$, $\ell = 6$, and three copies of $\mathcal{Q}_3$.

The Lagrangian is

$$\mathcal{L}(x, \mu, \lambda, t) = \frac{1}{2}x^\top Qx + q(t)^\top x + \mu^\top (Gx + \omega_{\text{ext}}(t)) - \lambda^\top x, \tag{3.1}$$

where $\mu \in \mathbb{R}^\ell$ is the equality multiplier and $\lambda \in \mathcal{K}$ is the cone multiplier. Since $\mathcal{K}$ is self-dual, the KKT conditions are

$$Qx + q(t) + G^\top \mu - \lambda = 0, \tag{3.2a}$$

$$Gx + \omega_{\text{ext}}(t) = 0, \tag{3.2b}$$

$$x \in \mathcal{K}, \quad \lambda \in \mathcal{K}, \quad \langle x, \lambda \rangle = 0. \tag{3.2c}$$

Define the primal–dual variable

$$z = (x^\top, \mu^\top, \lambda^\top)^\top \in \mathbb{R}^{2n+\ell}. \tag{3.3}$$

Let $\Pi_{\mathcal{K}}$ denote the Euclidean projector onto $\mathcal{K}$. Moreau's decomposition for the self-dual cone gives the natural residual

$$\phi_{\text{NR}}(x, \lambda) = x - \Pi_{\mathcal{K}}(x - \lambda), \tag{3.4}$$

which satisfies

$$\phi_{\text{NR}}(x, \lambda) = 0 \iff x \in \mathcal{K}, \quad \lambda \in \mathcal{K}, \quad \langle x, \lambda \rangle = 0. \tag{3.5}$$

Thus, the nonsmoothness of the KKT system is confined to the metric projector in (3.4).

3.2. Smoothed projection and KKT residual

For a second-order cone

$$\mathcal{K}_i = \{(u_0, \bar{u}^\top)^\top \in \mathbb{R} \times \mathbb{R}^{m_i-1} \mid \|\bar{u}\| \leq u_0\},$$

let $v = (v_0, \bar{v}^\top)^\top$, and define

$$r_\varepsilon(\bar{v}) = \sqrt{\|\bar{v}\|^2 + \varepsilon^2}, \quad p_\varepsilon(s) = \frac{1}{2} \left(s + \sqrt{s^2 + \varepsilon^2} \right), \quad (3.6)$$

where $\varepsilon > 0$. Then $p_\varepsilon(s) \rightarrow \max\{s, 0\}$ as $\varepsilon \downarrow 0$, and

$$0 < p'_\varepsilon(s) < 1.$$

Set

$$\rho_1^\varepsilon(v) = v_0 + r_\varepsilon(\bar{v}), \quad \rho_2^\varepsilon(v) = v_0 - r_\varepsilon(\bar{v})$$

and

$$c_1^\varepsilon(v) = \frac{1}{2} \begin{bmatrix} 1 \\ \bar{v}/r_\varepsilon(\bar{v}) \end{bmatrix}, \quad c_2^\varepsilon(v) = \frac{1}{2} \begin{bmatrix} 1 \\ -\bar{v}/r_\varepsilon(\bar{v}) \end{bmatrix}.$$

The smoothed projector onto $\mathcal{K}_i$ is

$$\Pi_{\mathcal{K}_i}^\varepsilon(v) = p_\varepsilon(\rho_1^\varepsilon(v))c_1^\varepsilon(v) + p_\varepsilon(\rho_2^\varepsilon(v))c_2^\varepsilon(v). \quad (3.7)$$

For $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_m$, the product projector is defined blockwise:

$$\Pi_{\mathcal{K}}^\varepsilon(v) = \left((\Pi_{\mathcal{K}_1}^\varepsilon(v_1))^\top, \dots, (\Pi_{\mathcal{K}_m}^\varepsilon(v_m))^\top \right)^\top. \quad (3.8)$$

For every $\varepsilon > 0$, $\Pi_{\mathcal{K}}^\varepsilon$ is continuously differentiable and converges pointwise to $\Pi_{\mathcal{K}}$ as $\varepsilon \downarrow 0$.

Define the smoothed natural residual by

$$\phi_{\text{NR}}^\varepsilon(x, \lambda) = x - \Pi_{\mathcal{K}}^\varepsilon(x - \lambda). \quad (3.9)$$

Let

$$P_\varepsilon(v) = D\Pi_{\mathcal{K}}^\varepsilon(v). \quad (3.10)$$

Since the projector argument is $x - \lambda$,

$$D\phi_{\text{NR}}^\varepsilon(x, \lambda)[\delta x, \delta \lambda] = (I - P_\varepsilon(x - \lambda))\delta x + P_\varepsilon(x - \lambda)\delta \lambda. \quad (3.11)$$

The smoothed KKT residual is

$$\mathcal{R}_\varepsilon(z, t) = \begin{bmatrix} Qx + q(t) + G^\top \mu - \lambda \\ Gx + \omega_{\text{ext}}(t) \\ \phi_{\text{NR}}^\varepsilon(x, \lambda) \end{bmatrix}. \quad (3.12)$$

A solution of

$$\mathcal{R}_\varepsilon(z, t) = 0 \quad (3.13)$$

is denoted by

$$z_\varepsilon^*(t) = ((x_\varepsilon^*(t))^\top, (\mu_\varepsilon^*(t))^\top, (\lambda_\varepsilon^*(t))^\top)^\top.$$

The Jacobian with respect to $z = (x, \mu, \lambda)$ is

$$\mathcal{J}_\varepsilon(z, t) = \begin{bmatrix} Q & G^\top & -I \\ G & 0 & 0 \\ I - P_\varepsilon(x - \lambda) & 0 & P_\varepsilon(x - \lambda) \end{bmatrix}. \quad (3.14)$$

The associated residual merit function is

$$\Psi_\varepsilon(z, t) = \frac{1}{2} \|\mathcal{R}_\varepsilon(z, t)\|^2, \quad (3.15)$$

with the gradient

$$\nabla_z \Psi_\varepsilon(z, t) = \mathcal{J}_\varepsilon(z, t)^\top \mathcal{R}_\varepsilon(z, t). \quad (3.16)$$

The merit function is used only to quantify residual magnitude. The time-varying model developed later is based on residual cancellation, rather than frozen-time gradient descent of Ψ_ε .

3.3. Local residual regularity and the moving path

We first establish pointwise nonsingularity of the smoothed KKT Jacobian for the strongly convex product-SOC model.

Lemma 3.1 (Pointwise nonsingularity of the smoothed KKT Jacobian). *Suppose that $Q = Q^\top > 0$ and $\text{rank}(G) = \ell$. For every fixed $\varepsilon > 0$, the derivative of the smoothed product-cone projector satisfies*

$$0 < P_\varepsilon(v) < I, \quad \text{for every finite } v \in \mathbb{R}^n.$$

Consequently, $\mathcal{J}_\varepsilon(z, t)$ is nonsingular for every finite z and $t \geq 0$.

Proof. For one SOC block, write $v = (v_0, \bar{v}^\top)^\top$ and define

$$r = \sqrt{\|\bar{v}\|^2 + \varepsilon^2}, \quad w = \frac{\bar{v}}{r}, \quad s_\pm = v_0 \pm r.$$

Since $\varepsilon > 0$, $\|w\| < 1$. Let

$$d_\pm = p'_\varepsilon(s_\pm), \quad a = \frac{d_+ + d_-}{2}, \quad b = \frac{d_+ - d_-}{2},$$

and

$$c = \frac{p_\varepsilon(s_+) - p_\varepsilon(s_-)}{2r}.$$

Direct differentiation gives

$$P_\varepsilon(v) = \begin{bmatrix} a & bw^\top \\ bw & cI + (a - c)ww^\top \end{bmatrix}.$$

Because $0 < p'_\varepsilon(s) < 1$, $d_\pm \in (0, 1)$, and the mean value theorem gives $c \in (0, 1)$. For $h = (h_0, \bar{h}^\top)^\top$ and $\eta = w^\top \bar{h}$,

$$h^\top P_\varepsilon(v) h = \begin{bmatrix} h_0 \\ \eta \end{bmatrix}^\top \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} h_0 \\ \eta \end{bmatrix} + c(\|\bar{h}\|^2 - \eta^2).$$

The 2×2 matrix has eigenvalues d_+ and d_- , while $\eta^2 \leq \|w\|^2 \|\bar{h}\|^2 < \|\bar{h}\|^2$ for $\bar{h} \neq 0$. Hence $P_\varepsilon(v) > 0$. Applying the same argument to $I - P_\varepsilon(v)$ yields $I - P_\varepsilon(v) > 0$, and therefore $0 < P_\varepsilon(v) < I$. The product-SOC result follows blockwise.

Now set $P = P_\varepsilon(x - \lambda)$ and suppose $(u, v, \xi) \in \ker \mathcal{J}_\varepsilon(z, t)$. Then

$$Qu + G^\top v - \xi = 0, \quad Gu = 0, \quad (I - P)u + P\xi = 0.$$

Since $P > 0$,

$$\xi = -P^{-1}(I - P)u,$$

and hence

$$(Q + P^{-1} - I)u + G^\top v = 0.$$

Premultiplication by $u^\top$ and $Gu = 0$ give

$$u^\top(Q + P^{-1} - I)u = 0.$$

Because $0 < P < I$, $P^{-1} - I > 0$; together with $Q > 0$, this implies $u = 0$. Thus $\xi = 0$ and $G^\top v = 0$. Since G has full row rank, $v = 0$. Therefore $\ker \mathcal{J}_\varepsilon(z, t) = \{0\}$. $\square$

The lemma provides the pointwise Jacobian nonsingularity used in the following local residual characterization.

Proposition 3.1 (Local residual regularity). *Fix $t \geq 0$ and $\varepsilon > 0$, and let $\mathcal{R}_\varepsilon(z_\varepsilon^*(t), t) = 0$. If $\mathcal{J}_\varepsilon(z_\varepsilon^*(t), t)$ is nonsingular, then there exist a neighborhood $\mathcal{U}_\varepsilon(t)$ of $z_\varepsilon^*(t)$ and constants $0 < m_\varepsilon \leq M_\varepsilon$ such that*

$$m_\varepsilon \|z - z_\varepsilon^*(t)\| \leq \|\mathcal{R}_\varepsilon(z, t)\| \leq M_\varepsilon \|z - z_\varepsilon^*(t)\|, \quad z \in \mathcal{U}_\varepsilon(t). \quad (3.17)$$

After shrinking the neighborhood if necessary,

$$\nabla_z \Psi_\varepsilon(z, t) = 0 \iff \mathcal{R}_\varepsilon(z, t) = 0.$$

Proof. Let $\bar{z} = z_\varepsilon^*(t)$. By the continuity and nonsingularity of $\mathcal{J}_\varepsilon(\bar{z}, t)$, the averaged Jacobian

$$\int_0^1 \mathcal{J}_\varepsilon(\bar{z} + s(z - \bar{z}), t) \, ds$$

remains nonsingular in a sufficiently small neighborhood of $\bar{z}$, with uniformly bounded largest and smallest singular values. The fundamental theorem of calculus then gives (3.17). The stationary-point equivalence follows from (3.16) and the local nonsingularity of $\mathcal{J}_\varepsilon$. $\square$

For the time-varying problem, let $\varepsilon : [0, \infty) \rightarrow (0, \infty)$ be continuously differentiable and consider

$$\mathcal{R}_{\varepsilon(t)}(z_\varepsilon^*(t), t) = 0. \quad (3.18)$$

Assumption 3.1 (Regular moving KKT path). For every $t \geq 0$, system (3.18) admits a locally unique solution $z_\varepsilon^*(t)$. The residual is continuously differentiable in a neighborhood of this path, and there exists $M_J > 0$ such that

$$\|\mathcal{J}_{\varepsilon(t)}^{-1}(z_\varepsilon^*(t), t)\| \leq M_J, \quad t \geq 0. \quad (3.19)$$

Lemma 3.1 guarantees pointwise nonsingularity under $Q > 0$, $\text{rank}(G) = \ell$, and $\varepsilon(t) > 0$, whereas Assumption 3.1 requires the stronger time-uniform inverse bound along the moving path. A sufficient setting is a compact path together with

$$0 < \underline{\varepsilon} \leq \varepsilon(t) \leq \bar{\varepsilon} < \infty,$$

for which the continuity of the inverse yields a finite uniform bound. If $\varepsilon(t) \rightarrow 0$, pointwise nonsingularity alone does not ensure such uniform boundedness, so the latter is retained explicitly in the assumption.

With z held fixed, define the total explicit time derivative

$$\mathcal{R}_t(z, t) := \frac{\partial}{\partial t} \mathcal{R}_{\varepsilon(t)}(z, t). \quad (3.20)$$

For constant matrices Q and G ,

$$\mathcal{R}_t(z, t) = \begin{bmatrix} \dot{q}(t) \\ \dot{\omega}_{\text{ext}}(t) \\ \partial_{\varepsilon} \phi_{\text{NR}}^{\varepsilon}(x, \lambda) \big|_{\varepsilon=\varepsilon(t)} \dot{\varepsilon}(t) \end{bmatrix}. \quad (3.21)$$

Differentiating (3.18) gives

$$\dot{z}_{\varepsilon}^*(t) = -\mathcal{J}_{\varepsilon(t)}^{-1}(z_{\varepsilon}^*(t), t) \mathcal{R}_t(z_{\varepsilon}^*(t), t). \quad (3.22)$$

Consequently,

$$\|\dot{z}_{\varepsilon}^*(t)\| \leq M_J \|\mathcal{R}_t(z_{\varepsilon}^*(t), t)\|.$$

When finite, define the path-speed bound

$$\nu_{\varepsilon} = \sup_{t \geq 0} \|\dot{z}_{\varepsilon}^*(t)\|. \quad (3.23)$$

3.4. Smoothing bias

Let

$$\mathcal{R}_0(z, t) = \begin{bmatrix} Qx + q(t) + G^{\top} \mu - \lambda \\ Gx + \omega_{\text{ext}}(t) \\ \phi_{\text{NR}}(x, \lambda) \end{bmatrix} \quad (3.24)$$

denote the nonsmoothed KKT residual, and let $z^*(t)$ satisfy

$$\mathcal{R}_0(z^*(t), t) = 0.$$

The scalar smoothing function satisfies

$$0 \leq p_{\varepsilon}(s) - \max\{s, 0\} \leq \frac{\varepsilon}{2}.$$

Accordingly, on every bounded neighborhood relevant to the solution path, there exists $c_{\Pi} > 0$ such that

$$\|\Pi_{\mathcal{K}}^{\varepsilon}(v) - \Pi_{\mathcal{K}}(v)\| \leq c_{\Pi} \varepsilon, \quad (3.25)$$

and hence

$$\|\mathcal{R}_{\varepsilon(t)}(z, t) - \mathcal{R}_0(z, t)\| \leq c_{\Pi} \varepsilon(t). \quad (3.26)$$

Assumption 3.2 (Uniform local error bound). There exist a uniform tubular neighborhood of $z^*(t)$ and a constant $\kappa_0 > 0$ such that

$$\|z - z^*(t)\| \leq \kappa_0 \|\mathcal{R}_0(z, t)\| \quad (3.27)$$

for every point z in that neighborhood and every $t \geq 0$.

Proposition 3.2 (Smoothing-bias estimate). Suppose that Assumption 3.2 holds and that $z_\varepsilon^*(t)$ lies in the corresponding error-bound neighborhood. Then

$$\|z_\varepsilon^*(t) - z^*(t)\| \leq C_\varepsilon \varepsilon(t), \quad C_\varepsilon := \kappa_0 c_\Pi. \quad (3.28)$$

Proof. Using $\mathcal{R}_{\varepsilon(t)}(z_\varepsilon^*(t), t) = 0$, the local error bound and (3.26) give

$$\begin{aligned} \|z_\varepsilon^*(t) - z^*(t)\| &\leq \kappa_0 \|\mathcal{R}_0(z_\varepsilon^*(t), t)\| \\ &= \kappa_0 \|\mathcal{R}_0(z_\varepsilon^*(t), t) - \mathcal{R}_{\varepsilon(t)}(z_\varepsilon^*(t), t)\| \\ &\leq \kappa_0 c_\Pi \varepsilon(t). \end{aligned}$$

□

Let

$$\varepsilon_\infty = \limsup_{t \rightarrow \infty} \varepsilon(t).$$

Then Proposition 3.2 yields

$$\limsup_{t \rightarrow \infty} \|z_\varepsilon^*(t) - z^*(t)\| \leq C_\varepsilon \varepsilon_\infty. \quad (3.29)$$

Thus, the final tracking bound must retain a smoothing term whenever the schedule has a positive limiting floor; this term vanishes only when $\varepsilon(t) \rightarrow 0$.

4. Event-triggered sparse anchor-assisted residual-cancellation neurodynamics

This section embeds the residual-cancellation mechanism into a redundant architecture consisting of one anchor and N_s delayed solver units. Each unit evaluates the complete smoothed KKT residual. The anchor follows the moving smoothed KKT path, while delayed diffusive coupling and event-triggered Top- p corrections maintain consistency among the replicated solvers.

4.1. Residual-cancellation field

Let

$$z = (x^\top, \mu^\top, \lambda^\top)^\top \in \mathbb{R}^{d_z}, \quad d_z = 2n + \ell,$$

and recall the smoothed residual $\mathcal{R}_{\varepsilon(t)}(z, t)$, its Jacobian $\mathcal{J}_{\varepsilon(t)}(z, t) = \nabla_z \mathcal{R}_{\varepsilon(t)}(z, t)$, and the total explicit time derivative $\mathcal{R}_t(z, t)$ introduced in Section 3.

Assumption 4.1 (Uniform residual regularity). There exists a forward-invariant domain $\mathcal{D} \subset \mathbb{R}^{d_z}$ containing the moving smoothed KKT path and all node trajectories considered below. For every $z \in \mathcal{D}$

and $t \geq 0$, $\mathcal{J}_{\varepsilon(t)}(z, t)$ is nonsingular. Moreover, there exist positive constants $M_J, L_J, L_R, L_t, \bar{R}, \bar{R}_t$ such that, for all $z, z_1, z_2 \in \mathcal{D}$,

$$\begin{aligned}\|\mathcal{J}_{\varepsilon(t)}^{-1}(z, t)\| &\leq M_J, \\ \|\mathcal{J}_{\varepsilon(t)}(z_1, t) - \mathcal{J}_{\varepsilon(t)}(z_2, t)\| &\leq L_J\|z_1 - z_2\|, \\ \|\mathcal{R}_{\varepsilon(t)}(z_1, t) - \mathcal{R}_{\varepsilon(t)}(z_2, t)\| &\leq L_R\|z_1 - z_2\|, \\ \|\mathcal{R}_t(z_1, t) - \mathcal{R}_t(z_2, t)\| &\leq L_t\|z_1 - z_2\|, \\ \|\mathcal{R}_{\varepsilon(t)}(z, t)\| &\leq \bar{R}, \quad \|\mathcal{R}_t(z, t)\| \leq \bar{R}_t.\end{aligned}\tag{4.1}$$

For the present smoothed strongly convex SOC model, Lemma 3.1 shows that

$$Q > 0, \quad \text{rank}(G) = \ell, \quad \varepsilon(t) > 0$$

are sufficient for pointwise nonsingularity of $\mathcal{J}_{\varepsilon(t)}(z, t)$ at every finite z . The uniform inverse bound in (4.1) is stronger. In particular, if $\mathcal{D}$ is compact and

$$0 < \underline{\varepsilon} \leq \varepsilon(t) \leq \bar{\varepsilon} < \infty,$$

then continuity of $\mathcal{J}_{\varepsilon}(z)^{-1}$ on $\mathcal{D} \times [\underline{\varepsilon}, \bar{\varepsilon}]$ implies a finite uniform bound M_J . The remaining constants in Assumption 4.1 require the corresponding uniform Lipschitz and boundedness properties of the Jacobian, residual, and explicit time derivative. If $\varepsilon(t) \rightarrow 0$, pointwise nonsingularity alone does not guarantee uniform inverse boundedness; hence the uniform regularity assumption is retained for the subsequent analysis.

Define the residual-cancellation field

$$\mathcal{F}_{\varepsilon(t)}(z, t) = -\mathcal{J}_{\varepsilon(t)}^{-1}(z, t)[\gamma\mathcal{R}_{\varepsilon(t)}(z, t) + \mathcal{R}_t(z, t)], \quad \gamma > 0.\tag{4.2}$$

Equivalently, $\mathcal{F}_{\varepsilon(t)}(z, t)$ is obtained by solving

$$\mathcal{J}_{\varepsilon(t)}(z, t)v = -\gamma\mathcal{R}_{\varepsilon(t)}(z, t) - \mathcal{R}_t(z, t).$$

If $\dot{z} = \mathcal{F}_{\varepsilon(t)}(z, t)$, the chain rule gives

$$\frac{d}{dt}\mathcal{R}_{\varepsilon(t)}(z(t), t) = -\gamma\mathcal{R}_{\varepsilon(t)}(z(t), t),\tag{4.3}$$

so the explicit residual motion is cancelled rather than absorbed into a tracking disturbance.

A standard inverse-perturbation estimate under Assumption 4.1 yields

$$\|\mathcal{F}_{\varepsilon(t)}(z_1, t) - \mathcal{F}_{\varepsilon(t)}(z_2, t)\| \leq L_F\|z_1 - z_2\|,\tag{4.4}$$

where one admissible constant is

$$L_F = M_J(\gamma L_R + L_t) + M_J^2 L_J(\gamma \bar{R} + \bar{R}_t).\tag{4.5}$$

Model uncertainty, derivative approximation errors, and numerical integration errors are represented in residual coordinates by

$$\tilde{d}_i(t) = d_i(t) + e_{\partial,i}(t) + e_{h,i}(t), \quad i = 0, 1, \dots, N_s,\tag{4.6}$$

with bounds

$$\|\tilde{d}_0(t)\| \leq \bar{d}_0^{\text{eff}}, \quad \|\tilde{d}_i(t)\| \leq \bar{d}_i^{\text{eff}}, \quad i = 1, \dots, N_s.\tag{4.7}$$

4.2. Anchor and delayed solver units

Let $z_0(t) \in \mathbb{R}^{d_z}$ denote the anchor state. Its dynamics are

$$\dot{z}_0(t) = \mathcal{F}_{\varepsilon(t)}(z_0(t), t) + \mathcal{J}_{\varepsilon(t)}^{-1}(z_0(t), t) \widetilde{d}_0(t), \quad (4.8)$$

and hence

$$\frac{d}{dt} \mathcal{R}_{\varepsilon(t)}(z_0(t), t) = -\gamma \mathcal{R}_{\varepsilon(t)}(z_0(t), t) + \widetilde{d}_0(t). \quad (4.9)$$

Let $z_{\varepsilon}^*(t)$ denote the moving smoothed KKT path and set

$$\mathcal{R}_{\varepsilon(t)}(z_{\varepsilon}^*(t), t) = 0, \quad e_0(t) = z_0(t) - z_{\varepsilon}^*(t). \quad (4.10)$$

Let $z_i(t) \in \mathbb{R}^{d_z}$, $i = 1, \dots, N_s$, denote the replicated solver states. The communication weights and delays satisfy

$$A = [a_{ij}] \in \mathbb{R}^{N_s \times N_s}, \quad a_{ij} \geq 0, \quad a_{ii} = 0, \quad \sum_{j=1}^{N_s} a_{ij} > 0, \quad (4.11)$$

$$0 \leq \tau_{ij}(t) \leq \bar{\tau}, \quad i, j = 1, \dots, N_s.$$

Thus, every solver receives at least one continuous coupling input. Between impulse instants, the i th solver evolves according to

$$\begin{aligned} \dot{z}_i(t) = & \mathcal{F}_{\varepsilon(t)}(z_i(t), t) + \alpha \sum_{j=1}^{N_s} a_{ij} [z_j(t - \tau_{ij}(t)) - z_i(t)] \\ & + \mathcal{J}_{\varepsilon(t)}^{-1}(z_i(t), t) \widetilde{d}_i(t), \quad t \neq t_k, \end{aligned} \quad (4.12)$$

where $\alpha > 0$.

The initial histories are prescribed by

$$z_i(s) = \varphi_i(s), \quad s \in [-\bar{\tau}, 0], \quad i = 0, 1, \dots, N_s, \quad (4.13)$$

where each φ_i is continuous and takes values in $\mathcal{D}$.

Assumption 4.1 and (4.7) imply the explicit anchor speed bound

$$\|\dot{z}_0(t)\| \leq \nu_0, \quad \nu_0 := M_J(\gamma \bar{R} + \bar{R}_t + \bar{d}_0^{\text{eff}}). \quad (4.14)$$

Assumption 4.2 (Anchor-history regularity). The prescribed anchor history φ_0 satisfies

$$\|\varphi_0(s) - \varphi_0(r)\| \leq \nu_0 |s - r|, \quad s, r \in [-\bar{\tau}, 0],$$

with $\varphi_0(0) = z_0(0)$, where ν_0 is defined in (4.14). The solver histories φ_i , $i = 1, \dots, N_s$, remain continuous and take values in $\mathcal{D}$ as specified in (4.13).

4.3. Event-triggered sparse correction

Define the node-to-anchor errors and weighted synchronization energy by

$$r_i(t) = z_i(t) - z_0(t), \quad V_s(t) = \sum_{i=1}^{N_s} \pi_i \|r_i(t)\|^2, \quad \pi_i > 0. \quad (4.15)$$

Let $0 < \Delta_{\min} < \Delta_{\max}$, set $t_0 = 0$ and $V_s(t_0^+) = V_s(0)$, and define, for $k \geq 0$,

$$t_{k+1} = \inf \left\{ t \geq t_k + \Delta_{\min} : V_s(t) \geq \vartheta \exp[-\lambda_e(t - t_k)] V_s(t_k^+) + \varrho_R \|\mathcal{R}_{\varepsilon(t)}(z_0(t), t)\|^2 + \varrho_d \right\} \wedge (t_k + \Delta_{\max}), \quad (4.16)$$

where $\vartheta > 1$, $\lambda_e > 0$, $\varrho_R \geq 0$, $\varrho_d \geq 0$, and $\inf \emptyset := +\infty$. The factor ϑ introduces post-correction hysteresis, $\Delta_{\min}$ imposes a positive refractory interval, and $\Delta_{\max}$ prevents correction from being postponed indefinitely.

Proposition 4.1 (Positive inter-event time and exclusion of Zeno behavior). *Let the event sequence $\{t_k\}_{k \geq 0}$ be generated by (4.16). Then every pair of consecutive event times satisfies*

$$\Delta_{\min} \leq t_{k+1} - t_k \leq \Delta_{\max}, \quad k \geq 0. \quad (4.17)$$

Consequently, the event sequence has no finite accumulation point, and hence Zeno behavior is excluded.

Proof. Let $\widehat{t}_{k+1}$ denote the infimum appearing in (4.16). Since every candidate time in the defining set satisfies $t \geq t_k + \Delta_{\min}$, one has

$$\widehat{t}_{k+1} \geq t_k + \Delta_{\min},$$

where $\widehat{t}_{k+1} = +\infty$ if the triggering set is empty. On the other hand, the timeout mechanism imposes the forced-update time $t_k + \Delta_{\max}$. Therefore,

$$t_{k+1} = \min\{\widehat{t}_{k+1}, t_k + \Delta_{\max}\}.$$

Because $0 < \Delta_{\min} < \Delta_{\max}$, it follows immediately that

$$\Delta_{\min} \leq t_{k+1} - t_k \leq \Delta_{\max}.$$

Since $t_0 = 0$, repeated application of the lower bound gives

$$t_k \geq k\Delta_{\min}, \quad k \geq 0.$$

Hence, for any finite $T > 0$, the number of triggering instants contained in $[0, T]$ satisfies

$$N(T) := \#\{k \geq 1 : t_k \leq T\} \leq \left\lfloor \frac{T}{\Delta_{\min}} \right\rfloor < \infty.$$

Thus infinitely many triggering instants cannot occur within any finite time interval, which excludes Zeno behavior. This conclusion follows directly from the imposed positive dwell-time constraint and does not depend on the synchronization stability condition developed in Section 5. $\square$

At each $t_k, k \geq 1$, let $\mathcal{P}_k$ contain the indices of the p largest values in $\{\pi_i \|r_i(t_k^-)\|^2\}_{i=1}^{N_s}$, where $1 \leq p \leq N_s$. Ties are resolved arbitrarily. The jump rule is

$$z_i(t_k^+) = \begin{cases} z_i(t_k^-) - \beta_i^k [z_i(t_k^-) - z_0(t_k^-)], & i \in \mathcal{P}_k, \\ z_i(t_k^-), & i \notin \mathcal{P}_k, \end{cases} \quad z_0(t_k^+) = z_0(t_k^-), \quad (4.18)$$

where

$$0 < \beta_{\min} \leq \beta_i^k \leq \beta_{\max} < 2. \quad (4.19)$$

Lemma 4.1 (Top- p coverage). *For every impulse instant t_k ,*

$$\sum_{i \in \mathcal{P}_k} \pi_i \|r_i(t_k^-)\|^2 \geq \frac{p}{N_s} V_s(t_k^-). \quad (4.20)$$

Proof. Arrange $y_i = \pi_i \|r_i(t_k^-)\|^2$ as $y_{(1)} \geq \dots \geq y_{(N_s)}$. The average of the largest p values is no smaller than the average of all N_s values; hence $p^{-1} \sum_{i=1}^p y_{(i)} \geq N_s^{-1} \sum_{i=1}^{N_s} y_{(i)}$, which proves (4.20). $\square$

At an impulse instant,

$$r_i(t_k^+) = \begin{cases} (1 - \beta_i^k) r_i(t_k^-), & i \in \mathcal{P}_k, \\ r_i(t_k^-), & i \notin \mathcal{P}_k. \end{cases}$$

Therefore,

$$\begin{aligned} V_s(t_k^+) &= V_s(t_k^-) - \sum_{i \in \mathcal{P}_k} \pi_i \beta_i^k (2 - \beta_i^k) \|r_i(t_k^-)\|^2 \\ &\leq q V_s(t_k^-), \quad q := 1 - \frac{p}{N_s} \beta_{\min} (2 - \beta_{\max}) \in [0, 1). \end{aligned} \quad (4.21)$$

For the logarithmic impulsive-delay analysis used below, we additionally require

$$\frac{p}{N_s} \beta_{\min} (2 - \beta_{\max}) < 1,$$

so that $0 < q < 1$. The excluded boundary case $q = 0$ occurs when $p = N_s$ and $\beta_{\min} = \beta_{\max} = 1$, corresponding to complete instantaneous resetting of all solver states to the anchor. Since the subsequent analysis involves both $\ln q$ and q^{-1} , it is stated only for $0 < q < 1$.

Thus, the Top- p rule provides an explicit coverage ratio and requires no additional sparse-coverage assumption.

5. Stability and tracking analysis

This section establishes local anchor tracking, delayed node-to-anchor synchronization, and a complete tracking estimate relative to the nonsmoothed moving KKT path.

5.1. Anchor tracking

Proposition 3.1 establishes a pointwise local residual error bound at a fixed time under nonsingularity of the smoothed KKT Jacobian. For the moving-path tracking analysis, a uniform estimate with an explicit constant along the prescribed regularity tube is required. The following lemma provides this uniform counterpart.

Choose $\rho_R > 0$ such that, for every $t \geq 0$, the line segment joining $z_\varepsilon^*(t)$ and any point z satisfying

$$\|z - z_\varepsilon^*(t)\| \leq \rho_R$$

is contained in the regularity domain $\mathcal{D}$, and assume

$$M_J L_J \rho_R \leq 1. \quad (5.1)$$

Lemma 5.1 (Uniform local residual error bound). *Under Assumption 4.1 and (5.1),*

$$\|z - z_\varepsilon^*(t)\| \leq \kappa_R \|\mathcal{R}_{\varepsilon(t)}(z, t)\|, \quad \kappa_R := 2M_J, \quad (5.2)$$

whenever

$$\|z - z_\varepsilon^*(t)\| \leq \rho_R.$$

Proof. By the fundamental theorem of calculus,

$$\mathcal{R}_{\varepsilon(t)}(z, t) = \overline{\mathcal{J}}_{\varepsilon(t)}(z, t)(z - z_\varepsilon^*(t)),$$

where $\overline{\mathcal{J}}_{\varepsilon(t)}$ is the Jacobian averaged along the segment joining $z_\varepsilon^*(t)$ and z . Its distance from $\mathcal{J}_{\varepsilon(t)}(z_\varepsilon^*(t), t)$ is bounded by $L_J \|z - z_\varepsilon^*(t)\|/2$. Hence

$$\|\mathcal{J}_{\varepsilon(t)}^{-1}(z_\varepsilon^*(t), t)[\overline{\mathcal{J}}_{\varepsilon(t)} - \mathcal{J}_{\varepsilon(t)}(z_\varepsilon^*(t), t)]\| \leq \frac{M_J L_J \rho_R}{2} \leq \frac{1}{2}.$$

The Banach perturbation lemma gives $\|\overline{\mathcal{J}}_{\varepsilon(t)}^{-1}\| \leq 2M_J$, which proves (5.2). $\square$

Define

$$R_{a,0} := \|\mathcal{R}_{\varepsilon(0)}(z_0(0), 0)\|. \quad (5.3)$$

Theorem 5.1 (Anchor tracking). *Suppose that Assumption 4.1 holds, $z_0(0)$ lies in the residual-error tube, and*

$$\kappa_R \max \left\{ R_{a,0}, \frac{\bar{d}_0^{\text{eff}}}{\gamma} \right\} < \rho_R. \quad (5.4)$$

Then the anchor remains in the tube and satisfies

$$\|\mathcal{R}_{\varepsilon(t)}(z_0(t), t)\| \leq \exp(-\gamma t) R_{a,0} + \frac{\bar{d}_0^{\text{eff}}}{\gamma} (1 - \exp(-\gamma t)), \quad (5.5)$$

$$\|z_0(t) - z_\varepsilon^*(t)\| \leq \kappa_R \exp(-\gamma t) R_{a,0} + \frac{\kappa_R \bar{d}_0^{\text{eff}}}{\gamma} (1 - \exp(-\gamma t)), \quad (5.6)$$

and

$$\limsup_{t \rightarrow \infty} \|z_0(t) - z_\varepsilon^*(t)\| \leq \frac{\kappa_R \bar{d}_0^{\text{eff}}}{\gamma}. \quad (5.7)$$

Proof. Equation (4.9) and the variation-of-constants formula give

$$\mathcal{R}_{\varepsilon(t)}(z_0(t), t) = \exp(-\gamma t) \mathcal{R}_{\varepsilon(0)}(z_0(0), 0) + \int_0^t \exp[-\gamma(t-s)] \tilde{d}_0(s) \, ds,$$

which yields (5.5). If the anchor reached the tube boundary for the first time at $t = t_e$, then Lemma 5.1 and (5.4) would imply

$$\|z_0(t_e) - z_\varepsilon^*(t_e)\| \leq \kappa_R \max \left\{ R_{a,0}, \frac{\bar{d}_0^{\text{eff}}}{\gamma} \right\} < \rho_R,$$

which is a contradiction. Thus the anchor remains in the tube, and (5.6)–(5.7) follow. $\square$

5.2. Impulsive delay comparison

Impulsive delay inequalities and generalized Halanay estimates provide standard tools for combining delayed continuous evolution with discrete contraction [31, 36]. The following bounded-input comparison lemma is adapted to the persistent Top- p contraction required here.

Lemma 5.2 (Impulsive delay comparison). *Let $V : [-\bar{\tau}, \infty) \rightarrow \mathbb{R}_+$ be continuous on every $[t_k, t_{k+1})$ and admit finite one-sided limits at each impulse. Suppose that*

$$D^+V(t) \leq aV(t) + b \sup_{t-\bar{\tau} \leq s \leq t} V(s) + u(t), \quad t \neq t_k, \quad (5.8)$$

where $a \in \mathbb{R}$, $b \geq 0$, $u(t) \geq 0$, and that

$$V(t_k^+) \leq qV(t_k^-), \quad q \in (0, 1), \quad 0 < \Delta_{\min} \leq t_{k+1} - t_k \leq \Delta_{\max}. \quad (5.9)$$

Define

$$c := -\left(a + \frac{\ln q}{\Delta_{\max}}\right), \quad V_h := \sup_{-\bar{\tau} \leq s \leq 0} V(s). \quad (5.10)$$

If there exists $\lambda \in (0, c)$ such that

$$\lambda + a + q^{-1}b \exp(\lambda \bar{\tau}) + \frac{\ln q}{\Delta_{\max}} < 0, \quad (5.11)$$

then

$$V(t) \leq C_{\lambda,0} \exp(-\lambda t) V_h + C_{\lambda,u} \sup_{0 \leq s \leq t} \exp[-\lambda(t-s)] u(s), \quad (5.12)$$

where

$$\theta_\lambda := \frac{q^{-1}b \exp(\lambda \bar{\tau})}{c - \lambda} \in (0, 1), \quad C_{\lambda,0} := \frac{q^{-1}}{1 - \theta_\lambda}, \quad C_{\lambda,u} := \frac{q^{-1}}{(c - \lambda)(1 - \theta_\lambda)}. \quad (5.13)$$

Proof. Let $N(t, s)$ be the number of impulses in $(s, t]$. The upper inter-impulse bound gives

$$q^{N(t,s)} \leq q^{-1} \exp\left(\frac{\ln q}{\Delta_{\max}}(t-s)\right).$$

Iterating the flow comparison and jump inequality therefore yields

$$V(t) \leq q^{-1} \exp(-ct) V_h + q^{-1} \int_0^t \exp[-c(t-s)] \left[b \sup_{s-\bar{\tau} \leq r \leq s} V(r) + u(s) \right] ds.$$

Set

$$Z(t) := \max \left\{ V_h, \sup_{0 \leq s \leq t} \exp(\lambda s) V(s) \right\}, \quad U(t) := \sup_{0 \leq s \leq t} \exp(\lambda s) u(s).$$

Since

$$\exp(\lambda s) \sup_{s-\bar{\tau} \leq r \leq s} V(r) \leq \exp(\lambda \bar{\tau}) Z(t),$$

multiplication by $\exp(\lambda t)$ gives

$$\exp(\lambda t) V(t) \leq q^{-1} V_h + \theta_\lambda Z(t) + \frac{q^{-1}}{c - \lambda} U(t).$$

Condition (5.11) implies $\theta_\lambda < 1$. Taking the supremum over $[0, t]$ proves (5.12). $\square$

5.3. Delayed node-to-anchor synchronization

For $t \neq t_k$, subtracting (4.8) from (4.12) gives

$$\begin{aligned} \dot{r}_i(t) &= \mathcal{F}_{\varepsilon(t)}(z_i(t), t) - \mathcal{F}_{\varepsilon(t)}(z_0(t), t) + \alpha \sum_{j=1}^{N_s} a_{ij} [r_j(t - \tau_{ij}(t)) - r_i(t) + \delta_{ij}(t)] \\ &\quad + \mathcal{J}_{\varepsilon(t)}^{-1}(z_i(t), t) \tilde{d}_i(t) - \mathcal{J}_{\varepsilon(t)}^{-1}(z_0(t), t) \tilde{d}_0(t), \end{aligned} \quad (5.14)$$

where

$$\delta_{ij}(t) := z_0(t - \tau_{ij}(t)) - z_0(t).$$

Under Assumption 4.2, the delay-induced anchor mismatch satisfies

$$\|\delta_{ij}(t)\| \leq \bar{\tau} \nu_0, \quad t \geq 0.$$

Indeed, let $\sigma = t - \tau_{ij}(t)$. If $\sigma \geq 0$, then (4.14) gives

$$\begin{aligned} \|\delta_{ij}(t)\| &= \|z_0(\sigma) - z_0(t)\| \\ &\leq \int_{\sigma}^t \|\dot{z}_0(\ell)\| \, d\ell \\ &\leq \nu_0(t - \sigma) \\ &= \nu_0 \tau_{ij}(t) \leq \bar{\tau} \nu_0. \end{aligned}$$

If $\sigma < 0$, then $z_0(\sigma) = \varphi_0(\sigma)$, and Assumption 4.2, together with $\varphi_0(0) = z_0(0)$, yields

$$\begin{aligned} \|\delta_{ij}(t)\| &= \|\varphi_0(\sigma) - z_0(t)\| \\ &\leq \|\varphi_0(\sigma) - \varphi_0(0)\| + \|z_0(0) - z_0(t)\| \\ &\leq \nu_0(-\sigma) + \nu_0 t \\ &= \nu_0 \tau_{ij}(t) \leq \bar{\tau} \nu_0. \end{aligned}$$

Hence, the same delay-mismatch estimate holds for all $t \geq 0$, including the initial interval in which the delayed argument lies in $[-\bar{\tau}, 0]$.

Recall

$$\bar{V}_s(t) := \sup_{t-\bar{\tau} \leq s \leq t} V_s(s),$$

and define

$$\begin{aligned} d_i^A &:= \sum_{j=1}^{N_s} a_{ij}, & d_{\min} &:= \min_i d_i^A, \\ a_{\text{del}}^\pi &:= \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \frac{\pi_i}{\pi_j} a_{ij}, & a_\Sigma^\pi &:= \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \pi_i a_{ij}, \\ \pi_\Sigma &:= \sum_{i=1}^{N_s} \pi_i, & \pi_{\min} &:= \min_i \pi_i. \end{aligned} \quad (5.15)$$

By (4.11), $d_{\min} > 0$. Choose $\vartheta_\tau > 0$, $\vartheta_\delta > 0$, $\vartheta_\tau + \vartheta_\delta < 2$, and $\vartheta_d > 0$, and define

$$\begin{aligned} a_s &:= 2L_F - \alpha(2 - \vartheta_\tau - \vartheta_\delta)d_{\min} + \vartheta_d, \\ b_s &:= \frac{\alpha}{\vartheta_\tau} a_{\text{del}}^\pi, \quad c_\tau := \frac{\alpha}{\vartheta_\delta} a_\Sigma^\pi, \quad c_d := \frac{M_J^2}{\vartheta_d} \pi_\Sigma. \end{aligned} \quad (5.16)$$

Lemma 5.3 (Synchronization differential estimate). *Under Assumptions 4.1 and 4.2, for $t \neq t_k$,*

$$D^+ V_s(t) \leq a_s V_s(t) + b_s \bar{V}_s(t) + B_s, \quad (5.17)$$

where

$$B_s := c_\tau \bar{\tau}^2 \nu_0^2 + c_d (\bar{d}^{\text{eff}} + \bar{d}_0^{\text{eff}})^2. \quad (5.18)$$

Proof. Differentiating V_s along (5.14) produces three contributions. By (4.4),

$$2 \sum_i \pi_i r_i^\top [\mathcal{F}_{\varepsilon(t)}(z_i, t) - \mathcal{F}_{\varepsilon(t)}(z_0, t)] \leq 2L_F V_s.$$

Young's inequality [38] and the delay-mismatch estimate above give

$$2\alpha \sum_{i,j} \pi_i a_{ij} r_i^\top [r_j(t - \tau_{ij}) - r_i + \delta_{ij}] \leq -\alpha(2 - \vartheta_\tau - \vartheta_\delta)d_{\min} V_s + b_s \bar{V}_s + c_\tau \bar{\tau}^2 \nu_0^2.$$

Finally,

$$\left\| \mathcal{J}_{\varepsilon(t)}^{-1}(z_i, t) \tilde{d}_i - \mathcal{J}_{\varepsilon(t)}^{-1}(z_0, t) \tilde{d}_0 \right\| \leq M_J (\bar{d}^{\text{eff}} + \bar{d}_0^{\text{eff}}),$$

and another application of Young's inequality [38] yields

$$2 \sum_i \pi_i \|r_i\| \|\xi_i\| \leq \vartheta_d V_s + c_d (\bar{d}^{\text{eff}} + \bar{d}_0^{\text{eff}})^2.$$

Summing these estimates proves (5.17). $\square$

Theorem 5.2 (Delayed impulsive synchronization). *Suppose that Assumptions 4.1 and 4.2 hold and that the correction parameters are chosen such that $0 < q < 1$. Define*

$$c_s := -\left(a_s + \frac{\ln q}{\Delta_{\max}}\right). \quad (5.19)$$

If there exists $\lambda_s \in (0, c_s)$ such that

$$\lambda_s + a_s + q^{-1} b_s \exp(\lambda_s \bar{\tau}) + \frac{\ln q}{\Delta_{\max}} < 0, \quad (5.20)$$

define

$$\theta_s := \frac{q^{-1} b_s \exp(\lambda_s \bar{\tau})}{c_s - \lambda_s} \in (0, 1), \quad C_{s,0} := \frac{q^{-1}}{1 - \theta_s}, \quad C_{s,u} := \frac{q^{-1}}{(c_s - \lambda_s)(1 - \theta_s)}. \quad (5.21)$$

Then, for all $t \geq 0$,

$$V_s(t) \leq C_{s,0} \exp(-\lambda_s t) V_{s,0} + C_{s,u} B_s, \quad V_{s,0} := \sup_{-\bar{\tau} \leq s \leq 0} V_s(s), \quad (5.22)$$

and

$$\max_{1 \leq i \leq N_s} \|z_i(t) - z_0(t)\| \leq \sqrt{\frac{C_{s,0}}{\pi_{\min}}} \exp\left(-\frac{\lambda_s t}{2}\right) \sqrt{V_{s,0}} + E_s, \quad (5.23)$$

where

$$E_s := \sqrt{\frac{C_{s,u}}{\pi_{\min}}} \left[\sqrt{c_\tau} \bar{\tau} v_0 + \sqrt{c_d} (\bar{d}^{\text{eff}} + \bar{d}_0^{\text{eff}}) \right]. \quad (5.24)$$

Proof. By Proposition 4.1, the event sequence satisfies the prescribed dwell-time bounds. Together with Lemma 5.3 and (4.21), the hypotheses of Lemma 5.2 are satisfied with $V = V_s$, $a = a_s$, $b = b_s$, and $u = B_s$, which proves (5.22). Since $V_s(t) \geq \pi_{\min} \max_i \|z_i(t) - z_0(t)\|^2$, (5.23) follows from $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. $\square$

Remark 5.1. The threshold parameters $\vartheta, \lambda_e, \varrho_R, \varrho_d$ determine the realized correction schedule but do not enter the sufficient condition (5.20). Theorem 5.2 therefore provides a schedule-robust sufficient guarantee for impulse sequences satisfying the prescribed dwell-time and jump-contraction conditions. The Top- p rule determines which solver units are corrected and yields the contraction factor q , whereas the event-triggering rule determines when the corrections are applied according to the evolving synchronization error and anchor residual. The present analysis does not claim a stronger stability guarantee for this trigger than for other admissible impulse schedules satisfying the same conditions. The condition may be conservative because of the Young and delay-supremum estimates. As shown in Section 6.6, the resulting conservative bound does not certify the nominal numerical setting, which does not imply instability because the condition is sufficient rather than necessary.

Remark 5.2 (Relation to recent synchronization frameworks). Recent studies have addressed predefined-time multilayer synchronization [20] and privacy-preserving event-triggered synchronization of delayed multi-agent systems [39]. In contrast, the present solver units synchronize to an optimization-generated anchor governed by a moving SOC-constrained KKT residual rather than to a prescribed network trajectory or consensus state. The resulting framework integrates moving KKT-path tracking, delayed solver replication, and sparse Top- p correction, while separating anchor error, delay mismatch, implementation disturbance, and smoothing bias. Its limitations are that it does not provide predefined-time or privacy guarantees and requires uniform residual regularity, Jacobian-based linear solves, and a reliable supervisory anchor. These distinctions clarify the scope of the proposed method without implying performance dominance over the cited synchronization approaches.

5.4. Complete tracking estimate

Let $z^*(t)$ denote a nonsmoothed KKT solution path. Under the assumptions of Proposition 3.2, the smoothing-bias estimate gives

$$\|z_\varepsilon^*(t) - z^*(t)\| \leq C_\varepsilon \varepsilon(t). \quad (5.25)$$

Moreover,

$$\|z_i(t) - z^*(t)\| \leq \|z_i(t) - z_0(t)\| + \|z_0(t) - z_\varepsilon^*(t)\| + \|z_\varepsilon^*(t) - z^*(t)\|. \quad (5.26)$$

Theorem 5.3 (Complete tracking bound). *Suppose that the conditions of Theorems 5.1 and 5.2 hold. Assume additionally that Assumption 3.2 holds and that $z_\varepsilon^*(t)$ remains in the corresponding uniform*

error-bound neighborhood of $z^*(t)$ for all $t \geq 0$. Then

$$\begin{aligned} \max_{1 \leq i \leq N_s} \|z_i(t) - z^*(t)\| &\leq \sqrt{\frac{C_{s,0}}{\pi_{\min}}} \exp\left(-\frac{\lambda_s t}{2}\right) \sqrt{V_{s,0}} + \kappa_R \exp(-\gamma t) R_{a,0} \\ &\quad + E_s + \frac{\kappa_R \bar{d}_0^{\text{eff}}}{\gamma} (1 - \exp(-\gamma t)) + C_\varepsilon \varepsilon(t). \end{aligned} \quad (5.27)$$

If $\varepsilon_\infty := \limsup_{t \rightarrow \infty} \varepsilon(t)$, then

$$\limsup_{t \rightarrow \infty} \max_{1 \leq i \leq N_s} \|z_i(t) - z^*(t)\| \leq E_s + \frac{\kappa_R \bar{d}_0^{\text{eff}}}{\gamma} + C_\varepsilon \varepsilon_\infty. \quad (5.28)$$

Proof. Apply (5.23), (5.6), and (5.25) to (5.26); taking the limit superior gives (5.28). $\square$

Corollary 5.1 (Ideal exponential tracking). *Under the conditions of Theorem 5.3, if*

$$\bar{d}_0^{\text{eff}} = \bar{d}^{\text{eff}} = 0, \quad \bar{\tau} v_0 = 0, \quad \varepsilon(t) \rightarrow 0,$$

then

$$\lim_{t \rightarrow \infty} \max_{1 \leq i \leq N_s} \|z_i(t) - z^*(t)\| = 0.$$

If, in addition, there exist constants $C_\varepsilon^0 > 0$ and $\lambda_\varepsilon > 0$ such that

$$\varepsilon(t) \leq C_\varepsilon^0 \exp(-\lambda_\varepsilon t),$$

then there exists $C > 0$ such that

$$\max_{1 \leq i \leq N_s} \|z_i(t) - z^*(t)\| \leq C \exp(-\lambda_* t), \quad \lambda_* := \min\left\{\frac{\lambda_s}{2}, \gamma, \lambda_\varepsilon\right\} > 0. \quad (5.29)$$

The complete estimate separates delay-induced mismatch, implementation disturbance, anchor residual error, and smoothing bias. All conclusions hold on the forward-invariant regularity domain $\mathcal{D}$ and therefore do not constitute unrestricted global convergence claims.

6. Numerical experiments and robotic applications

This section evaluates the proposed anchor–solver neurodynamics on time-varying contact-force allocation problems. The experiments examine tracking accuracy, replica synchronization, sparse correction, fault recovery, and numerical-integration sensitivity, with both internal ablations and published neurodynamic baselines.

6.1. Experimental setting and evaluation metrics

The experiments were implemented in Python with NumPy, SciPy, and Matplotlib. CVXPY with CLARABEL was used only to compute frozen-time SOC program reference solutions. The full delayed hybrid system uses explicit Euler integration, while single-node ordinary differential equation (ODE) comparisons use Radau with independent backward differentiation formula (BDF) checks. The explicit residual-time derivative is approximated at fixed z by a centered finite difference, or a forward

difference near $t = 0$. Comparisons use matched initial states for common variables; suite-specific seeds, initialization, and constant negative-time histories are reported in Appendix A.

The nominal architecture has one anchor and $N_s = 5$ solver units, with $p = 2$ units selected at each correction. The delay and smoothing schedules are

$$\tau_{ij}(t) = 0.02[1 + 0.5 \sin(0.8t)], \quad 0 \leq \tau_{ij}(t) \leq 0.03,$$

$$\varepsilon(t) = 10^{-5} + (10^{-2} - 10^{-5})\exp(-0.4t).$$

The nominal parameters provide moderately sparse correction and a nontrivial state-dependent triggering interval. In particular, $p = 2$ corrects 40% of the $N_s = 5$ solver units at each correction instant and, together with $\beta = 0.65$, gives

$$q = 1 - \frac{p}{N_s}\beta(2 - \beta) = 0.649.$$

Moreover, $\Delta_{\min} = 0.04 = 16h$ enforces a positive refractory interval, whereas $\Delta_{\max} = 0.25 = 100h$ prevents indefinitely delayed correction while retaining state-dependent event timing. The remaining values in Table 1 define a representative numerical setting rather than an optimized parameter set. Their effects are examined further in Section 6.6; these nominal values are not claimed to numerically certify the conservative sufficient condition of Theorem 5.2. The external baseline comparison uses independently calibrated gains stated below.

For $z = (x^\top, \mu^\top, \lambda^\top)^\top \in \mathbb{R}^{d_z}$, define

$$\begin{aligned} r_R(t) &= \|\mathcal{R}_{\varepsilon(t)}(z(t), t)\|, & r_{\text{norm}}(t) &= \frac{r_R(t)}{\sqrt{d_z}}, \\ r_h(t) &= \|Gx(t) + \omega_{\text{ext}}(t)\|, & r_K(t) &= \text{dist}(x(t), \mathcal{K}), \\ r_c(t) &= |\langle x(t), \lambda(t) \rangle|, & e_{\text{syn}}(t) &= \max_{1 \leq i \leq N_s} \|z_i(t) - z_0(t)\|. \end{aligned} \quad (6.1)$$

Table 1. Nominal numerical and event-trigger parameters.

Parameter	Value	Parameter	Value
Horizon T	8	Step size h	2.5×10^{-3}
Solver units N_s	5	Selected units p	2
Coupling gain α	0.30	Correction gain β	0.65
Residual gain γ	5.0	Difference step	10^{-4}
$\Delta_{\min}$	0.04	$\Delta_{\max}$	0.25
Trigger factor ϑ	1.10	Trigger decay λ_e	0.50
ϱ_R	10^{-2}	ϱ_d	3×10^{-7}
$\varepsilon(0)$	10^{-2}	$\varepsilon_{\min}$	10^{-5}

For replicated methods, r_R is averaged over solver units in the nominal tables, while r_h , r_K , and r_c are evaluated at the anchor. Fault studies use the maximum node residual.

The aggregate squared correction magnitude is

$$C_{\text{corr}} = \sum_k \sum_{i \in \mathcal{P}_k} \|z_i(t_k^+) - z_i(t_k^-)\|^2. \quad (6.2)$$

We also report the correction count N_{trig} and corrected-node update count N_{upd} . These metrics describe discrete correction activity only and do not measure total communication, runtime, or energy consumption.

6.2. Contact-force allocation and reference accuracy

Application A1 is the three-finger problem in Section 2, with cone Q_3^3 ; A2 uses four contacts and Q_3^4 . Both retain fixed contact geometry while the external wrench and preferred force distribution vary, consistent with the constant equality matrix in the analysis. Appendix A specifies the geometry, friction coefficients, and time-varying data. Table 2 summarizes the terminal results for the three-finger and four-contact force-allocation problems.

Table 2. Terminal results for contact-force allocation problems.

Case	d_z	r_R	r_{norm}	r_h	r_K	r_c	e_{syn}
A1: three-finger	24	5.76×10^{-5}	1.18×10^{-5}	1.04×10^{-5}	$< 10^{-12}$	1.31×10^{-7}	4.47×10^{-4}
A2: four-contact	30	5.43×10^{-5}	9.91×10^{-6}	4.50×10^{-6}	$< 10^{-12}$	1.74×10^{-7}	5.08×10^{-5}

Both applications attain normalized residuals of order 10^{-5} , negligible cone violation, and small synchronization errors.

Figures 1 and 2 show the force responses. In A2, every correction occurs at Δ_{max} , so this example demonstrates force-allocation accuracy and replica consistency rather than adaptive event timing.

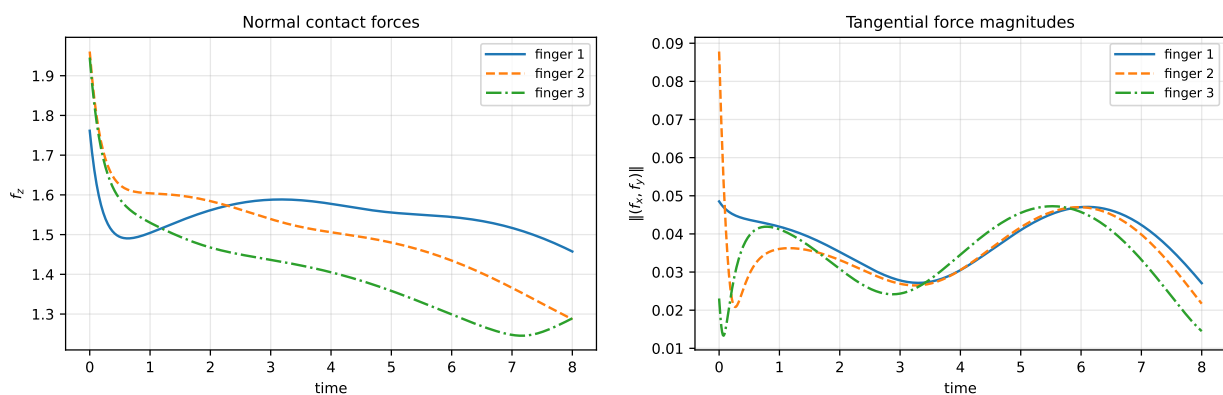


Figure 1. Three-finger contact-force evolution: Normal forces and tangential-force magnitudes.

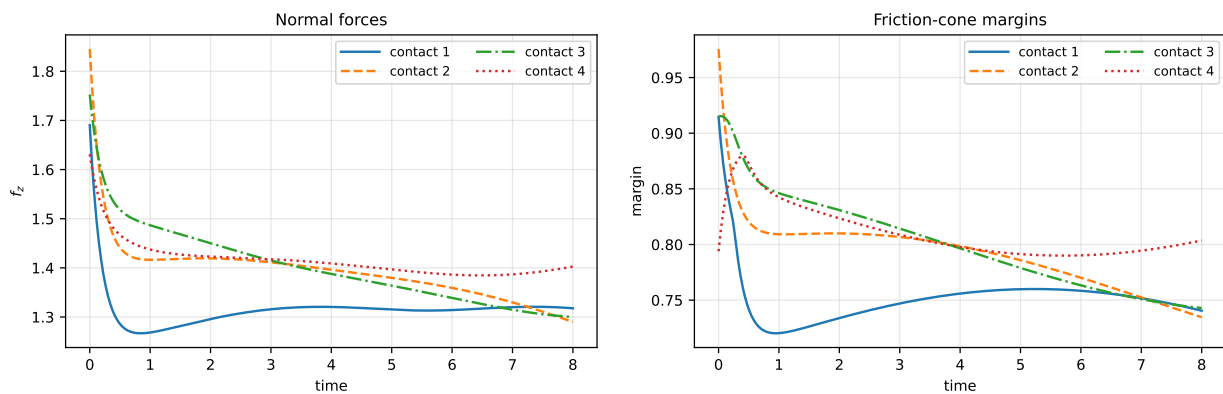


Figure 2. Four-contact force redistribution: Normal forces and friction-cone margins.

The anchor primal state is compared with CLARABEL solutions at 81 uniformly spaced times using

$$E_{\text{ref}}(t_k) = \frac{\|x_{\text{NN}}(t_k) - x_{\text{ref}}(t_k)\|}{1 + \|x_{\text{ref}}(t_k)\|}. \quad (6.3)$$

Table 3 reports the 61 samples with $t_k \geq 2$, excluding the larger initialization transient. The maximum sampled post-transient errors are below 10^{-5} in both applications.

Table 3. Post-transient comparison with frozen-time CLARABEL solutions.

Case	Samples	$E_{\text{ref}}^{\text{avg}}$	$E_{\text{ref}}^{\text{max}}$	$E_{\text{ref}}^{\text{term}}$
A1	61	3.82×10^{-6}	9.49×10^{-6}	3.85×10^{-6}
A2	61	2.10×10^{-6}	9.04×10^{-6}	2.26×10^{-6}

6.3. Comparison with published neurodynamic methods

To complement the internal ablation study, we compare the proposed method with two published conic neurodynamic models: The cone projection (CP) network of Ko et al. [15] and the smoothed metric-projector network of Sun et al. [29]. Both are applied to the same time-varying A1/A2 data using their published state equations, including the dynamic smoothing variable in the Sun model. Since their original analyses do not establish the moving-path residual-cancellation guarantee studied here, their use is interpreted as a numerical adaptation to common time-varying conic data.

To reduce tuning bias, gains are calibrated on data shifted by 11 time units using seed 1900, while the reported tests use seed 1901. For each method, the smallest candidate gain whose calibration score does not exceed 10^{-2} is selected, where the score combines the tail mean relative primal error, normalized equality residual, and normalized cone violation. The selected gains (Ko2011, Sun2021, proposed core) are (30, 1000, 3) for A1 and (10, 10, 3) for A2; the replicated proposed method uses the same core gain.

Table 4 reports mean relative primal errors over the 61 samples with $t \geq 2$, using the worst solver unit for the replicated method. Under the above calibration protocol, the proposed core and replicated models attain mean errors of order 10^{-5} , compared with 10^{-3} – 10^{-2} for the selected baselines. This is a mean-error comparison rather than a uniform-error claim; the sampled maxima of the proposed method are 3.53×10^{-4} – 3.88×10^{-4} .

Table 4. Mean post-transient relative primal errors for published conic neurodynamic baselines and the proposed method.

Method	A1	A2
Ko et al. [15]	9.26×10^{-3}	1.25×10^{-3}
Sun et al. [29]	1.16×10^{-2}	2.89×10^{-3}
Proposed core	2.23×10^{-5}	2.45×10^{-5}
Proposed replicated	4.75×10^{-5}	4.31×10^{-5}

The comparison is intended to assess numerical tracking accuracy under common time-varying conic data, not theoretical equivalence. Because the models have different state dimensions and integration costs, no runtime advantage is claimed. Numerical-integration checks are reported in Section 6.6, and CLARABEL is used only as a frozen-time reference solver.

6.4. Ablation and event-triggered correction

The deterministic ablations share their common initial states. No-feedforward removes only $\partial_t \mathcal{R}_{\varepsilon(t)}$, retaining the Jacobian solve and residual-decay term. Single-node ZNN retains the full residual-cancellation law but removes replication, coupling, and correction. The NR and SOC-FB variants use their respective smoothed residual-gradient dynamics.

Periodic sparse correction uses the proposed method's mean inter-event interval and the same p ; dense periodic correction uses that period but corrects every node. No sparse correction retains delayed coupling without anchor corrections. No-delay sets $\tau_{ij} = 0$, and constant-smoothing fixes $\varepsilon = 10^{-5}$. The proposed and periodic sparse runs have identical correction counts, isolating the effect of correction timing. Table 5 summarizes the principal ablation results for three-finger grasping.

Table 5. Principal ablation results for three-finger grasping.

Method	r_R	e_{syn}	N_{trig}	N_{upd}	C_{corr}
Proposed	5.87×10^{-5}	4.52×10^{-4}	56	112	9.82×10^{-2}
Periodic sparse	8.34×10^{-5}	6.74×10^{-4}	56	112	3.44×10^{-1}
No-feedforward	2.35×10^{-2}	6.49×10^{-4}	32	64	9.82×10^{-2}
No sparse correction	1.35×10^{-4}	7.56×10^{-4}	0	0	0
Dense periodic	4.25×10^{-5}	2.40×10^{-4}	56	280	6.10×10^{-1}
Single-node ZNN	1.17×10^{-5}	–	0	0	0

Removing feedforward increases the terminal and tail-average residuals by factors of approximately 400 and 360, respectively. At the matched budget $N_{\text{trig}} = 56$, $N_{\text{upd}} = 112$, the proposed scheme reduces C_{corr} by 71.5% relative to periodic sparse correction, with smaller terminal residual and synchronization errors. Against dense periodic correction, corrected-node updates decrease by 60% and squared correction magnitude by 83.9%. These are correction-activity comparisons, not total communication or energy savings.

State-dependent timing therefore has a numerical benefit in this matched-budget test; it does not imply a stronger stability guarantee than another schedule satisfying the same dwell-time and contraction

conditions. Single-node ZNN has the smallest fault-free residual. Replication provides computational redundancy and supports consistency recovery among the delayed solver units. Figure 3 compares the common NR-residual trajectories of the principal ablation variants.

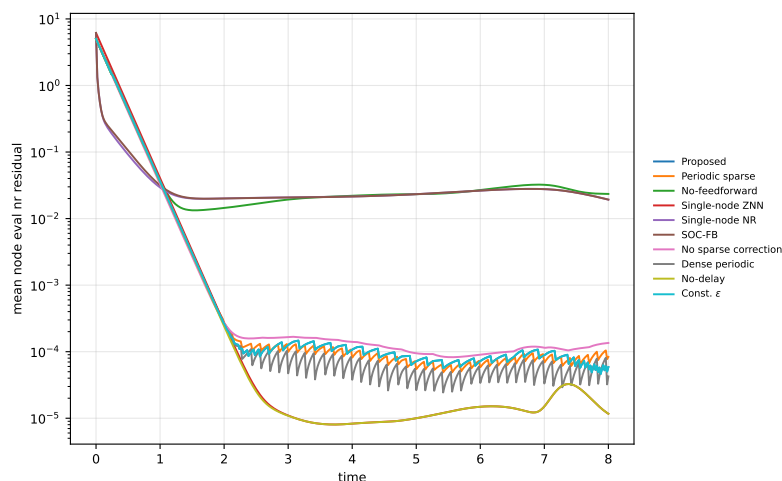


Figure 3. Common NR-residual trajectories for principal ablation variants.

Figure 4 shows the distribution of the nominal inter-event intervals. The 56 nominal inter-event intervals satisfy

$$0.045 \leq t_{k+1} - t_k \leq 0.25.$$

Of the corrections, 60.7% are threshold-triggered and 39.3% are forced by the upper dwell-time bound. Along the nominal anchor trajectory,

$$\min_t \sigma_{\min}(\mathcal{J}_{\varepsilon(t)}(z_0(t), t)) = 2.108 \times 10^{-2}, \quad \max_t \kappa(\mathcal{J}_{\varepsilon(t)}(z_0(t), t)) = 2.523 \times 10^2.$$

These sampled diagnostics support numerical regularity along the simulated path, but do not establish the uniform inverse bound required by Assumption 4.1 over the full regularity domain.

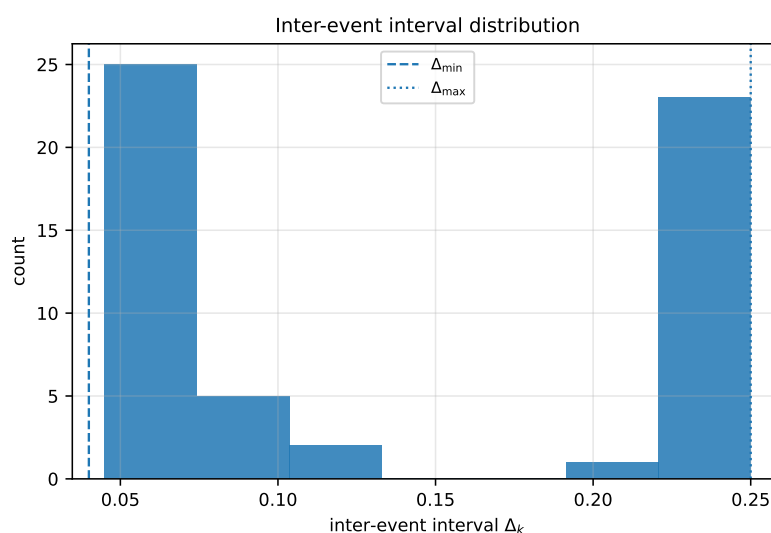


Figure 4. Distribution of nominal inter-event intervals; vertical lines indicate $\Delta_{\min}$ and $\Delta_{\max}$.

6.5. Fault recovery and robustness

Fault robustness is evaluated on application A1 using the proposed method (P), periodic sparse correction (PS), and the replicated system without sparse correction (NC). All methods use seed 980 and matched fault directions. At $t_f = 4$, impulsive perturbations affect one, two, or three solver units, or the anchor. For m simultaneously affected solvers, each perturbation has norm $A/\sqrt{m}$, so the aggregate injected norm is A . We test $A \in \{0.5, 1, 2\}$ with five directions generated from seeds 100983–100987. We also consider a solver drift over $[4, 5]$ with integrated bias norm A , and a temporary anchor outage over $[4, 5]$, during which the anchor is frozen and its corrections are suspended. The periodic correction interval is calibrated from a fault-free run and then fixed for all fault tests.

Let $R_{\max}$ denote the maximum NR residual among the replicated solver units and R_{pk} its largest recorded post-correction value after fault onset. A solver is regarded as available when its residual is at most 10^{-3} . The settling time T_{set} is the earliest recorded time after which $R_{\max} \leq 10^{-3}$ through the end of the simulation, with at least 0.04 s of final confirmation. Time is measured from t_f for impulsive faults and from $t = 5$ for drift and outage cases.

Table 6 summarizes the statistical comparison of P, PS, and NC at $A = 1$.

Table 6. Fault-response statistics at $A = 1$, averaged over five paired fault directions; anchor outage is one deterministic run. P: proposed; PS: periodic sparse; NC: no sparse correction.

Fault	Post-correction peak R_{pk}			Settling time T_{set} (s)		
	P	PS	NC	P	PS	NC
One solver	0.690	1.965	1.965	0.750	0.642	1.430
Two solvers	0.638	1.824	1.824	0.750	0.713	1.417
Three solvers	1.128	1.489	1.489	0.935	0.777	1.374
Solver drift	0.096	0.235	0.371	0.465	0.415	1.116
Anchor impulse	1.282	0.656	1.40×10^{-4}	1.500	1.508	0.000
Anchor outage	0.075	0.039	1.40×10^{-4}	0.928	0.943	0.000

In total, 228 simulations were completed: 225 amplitude–direction–method combinations and three outage runs. All runs satisfied the final settling criterion, with terminal node residuals below 1.36×10^{-4} . For NC under anchor-only disturbances, $T_{\text{set}} = 0$ means that solver residuals never leave the prescribed accuracy region. These results therefore provide finite-horizon robustness evidence rather than a general fault-tolerance guarantee.

For trajectory-level illustration, Figure 5 additionally retains the nonreplicated single-node ZNN as a qualitative reference.

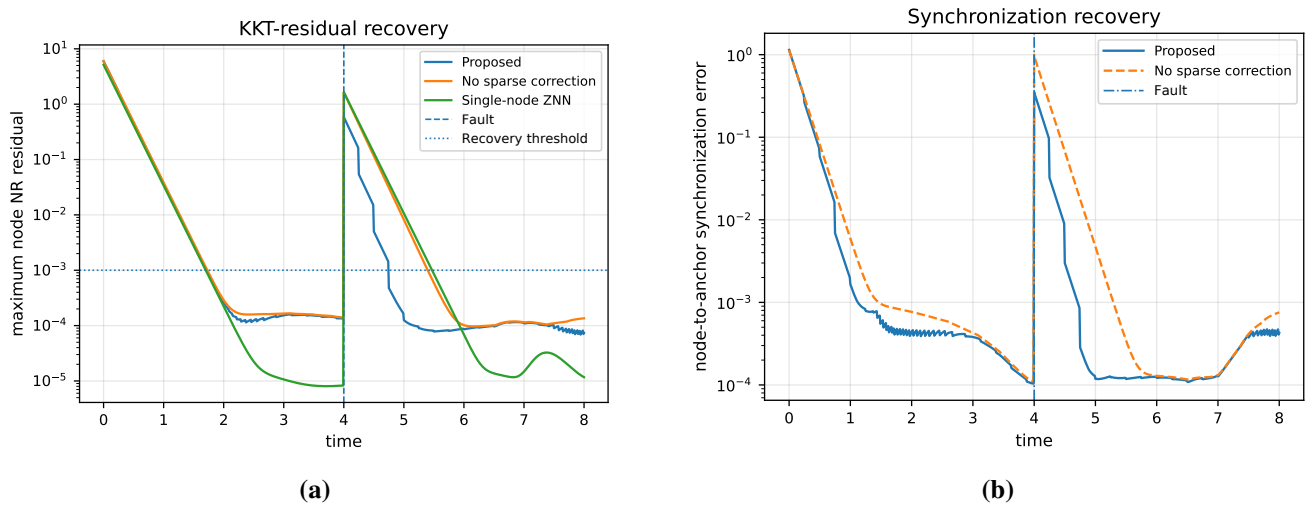


Figure 5. Representative response to a single solver-state fault at $t_f = 4$: (a) maximum-node NR residual and (b) node-to-anchor synchronization error.

In the representative single-solver fault, the proposed method reduces the maximum-node NR residual from approximately 1.647 immediately after injection to a largest recorded post-correction value of 0.586, while restoring node-to-anchor consistency.

The stress tests show increasing recovery difficulty with fault severity. For the three-solver impulse with $A = 1$, P gives $R_{pk} = 1.128 \pm 0.200$ and $T_{set} = 0.935 \pm 0.044$ s, while its mean settling time increases from 0.785 to 1.000 s as A rises from 0.5 to 2. For solver impulses and drift, P settles sooner than NC and has lower mean residual peaks than both comparison schemes, although PS is faster in some cases. Correction activity is likewise scenario-dependent; for example, at $A = 1$, C_{corr} is 0.501 versus 0.366 for P and PS under a single-solver impulse, but 0.101 versus 0.303 under drift. Hence, no universal reduction in recovery time, aggregate squared correction magnitude, or corrected-node update count is claimed.

Replication provides redundancy but not uninterrupted availability under arbitrary impulsive faults. During the $A = 1$ drift tests, at least four solver units remain available for every proposed run, whereas both comparison schemes temporarily lose all five. Anchor disturbances also expose a structural limitation: a corrupted or stale anchor can transfer error to otherwise healthy solvers. Recovery after a temporary outage therefore relies on restoration of a reliable anchor, and permanent supervisory loss remains outside the scope of the present architecture.

6.6. Sensitivity, numerical robustness, and limitations

Table 7 reports the sensitivity of the residual and synchronization errors to communication delays. Increasing the communication delay raises both residual and synchronization floors, whereas the zero-delay case yields near-machine-precision synchronization.

Table 7. Sensitivity to communication-delay amplitude.

τ_0	r_R	r_{norm}	e_{syn}	N_{trig}
0	1.20×10^{-5}	2.45×10^{-6}	4.07×10^{-14}	32
0.02	5.90×10^{-5}	1.20×10^{-5}	4.52×10^{-4}	56
0.05	1.23×10^{-4}	2.51×10^{-5}	1.02×10^{-3}	93
0.10	2.85×10^{-4}	5.82×10^{-5}	2.19×10^{-3}	135

The sufficient condition in Theorem 5.2 does not certify the nominal parameters. Indeed,

$$q = 0.649, \quad \frac{\ln q}{\Delta_{\max}} \approx -1.72929.$$

Since $\|\mathcal{J}^{-1}\|_2 \|\mathcal{J}\|_2 \geq 1$, $M_J L_R \geq 1$ and hence $L_F \geq \gamma = 5$. With $\alpha = 0.30$ and $d_{\min} = 1$,

$$a_s > 2L_F - 2\alpha d_{\min} \geq 9.4, \quad a_s + \frac{\ln q}{\Delta_{\max}} > 7.67071 > 0,$$

so $c_s < 0$. This failure of a conservative sufficient condition does not imply instability; the theorem uses uniform Lipschitz, Young-type, and delay-supremum bounds that may discard contractive directions of the actual field. The simulations therefore assess finite-horizon behavior rather than verify all uniform assumptions.

Table 8 reports the sensitivity to the upper dwell-time bound. Increasing $\Delta_{\max}$ reduces correction activity and C_{corr} , while the terminal residual remains of order 10^{-5} .

Table 8. Sensitivity to the upper dwell-time bound.

$\Delta_{\max}$	r_R	r_{norm}	e_{syn}	N_{trig}	N_{upd}	C_{corr}
0.10	4.9×10^{-5}	1.0×10^{-5}	4.08×10^{-4}	93	186	4.49×10^{-1}
0.20	5.0×10^{-5}	1.0×10^{-5}	4.13×10^{-4}	63	126	1.28×10^{-1}
0.30	6.3×10^{-5}	1.3×10^{-5}	4.69×10^{-4}	52	104	4.12×10^{-2}
0.50	4.7×10^{-5}	1.0×10^{-5}	3.97×10^{-4}	41	82	4.74×10^{-3}

Reducing the smoothing floor from 10^{-2} to 10^{-3} reduces terminal complementarity error from 7.50×10^{-5} to 1.40×10^{-6} . Relative to an $\varepsilon = 10^{-8}$ reference, the corresponding primal smoothing bias decreases from 4.36×10^{-7} to 4.36×10^{-9} , after which the offline accuracy floor is reached.

Numerical refinement of the full delayed hybrid system uses six Euler step sizes, with nearest-grid delay evaluation and constant initial histories. Table 9 therefore reflects combined integration and delay-sampling effects; the finest run is used only as a numerical self-reference.

Table 9. Full-system Euler refinement against $h_* = 3.125 \times 10^{-4}$. D_a : terminal relative anchor-primal difference; $\overline{D}_x$: time-averaged maximum relative node-primal difference; $\overline{R}_{\max}$: tail mean maximum node residual.

h	D_a	$\overline{D}_x$	$\overline{R}_{\max}$	N_{trig}
10^{-2}	1.49×10^{-5}	3.95×10^{-4}	1.09×10^{-4}	57
5×10^{-3}	7.21×10^{-6}	1.66×10^{-4}	9.94×10^{-5}	57
2.5×10^{-3}	3.37×10^{-6}	5.68×10^{-5}	9.96×10^{-5}	56
1.25×10^{-3}	1.45×10^{-6}	2.33×10^{-5}	1.01×10^{-4}	56
6.25×10^{-4}	4.82×10^{-7}	7.89×10^{-6}	1.01×10^{-4}	56
3.125×10^{-4}	–	–	1.02×10^{-4}	56

State differences decrease under refinement, while $\overline{R}_{\max}$ remains near 10^{-4} and N_{trig} stabilizes at 56 for $h \leq 2.5 \times 10^{-3}$. Event sequences need not converge monotonically: their Hausdorff distances from the finest event set are 7.19×10^{-3} and 1.03×10^{-2} s for $h = 1.25 \times 10^{-3}$ and 6.25×10^{-4} , respectively. Thus, stable residual statistics do not imply identical event times.

For the published-baseline comparison, Radau and BDF agree within 5×10^{-9} in relative primal norm on the common sampling grid. For the replicated comparison, halving $h = 1.25 \times 10^{-3}$ gives post-transient differences of 3.59×10^{-6} on A1 and 2.22×10^{-6} on A2, both below the prescribed 10^{-5} tolerance.

The residual-cancellation core is also examined on condensed SOC trajectory problems with increasing KKT dimension. Table 10 summarizes the residual and Jacobian-conditioning results for these problems.

Table 10. Conditioning and computational sensitivity.

Horizon	d_z	r_{norm}	$\min \sigma_{\min}(\mathcal{J}_\varepsilon)$	$\max \kappa(\mathcal{J}_\varepsilon)$
10	83	4.40×10^{-5}	8.65×10^{-2}	2.24×10^1
20	163	8.90×10^{-5}	4.29×10^{-2}	3.43×10^2
40	323	1.97×10^{-3}	8.65×10^{-2}	2.47×10^3

Independent no-delay core checks compare Euler with a Radau reference (relative/absolute tolerances $10^{-9}/10^{-11}$) and BDF at $10^{-10}/10^{-12}$. The maximum sampled relative primal differences between BDF and Radau are 3.71×10^{-10} at $d_z = 83$ and 9.71×10^{-10} at $d_z = 323$. At $d_z = 323$, reducing the Euler step from 0.01 to 0.00125 decreases the maximum relative primal difference from 8.79×10^{-3} to 1.08×10^{-3} .

A further Radau tolerance-tightening test succeeds at $d_z = 83$ but exceeds the 1800-s limit at $d_z = 323$. Thus, the larger case has cross-integrator agreement but not a completed additional tolerance check. These no-delay tests indicate increasing conditioning and computational sensitivity; they neither establish large-scale scalability nor show that Jacobian conditioning alone implies stiffness.

7. Conclusions

This paper has developed an event-triggered sparse anchor-assisted residual-cancellation neurodynamic framework for moving KKT-path tracking in time-varying strongly convex optimization under product SOC constraints. An NR-smoothed KKT residual and its explicit temporal derivative generate the residual-cancellation dynamics, while an optimization-driven anchor coordinates delayed replicated solver units through continuous coupling and event-triggered Top- p sparse corrections. The resulting analysis combines anchor tracking, delayed synchronization, implementation disturbance, and smoothing bias within a unified tracking estimate.

The framework offers three main advantages. Residual-time cancellation reduces moving-path lag relative to the no-feedforward variant; anchor–replica coordination provides computational redundancy and state-consistency recovery; and the Top- p mechanism restricts correction to selected solver units. In the matched-budget experiment, the proposed trigger yields a smaller aggregate squared correction magnitude than periodic sparse correction, while published conic neurodynamic baselines show larger mean post-transient primal tracking errors under the adopted calibration protocol.

The results also have clear limitations. The theoretical guarantees are local and rely on uniform residual regularity, while the impulsive-delay condition is conservative and does not certify the nominal simulation parameters. The method requires repeated Jacobian-based linear solves and becomes more sensitive to conditioning as the KKT dimension increases. Replication does not reduce the complexity of an individual conic solve, and the reported correction metrics do not measure total communication, runtime, or energy. Fault performance is scenario dependent, and reliable operation also depends on restoration of a trustworthy supervisory anchor.

Future work will investigate less-conservative regularity and synchronization conditions, preconditioned or variable-metric residual-cancellation dynamics, and higher-order or implicit discretization for larger and poorly conditioned problems. Adaptive triggering, predictor-based delay compensation, privacy-aware communication, redundant or distributed anchors, and hardware validation in contact-rich manipulation are also important directions.

Author contributions

Xu Wu: Conceptualization, methodology, formal analysis, software, validation, visualization, and writing—original draft; Juhe Sun: Conceptualization, methodology, supervision, funding acquisition, and writing—review and editing; Li Wang: Formal analysis, validation, supervision, and writing—review and editing; Yunlai Zhong: Software, investigation, data curation, visualization, and writing—review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors used ChatGPT(OpenAI) for English-language editing throughout the manuscript. All content was reviewed and approved by the authors.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 11801381, and by the Liaoning Provincial Department of Education under Grant Nos. JYTMS20230279 and LJ222510143003.

Data availability

No external datasets were used in this study. The numerical results reported in this article were generated by the authors. The code and numerical data supporting the findings are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare no conflicts of interest.

References

1. F. Alizadeh, D. Goldfarb, Second-order cone programming, *Math. Program.*, **95** (2003), 3–51. <https://doi.org/10.1007/s10107-002-0339-5>
2. F. W. Alsaade, M. S. Al-Zahrani, F. E. Alsaadi, Event-driven bi-fidelity control for efficient and reliable robotic manipulation, *Results Eng.*, **26** (2025), 105374. <https://doi.org/10.1016/j.rineng.2025.105374>
3. A. Bicchi, On the closure properties of robotic grasping, *Int. J. Robot. Res.*, **14** (1995), 319–334.
4. J. F. Bonnans, H. Ramírez C., Perturbation analysis of second-order cone programming problems, *Math. Program.*, **104** (2005), 205–227. <https://doi.org/10.1007/s10107-005-0613-4>
5. J. S. Chen, *SOC functions and their applications*, Singapore: Springer, 2019. <https://doi.org/10.1007/978-981-13-4077-2>
6. J. S. Chen, S. Pan, A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs, *Pac. J. Optim.*, **8** (2012), 33–74.
7. J. S. Chen, P. Tseng, An unconstrained smooth minimization reformulation of the second-order cone complementarity problem, *Math. Program.*, **104** (2005), 293–327. <https://doi.org/10.1007/s10107-005-0617-0>
8. F. Facchinei, J. S. Pang, *Finite-dimensional variational inequalities and complementarity problems*, New York: Springer, 2003. <https://doi.org/10.1007/b97543>
9. C. Ferrari, J. Canny, Planning optimal grasps, In: *Proceedings 1992 IEEE international conference on robotics and automation*, France: IEEE, 1992, 2290–2295. <https://doi.org/10.1109/ROBOT.1992.219918>
10. M. Fukushima, Z. Q. Luo, P. Tseng, Smoothing functions for second-order-cone complementarity problems, *SIAM J. Optim.*, **12** (2002), 436–460. <https://doi.org/10.1137/S1052623400380365>

11. D. Gerontitis, R. Behera, Y. Shi, P. S. Stanimirović, A robust noise-tolerant zeroing neural network for solving time-varying linear matrix equations, *Neurocomputing*, **508** (2022), 254–274. <https://doi.org/10.1016/j.neucom.2022.08.036>
12. P. T. Harker, J. S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math. Program.*, **48** (1990), 161–220. <https://doi.org/10.1007/BF01582255>
13. L. He, H. Cheng, Y. Zhang, Event-triggered discrete-time ZNN algorithm for distributed optimization with time-varying objective functions, *Electronics*, **14** (2025), 1359. <https://doi.org/10.3390/electronics14071359>
14. J. J. Hopfield, D. W. Tank, “Neural” computation of decisions in optimization problems, *Biol. Cybern.*, **52** (1985), 141–152. <https://doi.org/10.1007/BF00339943>
15. C. H. Ko, J. S. Chen, C. Y. Yang, Recurrent neural networks for solving second-order cone programs, *Neurocomputing*, **74** (2011), 3646–3653. <https://doi.org/10.1016/j.neucom.2011.07.009>
16. H. Li, X. Cheng, P. van Heijster, S. Qin, Event-triggered control for distributed time-varying optimization, *ISA Trans.*, **167** (2025), 1078–1084. <https://doi.org/10.1016/j.isatra.2025.09.025>
17. J. Li, Y. Rong, Z. Li, S. Li, L. Qu, A high-order integration-enhanced zeroing neural network for time-varying convex quadratic program against nonlinear noise, *IEEE Access*, **12** (2024), 22098–22109. <https://doi.org/10.1109/ACCESS.2024.3364377>
18. M. Li, H. Zhang, A. Song, Neural dynamics models for time-varying algebraic equations and optimizations with robotic applications: A survey, *Neurocomputing*, **696** (2026), 134059. <https://doi.org/10.1016/j.neucom.2026.134059>
19. Z. Li, S. Sastry, Task-oriented optimal grasping by multifingered robot hands, *IEEE J. Robot. Autom.*, **4** (1988), 32–44. <https://doi.org/10.1109/56.769>
20. H. Liu, D. Tong, Q. Chen, W. Zhou, Adaptive predefined-time intralayer and interlayer synchronization of multilayer networks, *IEEE Trans. Syst. Man Cybern. Syst.*, 2026, 1–12. <https://doi.org/10.1109/TSMC.2026.3713217>
21. L. Liu, H. Bao, Event-triggered impulsive synchronization of coupled delayed memristive neural networks under dynamic and static conditions, *Neurocomputing*, **504** (2022), 109–122. <https://doi.org/10.1016/j.neucom.2022.06.098>
22. X. Liu, K. Zhang, W. C. Xie, Pinning impulsive synchronization of reaction–diffusion neural networks with time-varying delays, *IEEE Trans. Neural Netw. Learn. Syst.*, **28** (2017), 1055–1067. <https://doi.org/10.1109/TNNLS.2016.2518479>
23. M. S. Lobo, L. Vandenbergh, S. Boyd, H. Lebet, Applications of second-order cone programming, *Linear Algebra Appl.*, **284** (1998), 193–228. [https://doi.org/10.1016/S0024-3795\(98\)10032-0](https://doi.org/10.1016/S0024-3795(98)10032-0)
24. J. Lu, D. W. C. Ho, J. Cao, A unified synchronization criterion for impulsive dynamical networks, *Automatica*, **46** (2010), 1215–1221. <https://doi.org/10.1016/j.automatica.2010.04.005>
25. X. Miao, J. S. Chen, C. H. Ko, A smoothed NR neural network for solving nonlinear convex programs with second-order cone constraints, *Inf. Sci.*, **268** (2014), 255–270. <https://doi.org/10.1016/j.ins.2013.10.017>

26. R. M. Murray, Z. Li, S. S. Sastry, *A mathematical introduction to robotic manipulation*, CRC Press, 1994.
27. A. Simonetto, E. Dall'Anese, S. Paternain, G. Leus, G. B. Giannakis, Time-varying convex optimization: Time-structured algorithms and applications, *Proc. IEEE*, **108** (2020), 2032–2048. <https://doi.org/10.1109/JPROC.2020.3003156>
28. J. Sun, J. S. Chen, C. H. Ko, Neural networks for solving second-order cone constrained variational inequality problem, *Comput. Optim. Appl.*, **51** (2012), 623–648. <https://doi.org/10.1007/s10589-010-9359-x>
29. J. Sun, W. Fu, J. H. Alcantara, J. S. Chen, A neural network based on the metric projector for solving SOCCVI problem, *IEEE Trans. Neural Netw. Learn. Syst.*, **32** (2021), 2886–2900. <https://doi.org/10.1109/TNNLS.2020.3008661>
30. H. Wahballa, A. Ahmed, J. Duan, X. Chen, L. Weining, Force tracking in robotic control systems using an online work object stiffness hybrid impedance PI control approach, *Results Eng.*, **26** (2025), 105520. <https://doi.org/10.1016/j.rineng.2025.105520>
31. Q. Wu, H. Zhang, L. Xiang, J. Zhou, A generalized Halanay inequality on impulsive delayed dynamical systems and its applications, *Chaos Solitons Fract.*, **45** (2012), 56–62. <https://doi.org/10.1016/j.chaos.2011.09.010>
32. Y. Xia, J. Wang, A general projection neural network for solving monotone variational inequalities and related optimization problems, *IEEE Trans. Neural Netw.*, **15** (2004), 318–328. <https://doi.org/10.1109/TNN.2004.824252>
33. Y. Xia, J. Wang, L. M. Fok, Grasping-force optimization for multifingered robotic hands using a recurrent neural network, *IEEE Trans. Robot. Autom.*, **20** (2004), 549–554. <https://doi.org/10.1109/TRA.2004.824946>
34. Y. Xia, J. Wang, A recurrent neural network for solving nonlinear convex programs subject to linear constraints, *IEEE Trans. Neural Netw.*, **16** (2005), 379–386. <https://doi.org/10.1109/TNN.2004.841779>
35. Y. Xia, Q. Liu, J. Wang, A. Cichocki, A survey of neurodynamic optimization, *IEEE Trans. Emerg. Top. Comput. Intell.*, **8** (2024), 2677–2696. <https://doi.org/10.1109/TETCI.2024.3369667>
36. D. Xu, Z. Yang, Impulsive delay differential inequality and stability of neural networks, *J. Math. Anal. Appl.*, **305** (2005), 107–120. <https://doi.org/10.1016/j.jmaa.2004.10.040>
37. X. Yang, J. Cao, J. Lu, Stochastic synchronization of complex networks with nonidentical nodes via hybrid adaptive and impulsive control, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **59** (2012), 371–384. <https://doi.org/10.1109/TCSI.2011.2163969>
38. X. Yang, J. Cao, Z. Yang, Synchronization of coupled reaction–diffusion neural networks with time-varying delays via pinning-impulsive controller, *SIAM J. Control Optim.*, **51** (2013), 3486–3510. <https://doi.org/10.1137/120897341>
39. X. Ye, Q. Chen, D. Tong, Event-triggered exponential synchronization for delayed multi-agent systems with privacy-preserving, *Commun. Nonlinear Sci. Numer. Simul.*, **162** (2026), 110444. <https://doi.org/10.1016/j.cnsns.2026.110444>

40. K. Zhang, E. Braverman, Event-triggered impulsive control for nonlinear systems with actuation delays, *IEEE Trans. Autom. Control*, **68** (2023), 540–547. <https://doi.org/10.1109/TAC.2022.3142127>
41. Y. Zhang, S. S. Ge, Design and analysis of a general recurrent neural network model for time-varying matrix inversion, *IEEE Trans. Neural Netw.*, **16** (2005), 1477–1490. <https://doi.org/10.1109/TNN.2005.857946>

A. Reproducibility details

The contact-force applications use fixed contact geometries. The numerical coordinates are reported in the internally consistent length units used in the implementation. The local contact frames are aligned with the object frame, i.e., $R_i = I_3$. The main contact parameters are summarized in Table 11.

Table 11. Contact parameters for the two force-allocation applications.

Case	Contact positions $p_i^\top$	R_i	θ_i
A1	(0.12, 0, 0)	I_3	0.55
	(−0.06, 0.104, 0)		
	(−0.06, −0.104, 0)		
A2	(0.25, 0, 0)	I_3	0.60
	(0, 0.25, 0)		
	(−0.25, 0, 0)		
	(0, −0.25, 0)		

For A1,

$$Q_{A1} = I_3 \otimes \text{diag}(0.55^{-2}, 1, 1) + 0.05I_9.$$

For $k = 0, 1, 2$, define

$$u_k(t) = \begin{bmatrix} 0.05 \sin(0.6t + 0.9k) \\ 0.04 \cos(0.5t + 0.7k) \end{bmatrix}, \quad s_k(t) = 0.75 + 0.08 \sin(0.4t + k) + \|u_k(t)\|,$$

and

$$\widehat{x}_{A1}(t) = \text{col}_{k=0}^2(s_k(t), u_k(t)).$$

The time-varying data are generated by

$$q_{A1}(t) = -0.20Q_{A1}\widehat{x}_{A1}(t), \quad \omega_{\text{ext},A1}(t) = -G_{A1}\widehat{x}_{A1}(t).$$

For A2, $Q_{A2} = I_{12}$. For $k = 0, 1, 2, 3$,

$$u_k(t) = \begin{bmatrix} 0.04 \sin(0.5t + k) \\ 0.04 \cos(0.4t + 1.3k) \end{bmatrix}, \quad s_k(t) = 0.70 + 0.05k + 0.06 \sin(0.3t + k) + \|u_k(t)\|,$$

with

$$x_{\text{ref},A2}(t) = \text{col}_{k=0}^3(s_k(t), u_k(t)).$$

The corresponding problem data are

$$q_{A2}(t) = -x_{\text{ref},A2}(t), \quad \omega_{\text{ext},A2}(t) = -G_{A2}x_{\text{ref},A2}(t).$$

The random seeds used in the numerical experiments are summarized in Table 12. All pseudorandom numbers are generated using NumPy's default random-number generator initialized with the corresponding seed. For the contact-force applications, the initial conditions are generated from a common primal–dual base state. For each three-dimensional primal SOC block, let

$$u_j^{\text{base}} = 0.05 \eta_j, \quad x_{\text{base},j} = \text{col}(1 + \|u_j^{\text{base}}\|, u_j^{\text{base}}),$$

where the entries of $\eta_j \in \mathbb{R}^2$ are sampled independently from the standard normal distribution. Hence,

$$x_{\text{base},j} \in \text{int } \mathcal{Q}_3.$$

The equality and cone multipliers of z_{base} are initialized at zero.

Table 12. Random seeds used in the numerical experiments.

Experiment suite	Random seed
A1: three-finger force allocation	101
A2: four-contact force allocation	102
Main ablation study	200
Delay sensitivity	310
Smoothing sensitivity	$370 + 1000r$, $r = 0, \dots, 9$
Conditioning/dimension sensitivity	400
Upper dwell-time sensitivity	950
Step-size sensitivity	970
Fault-recovery state initialization	980
Fault-direction generation	100983–100987
Published-baseline calibration	1900
Published-baseline testing	1901

The anchor and solver states are then generated according to

$$z_0(0) = z_{\text{base}} + 0.05 \xi_0, \quad z_i(0) = z_{\text{base}} + 0.20 \xi_i, \quad i = 1, \dots, N_s,$$

where the entries of ξ_0 and ξ_i are sampled independently from the standard normal distribution. To reproduce the reported random realizations exactly, the random draws follow the same fixed order as in the implementation: An initial n -dimensional standard-normal draw is generated first, followed by the random variables used to construct the primal SOC blocks, the anchor perturbation, and finally the solver perturbations in solver-index order. Within each comparative experiment suite, all deterministic variants use the same seed and therefore share identical initial anchor and solver states.

Constant initial histories

$$\varphi_i(s) = z_i(0), \quad s \in [-\bar{\tau}, 0], \quad i = 0, 1, \dots, N_s,$$

are used for the delayed states. In particular, the anchor history φ_0 is constant on $[-\bar{\tau}, 0]$ and therefore satisfies Assumption 4.2 with the Lipschitz constant zero.

The replicated solver network uses

$$a_{ij} = \frac{1}{N_s - 1} \quad (i \neq j), \quad a_{ii} = 0, \quad \pi_i = 1.$$

Unless otherwise specified in Sections 6.3, 6.5, and 6.6, the nominal numerical, smoothing, and event-trigger parameters are those listed in Table 1.



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)