



*Research article***Advancing group decision-making with an interval-valued intuitionistic fuzzy game model and a novel score function: application to sustainable AC system ranking****Sandeep Kumar¹, Reshu Tyagi¹ and Saiful R Mondal^{2,*}**¹ Department of Mathematics, Ch. Charan Singh University, Meerut-250004, Uttar Pradesh, India² Department of Mathematics and Statistics, College of Science, King Faisal University, Al Ahsa 31982, Saudi Arabia* **Correspondence:** Email: smondal@kfu.edu.sa.

Abstract: This paper develops a game-theoretic framework for multiple-attribute group decision-making (MAGDM) under an interval-valued intuitionistic fuzzy (IVIF) environment. First, a generalized score function (GSF) is proposed for intuitionistic fuzzy numbers (IFNs) and extended to interval-valued intuitionistic fuzzy numbers (IVIFNs). By incorporating indeterminacy through two flexible parameters, the GSF improves the discrimination and sensitivity of alternative evaluations under uncertain information. Its effectiveness is validated through seven benchmark cases in which the existing score and accuracy functions fail to distinguish the considered IFNs and IVIFNs. In all cases, the GSF yields distinct rankings, resolving the ambiguities. An IVIF MAGDM algorithm is then developed for problems with unknown attribute weights. The aggregated IVIF decision matrix is formulated as the payoff matrix of a two-player zero-sum matrix game (TMG), enabling determination of the attribute weights from assessments of multiple decision-makers. The game models are defined for membership intervals excluding 1 and nonmembership intervals excluding 0, while these boundary cases lie outside the admissible domain of the formulation. An IVIF expected score function is introduced to evaluate the overall performance of alternatives, followed by the GSF to obtain their final ranking. The framework preserves the IVIF structure throughout the decision-making process, with scalarization restricted to the final ranking stage, thereby retaining the inherent uncertainty and hesitation of the original decision assessments. The applicability of the approach is demonstrated through a sustainable air-conditioning system selection problem. Comparative analysis with four existing approaches shows exact agreement with one method and slight differences from the others. Sensitivity analysis with respect to the preference parameter ω confirms that the obtained ranking remains unchanged for $\omega = 0.2, 0.5$, and 0.8 , indicating 100% ranking stability. These findings demonstrate the discrimination capability of the GSF and the robustness and reliability of the IVIF MAGDM framework.

Keywords: MAGDM problems; generalized score function; IVIFN; IVIF TMG model

Mathematics Subject Classification: 03E72, 90B50, 91A80, 91B06

1. Introduction

As the field of decision science continues to make progress, the pursuit of more precise decision-making within a manageable time frame has underscored the involvement of decision-makers (DMs) from multifarious professional backgrounds. This interdisciplinary collaboration aims to mitigate the hardships inherent in complex decision-making process. In these scenarios, the theory of MAGDM (multiple attribute group decision-making) assumes a pivotal role [1, 2]. When addressing practical MAGDM problems, several crucial considerations come to the forefront. First, there is the challenge of accurately representing evaluation information to better reflect human thought processes and preferences. Second, the task of determining attribute weights and alternative weights is of paramount importance. To tackle the issue raised in the first point, intuitionistic fuzzy set (IFS) theory [3] offers a reliable framework. This framework was later extended by scholars [4] to develop the theory of an IVIF set (IVIFS). In this theory, each element is represented by interval numbers for three degree-membership (MD), nonmembership (ND), and indeterminacy (ID) instead of using precise crisp values. In relation to the second issue, it is worth noting that prior literature offers a variety of methods and theories [5–7] and [8–10] for determining attribute weights and alternative weights. In the context of decision-making problems (DMPs), a trustworthy approach involves the utilization of a two-player zero-sum matrix game (TMG) [6, 11–13]. What sets this approach apart is its distinct advantage, which lies in the fact that DMs can concentrate their attention solely on the behavior exhibited by the decision matrices.

1.1. Review of existing studies

This subsection explores recent advancements in MAGDM methods within the IVIFS framework. Park et al. [7] solved an IVIF MAGDM problem with partially unknown attribute weights. In their procedure, they provided an extension of the TOPSIS (technique for order preference by similarity to the ideal solution) method. Wang et al. [14] applied the TOPSIS method to address multiple-attribute decision-making (MADM) problems in which both the attribute values and the attribute weights are expressed as IVIFNs. In it, a quadratic program is built up to compute a unified weight information for finding the best alternative(s). Wan and Li [15] provided a LINMAP (linear programming technique for multidimensional analysis of preference) method to solve a heterogeneous MADM problem, in which the attribute weights are described with incomplete information. Garg, and Yang et al. [16, 17] introduced a generalized form of the improved score function (SF) [18] to multi-criteria decision-making (MCDM) problems in IVIF context. In such problems, the completely unknown weights are obtained by using the Shannon entropy method. Using the concept of linear programming, [19] presented an algorithm to solve MADM problems with IVIF information. Wang and Chen [20] provided an extension to the TOPSIS method for solving an IVIF MADM problem, where the linear programming technique is applied to compute the optimal attribute weights. Wang and Chen [21] introduced a linear programming method to solve IVIF MADM problems based on a novel SF for

IVIF values. The weight information in this DMP is incomplete. Kumar and Garg [22] studied an MADM problem under the IVIF framework on the basis of set pair analysis. In this research, the authors derived the connection numbers in order to extend the TOPSIS method to rank the alternatives. Nayagam and others [23] built a non-hesitance SF based algorithm for the MCDM in an IVIF context. Selvaraj and Majumdar [24] developed an IVIF TOPSIS methodology to decision-making based on a new NM SF. Kumar et al. [25] put forward a function to make a preference order between two IVIFNs. They also formulated an algorithm FOR MAGDM based on this function. Chen and Tsai [26] proposed a novel score function for IVIF values. Subsequently, they developed a solution procedure for MADM problems based on the mean and variance of the resulting score matrix. Wang et al. [27] designed the solution steps for a DMP by using the Jensen-Shannon operator. Chen and Hsu [28] proposed a five-step algorithm for determining attribute weights in IVIF-based MADM problems, utilizing a new SF and a nonlinear optimization model.

Apart from these studies, the TMG theory has been applied as an exquisite tool by the researchers in the formation of the algorithm solution steps to decision-making under an uncertain environment. Chen and Larbani [6] used a TMG approach to solve a fuzzy MADM problem. This method is based on a novel idea for defining the fuzzy matrices. For building an MADM algorithm under an intuitionistic fuzzy framework, the authors Xu et al. [11] applied a TMG approach with a new value index. Kumar and Kumar [13] developed a new TMG approach to solve an IVIF MAGDM problem with unknown attribute weights. They introduced a new generalized order function for IVIFNs to find the best alternative(s). Jana and Roy [29] used a linguistic Pythagorean hesitant fuzzy matrix game approach for solving MCDM problems. Wu et al. [30] investigated the distributed zero-sum differential game adaptive control problem for six-rotor unmanned aerial vehicles under denial-of-service attacks, using a self-triggered control mechanism. Jiafu et al. [31] proposed an improved IVIF-based group decision-making method incorporating consensus adjustment and prospect theory to evaluate online knowledge payment products from the perspective of the value of consumer experience. Nirmalsingh and Robinson [32] solved an intuitionistic fuzzy DMP by using matrix game method. Deng et al. [33] studied the optimal control problem for Takagi–Sugeno fuzzy systems by using TMG approach. Zeng et al. [34] produced a novel approach which integrates Type-2 fuzzy logic controllers, digital twin technology, and neural networks for comprehensive management of hydropower systems. Kumar and Garg [35] constructed a two-level approach for handling multiobjective fuzzy matrix games. Xie et al [36] proposed an IVIF MAGDM approach based on projection measurement extended Vlsekriterijumska Optimizacija I Kompromisno Resenje (VIKOR).

Table 1. Categorizing literature reviews: an overview.

Name of the MADM/MAGDM approaches	References
TMG-based approaches	[6, 11, 13, 29, 32]
TOPSIS approach	[7, 14, 22, 24, 37], [38–40]
LINMAP approach	[15, 19–21, 41], [42–44]
Entropy method	[16, 45], [46–48], [49–51]
VIKOR approach	[36, 52], [53–55], [56–58]
Other MADM/MAGDM approaches	[23, 25, 31, 59, 60], [61–63]

According to the reviewed literature, existing MADM/MAGDM approaches can be broadly

classified as shown in Table 1. While methods such as TOPSIS, LINMAP, entropy, VIKOR, and other approaches have been extensively studied in fuzzy decision-making, TMG-based approaches remain comparatively limited. Moreover, the integration of TMG with IVIF information and an effective ranking mechanism for IVIFNs remains insufficiently explored. To address this gap, the present study develops a TMG-based IVIF MAGDM algorithm incorporating a GSF for ranking IVIFNs.

The aggregation operators are actively used to fuse the decision information into single ones. Up until now, various such operators have been revealed by the scholars [64–66] to solve the group DMPs. In this paper, the IVIF hybrid geometric (IIFHG) operator is used to integrate the IVIFNs due to its wide applicability.

1.2. Motivation of the present research

After analyzing the above-mentioned studies, the following issues are encountered in the present study.

- (1) how can we compute the weight calculations to MAGDM by selecting a suitable technique under IVIFS environment? Owing to factors such as time constraints, limited information, uncertainty, and insufficient knowledge, determining unified weights becomes a challenging task.
- (2) A SF is required for IVIFNs that involves the notions of the MD, the ND and the ID of the IVIFNs, so that alternatives can be ranked in a reasonable and accurate manner.
- (3) How can the preservation of original IVIF information be ensured throughout the decision-making process, given that any loss of this information may adversely affect the final ranking of alternatives?

To address the aforementioned issues, this study presents an effective methodology, which is described in detail in the subsequent subsection.

1.3. Details of the present research

The present study is structured in two stages. First, a GSF is introduced for IFNs, and subsequently, it is extended to IVIFNs. Any IFS (or IVIFS) describes the uncertainty in terms of MD, ND, and ID. Considering all these degrees with two parameters λ_1 and λ_2 , the proposed SFs are constructed with their properties. Another advantage of the proposed GSFs is that both functions have flexible parameters λ_1 and λ_2 that are connected by the relation $\lambda_1\lambda_2 = 1$; $\lambda_1 \in (0, 1]$. By varying their values, these parameters can significantly contribute to the development of an effective decision-making process. After that, a new algorithm is introduced to solve an IVIFS MAGDM problem without any kind of weight information. In it, the IIFHG operator is used to determine the joint information, while a TMG model with IVIFS data [67] is used to find attribute weights. This TMG model is defined for membership intervals excluding 1 and nonmembership intervals excluding 0, while these boundary cases lie outside the admissible domain of the formulation. On the basis of the optimal solution of this TMG model, we define IVIF expected SF with respect to each alternative. The value of this function for each alternative is represented by an IVIFN. In order to rank the alternatives, these calculated IVIFNs are then defuzzified by utilizing the proposed SF for IVIFNs. An important methodological advantage of the proposed approach is its information-preserving treatment of IVIF assessments. Unlike approaches that transform IVIF information into crisp scores at an early

stage, the proposed framework maintains the IVIF structure from the construction of the collective decision matrix through normalization, game-theoretic weight determination, and calculation of the IVIF expected score (IIFES). The conversion of IIFESs into scalar values is postponed until the final stage through the proposed GSF. In this way, the interval-valued membership and non-membership information, together with the associated uncertainty and hesitation, is retained during the major stages of the decision process. This delayed scalarization reduces the risk of premature information loss and provides a more faithful basis for comparing alternatives, which can contribute to more reliable ranking outcomes. In the end, an illustration is given to demonstrate the present algorithm. The advantages of the current study over the existing ones are also provided.

1.4. Contribution and structure of the paper

The prime contributions of this research are pointed out as follows.

- To introduce the SFs for IFNs and IVIFNs, several robust properties of these functions are also presented. In developing these functions, the concepts of the λ_1 -degree of belongingness and the λ_2 -degree of non-belongingness are utilized by incorporating the MD, ND, and ID through two adjustable parameters, λ_1 and λ_2 ($\lambda_1\lambda_2 = 1$). The inclusion of these parameters offers enhanced flexibility, as varying the value of λ_1 within the interval $(0, 1]$ enables more effective ranking and discrimination among alternatives.
- Quantitative analyses on multiple benchmark examples reveal that the proposed GSFs ensure complete improvement in decisions' consistency and fully eliminates ranking ambiguities relative to existing SFs.
- An algorithm is introduced to address the IVIF MAGDM problem based on the max–min principle and the proposed SF. In this framework, an IVIF expected SF is formulated to compute the aggregated value of each alternative in the form of an IVIFN. The proposed algorithm efficiently preserves the complete IVIF information while ensuring high computational efficiency.
- The proposed algorithm is demonstrated through a real-world numerical example, and a comprehensive comparative analysis is conducted against existing methods to validate its effectiveness and superiority.

The decoration of the research work is as follows. In Section 2, we provide some preliminaries regarding the present research work. In Section 3, new GSFs are constructed with their properties. The influence of the parameters λ_1 and λ_2 on the proposed SF values are also shown through a number of examples in this section. In Section 4, a brief study of Li's IVIF game model is given. An algorithm for solving the present MAGDM problem is developed in Section 5. Section 6 presents a numerical example to demonstrate the applicability of the proposed algorithm, while Section 7 highlights the superiority of the established results compared with existing literature. Finally, the conclusion is covered in Section 8.

2. Auxiliary facts

This section presents the preliminaries regarding the IFS, IVIFS, and their measurement functions.

Definition 1. [3] In a fixed finite set U , an IFS I is defined as

$$I = \{\langle x, \xi_I(x), \zeta_I(x) \rangle, x \in U\},$$

where $\xi_I(x)$ and $\zeta_I(x)$ are the MD and ND of element $x \in U$ of the set I , respectively, such that $\xi_I : U \rightarrow [0, 1]$, $\zeta_I : U \rightarrow [0, 1]$ with $0 \leq \xi_I(x) + \zeta_I(x) \leq 1$.

The ID of element $x \in U$ of the set I is given by

$$\pi_I(x) = 1 - \xi_I(x) - \zeta_I(x).$$

For a given IFS I and for a given $x \in U$, the pair $\langle \xi, \zeta \rangle$ is called an IFN, where $\xi \in [0, 1]$, $\zeta \in [0, 1]$, and $\xi + \zeta \in [0, 1]$. For convenience, an IFN is denoted by α .

For the IFNs $\alpha = \langle \xi, \zeta \rangle$, $\alpha_1 = \langle \xi_1, \zeta_1 \rangle$, and $\alpha_2 = \langle \xi_2, \zeta_2 \rangle$, some operational results [68] are given as follows:

$$(i) \alpha_1 \oplus \alpha_2 = \langle 1 - (1 - \xi_1)(1 - \xi_2), \zeta_1 \zeta_2 \rangle,$$

$$(ii) \alpha_1 \otimes \alpha_2 = \langle \xi_1 \xi_2, 1 - (1 - \zeta_1)(1 - \zeta_2) \rangle,$$

$$(iii) \gamma \alpha = \langle 1 - (1 - \xi)^\gamma, \zeta^\gamma \rangle; \gamma > 0,$$

$$(iv) \alpha^\gamma = \langle \xi^\gamma, 1 - (1 - \zeta)^\gamma \rangle; \gamma > 0.$$

Definition 2. [69, 70] For an IFN $\alpha = \langle \xi, \zeta \rangle$, the SF S_α and the accuracy function (AF) A_α are defined as

$$S_\alpha = \xi - \zeta; \quad S_\alpha \in [-1, 1],$$

and

$$A_\alpha = \xi + \zeta; \quad A_\alpha \in [0, 1],$$

respectively.

When IFN α reduces into a fuzzy number, i.e., $\xi + \zeta = 1$, then $SF S_\alpha = 2\xi - 1 = 1 - 2\zeta$.

Definition 3. [4] In a fixed finite set U , an IVIFS $\tilde{I}$ is the set of triplets in which the MD and ND of $x \in U$ are represented by interval numbers. These interval numbers are subsets of $[0, 1]$. That is, the set $\tilde{I}$ can be propagated as

$$\begin{aligned} \tilde{I} &= \{\langle x, \xi_{\tilde{I}}(x), \zeta_{\tilde{I}}(x) \rangle, x \in U\} \\ &= \{\langle x, [\Omega_{\tilde{I}}(x), \Theta_{\tilde{I}}(x)], [\vartheta_{\tilde{I}}(x), \varrho_{\tilde{I}}(x)] \rangle, x \in U\}, \end{aligned}$$

where $0 \leq \Omega_{\tilde{I}} \leq \Theta_{\tilde{I}} \leq 1$, $0 \leq \vartheta_{\tilde{I}} \leq \varrho_{\tilde{I}} \leq 1$, and $\Theta_{\tilde{I}}(x) + \varrho_{\tilde{I}}(x) \leq 1$.

The interval of indeterminacy of IVIFS $\tilde{I}$ of U [71] is represented as

$$\begin{aligned} \pi_{\tilde{I}}(x) &= 1 - \xi_{\tilde{I}}(x) - \zeta_{\tilde{I}}(x) \\ &= [1 - \Theta_{\tilde{I}}(x) - \varrho_{\tilde{I}}(x), 1 - \Omega_{\tilde{I}}(x) - \vartheta_{\tilde{I}}(x)]. \end{aligned}$$

For a fixed value of $x \in U$, the pair $\langle \xi, \zeta \rangle$ is known as IVIFN [64]. Generally, the pair $\langle \xi, \zeta \rangle$ is shown by $\langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$, where $[\Omega, \Theta] \subseteq [0, 1]$, $[\vartheta, \varrho] \subseteq [0, 1]$. For convenience, an IVIFN is denoted by $\tilde{\alpha}$.

Definition 4. [64, 72] For IVIFNs $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$, $\tilde{\alpha}_1 = \langle [\Omega_1, \Theta_1], [\vartheta_1, \varrho_1] \rangle$, and $\tilde{\alpha}_2 = \langle [\Omega_2, \Theta_2], [\vartheta_2, \varrho_2] \rangle$, the operational results are as follows:

- (i) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \langle [\Omega_1 + \Omega_2 - \Omega_1\Omega_2, \Theta_1 + \Theta_2 - \Theta_1\Theta_2], [\vartheta_1\vartheta_2, \varrho_1\varrho_2] \rangle$,
- (ii) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \langle [\Omega_1\Omega_2, \Theta_1\Theta_2], [\vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2, \varrho_1 + \varrho_2 - \varrho_1\varrho_2] \rangle$,
- (iii) $\tilde{\alpha}^\gamma = \langle [(\Omega)^\gamma, (\Theta)^\gamma], [1 - (1 - \vartheta)^\gamma, 1 - (1 - \varrho)^\gamma] \rangle$; $\gamma > 0$.

It is notable that the outcomes above are likewise IVIFNs.

Definition 5. Let $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$ be an IVIFN. Then the SF and $\mathcal{AF}$, given by different researchers, are as follows:

- Xu et al.'s SF and AF [64] are

$$S_{\tilde{\alpha}} = \frac{\Omega - \vartheta + \Theta - \varrho}{2}; \quad S_{\tilde{\alpha}} \in [-1, 1],$$

and

$$A_{\tilde{\alpha}} = \frac{\Omega + \vartheta + \Theta + \varrho}{2}; \quad A_{\tilde{\alpha}} \in [0, 1].$$

- Ye's novel AF [73] is

$$M_{\tilde{\alpha}} = \frac{\Omega - (1 - \Omega - \vartheta) + \Theta - (1 - \Theta - \varrho)}{2}; \quad M_{\tilde{\alpha}} \in [-1, 1].$$

- Nayagam et al.'s another AF [74] is

$$L_{\tilde{\alpha}} = \frac{\Omega + \Theta - \varrho(1 - \Theta) - \vartheta(1 - \Omega)}{2}; \quad L_{\tilde{\alpha}} \in [-1, 1].$$

- Bai's improved SF [18] is

$$IS_{\tilde{\alpha}} = \frac{\Omega + \Omega(1 - \Omega - \vartheta) + \Theta + \Theta(1 - \Theta - \varrho)}{2}; \quad IS_{\tilde{\alpha}} \in [0, 1].$$

- Garg's generalized improved SF [16]:

$$GIS_{\tilde{\alpha}} = \left(\frac{\Omega + \Theta}{2} \right) + k_1\Omega(1 - \Omega - \vartheta) + k_2\Theta(1 - \Theta - \varrho); \quad GIS_{\tilde{\alpha}} \in [0, 1],$$

where $k_1 + k_2 = 1$; $k_1, k_2 \geq 0$.

- Nayagam et al.'s non-hesitance SF [23] is

$$J_{\tilde{\alpha}} = \frac{\Omega + \Theta + \vartheta - \varrho + \Omega\Theta + \vartheta\varrho}{3}; \quad J_{\tilde{\alpha}} \in [-1, 1].$$

- Selvaraj and Majumdar's nonmembership SF [24] is

$$N_M(\tilde{\alpha}) = \frac{\Omega - \Theta + \vartheta + \varrho + \Omega\Theta + \vartheta\varrho}{3}; \quad N_M(\tilde{\alpha}) \in [-1, 1].$$

- Kumar et al.'s order function [25] is

$$\psi_{\tilde{\alpha}} = \frac{\psi_{1,\tilde{\alpha}} + \psi_{3,\tilde{\alpha}}}{2}; \quad \psi_{\tilde{\alpha}} \in [0, 1],$$

where $\psi_{1,\tilde{\alpha}}$ and $\psi_{3,\tilde{\alpha}}$ are the first and second order functions, respectively.

Definition 6. [64] For comparing two IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, the existing law is given as

- (i) If $S_{\tilde{\alpha}_1} < S_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_2 > \tilde{\alpha}_1$.
- (ii) If $S_{\tilde{\alpha}_1} > S_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
- (iii) If the SF values for $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are equal, then
 - (i)' If $A_{\tilde{\alpha}_1} < A_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_2 > \tilde{\alpha}_1$.
 - (ii)' If $A_{\tilde{\alpha}_1} > A_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.
 - (iii)' If $A_{\tilde{\alpha}_1} = A_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_1$ is equivalent to $\tilde{\alpha}_2$.

It is to be noted that the notation ' $>$ ' means "prefers to".

Definition 7. [65, 72] Suppose $\tilde{\alpha}_h = \langle [\Omega_h, \Theta_h], [\vartheta_h, \varrho_h] \rangle$, ($h = 1$ to m) are m IVIFNs. To aggregate these IVIFNs, the IIFHG operator θ is given as follows:

$$\begin{aligned} \theta_{a,\delta}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_m) &= \left(\dot{\tilde{\alpha}}_{\rho(1)} \right)^{a_1} \otimes \left(\dot{\tilde{\alpha}}_{\rho(2)} \right)^{a_2} \otimes \dots \otimes \left(\dot{\tilde{\alpha}}_{\rho(m)} \right)^{a_m} \\ &= \left\langle \left[\prod_h \left(\dot{\Omega}_{\rho(h)} \right)^{a_h}, \prod_h \left(\dot{\Theta}_{\rho(h)} \right)^{a_h} \right], \right. \\ &\quad \left. \left[\left(1 - \prod_h \left(1 - \dot{\vartheta}_{\rho(h)} \right)^{a_h} \right), \left(1 - \prod_h \left(1 - \dot{\varrho}_{\rho(h)} \right)^{a_h} \right) \right] \right\rangle \end{aligned}$$

where the weight vector of the operator θ is given as $a = (a_1, a_2, \dots, a_m)^T$ with $a_h > 0$ ($h = 1$ to m) and $\sum_h a_h = 1$ and $\dot{\tilde{\alpha}}_{\rho(h)} = \langle [\dot{\Omega}_{\rho(h)}, \dot{\Theta}_{\rho(h)}], [\dot{\vartheta}_{\rho(h)}, \dot{\varrho}_{\rho(h)}] \rangle$ is the h^{th} largest of the weighted IVIFN $\dot{\tilde{\alpha}}_h$ where $\dot{\tilde{\alpha}}_h = (\tilde{\alpha}_h)^{m\delta_h}$; $\delta = (\delta_1, \delta_2, \dots, \delta_m)^T$ is the weight vector for the IVIFNs $\tilde{\alpha}_h$ ($h = 1$ to m).

3. New GSFs for IFNs and IVIFNs

This section is organized into four subsections. Subsection 3.1 presents the limitations of the existing SFs. In Subsection 3.2, a novel GSF is introduced for comparing two or more IFNs, along with its fundamental properties. A comparative law based on the proposed GSF values is also established. Subsection 3.3 extends the proposed GSF to the decision-making process involving IVIFNs. Finally, the last subsection presents the key advantages and merits of the proposed GSFs.

3.1. Shortcomings of the existing SFs values

The existing SFs for IFNs and IVIFNs cannot accurately rank the alternatives in a MADM/MAGDM problem, as demonstrated by the following numerical examples.

Example 3.1. Suppose that $\alpha_1 = \langle 0.55, 0.35 \rangle$ and $\alpha_2 = \langle 0.60, 0.40 \rangle$ are two IFNs. The SF values [69] are given as $S_{\alpha_1} = S_{\alpha_2} = 0.20$. This shows that this SF is unable to rank IFNs α_1 and α_2 . On the other hand, the AF values [70] are computed as $A_{\alpha_1} = 0.90$ and $A_{\alpha_2} = 1.0$. These values show that $\alpha_2 > \alpha_1$. On account of the improved SF formula [18] IS_{α} , the computed score values are $IS_{\alpha_1} = 0.605$ and $IS_{\alpha_2} = 0.600$. These values indicate that $\alpha_1 > \alpha_2$. In such a case, the DM is unable to determine a reasonable ranking between α_1 and α_2 .

Example 3.2. In this example, consider two IVIFNs $\tilde{\alpha}_1 = \langle [0, 0.50], [0.10, 0.20] \rangle$ and $\tilde{\alpha}_2 = \langle [0.20, 0.30], [0.15, 0.15] \rangle$. The SF values [64] are given as $S_{\tilde{\alpha}_1} = S_{\tilde{\alpha}_2} = 0.10$ and the AF values [64] are computed as $A_{\tilde{\alpha}_1} = A_{\tilde{\alpha}_2} = 0.40$. However, utilizing the formula as examined by [73], the novel AF values for $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are $M_{\tilde{\alpha}_1} = M_{\tilde{\alpha}_2} = -0.35$, respectively. By using the order function as developed by [25], the order function values corresponding to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are calculated as $\psi_{\tilde{\alpha}_1} = \psi_{\tilde{\alpha}_2} = 0.58$.

Example 3.3. Consider the IFS information in terms of two IVIFNs $\tilde{\alpha}_1 = \langle [0, 0.50], [0, 0.30] \rangle$ and $\tilde{\alpha}_2 = \langle [0, 0.40], [0.10, 0.10] \rangle$. Using the formula for SF [64], we get $S_{\tilde{\alpha}_1} = S_{\tilde{\alpha}_2} = 0.10$; whereas, by using the improved SF [18], we have $IS_{\tilde{\alpha}_1} = IS_{\tilde{\alpha}_2} = 0.30$. Therefore, once more, we draw the conclusion that the existing SFs are unable to provide a rational ranking for a group of options when making decisions.

Example 3.4. Assume that $\tilde{\alpha}_1 = \langle [0, 0.60], [0, 0] \rangle$ and $\tilde{\alpha}_2 = \langle [0, 0.70], [0, 0.10] \rangle$ are the two IVIFS decision information provided. By using the SF given by [64], the computed values for these IVIFNs are $S_{\tilde{\alpha}_1} = S_{\tilde{\alpha}_2} = 0.30$. On the other hand, for these IVIFNs, the improved SF values [18] are given as $IS_{\tilde{\alpha}_1} = IS_{\tilde{\alpha}_2} = 0.42$ and the calculated nonmembership SF values [24] are $N_M(\tilde{\alpha}_1) = N_M(\tilde{\alpha}_2) = -0.20$. Therefore, these existing SFs cannot provide a comparison between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

Example 3.5. Consider two IVIFNs $\tilde{\alpha}_1 = \langle [0, 0.50], [0.10, 0.40] \rangle$ and $\tilde{\alpha}_2 = \langle [0.20, 0.30], [0.20, 0.20] \rangle$. For $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, the computed values of AF [74] are $L_{\tilde{\alpha}_1} = L_{\tilde{\alpha}_2} = 0.10$. Therefore, this function has been failed to compare $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

Example 3.6. This example considers two IVIFNs $\tilde{\alpha}_1 = \langle [0, 0], [0.20, 0.50] \rangle$ and $\tilde{\alpha}_2 = \langle [0, 0], [0.30, 0.60] \rangle$. The improved SF values, given by Bai's improved SF [18] are $IS_{\tilde{\alpha}_1} = IS_{\tilde{\alpha}_2} = 0$, and by using Garg's generalized improved SF [16], the values are $GIS_{\tilde{\alpha}_1} = GIS_{\tilde{\alpha}_2} = 0$. These results indicate that the considering functions are unable to put forward a ranking order between IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

Example 3.7. Consider two IVIFNs $\tilde{\alpha}_1 = \langle [0.10, 0.40], [0.10, 0.20] \rangle$ and $\tilde{\alpha}_2 = \langle [0.20, 0.25], [0.20, 0.30] \rangle$. The nonhesitance SF values [23] for $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are $J_{\tilde{\alpha}_1} = J_{\tilde{\alpha}_2} = 0.1533$. Therefore, this function has failed to compare $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

3.2. Proposed new GSF for IFNs

Considering the shortcomings highlighted in the preceding examples, it becomes necessary to incorporate additional concepts into the formulation of SFs for both IFNs and IVIFNs. To address this issue comprehensively, a new GSF is proposed as follows.

Definition 8. For an IFN $\alpha = \langle \xi, \zeta \rangle$, we define the two concepts as follows.

(c_0) $\xi - \xi^{\lambda_1}(1 - \xi - \zeta)$ is called the λ_1 -degree of belongingness of an element $x \in U$ to α , where $\lambda_1 > 0$.

(c_1) $\zeta - \zeta^{\lambda_2}(1 - \xi - \zeta)$ is called the λ_2 -degree of non-belongingness of an element $x \in U$ to α , where $\lambda_2 > 0$.

Here, the parameters λ_1 and $\lambda_2 (\neq \lambda_1)$ are connected by the relation $\lambda_1 \lambda_2 = 1$.

Remark 3.1. For IFN $\alpha = \langle 1, 0 \rangle$, the degree of belongingness of an element $x \in U$ is 1, while the degree of non-belongingness of this element is 0 by using the concepts (c_0) – (c_1) . In order to calculate such degrees, the parameters λ_1 and λ_2 do not have any role. Similarly, such degrees can be evaluated for IFN $\alpha = \langle 0, 1 \rangle$ without any importance of the parameters λ_1 and λ_2 .

On the basis of the concepts (c_0) – (c_1) , a GSF for an IFN $\alpha = \langle \xi, \zeta \rangle$ is defined as

$$S_{\alpha}^{new} = (\xi - \zeta) - (\xi^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi - \zeta). \quad (3.1)$$

Theorem 3.1. For an arbitrary IFN $\alpha = \langle \xi, \zeta \rangle$, the proposed GSF S_{α}^{new} increases monotonically with the increase of the parameter $\lambda_1 \in (0, 1)$.

Proof. See Appendix A. □

Theorem 3.2. For two IFNs $\alpha_1 = \langle \xi_1, \zeta_1 \rangle$ and $\alpha_2 = \langle \xi_2, \zeta_2 \rangle$ with $\xi_1 < \xi_2$ and $\zeta_1 > \zeta_2$, then $S_{\alpha_1}^{new} < S_{\alpha_2}^{new}$ for $\lambda_1 \in (0, 1)$.

Proof. See Appendix B. □

Theorem 3.3. For an arbitrary IFN $\alpha = \langle \xi, \zeta \rangle$, the proposed GSF S_{α}^{new} increases monotonically with the ξ and decreases monotonically with the ζ .

Proof. See Appendix C. □

Theorem 3.4. If we let $\alpha = \langle \xi, \zeta \rangle$ be an arbitrary IFN, then $-1 \leq S_{\alpha}^{new} \leq 1$.

Proof. See Appendix D. □

Theorem 3.5. The following properties (Ppys) of the GSF S_{α}^{new} are as follows.

Ppy 1: For any IFN $\alpha = \langle \xi, 0 \rangle$, the new GSF $S_{\alpha}^{new} \in [0, 1]$ and for $\alpha = \langle 0, \zeta \rangle$, $S_{\alpha}^{new} \in [-1, 0]$.

Ppy 2: If IFN $\alpha = \langle 1, 0 \rangle$, then $S_{\alpha}^{new} = 1$.

Ppy 3: If IFN $\alpha = \langle 0, 1 \rangle$, then $S_{\alpha}^{new} = -1$.

Ppy 4: For a subset (fuzzy) $\alpha = \langle \xi, 1 - \xi \rangle$, the new GSF has the value $S_{\alpha}^{new} = 2\xi - 1$.

Ppy 5: If $\lambda_1 \rightarrow 0$, then $\lambda_2 \rightarrow \infty$ and the value of new GSF $S_{\alpha}^{new} \rightarrow 2\xi - 1$ for any IFN $\alpha = \langle \xi, \zeta \rangle$.

Proof. The proof is easy to derive. □

Definition 9. Two IFNs α_1 and α_2 follow the relation $\alpha_1 > \alpha_2$ when $S_{\alpha_1}^{new} > S_{\alpha_2}^{new}$ for $0 < \lambda_1 < 1$ holds.

3.3. A new GSF for IVIFNs

In this section, we define the new GSF for the IVIFNs.

Definition 10. Suppose that $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$ is an IVIFN. We define the following concepts (c₂)–(c₃).

(c₂) $\Omega - \Omega^{\lambda_1}(1 - \Omega - \vartheta)$ and $\Theta - \Theta^{\lambda_1}(1 - \Theta - \varrho)$ are called the λ_1 -degrees of belongingness of an element $x \in U$ relative to $\tilde{\alpha}$, where $\lambda_1 > 0$.

(c₃) $\vartheta - \vartheta^{\lambda_2}(1 - \Omega - \vartheta)$ and $\varrho - \varrho^{\lambda_2}(1 - \Theta - \varrho)$ are called the λ_2 -degrees of non-belongingness of an element $x \in U$ relative to $\tilde{\alpha}$, where $\lambda_2 > 0$.

Here, the parameters λ_1 and $\lambda_2 (\neq \lambda_1)$ are connecting by the relation $\lambda_1 \lambda_2 = 1$.

Using the concepts (c₂)–(c₃), a new GSF for an IVIFN $\tilde{\alpha}$ is formulated in the following way:

$$S_{\tilde{\alpha}}^{new} = \frac{1}{2} \left(\left(\Omega - \Omega^{\lambda_1}(1 - \Omega - \vartheta) + \Theta - \Theta^{\lambda_1}(1 - \Theta - \varrho) \right) - \left(\vartheta - \vartheta^{\lambda_2}(1 - \Omega - \vartheta) + \varrho - \varrho^{\lambda_2}(1 - \Theta - \varrho) \right) \right)$$

which can be written as

$$S_{\tilde{\alpha}}^{new} = \frac{1}{2} \left((\Omega - \vartheta) + (\Theta - \varrho) - (\Omega^{\lambda_1} - \vartheta^{\lambda_2})(1 - \Omega - \vartheta) - (\Theta^{\lambda_1} - \varrho^{\lambda_2})(1 - \Theta - \varrho) \right). \quad (3.2)$$

Remark 3.2. Particularly, when $\Omega = \Theta = \xi$ (say) and $\vartheta = \varrho = \zeta$ (say), an IVIFN $\tilde{\alpha}$ is reduced into an IFN $\alpha = \langle \xi, \zeta \rangle$ and then the function $S_{\tilde{\alpha}}^{new}$ is reduced into the function S_{α}^{new} of an IFN α .

Theorem 3.6. For an arbitrary IVIFN $\tilde{\alpha}$, the proposed GSF $S_{\tilde{\alpha}}^{new}$ increases monotonically with an increase in the parameter $\lambda_1 \in (0, 1)$.

Proof. See Appendix E. □

Theorem 3.7. For an arbitrary IVIFN $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$, the proposed GSF $S_{\tilde{\alpha}}^{new}$ increases monotonically with Ω, Θ and decreases monotonically with ϑ, ϱ .

Proof. See Appendix F. □

Theorem 3.8. If we let $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$ be an arbitrary IVIFN, then $-1 \leq S_{\tilde{\alpha}}^{new} \leq 1$.

Proof. See Appendix G. □

Theorem 3.9. The following properties (Ppys) of the GSF $S_{\tilde{\alpha}}^{new}$ are listed as follows.

Ppy 1: For any IVIFN $\tilde{\alpha} = \langle [\Omega, \Theta], [0, 0] \rangle$, the GSF $S_{\tilde{\alpha}}^{new} \in [0, 1]$, and for $\tilde{\alpha} = \langle [0, 0], [\vartheta, \varrho] \rangle$, $S_{\tilde{\alpha}}^{new} \in [-1, 0]$.

Ppy 2: For the largest IVIFN $\tilde{\alpha} = \langle [1, 1], [0, 0] \rangle$, the GSF $S_{\tilde{\alpha}}^{new} = 1$.

Ppy 3: For the smallest IVIFN $\tilde{\alpha} = \langle [0, 0], [1, 1] \rangle$, the GSF $S_{\tilde{\alpha}}^{new} = -1$.

Ppy 4: For a subset $\tilde{\alpha} = \langle [\Omega, \Theta], [1 - \Theta, 1 - \Omega] \rangle$, the proposed GSF is

$$S_{\tilde{\alpha}}^{new} = \frac{1}{2} \left(2(\Omega + \Theta - 1) - (\Theta - \Omega)(\Omega^{\lambda_1} - (1 - \Theta)^{\lambda_2}) - (\Omega - \Theta)(\Theta^{\lambda_1} - (1 - \Omega)^{\lambda_2}) \right).$$

In particular, if $\Omega = \Theta$, then $S_{\tilde{\alpha}}^{new} = 2\Omega - 1$.

Ppy 5: For any IVIFN $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$, if $\lambda_1 \rightarrow 0$, then $\lambda_2 \rightarrow \infty$ and the value of proposed GSF approaches to $\Theta + \Omega - 1$.

Proof. The proof is easy to derive. $\square$

Definition 11. Two IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ follow the relation $\tilde{\alpha}_1 > \tilde{\alpha}_2$ when $S_{\tilde{\alpha}_1}^{new} > S_{\tilde{\alpha}_2}^{new}$ for $0 < \lambda_1 < 1$ holds.

In some cases, it may be possible that the SF values to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are equal for a value of $\lambda_1 \in (0, 1]$, then we check the comparability of such IVIFNs by corresponding to the distinct SF values for $\lambda_1 \in (0, 1]$.

Theorem 3.10. For two IVIFNs $\tilde{\alpha}_1 = \langle [\Omega_1, \Theta_1], [\vartheta_1, \varrho_1] \rangle$ and $\tilde{\alpha}_2 = \langle [\Omega_2, \Theta_2], [\vartheta_2, \varrho_2] \rangle$, with $\Omega_1 < \Omega_2$, $\Theta_1 < \Theta_2$, $\vartheta_1 > \vartheta_2$, and $\varrho_1 > \varrho_2$, $S_{\tilde{\alpha}_1}^{new} < S_{\tilde{\alpha}_2}^{new}$ for $\lambda_1 \in (0, 1)$.

Proof. The proofs outlines are similar to those in Theorem 3.2. $\square$

3.4. Merits of the proposed GSFs

To highlight the advantages of the proposed functions, we revisit the examples presented in Section 3.1 and discuss their results as follows.

Example 3.8. We reconsider the IFNs α_1 and α_2 as given in Example 3.1 to measure through the proposed GSF given in (3.1). For this, we vary the value of $\lambda_1 \in (0, 1)$ and hence the resultant GSF values $S_{\alpha_1}^{new}$ and $S_{\alpha_2}^{new}$ are given in Table 2. From this, it is recorded that $S_{\alpha_1}^{new} < S_{\alpha_2}^{new}$ and hence $\alpha_2 > \alpha_1$.

Table 2. The values of $S_{\alpha_1}^{new}$ and $S_{\alpha_2}^{new}$ for different values of λ_1 in Example 3.8.

λ_1	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	0.106	0.112	0.119	0.129	0.138	0.148	0.156	0.165	0.173
$S_{\alpha_2}^{new}$	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200

Example 3.9. Let $\alpha_1 = \langle 0.50, 0.50 \rangle$ and $\alpha_2 = \langle 0.30, 0.30 \rangle$ be two IFNs. We compute the GSF values $S_{\alpha_1}^{new}$ and $S_{\alpha_2}^{new}$ by varying the values of $\lambda_1 \in (0, 1)$ and their results are listed in Table 3. It is concluded that the relation $S_{\alpha_1}^{new} > S_{\alpha_2}^{new}$ holds for all values of λ_1 and thus $\alpha_1 > \alpha_2$.

Table 3. The values of $S_{\alpha_1}^{new}$ and $S_{\alpha_2}^{new}$ for different values of λ_1 in Example 3.9.

λ_1	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	0	0	0	0	0	0	0	0	0
$S_{\alpha_2}^{new}$	-0.355	-0.313	-0.272	-0.227	-0.183	-0.141	-0.101	-0.064	-0.031

Example 3.10. Consider a real number ϵ under the condition $0 \leq \epsilon \leq 0.5$. With respect to such a real number ϵ , we contrive two IVIFNs, namely $\tilde{\alpha}_1 = \langle [\epsilon, \epsilon], [\epsilon, \epsilon] \rangle$ and $\tilde{\alpha}_2 = \langle [0.50 - \epsilon, 0.50 - \epsilon], [0.50 - \epsilon, 0.50 - \epsilon] \rangle$. These IVIFNs become $\tilde{\alpha}_1 = \langle [0.20, 0.20], [0.20, 0.20] \rangle$ and $\tilde{\alpha}_2 = \langle [0.30, 0.30], [0.30, 0.30] \rangle$ for $\epsilon = 0.20$. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ are computed through the proposed GSF given in (3.2) by varying the parameter $\lambda_1 \in (0, 1)$, and we construct Table 4 and, consequently, find the inequality $S_{\tilde{\alpha}_1}^{new} < S_{\tilde{\alpha}_2}^{new}$ for all λ_1 . Thus we obtain the relation $\tilde{\alpha}_2 > \tilde{\alpha}_1$.

Table 4. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.10.

λ_1	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	-0.5108	-0.4347	-0.3674	-0.3045	-0.2443	-0.1874	-0.1343	-0.0853	-0.0406
$S_{\alpha_2}^{new}$	-0.3546	-0.3134	-0.2715	-0.2274	-0.1831	-0.1405	-0.1006	-0.0639	-0.0304

Example 3.11. By taking two pieces of decision information in terms of IVIFNs as $\tilde{\alpha}_1 = \langle [0.10, 0.20], [0.10, 0.30] \rangle$ and $\tilde{\alpha}_2 = \langle [0, 0], [0.10, 0.20] \rangle$, the calculated GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ are put forward in Table 5 for distinct values of $\lambda_1 \in (0, 1)$ in order to determine the relationship between these IVIFNs. These results imply that $S_{\tilde{\alpha}_1}^{new} < S_{\tilde{\alpha}_2}^{new}$. Therefore, $\tilde{\alpha}_2 > \tilde{\alpha}_1$.

Table 5. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.11.

λ_1	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	-0.5805	-0.4829	-0.4000	-0.3270	-0.2618	-0.2034	-0.1512	-0.1044	-0.0625
$S_{\alpha_2}^{new}$	-0.1500	-0.1499	-0.1479	-0.1414	-0.1295	-0.1129	-0.0931	-0.0712	-0.0483

Example 3.12. The aim of this example is to measure the IVIFNs $\tilde{\alpha}_1 = \langle [0.10, 0.20], [0.30, 0.40] \rangle$ and $\tilde{\alpha}_2 = \langle [0.20, 0.30], [0.10, 0.30] \rangle$. For it, the resulting SF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for several values of $\lambda_1 \in (0, 1)$ are given in Table 6 and get $S_{\tilde{\alpha}_1}^{new} < S_{\tilde{\alpha}_2}^{new}$. Thus $\tilde{\alpha}_2 > \tilde{\alpha}_1$.

Table 6. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.12.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	-0.6085	-0.5315	-0.4589	-0.3895	-0.3253	-0.2677	-0.2169	-0.1725	-0.1338
$S_{\alpha_2}^{new}$	-0.4253	-0.3604	-0.3016	-0.2465	-0.1946	-0.1459	-0.1007	-0.0588	-0.0203

Example 3.13. In order to overcome the issue of ranking order between IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as given in Example 3.2, the proposed GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ are given in Table 7 by varying the values of $\lambda_1 \in (0, 1)$. From these results, we get $S_{\tilde{\alpha}_1}^{new} > S_{\tilde{\alpha}_2}^{new}$. Thus $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

Table 7. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.13.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	-0.040	-0.031	-0.021	-0.010	0.004	0.021	0.039	0.059	0.079
$S_{\alpha_2}^{new}$	-0.420	-0.352	-0.291	-0.235	-0.182	-0.132	-0.084	-0.039	0.003

Example 3.14. For handling the issue of the measurement of the IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as given in Example 3.3, the computed GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ have been shown in Table 8 for different values of $\lambda_1 \in (0, 1)$ and hence get $S_{\tilde{\alpha}_1}^{new} > S_{\tilde{\alpha}_2}^{new}$. Consequently, $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

Table 8. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.14.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\tilde{\alpha}_1}^{new}$	0.007	0.013	0.021	0.029	0.038	0.047	0.056	0.065	0.073
$S_{\tilde{\alpha}_2}^{new}$	-0.128	-0.108	-0.090	-0.071	-0.051	-0.029	-0.006	0.019	0.044

Example 3.15. For obtaining a reasonable ranking between two IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as shown in Example 3.4, the proposed GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ are calculated by varying the value of $\lambda_1 \in (0, 1)$ and are shown in Table 9. Such resulting values converge in the form of a relation $S_{\tilde{\alpha}_1}^{new} < S_{\tilde{\alpha}_2}^{new}$. Thus $\tilde{\alpha}_2 > \tilde{\alpha}_1$.

Table 9. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.15.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\tilde{\alpha}_1}^{new}$	0.1100	0.1194	0.1284	0.1370	0.1451	0.1528	0.1602	0.1671	0.1737
$S_{\tilde{\alpha}_2}^{new}$	0.2035	0.2069	0.2102	0.2136	0.2173	0.2214	0.2258	0.2304	0.2352

Example 3.16. To calculate a ranking order between two IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as presented in Example 3.5, the proposed GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ are calculated by varying the value of $\lambda_1 \in (0, 1)$ and the result are shown in Table 10. Such data provide the relation $S_{\tilde{\alpha}_1}^{new} > S_{\tilde{\alpha}_2}^{new}$. Thus $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

Table 10. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.16.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\tilde{\alpha}_1}^{new}$	-0.0466	-0.0430	-0.0380	-0.0314	-0.0229	-0.0124	-0.005	0.0125	0.0261
$S_{\tilde{\alpha}_2}^{new}$	-0.4270	-0.3638	-0.3067	-0.2522	-0.1991	-0.1480	-0.0997	-0.0546	-0.0131

Example 3.17. To measure the IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as given in Example 3.6, the proposed GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ are calculated by varying the value of $\lambda_1 \in (0, 1)$ and the result are shown in Table 11. It gives $S_{\tilde{\alpha}_1}^{new} > S_{\tilde{\alpha}_2}^{new}$. Hence $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

Table 11. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.17.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\tilde{\alpha}_1}^{new}$	-0.3498	-0.3421	-0.3233	-0.2987	-0.2715	-0.2439	-0.2170	-0.1914	-0.1674
$S_{\tilde{\alpha}_2}^{new}$	-0.4488	-0.4336	-0.4072	-0.3770	-0.3465	-0.3176	-0.2909	-0.2667	-0.2448

Example 3.18. For computing an order relation between two IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ as presented in Example 3.7, the proposed GSF values $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of $\lambda_1 \in (0, 1)$ are given in Table 12 and we get the relation $S_{\tilde{\alpha}_1}^{new} > S_{\tilde{\alpha}_2}^{new}$. Hence $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

After observing the results above, we can see that the proposed GSFs presented in (3.1) and (3.2) provide a pertinent tool to appraise the performance of decision information.

Table 12. The values of $S_{\tilde{\alpha}_1}^{new}$ and $S_{\tilde{\alpha}_2}^{new}$ for different values of λ_1 in Example 3.18.

λ_1	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90
$S_{\alpha_1}^{new}$	-0.4002	-0.3188	-0.2513	-0.1930	-0.1410	-0.0936	-0.0501	-0.0102	0.0264
$S_{\alpha_2}^{new}$	-0.4763	-0.4123	-0.3531	-0.2954	-0.2394	-0.1864	-0.1371	-0.0919	-0.0509

4. The method of determining the weights via the TMG model

For solving MAGDM problems under uncertainty, DMs need to select a suitable technique for determining attributes and alternative weights. A TMG approach is here used for this aim. The present DMP is formulated as a TMG model (Li's model) [67], in which the values of utility function are represented by IVIFNs. In this paper, the fused alternative attribute matrix is utilized as a payoff matrix and Player 1 is the group of DMs while nature is Player 2. The goal of DMs is to maximize their expected payoffs while nature selects to play a strategy against a group of DMs to minimize the loss. First of all, we provide some basic results related to a TMG model as given below.

Let R_+^n denote the non-negative part of n -dimensional Euclidean space R^n . Let $P = \{i\}_{i=1}^p$ and $Q = \{j\}_{j=1}^q$ be sets of pure strategies for Players 1 and 2, respectively. Then, the spaces regarding the mixed strategies for Players 1 and 2 are defined as

$$\Delta_p = \left\{ x = (x_1, x_2, \dots, x_p) \in R_+^p \mid \sum_i x_i = 1, i \in P \right\},$$

and

$$\Delta_q = \left\{ y = (y_1, y_2, \dots, y_q) \in R_+^q \mid \sum_j y_j = 1, j \in Q \right\},$$

respectively.

Here, element $x(y)$ in $\Delta_p(\Delta_q)$ is a probability distribution over $P(Q)$.

Definition 12. [67] A TMG with IVIF payoffs is defined as

$$\mathcal{IFG} = (P, \Delta_p; Q, \Delta_q; A),$$

where $A = [\tilde{a}_{ij}]_{p \times q}$ is payoff matrix in which ij^{th} entry $\tilde{a}_{ij}$ is represented by an IVIFN, i.e., $\tilde{a}_{ij} = \langle [\Omega_{ij}, \Theta_{ij}], [\vartheta_{ij}, \varrho_{ij}] \rangle$. The term 'zero-sum' is defined in intuitionistic fuzzy sense [67].

Remark 4.1. The solution of the matrix game $\mathcal{IFG}$ always exists with respect to a given weight $\omega \in [0, 1]$.

Definition 13. [67] For a pair $(x, y) \in \Delta_p \times \Delta_q$, the IVIF-valued function $E(x, y)$ represents the expected payoff for Player 1 in $\mathcal{IFG}$. It is defined as follows:

$$\begin{aligned} E(x, y) &= x^T A y \\ &= \left\langle \left[1 - \prod_j \prod_i (1 - \Omega_{ij})^{x_i y_j}, 1 - \prod_j \prod_i (1 - \Theta_{ij})^{x_i y_j} \right] \right\rangle, \end{aligned}$$

$$\left[\prod_j \prod_i (\vartheta_{ij})^{x_i y_j}, \prod_j \prod_i (\varrho_{ij})^{x_i y_j} \right] \Bigg\rangle.$$

Here, i and j run over from 1 to p and 1 to q , respectively.

In this game, a group of DMs (Player 1) follows the max–min principle of game theory in which this group expects that nature (Player 2) behaves strategically against their actions. The optimal solution of Player 1 gives alternative weights, while optimal solution for Player 2 provides attribute weights. In order to calculate such solutions, based on the max–min principle, two models are described as follows.

Optimization model for Player 1

The expected gain minimum ‘ $\min_{y \in \Delta_q} E(x, y)$ ’ for Player 1 is defined as

$$\begin{aligned} \kappa_1 &= \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle \\ &= \left\langle \min_{y \in \Delta_q} \left\{ 1 - \prod_j \prod_i (1 - \Omega_{ij})^{x_i y_j}, 1 - \prod_j \prod_i (1 - \Theta_{ij})^{x_i y_j} \right\}, \right. \\ &\quad \left. \max_{y \in \Delta_q} \left\{ \left[\prod_j \prod_i (\vartheta_{ij})^{x_i y_j}, \prod_j \prod_i (\varrho_{ij})^{x_i y_j} \right] \right\} \right\rangle. \end{aligned}$$

Here, i and j are running over from 1 to p and 1 to q , respectively.

Since Player 1 is a maximizing player. The aim of Player 1 is to determine a mixed strategy $x^* \in \Delta_p$ that reaches the maximum IVIF value of κ_1 . This x^* is known as the optimal (maximin) strategy for Player 1. Thus, the gain floor (κ_1^*) for Player 1 is given by

$$\begin{aligned} \kappa_1^* &= \max \kappa_1 \\ &= \left\langle \max_{x \in \Delta_p} \min_{y \in \Delta_q} \left\{ 1 - \prod_j \prod_i (1 - \Omega_{ij})^{x_i y_j}, 1 - \prod_j \prod_i (1 - \Theta_{ij})^{x_i y_j} \right\}, \right. \\ &\quad \left. \min_{x \in \Delta_p} \max_{y \in \Delta_q} \left\{ \left[\prod_j \prod_i (\vartheta_{ij})^{x_i y_j}, \prod_j \prod_i (\varrho_{ij})^{x_i y_j} \right] \right\} \right\rangle. \end{aligned}$$

The maximin solution of Player 1 is equivalent to the solution of following nonlinear interval programming problems

$$\max \{[\Omega, \Theta]\}, \min \{[\vartheta, \varrho]\}$$

such that

$$\begin{cases} \left[1 - \prod_j \prod_i (1 - \Omega_{ij})^{x_i y_j}, 1 - \prod_j \prod_i (1 - \Theta_{ij})^{x_i y_j} \right] \geq [\Omega, \Theta], \\ \left[\prod_j \prod_i (\vartheta_{ij})^{x_i y_j}, \prod_j \prod_i (\varrho_{ij})^{x_i y_j} \right] \leq [\vartheta, \varrho], \\ \text{for any } y \in \Delta_q, \\ 0 \leq \Theta + \varrho \leq 1, \sum_i x_i = 1, \\ \Omega \geq 0, \Theta \geq 0, \vartheta \geq 0, \varrho \geq 0, x_i \geq 0, i = 1 \text{ to } p. \end{cases} \quad (4.1)$$

The objective functions of the aforementioned problem can be reformulated as follows:

$$\min \{[\ln(1 - \Theta), \ln(1 - \Omega)]\}, \min \{[\ln \vartheta, \ln \varrho]\},$$

for $0 \leq \Omega, \Theta < 1$ and $0 < \vartheta, \varrho \leq 1$.

The objective functions above can be transformed as

$$\min \{[\omega \ln(1 - \Theta) + (1 - \omega) \ln \vartheta, \omega \ln(1 - \Omega) + (1 - \omega) \ln \varrho]\}, \quad (4.2)$$

for $0 \leq \Omega, \Theta < 1$, and $0 < \vartheta, \varrho \leq 1$ and $\omega \in [0, 1]$ is a weight which is determined by the players. The parameter ω is a preference parameter that represents the relative emphasis placed on the membership and nonmembership information when evaluating an uncertain payoff. Specifically, in the transformed objective function, ω determines the contribution of the membership-related component, while $1 - \omega$ determines the contribution of the nonmembership-related component. Thus, ω provides the players with flexibility to reflect their decision priorities and attitude toward the two types of information. A larger value of ω indicates greater emphasis on membership information, whereas a smaller value gives relatively greater emphasis to nonmembership information. The intermediate values $0 < \omega < 1$ allow both components to be considered simultaneously, with $\omega = 0.5$ representing equal emphasis. For instance, in a practical DMP, if a player considers positive/supporting evidence to be more important in evaluating an uncertain outcome, a relatively larger value of ω can be adopted; conversely, if greater emphasis is placed on non-supporting evidence, a smaller value can be selected. Accordingly, the term “determined by the players” means that ω is selected to reflect the players’ relative preference between the two components of the intuitionistic fuzzy payoff, rather than being an additional strategic probability. In the solution process of a real-event-based numerical example, a sensitivity analysis has been conducted for checking the impact of the parameter ω on the stability of ranking order of given alternatives. Notably, despite distinct overall expected scores for each alternative at ω values of 0.2, 0.5, and 0.8, the ranking order of available alternatives remains consistent and their corresponding optimal solutions for Players 1 and 2 are summarized in section 6. This underscores the stability of ranking order concerning variations in ω .

The constraints in (4.1) can be written as

$$\begin{cases} \sum_j \sum_i [\omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln \vartheta_{ij}] x_i y_j \leq \omega \ln(1 - \Theta) + (1 - \omega) \ln \vartheta, \text{ for any } y \in \Delta_q, \\ \sum_j \sum_i [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln \varrho_{ij}] x_i y_j \leq \omega \ln(1 - \Omega) + (1 - \omega) \ln \varrho, \text{ for any } y \in \Delta_q, \\ 0 \leq \Theta + \varrho \leq 1, \sum_i x_i = 1, \\ \Omega \geq 0, \Theta \geq 0, \vartheta \geq 0, \varrho \geq 0, x_i \geq 0, i = 1 \text{ to } p. \end{cases} \quad (4.3)$$

except for $\Omega = 1, \Theta = 1, \Omega_{ij} = 1, \Theta_{ij} = 1, \vartheta = 0, \vartheta_{ij} = 0$, and/or $\varrho = 0, \varrho_{ij} = 0$.

Combining Eqs (4.2) and (4.3), we get

$$\begin{aligned}
& \min \{[\omega \ln(1 - \Theta) + (1 - \omega) \ln \vartheta, \omega \ln(1 - \Omega) + (1 - \omega) \ln \varrho]\} \\
& \text{s.t.} \\
& \begin{cases} \sum_j \sum_i [\omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln \vartheta_{ij}] x_i y_j \leq \omega \ln(1 - \Theta) + (1 - \omega) \ln \vartheta, \text{ for any } y \in \Delta_q, \\ \sum_j \sum_i [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln \varrho_{ij}] x_i y_j \leq \omega \ln(1 - \Omega) + (1 - \omega) \ln \varrho, \text{ for any } y \in \Delta_q, \\ 0 \leq \Theta + \varrho \leq 1, \sum_i x_i = 1, \\ \Omega \geq 0, \Theta \geq 0, \vartheta \geq 0, \varrho \geq 0, x_i \geq 0, i = 1 \text{ to } p. \end{cases} \quad (4.4)
\end{aligned}$$

Let $\theta^- = \omega \ln(1 - \Theta) + (1 - \omega) \ln \vartheta$ and $\theta^+ = \omega \ln(1 - \Omega) + (1 - \omega) \ln \varrho$. Obviously, $\theta^- \leq 0$ and $\theta^+ \leq 0$. Then (4.4) becomes

$$\begin{aligned}
& \min \{[\theta^-, \theta^+]\} \\
& \text{s.t.} \\
& \begin{cases} \sum_j \sum_i [\omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln \vartheta_{ij}] x_i y_j \leq \theta^-, \text{ for any } y \in \Delta_q, \\ \sum_j \sum_i [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln \varrho_{ij}] x_i y_j \leq \theta^+, \text{ for any } y \in \Delta_q, \\ \sum_i x_i = 1, \\ \theta^- < 0, \theta^+ < 0, x_i \geq 0, i = 1 \text{ to } p. \end{cases} \quad (4.5)
\end{aligned}$$

Using the procedure presented in (Li, 2010), (4.5) becomes

$$\begin{aligned}
& \min \{\theta^-, \min \{\theta^+\}\} \\
& \text{s.t.} \\
& \begin{cases} \sum_i [\omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln \vartheta_{ij}] x_i \leq \theta^-, j = 1 \text{ to } q, \\ \sum_i [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln \varrho_{ij}] x_i \leq \theta^+, j = 1 \text{ to } q, \\ \sum_i x_i = 1, \\ \theta^- < 0, \theta^+ < 0, x_i \geq 0, i = 1 \text{ to } p. \end{cases} \quad (4.6)
\end{aligned}$$

Then (4.6) can be written as

$$\begin{aligned}
& \min \{\theta^- + \theta^+\} \\
& \text{s.t.} \\
& \begin{cases} \sum_i [\omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln \vartheta_{ij}] x_i \leq \theta^-, j = 1 \text{ to } q, \\ \sum_i [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln \varrho_{ij}] x_i \leq \theta^+, j = 1 \text{ to } q, \\ \sum_i x_i = 1, \\ \theta^- < 0, \theta^+ < 0, x_i \geq 0, i = 1 \text{ to } p. \end{cases} \quad (4.7)
\end{aligned}$$

Let $\theta^- + \theta^+ = u$, Eq (4.7) becomes

min u
s.t.

$$\begin{cases} \sum_i [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln(\varrho_{ij}) + \omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln(\vartheta_{ij})] x_i \leq u, j = 1 \text{ to } q, \\ \sum_i x_i = 1, \\ u \leq 0, x_i \geq 0, i = 1 \text{ to } p, \end{cases} \quad (4.8)$$

except for $\Omega_{ij} = 1$, $\Theta_{ij} = 1$, $\vartheta_{ij} = 0$, and/or $\varrho_{ij} = 0$. Here $u = \omega \ln(1 - \Theta) + (1 - \omega) \ln \vartheta + \omega \ln(1 - \Omega) + (1 - \omega) \ln \varrho$.

Optimization model for Player 2

For Player 2, the value of ' $\max_{x \in \Delta_p} E(x, y)$ ' is defined as

$$\begin{aligned} \kappa_2 &= \langle [\tau^-, \tau^+], [\sigma^-, \sigma^+] \rangle \\ &= \left\langle \max_{x \in \Delta_p} \left\{ \left[1 - \prod_j \prod_i (1 - \Omega_{ij})^{x_i y_j}, 1 - \prod_j \prod_i (1 - \Theta_{ij})^{x_i y_j} \right] \right\}, \right. \\ &\quad \left. \min_{x \in \Delta_p} \left\{ \left[\prod_j \prod_i (\vartheta_{ij})^{x_i y_j}, \prod_j \prod_i (\varrho_{ij}^+)^{x_i y_j} \right] \right\} \right\rangle. \end{aligned}$$

Here, i and j run over from 1 to p and 1 to q , respectively.

Now the aim of the minimizing Player 2 is to evaluate a mixed strategy $y^* \in \Delta_q$ that achieves the minimum IVIF value of κ_2 . This y^* is the optimal (minimax) strategy for Player 2. Thus, the loss ceiling (κ_2^*) for Player 2 is given by

$$\begin{aligned} \kappa_2^* &= \min \kappa_2 \\ &= \left\langle \min_{y \in \Delta_q} \max_{x \in \Delta_p} \left\{ \left[1 - \prod_j \prod_i (1 - \Omega_{ij})^{x_i y_j}, 1 - \prod_j \prod_i (1 - \Theta_{ij})^{x_i y_j} \right] \right\}, \right. \\ &\quad \left. \max_{y \in \Delta_q} \min_{x \in \Delta_p} \left\{ \left[\prod_j \prod_i (\vartheta_{ij})^{x_i y_j}, \prod_j \prod_i (\varrho_{ij})^{x_i y_j} \right] \right\} \right\rangle. \end{aligned}$$

The solution of the minimax problem κ_2^* can be determined (by following the same procedure as described for Player 1) by solving the following problem:

max v
s.t.

$$\begin{cases} \sum_j [\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln(\zeta_{ij}^+) + \omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln(\vartheta_{ij})] y_j \geq v, \\ \sum_j y_j = 1, \\ v \leq 0, y_j \geq 0, j = 1 \text{ to } q, \end{cases} \quad (4.9)$$

except for $\Omega_{ij} = 1$, $\Theta_{ij} = 1$, $\vartheta_{ij} = 0$, and/or $\varrho_{ij} = 0$. Here, $v = \omega \ln(1 - \tau^+) + (1 - \omega) \ln \sigma^- + \omega \ln(1 - \tau^-) + (1 - \omega) \ln \sigma^+$. The weight $\omega \in [0, 1]$ is determined by the players.

5. Formulation of the present MAGDM problem and its solution steps

The present section provides an algorithm for solving an IVIF MAGDM problem without any kind of weight information. This algorithm is based on the concept of IVIF TMG theory and the proposed GSF for IVIFNs.

5.1. Description of the problem

In the present MAGDM problem, let $\{T_i, i = 1 \text{ to } m\}$ be a set of m alternatives and $\{E_j, j = 1 \text{ to } n\}$ be a set of n attributes. To calculate the reasonable ranking results, multiple DMs are assigned in a decision process. Suppose that the group of such DMs contains l members, say, d_1 to d_l . These DMs have different attributes, so there is a weight vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)^T$ for DMs. Here, $\sum_k \lambda_k = 1$ ($k = 1$ to l), $\lambda_k > 0$ for all k . The main features of the MAGDM problem are described above as follows.

- (i) All decision information given by k^{th} DM d_k have been put forward in the form of an IVIF decision matrix $\tilde{D}_{m \times n}^{(k)}(\tilde{d}_{ij}^{(k)}) = (\langle [\Omega_{ij}^{(k)}, \Theta_{ij}^{(k)}], [\vartheta_{ij}^{(k)}, \varrho_{ij}^{(k)}] \rangle)$, where $[\Omega_{ij}^{(k)}, \Theta_{ij}^{(k)}] \subset [0, 1]$, $[\vartheta_{ij}^{(k)}, \varrho_{ij}^{(k)}] \subset [0, 1]$ and $\Theta_{ij}^{(k)} + \varrho_{ij}^{(k)} \leq 1$. The interval numbers $[\Omega_{ij}^{(k)}, \Theta_{ij}^{(k)}]$ and $[\vartheta_{ij}^{(k)}, \varrho_{ij}^{(k)}]$ represent the degrees of satisfaction and dissatisfaction intervals, respectively, for an alternative T_i with respect to attribute E_j .
- (ii) For every k , the DM d_k keeps his/her focus only on the behavior of the decision matrix $\tilde{D}^{(k)}$.

The target of any decision process is mainly to locate the best alternative or to rank alternatives. Indeed, this target is closely related to the attribute weights. That is, evaluation of the attribute weights is the underlying problem in a decision-making process. A TMG theoretic approach is an effective technique for determining attribute weights in fuzzy MADM/MAGDM problems. This technique permits us to test the impact of the distortion of information on the stability of the outputs of the decision process by using the sensitivity analysis for the different parameters. Keeping in mind this issue, we have used Li's game model [67] for calculating the attribute weights of the present MAGDM problem.

For calculating a weight distribution over the sets $\{E_j, j = 1 \text{ to } n\}$ and $\{T_i, i = 1 \text{ to } m\}$ simultaneously through the considered game model, it is assumed that the group of DMs behaves according to the max–min principle (Wald criterion) [75]. Now the role of players can be defined as follows

Player 1: The group of DMs d_1 to d_l .

Player 2: Nature.

According to the max–min principle, Player 1 assumes that the purpose of Player 2 is to minimize their payoffs. It means that Player 2 plays strategically against Player 1. After selecting a mixed strategy x (a probability distribution) on the set $\{T_i, i = 1 \text{ to } m\}$ by Player 1, Player 2 selects a mixed strategy y on the set $\{E_j, j = 1 \text{ to } n\}$.

Using this concept, both kinds of weights can be calculated for the DMP considered here. To check the performance of an alternative on the basis of these weights, we define the IVIF expected SF as follows.

Definition 14. If the probability vectors $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ and $y^* = (y_1^*, y_2^*, \dots, y_n^*)^T$ represent the weight vectors for the sets of alternatives and attributes, respectively. Then, for the i^{th} alternative T_i ,

the overall expected SF value (IIFES ($T_i, \tilde{D}$)) is defined as

$$\text{IIFES} (T_i, \tilde{D}) = x_i^* \sum_j \tilde{d}_{ij} y_j^*$$

where $\tilde{D} = [\tilde{d}_{ij}]_{m \times n}$ and, i and j run over from 1 to m and 1 to n , respectively.

This value IIFES ($T_i, \tilde{D}$) can be calculated by using the Properties (i) and (iii) of Definition 4. The calculated IIFES ($T_i, \tilde{D}$) is given as follows:

$$\begin{aligned} \text{IIFES} (T_i, \tilde{D}) = & \left\langle \left[1 - \prod_j (1 - \Omega_{ij})^{x_i^* y_j^*}, 1 - \prod_j (1 - \Theta_{ij})^{x_i^* y_j^*} \right], \right. \\ & \left. \left[\prod_j (\vartheta_{ij})^{x_i^* y_j^*}, \prod_j (\varrho_{ij})^{x_i^* y_j^*} \right] \right\rangle \end{aligned} \quad (5.1)$$

where $\tilde{d}_{ij} = \langle [\Omega_{ij}, \Theta_{ij}], [\vartheta_{ij}, \varrho_{ij}] \rangle$ and j runs over from 1 to n .

From this definition, we find that the expected outcome of an alternative is obtained in the form of an IVIFN. To convert the set $\{T_i, i = 1 \text{ to } m\}$ into an ordered set, we calculate the SF value of the IIFESs value corresponding to each alternative.

5.2. Proposed MAGDM approach

In the light of Definition 14 and the proposed GSF given in (3.2), we used the following steps for solving the IVIF-MAGDM problem. The flowchart of this algorithm is given in Figure 1.

Step 1: The aim of this step is to construct the collective IVIF decision matrix. To do this, for k^{th} DM d_k , $\tilde{D}_{m \times n}^{(k)}(\tilde{d}_{ij}^{(k)}) = (\langle [\Omega_{ij}^{(k)}, \Theta_{ij}^{(k)}], [\vartheta_{ij}^{(k)}, \varrho_{ij}^{(k)}] \rangle)$ is the IVIF decision matrix. Then the collective IVIF decision matrix for all DMs, $\tilde{D}_{m \times n}(\tilde{d}_{ij}) = (\langle [\Omega_{ij}, \Theta_{ij}], [\vartheta_{ij}, \varrho_{ij}] \rangle)$, can be calculated by using the IIFHG operator θ (Definition 7). The calculated ij^{th} entry $\tilde{d}_{ij}$ of the matrix $\tilde{D}_{m \times n}$ is given as

$$\begin{aligned} \tilde{d}_{ij} = & \left\langle \left[\prod_k (\dot{\Omega}_{ij}^{(\rho(k))})^{a_k}, \prod_k (\dot{\Theta}_{ij}^{(\rho(k))})^{a_k} \right], \right. \\ & \left. \left[\left(1 - \prod_k (1 - \dot{\vartheta}_{ij}^{(\rho(k))})^{a_k} \right), \left(1 - \prod_k (1 - \dot{\varrho}_{ij}^{(\rho(k))})^{a_k} \right) \right] \right\rangle. \end{aligned}$$

Here, k runs over from 1 to l .

Step 2: In this step, the matrix $\tilde{D}$ is converted into a normalized matrix $\tilde{R} = [\tilde{r}_{ij}]_{m \times n}$.

Step 3: The aim of this step is to calculate the attribute weights and alternative weights for the group of DMs. The group of DMs needs to select a criterion for decision-making under uncertainty. In this paper, we assume that the group of DMs adopts Wald's criterion [75], which assumes that the group of DMs is reactionary and expects that nature is competing against them to minimize their payoffs. In this sense, the above-mentioned IVIF MAGDM problem can be cast as a two person zero-sum IVIF matrix game with Player 1 being the group of DMs and Player 2 being nature.

Since the group of DMs is cautious about the strategy selection of nature, they will choose the mixed strategy $x = (x_1, x_2, \dots, x_m)$, $x_i \geq 0, i = 1$ to m , $\sum_{i=1}^m x_i = 1$ over the set of alternatives $\{T_i, i = 1$ to $m\}$. The group of DMs views nature as a non cooperative player and to counter their mixed strategy choice, nature will select the mixed strategy $y = (y_1, y_2, \dots, y_n)$, $y_j \geq 0, j = 1$ to n , $\sum_{j=1}^n y_j = 1$ over the set of attributes $\{E_j, j = 1$ to $n\}$. Thus, we obtain the corresponding two person zero-sum IVIF game with the payoff matrix $\tilde{R} = [\tilde{r}_{ij}]_{m \times n}$ where IVIFN $\tilde{r}_{ij}$ is the payoff of the group of DMs when they choose the i^{th} strategy with probability x_i and nature selects the j^{th} strategy with probability y_j . The normalized collective decision matrix $\tilde{R}$ is used in the form of a payoff matrix for Li's game model. To solve it, we solve the following optimization problems given in the models in Section 4.

$$(M1) \quad \begin{array}{ll} \min & u \\ \text{s.t.} & \left\{ \begin{array}{l} \sum_i \left[\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln(\varrho_{ij}) + \omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln(\vartheta_{ij}) \right] x_i \leq u, \\ \sum_i x_i = 1, \\ u \leq 0, x_i \geq 0, i = 1 \text{ to } m, \end{array} \right. \end{array} \quad j = 1 \text{ to } n,$$

and

$$(M2) \quad \begin{array}{ll} \max & v \\ \text{s.t.} & \left\{ \begin{array}{l} \sum_j \left[\omega \ln(1 - \Omega_{ij}) + (1 - \omega) \ln(\varrho_{ij}) + \omega \ln(1 - \Theta_{ij}) + (1 - \omega) \ln(\vartheta_{ij}) \right] y_j \geq v, \\ \sum_j y_j = 1, \\ v \leq 0, y_j \geq 0, j = 1 \text{ to } n, \end{array} \right. \end{array} \quad i = 1 \text{ to } m,$$

Except for the cases $\Omega_{ij} = 1$, $\Theta_{ij} = 1$, $\vartheta_{ij} = 0$, and/or $\varrho_{ij} = 0$, the parameter $\omega \in [0, 1]$ is determined by the players in the models above.

After solving the models (M1) and (M2) for a given value of $\omega \in [0, 1]$, we get the optimal solutions x^* and y^* . The calculated vector $x^*(y^*)$ is the weight vector corresponding to the set $\{T_i, i = 1$ to $m\}$ ($\{E_j, j = 1$ to $n\}$).

Step 4: This step puts forward the IIFES value for each alternative. We determine the IIFES value for each alternative T_i , $IIFES(T_i, \tilde{R})$ given in Definition 14, corresponding to the decision matrix $\tilde{R}_{m \times n}(\tilde{r}_{ij}) = (\langle [\Omega_{ij}, \Theta_{ij}], [\vartheta_{ij}, \varrho_{ij}] \rangle)$. That is, we get

$$IIFES(T_i, \tilde{R}) = x_i^* \sum_{j=1}^n \tilde{r}_{ij} y_j^*.$$

Step 5: Determine the GSF value of the IVIF expected SF by utilizing the proposed GSF (Eq (3.2)) with respect to each alternative T_i .

Step 6: On the basis of calculated GSF values of the IVIF expected SF values, rank the alternatives T_i ($i = 1$ to m). According to the descending order of the GSF values of the IVIF expected SF values, the best alternative is chosen.

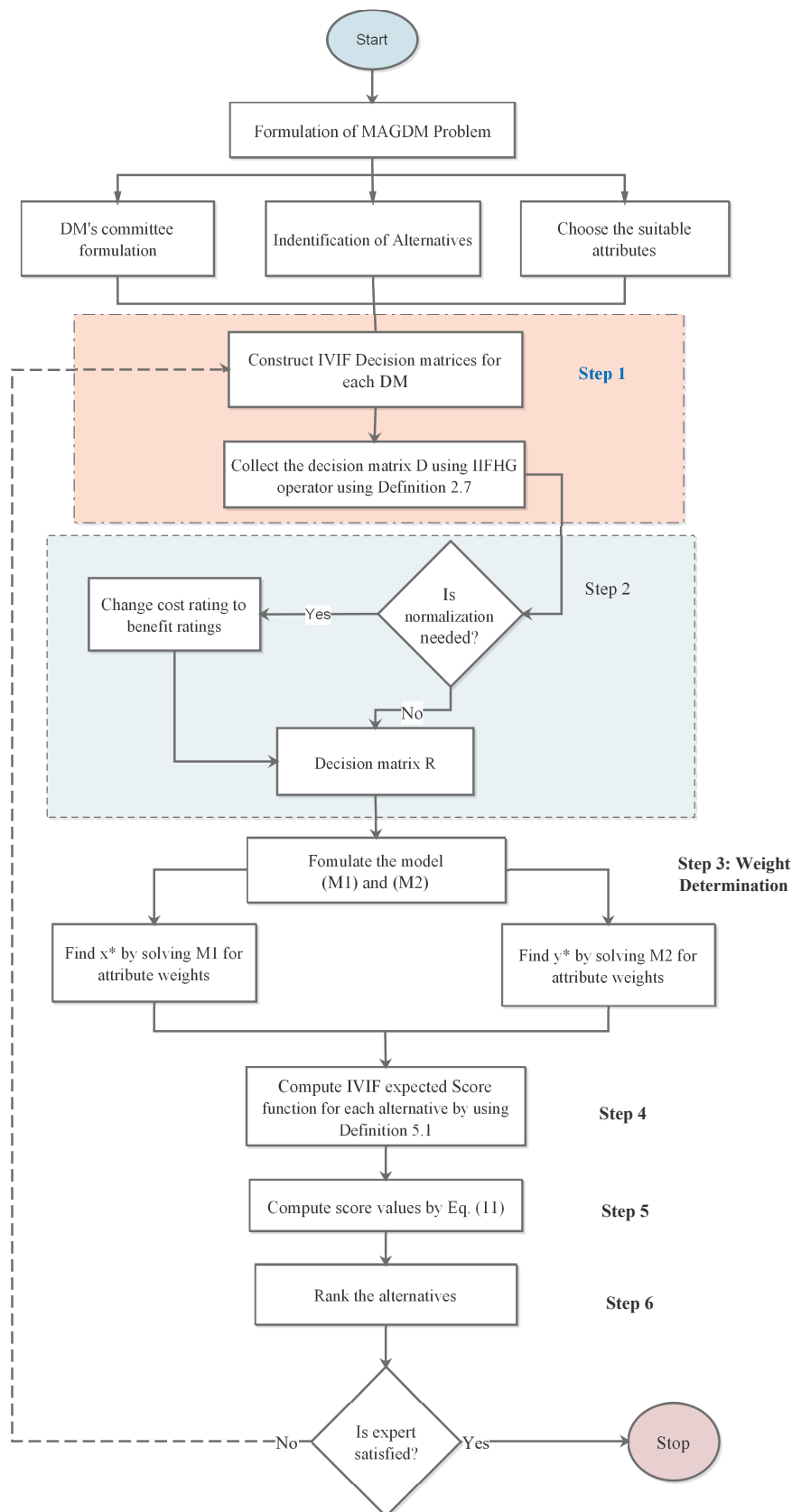


Figure 1. Flowchart of the proposed methodology.

5.3. Analysis of the proposed MAGDM algorithm

In this subsection, we give some advantages and disadvantages of the developed algorithm. The advantages are as follows.

- (1) The present algorithm outlines a step-by-step approach for addressing the MAGDM problem within the context of IVIF decision matrices when the attribute weights are unknown. In these specific MAGDM scenarios, DMs concentrate solely on the analysis of IVIF decision matrices, where each entry is represented by IVIFNs.
- (2) In this algorithm, we use an IVIF–TMG approach to assess the weight information for both attributes and alternatives concurrently. The key benefit of this approach lies in its ability to examine how information distortion affects the determination of outcomes in the decision-making process. This is accomplished through a sensitivity analysis applied to different parameters.
- (3) In most cases, the overall performance of alternatives is assessed solely on the basis of the attribute weights' information. However, in the present work, we take a more comprehensive approach by leveraging both the attribute and alternative weights to evaluate the overall performance of alternatives in conjunction with their attribute values. To achieve this, the IIFES function is built up for each alternative. This methodology allows to better meet a broader spectrum of demands within the context of fuzzy decision-making concerns.
- (4) The score values for the IIFES values (IVIFNs) are computed using the proposed GSF given in Eq (3.2). The advantage of this GSF lies in its ability to consider not only the MD, the ND, and the ID associated with an IVIFN but also takes into account two parameters. These parameters perform as regulators, serving to gauge the informational content of the IVIF decision.
- (5) The utilization of the SF in the algorithm's final step prevents the distortion of the original fuzzy information, leading to a more reasonable and accurate ranking.

The disadvantages of this approach are as follows.

- (a) The models outlined in (M1) and (M2) may not be suitable under certain conditions specified in (4.8) and (4.9) for determining the attribute and alternative weights.
- (b) The group of DMs behaves as a player only when there is no conflict in nature among the DMs.

6. Numerical example

This section presents a real-life example of a MAGDM problem [7] involving four alternatives, denoted $\{T_i, i = 1 \text{ to } 4\}$, to demonstrate the applicability of the proposed algorithm. In modern organizations, maintaining a comfortable and controlled indoor environment is essential for ensuring employees' well-being, operational efficiency, and overall productivity. In order to create ideal working conditions that improve focus and performance, air conditioning systems are essential for controlling temperature, humidity, and air quality. Additionally, in industries where equipment or processes are temperature-sensitive, reliable air conditioning systems are vital for maintaining consistent operational standards.

With this motivation, the organization aims to install a finite number of air conditioning systems equipped with specific attributes that meet both the performance and safety requirements. To achieve this, a committee is formed to study and select the most suitable contract offer among four available alternatives. Suppose that this committee consists of four experts. The considered evaluation attributes are as follows: E_1 : Coefficient of performance, E_2 : maintainability, E_3 : flexibility, E_4 : cost, and E_5 : safety.

A brief study of attributes E_1, E_2, E_3, E_4 , and E_5 is given below:

E_1 : This is a measure of how efficiently the air conditioning system can convert electrical energy into cooling output. The higher the coefficient of performance, the more efficient the system is, which means that it will consume less electricity to provide the same amount of cooling.

E_2 : This refers to how easy it is to maintain and repair the air conditioning system over time. A system with good maintainability will require less downtime for repairs and will generally have a longer lifespan.

E_3 : This attribute refers to how well the air conditioning system can adapt to different conditions and environments. A flexible system can adjust its cooling output according to changing conditions, such as the number of people in a room or changes in outdoor temperature.

E_4 : This refers to the initial cost of installing the air conditioning system, as well as any ongoing operating costs (such as electricity usage). A system with a lower cost may be more attractive to the organization, but it is important to balance this against other attributes.

E_5 : This attribute refers to how safe the air conditioning system is to operate and maintain. A safe system will have features such as automatic shutoff in the event of a malfunction, as well as clear warning signs and instructions for users.

The IVIF decision matrices $\tilde{D}_{4 \times 5}^{(k)}$ ($k = 1$ to 4) are given in Tables 13–16 in terms of the normalized form. Suppose that the weight vector with respect to DMs d_1 to d_4 is $\lambda = (0.3, 0.2, 0.3, 0.2)^T$.

Table 13. First IVIF normalized decision matrix $\tilde{D}^{(1)}$.

	E_1	E_2	E_3	E_4	E_5
T_1	$\langle [0.5, 0.6], [0.2, 0.3] \rangle$	$\langle [0.3, 0.5], [0.4, 0.5] \rangle$	$\langle [0.6, 0.7], [0.2, 0.3] \rangle$	$\langle [0.5, 0.7], [0.1, 0.2] \rangle$	$\langle [0.1, 0.4], [0.3, 0.5] \rangle$
T_2	$\langle [0.3, 0.4], [0.4, 0.6] \rangle$	$\langle [0.1, 0.3], [0.2, 0.4] \rangle$	$\langle [0.3, 0.4], [0.4, 0.5] \rangle$	$\langle [0.2, 0.4], [0.5, 0.6] \rangle$	$\langle [0.7, 0.8], [0.1, 0.2] \rangle$
T_3	$\langle [0.4, 0.5], [0.3, 0.5] \rangle$	$\langle [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [0.5, 0.8], [0.1, 0.2] \rangle$	$\langle [0.4, 0.6], [0.2, 0.3] \rangle$	$\langle [0.5, 0.6], [0.2, 0.3] \rangle$
T_4	$\langle [0.3, 0.5], [0.4, 0.5] \rangle$	$\langle [0.1, 0.2], [0.7, 0.8] \rangle$	$\langle [0.1, 0.2], [0.5, 0.8] \rangle$	$\langle [0.2, 0.3], [0.4, 0.6] \rangle$	$\langle [0.2, 0.3], [0.5, 0.6] \rangle$

Table 14. Second IVIF normalized decision matrix $\tilde{D}^{(2)}$.

	E_1	E_2	E_3	E_4	E_5
T_1	$\langle [0.4, 0.5], [0.2, 0.4] \rangle$	$\langle [0.3, 0.4], [0.4, 0.6] \rangle$	$\langle [0.6, 0.7], [0.1, 0.2] \rangle$	$\langle [0.5, 0.6], [0.1, 0.3] \rangle$	$\langle [0.1, 0.3], [0.3, 0.5] \rangle$
T_2	$\langle [0.3, 0.5], [0.4, 0.5] \rangle$	$\langle [0.1, 0.3], [0.3, 0.7] \rangle$	$\langle [0.3, 0.4], [0.4, 0.5] \rangle$	$\langle [0.2, 0.3], [0.6, 0.7] \rangle$	$\langle [0.6, 0.8], [0.1, 0.2] \rangle$
T_3	$\langle [0.4, 0.6], [0.3, 0.4] \rangle$	$\langle [0.6, 0.8], [0.1, 0.2] \rangle$	$\langle [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [0.4, 0.6], [0.3, 0.4] \rangle$	$\langle [0.5, 0.6], [0.2, 0.4] \rangle$
T_4	$\langle [0.3, 0.4], [0.4, 0.6] \rangle$	$\langle [0.1, 0.2], [0.6, 0.8] \rangle$	$\langle [0.1, 0.2], [0.7, 0.8] \rangle$	$\langle [0.3, 0.4], [0.4, 0.6] \rangle$	$\langle [0.2, 0.4], [0.5, 0.6] \rangle$

Table 15. Third IVIF normalized decision matrix $\tilde{D}^{(3)}$.

	E_1	E_2	E_3	E_4	E_5
T_1	$\langle [0.4, 0.7], [0.1, 0.2] \rangle$	$\langle [0.3, 0.5], [0.3, 0.4] \rangle$	$\langle [0.6, 0.7], [0.1, 0.2] \rangle$	$\langle [0.5, 0.6], [0.1, 0.3] \rangle$	$\langle [0.3, 0.5], [0.4, 0.5] \rangle$
T_2	$\langle [0.4, 0.5], [0.2, 0.4] \rangle$	$\langle [0.2, 0.4], [0.4, 0.5] \rangle$	$\langle [0.4, 0.5], [0.3, 0.4] \rangle$	$\langle [0.1, 0.2], [0.7, 0.8] \rangle$	$\langle [0.6, 0.7], [0.2, 0.3] \rangle$
T_3	$\langle [0.2, 0.4], [0.3, 0.4] \rangle$	$\langle [0.6, 0.8], [0.1, 0.2] \rangle$	$\langle [0.5, 0.7], [0.1, 0.3] \rangle$	$\langle [0.5, 0.7], [0.2, 0.3] \rangle$	$\langle [0.6, 0.8], [0.1, 0.2] \rangle$
T_4	$\langle [0.3, 0.4], [0.2, 0.4] \rangle$	$\langle [0.1, 0.2], [0.6, 0.8] \rangle$	$\langle [0.1, 0.3], [0.5, 0.7] \rangle$	$\langle [0.2, 0.3], [0.5, 0.7] \rangle$	$\langle [0.1, 0.2], [0.6, 0.8] \rangle$

Table 16. Fourth IVIF normalized decision matrix $\tilde{D}^{(4)}$.

	E_1	E_2	E_3	E_4	E_5
T_1	$\langle [0.6, 0.7], [0.2, 0.3] \rangle$	$\langle [0.3, 0.4], [0.3, 0.4] \rangle$	$\langle [0.7, 0.8], [0.1, 0.2] \rangle$	$\langle [0.5, 0.6], [0.1, 0.3] \rangle$	$\langle [0.1, 0.2], [0.5, 0.7] \rangle$
T_2	$\langle [0.4, 0.5], [0.4, 0.5] \rangle$	$\langle [0.1, 0.2], [0.2, 0.3] \rangle$	$\langle [0.3, 0.4], [0.5, 0.6] \rangle$	$\langle [0.2, 0.3], [0.4, 0.6] \rangle$	$\langle [0.6, 0.7], [0.1, 0.2] \rangle$
T_3	$\langle [0.4, 0.5], [0.3, 0.4] \rangle$	$\langle [0.6, 0.7], [0.1, 0.3] \rangle$	$\langle [0.5, 0.8], [0.1, 0.2] \rangle$	$\langle [0.4, 0.5], [0.2, 0.3] \rangle$	$\langle [0.5, 0.6], [0.3, 0.4] \rangle$
T_4	$\langle [0.3, 0.4], [0.4, 0.5] \rangle$	$\langle [0.1, 0.3], [0.6, 0.7] \rangle$	$\langle [0.1, 0.2], [0.5, 0.8] \rangle$	$\langle [0.2, 0.3], [0.4, 0.5] \rangle$	$\langle [0.3, 0.4], [0.5, 0.6] \rangle$

The following steps of the proposed algorithm will now be given to compute the best alternative.

Step 1: For the operator θ , the vector $a = (0.155, 0.345, 0.345, 0.155)^T$ is determined by utilizing the technique [76]. Then, the collective decision matrix $\tilde{D}$ (Table 17) is obtained from the matrices (Tables 13–16) by making the use of θ .

Table 17. Collective IVIF decision matrix $\tilde{D}$.

	E_1	E_2	E_3
T_1	$\langle [0.4385, 0.6199], [0.1549, 0.2848] \rangle$	$\langle [0.3000, 0.4573], [0.3404, 0.4710] \rangle$	$\langle [0.6116, 0.7117], [0.1089, 0.2083] \rangle$
T_2	$\langle [0.3502, 0.4797], [0.3114, 0.4681] \rangle$	$\langle [0.1138, 0.3010], [0.2511, 0.4773] \rangle$	$\langle [0.3379, 0.4387], [0.3872, 0.4887] \rangle$
T_3	$\langle [0.3516, 0.4906], [0.2940, 0.4214] \rangle$	$\langle [0.6395, 0.7711], [0.0980, 0.2263] \rangle$	$\langle [0.5213, 0.7804], [0.0980, 0.2083] \rangle$
T_4	$\langle [0.3000, 0.4170], [0.3114, 0.4887] \rangle$	$\langle [0.1000, 0.2103], [0.6012, 0.7678] \rangle$	$\langle [0.1000, 0.2366], [0.5577, 0.7569] \rangle$
	E_4	E_5	
T_1	$\langle [0.5000, 0.6395], [0.0980, 0.2567] \rangle$	$\langle [0.1323, 0.3623], [0.3747, 0.5482] \rangle$	
T_2	$\langle [0.1758, 0.3134], [0.5305, 0.6496] \rangle$	$\langle [0.6395, 0.7521], [0.1089, 0.2083] \rangle$	
T_3	$\langle [0.4387, 0.6252], [0.2263, 0.3262] \rangle$	$\langle [0.5452, 0.6502], [0.1770, 0.3005] \rangle$	
T_4	$\langle [0.2103, 0.3109], [0.4050, 0.5613] \rangle$	$\langle [0.1849, 0.3121], [0.5031, 0.6118] \rangle$	

Step 2: The information is already summarized in the normalized form.

Step 3: In this step, we develop a TMG set-up by taking the computed payoff decision matrix $\tilde{D}$. We solve the optimization models (M1) and (M2). For $\omega = 0.2, 0.5$, and 0.8 , we make a sensitivity analysis. For these values of ω , the optimal solutions for Players 1 and 2 are put forward in Table 18.

Table 18. Optimal results of Models (M1) and (M2) for different values of ω .

Variables	The values of ω		
	0.2	0.5	0.8
x_1^*	0.4602	0.4728	0.4900
x_2^*	0.2253	0.2148	0.2048
x_3^*	0.3145	0.3124	0.3052
x_4^*	0.0000	0.0000	0.0000
u^*	-2.2822	-1.9255	-1.5658
y_1^*	0.6147	0.6549	0.7062
y_2^*	0.0880	0.0723	0.0579
y_3^*	0.0000	0.0000	0.0000
y_4^*	0.0000	0.0000	0.0000
y_5^*	0.2973	0.2728	0.2359
v^*	-2.2822	-1.9255	-1.5658

Step 4: Using (5.1), the IVIF expected SF value for each T_i is evaluated with the help of the results presented in Tables 17 and 18. The evaluated values of $IIFES(T_i, \tilde{D})$ ($i = 1$ to 4) are provided in Table 19.

Table 19. IVIF expected score value of each alternative for different values of ω .

The values of ω	$IIFES(T_1, \tilde{D})$	$IIFES(T_2, \tilde{D})$	$IIFES(T_3, \tilde{D})$	$IIFES(T_4, \tilde{D})$
0.2	$\langle [0.1921, 0.3639], [0.4939, 0.6263] \rangle$	$\langle [0.1465, 0.2383], [0.7136, 0.7986] \rangle$	$\langle [0.2106, 0.3370], [0.6295, 0.7257] \rangle$	$\langle [0, 0], [1, 1] \rangle$
0.5	$\langle [0.2012, 0.3735], [0.4766, 0.6113] \rangle$	$\langle [0.1082, 0.2225], [0.7294, 0.8104] \rangle$	$\langle [0.2005, 0.3187], [0.6374, 0.7314] \rangle$	$\langle [0, 0], [1, 1] \rangle$
0.8	$\langle [0.2138, 0.3861], [0.4541, 0.5913] \rangle$	$\langle [0.1265, 0.2034], [0.7466, 0.8234] \rangle$	$\langle [0.1861, 0.2939], [0.6508, 0.7415] \rangle$	$\langle [0, 0], [1, 1] \rangle$

Step 5: Utilize the suggested the GSF given in (3.2) to obtain GSF values ($\lambda_1 = 0.6$) of $IIFES(T_i, \tilde{D})$ ($i = 1$ to 4). These GSF values are presented in Table 20. In Table 20, (.) represents the ranking order of alternatives.

Table 20. Score values of $IIFES(T_i, \tilde{D})$ for different values of ω .

The values of ω	$S_{IIFES(T_1, \tilde{D})}^{new}$	$S_{IIFES(T_2, \tilde{D})}^{new}$	$S_{IIFES(T_3, \tilde{D})}^{new}$	$S_{IIFES(T_4, \tilde{D})}^{new}$
0.2	-0.2742(1)	-0.5454(3)	-0.3946(2)	-1(4)
0.5	-0.2532(1)	-0.5764(3)	-0.4133(2)	-1(4)
0.8	-0.2259(1)	-0.5794(3)	-0.4412(2)	-1(4)

Step 6: From Table 20, it is concluded that the best alternative is T_1 for all considered values of ω .

7. Comparison and advantages of the proposed approach

This section involves a comparative analysis between the proposed algorithm and previous research. We also discussed the advantages of the suggested work.

7.1. Comparison with existing studies

The current subsection presents some comparisons related to the existing methods for exploring the effectivity of the developed results.

(1) Comparison with Park et al. [7]

Park et al. provided a TOPSIS-based method for handling an IVIF MAGDM problem. The attribute weights in this research problem are not completely known. For the same data presented in the numerical example (Section 6) with the optimal vector $w = (0.2069, 0.1252, 0.2604, 0.2447, 0.1627)^T$ for attributes, the ranking order of alternatives is $(T_3 > T_2 > T_1 > T_4)$. This sequence is different to the calculated sequence by our suggested algorithm.

(2) Comparison with Kumar and Kumar [13]

Kumar and Kumar presented a solution for a group DMP without prior knowledge of the weights under the IVIF framework. In order to calculate the attribute weights, they used a TMG approach and an SF for IVIFNs. Using the same data presented in the numerical example (Section 6), according to this approach, the ranking order of alternatives is $(T_3 > T_1 > T_2 > T_4)$. The calculated ranking by our proposed approach is different to that given by Kumar and Kumar in the sense that the place of alternative T_1 has been interchanged with the place of alternative T_3 only in ranking order.

(3) Comparison with Kumar et al. [25]

Kumar et al. introduced a new IVIFS-based method to deal with an MAGDM problem. This method is designed by using the ordered weighted averaging (OWA) operator [77] and a SF for IVIFNs. For the same data presented in the numerical example (Section 6), according to Kumar et al., the optimal attribute weight distribution $w = (0.1888, 0.1928, 0.2185, 0.2047, 0.1952)^T$. With the help of these weights, the ranking order of alternatives is $(T_3 > T_1 > T_2 > T_4)$. This sequence is different to the calculated sequence by our suggested TMG theory-based algorithm with only T_2 and T_4 are put in the same place.

(4) Comparison with Garg [16]

Garg proposed a method for solving an IVIF MADM problem in which the weight values (for attributes) are not known. In it, these values are obtained by the Shannon entropy method. To conduct a comparative analysis between the developed and Garg's methods, we consider a numerical problem presented in Section 6. By applying the solution steps of Garg method, the weights of the attributes are calculated as $w_1 = 0.2746$, $w_2 = 0.1757$, $w_3 = 0.2978$, $w_4 = 0.2351$, and $w_5 = 0.0168$ for $k_1 = k_2 = 0.5$ and then we obtain the final ranking as $(T_1 > T_3 > T_2 > T_4)$. This sequence is similar to the ranking sequence given by the proposed algorithm.

The main observations based on these results are given as follows (i) The best alternative obtained by the proposed approach is the same as the best alternative given by Garg's method. This shows the reliability of the mentioned algorithm. (ii) In the process of evaluating attribute weights, the formulated GSF is not utilized. (iii) For several values of $\omega \in [0, 1]$, we can calculate different sets of attribute weights with respect to the same problem. However, the use of these different sets of attribute weights does not change the ranking order of alternatives.

Further, we provide some characteristic comparisons of our developed approach and commonly used methods, which are listed in Table 21. In addition to these, we also give the difference between the proposed algorithm and the existing methods.

- (i) Both subjective and objective forms of attribute weight information are used in the approaches presented in Table 21. However, in our proposed algorithm, we have objective information about the attribute weights, i.e., the weights are determined from the calculated collective decision matrix, which does not depend on personal decision opinion. However, the proposed solution procedure is entirely different from the steps adopted in the existing approaches.
- (ii) The MAGDM approach in [13] is based on a crisp TMG framework, while the proposed algorithm integrates an IVIF TMG framework with a new GSF for IVIFNs. Consequently, the proposed approach preserves the IVIF structure throughout the decision-making process, restricting scalarization to the final ranking stage and thereby retaining the inherent uncertainty and hesitation of the original assessments.
- (iii) Spearman's rank correlation coefficient is used to quantify the agreement between the ranking generated by the proposed approach and those obtained by the existing methods. The corresponding coefficients are 0.4, 0.8, 0.8, and 1.0, respectively, as reported in Table 21. These results indicate moderate agreement with the first method, strong agreement with the second and third methods, and perfect agreement with the fourth method. Thus, the proposed approach demonstrates a high degree of consistency with existing ranking methods, with complete agreement observed in the fourth case.

Table 21. Ranking of alternatives by different approaches.

Approach	Attribute weight information	Ranking
Park et al. [7]	Partially unknown	$T_3 > T_2 > T_1 > T_4$
Kumar and Kumar [13]	IVIF form	$T_3 > T_1 > T_2 > T_4$
Kumar et al. [25]	Completely unknown	$T_3 > T_1 > T_2 > T_4$
Garg [16]	Completely unknown	$T_1 > T_3 > T_2 > T_4$
Our proposed approach	Completely unknown	$T_1 > T_3 > T_2 > T_4$

7.2. Advantages of the proposed approach

To demonstrate the effectiveness of the proposed approach, its major methodological advantages over the existing approaches are discussed as follows.

- (1) **More reliable determination of attribute weights:** Both the proposed approach and the method of Kumar and Kumar [13] are based on the TMG model for determining attribute weights. However, they differ in the construction of the payoff matrix. In the proposed approach, the

collective IVIF decision matrix is directly used as the payoff matrix in the TMG framework. Thus, the interval-valued membership and nonmembership information is retained during the weight determination process. In contrast, Kumar and Kumar [13] first transformed the collective IVIF decision matrix into a crisp score matrix and then used it as the payoff matrix. Such early conversion may lead to the loss of valuable uncertainty and hesitation information. Therefore, the proposed approach provides a more information-preserving mechanism for determining the attribute weights.

- (2) **Preservation of the original IVIF information:** A major advantage of the proposed approach is that it preserves the original IVIF information throughout the major stages of the decision-making process. As shown in Step 4, the overall evaluation of each alternative is obtained in the form of an IVIFN rather than being immediately converted into a crisp value. This overall IVIFN retains the membership, nonmembership, and associated uncertainty information of the alternative. In contrast, the approach of Kumar and Kumar converts the corresponding evaluation into a crisp value at an earlier stage. Consequently, the proposed approach avoids premature scalarization and reduces the possibility of information loss, thereby providing a more faithful representation of the DMs' uncertain preferences.
- (3) **Simultaneous consideration of attribute and alternative weights:** The proposed approach simultaneously incorporates attribute weights and alternative weights in the calculation of the IIFES. The resulting IIFES represents the overall evaluation of each alternative by integrating both the relative importance of the attributes and the performance of the alternatives. Hence, the final evaluation is obtained by considering both aspects within a unified framework rather than treating them independently. This provides a more comprehensive measure of the overall value of each alternative and facilitates a meaningful comparison among the available alternatives.
- (4) **Avoidance of intermediate defuzzification:** Garg [16] proposed a method of determining attribute weights within the IVIF environment in which a defuzzification procedure is used. Although defuzzification facilitates numerical computation, transforming IVIF information into crisp values may result in the loss of interval-valued uncertainty and hesitation information. In contrast, the proposed approach determines the weights without requiring an intermediate defuzzification process. Therefore, the uncertainty embedded in the original IVIF assessments is retained during weight determination, providing an important methodological advantage over approaches based on early defuzzification.
- (5) **More information-consistent ranking of alternatives:** The characteristics above directly contribute to the final ranking results. In the proposed framework, IVIF information is retained during the construction of the collective decision matrix, normalization, weight determination, and calculation of the IIFES. Scalarization is performed only at the final ranking stage using the proposed GSF. Therefore, the ranking is obtained after incorporating a larger portion of the uncertainty information contained in the original IVIF assessments. The differences between the ranking obtained by the proposed method and those obtained by existing approaches can consequently be explained by their different mechanisms for processing and preserving uncertain information.
- (6) **Improved discrimination capability through the proposed GSF:** The proposed GSF is used

at the final stage to convert the IVIF expected scores into comparable scalar values. Since the GSF is applied only after the overall IVIF evaluations have been obtained, it can utilize the information retained during the preceding stages. Moreover, the proposed GSF provides an effective mechanism for distinguishing alternatives with close or identical conventional score values and can help resolve ranking ambiguities. Thus, the proposed GSF contributes to obtaining a clearer and more discriminative ranking of the alternatives.

- (7) **Justification of the different ranking results:** It should be emphasized that the superiority of the proposed approach is not claimed merely because it produces a ranking order different from those of existing methods. Rather, the difference arises from the distinct treatment of IVIF information throughout the decision-making process. The proposed approach avoids premature conversion of IVIF information into crisp values, directly utilizes the collective IVIF decision matrix in the TMG framework, simultaneously considers attribute and alternative weights, and postpones scalarization until the final ranking stage. Therefore, the obtained ranking is based on a comparatively richer representation of the original uncertain information.

Finally, the comparative analysis demonstrates that the principal advantage of the proposed approach lies not simply in producing a different ranking order, but in the information-preserving and integrated mechanism through which the ranking is obtained. By retaining IVIF information during the major computational stages, avoiding intermediate defuzzification, incorporating both attribute and alternative weights, and applying the proposed GSF only at the final stage, the proposed framework provides a theoretically sound and more informative basis for ranking sustainable air conditioning systems. These characteristics explain and substantiate why the ranking obtained by the proposed method can be considered more reliable and robust than rankings generated by approaches involving early scalarization or defuzzification.

7.3. Impact of ω

To derive alternative weights, we solve the maximin problem of Player 1 whose solution is equal to the optimal solution of Model (M1) while the optimal solution of Model (M2) equal to the minimax problem of Player 2 provides the attribute weights. These models are nonlinear in $\omega \in [0, 1]$. In the solution process of a real-event-based numerical example, a sensitivity analysis has been conducted for checking the impact of the parameter ω on the stability of the ranking order of the given alternatives. Notably, despite distinct overall expected scores for each alternative at ω values of 0.2, 0.5, and 0.8, the ranking order of available alternatives remains consistent and their corresponding optimal solutions for Players 1 and 2 are summarized in Table 18. This underscores the stability of ranking order concerning variations in ω .

8. Conclusions

8.1. Main outcomes of the present research

The following is the primary contribution that this study produced.

- (1) In this paper, we study the concept of MAGDM problem with undetermined attribute weights under an IVIFS framework. We address uncertainties in our analysis by leveraging the features

of the IVIFS. To tackle this, a novel GSF is constructed for IFNs that amalgamates all the degrees associated with a given IFS. The main advantage of this function lies in its flexibility, as it encompasses two parameters, λ_1 and λ_2 , which are interrelated through the relation $\lambda_1\lambda_2 = 1$. By adjusting the values of λ_1 within the interval $(0, 1)$, DMs can enhance the effectiveness of their decision-making process. Furthermore, the construction of this GSF is extended to IVIFNs and provides a concise exploration of its basic properties.

- (2) A series of numerical examples (Examples 3.1–3.7) has been put forward to highlight the faults of existing SFs when applied to IFNs and IVIFNs. These functions demonstrate an inability to effectively quantify imprecise information. Moreover, the effectiveness of the newly formulated GSFs is demonstrated through examples (Examples 3.8–3.18). As a result, these proposed functions can be efficiently utilized for assessing information from two or more IFSs and IVIFSs in the decision-making process.
- (3) The algorithm for solving the MAGDM problem outlined above involves several key steps. This algorithm is based on an IVIF TMG model and the proposed GSF. An IVIF expected SF is defined to assess the overall value of each alternative in the form of an IVIFN. The benefits and limitations of this algorithm have also been provided. Additionally, the algorithm includes an analysis of the parameter-‘ ω ’ and its impact on the ranking results (Tables 18–20).
- (4) The proposed algorithm offers two main advantages: (i) It utilizes the Li’s game model to calculate the attribute and alternative weights, ensuring a reliable determination of these weights (Remark 4.1). (ii) It incorporates a novel GSF in the last step, preserving the original fuzzy information while enhancing the consistency of the ranking results by eliminating irregularities.
- (5) The application of the suggested MAGDM algorithm is examined through a numerical example. Furthermore, a comparative analysis with a number of previous studies is carried out to demonstrate the feasibility and reliability of the suggested approach.

8.2. Limitations and the future research directions

Although the present study provides an effective methodology for handling MAGDM problems in the context of IVIFSs, it still has certain limitations that will be addressed in future research.

- (1) In this paper, we used Models (M1) and (M2) to compute weight-related outcomes. However, it is essential to note that these models yield Nash equilibrium strategy results, except when $\Omega_{ij} = 1$, $\Theta_{ij} = 1$, $\vartheta_{ij} = 0$, and/or $\varrho_{ij} = 0$. Under these circumstances, the method is ineffective in determining the weight distribution.
- (2) For calculating the weighting results, we select a particular value of $\omega \in [0, 1]$ in Models (M1) and (M2). However, this paper does not provide an insight which shows that the value of parameter ω has no influence on the stability of the ranking order of the alternatives.
- (3) Even though the current MAGDM is built with IVIFS decision information, the information used in real-world DMPs includes heterogeneous information like various fuzzy number types, intervals, random numbers, and linguistic IVIF numbers. Hence, an area for future research is the extension of the current approach to address heterogeneous MAGDM challenges.

- (4) In the future, the aspiration of the authors is to extend the present methodology for intuitionistic-based interval Type-2 and picture fuzzy environments [78]. We also plan to broaden our research by defining additional generalized information measures for ranking various alternatives.

Author contributions

Sandeep Kumar: Conceptualization, Investigation, Methodology, Validation, Writing–original draft, Writing–review and editing, Supervision; Reshu Tyagi: Conceptualization, Investigation, Methodology, Validation, Writing–original draft, Writing–review and editing; Saiful R Mondal: Conceptualization, Investigation, Methodology, Validation, Writing–original draft, Writing–review and editing, Funding.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. T. Li, L. Zhang, Z. Zhang, The consistency and consensus analysis for group decision-making with incomplete linguistic interval-valued intuitionistic fuzzy preference relations, *Appl. Intell.*, **53** (2023), 23500–23521. <https://doi.org/10.1007/s10489-023-04605-5>
2. Y. Qin, Q. Qi, P. Shi, P. J. Scott, X. Jiang, A novel weighted averaging operator of linguistic interval-valued intuitionistic fuzzy numbers for cognitively inspired decision-making, *Cogn. Comput.*, **15** (2023), 2101–2119. <https://doi.org/10.1007/s12559-023-10167-y>
3. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96. [https://doi.org/10.1016/S0165-0114\(86\)80034-3](https://doi.org/10.1016/S0165-0114(86)80034-3)
4. K. Atanassov, G. Gargov, Interval-valued intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **31** (1989), 343–349. [https://doi.org/10.1016/0165-0114\(89\)90205-4](https://doi.org/10.1016/0165-0114(89)90205-4)
5. S.-J. Chen, C.-L. Hwang, *Fuzzy multiple attribute decision making: methods and applications*, Heidelberg: Springer, 1992. <https://doi.org/10.1007/978-3-642-46768-4>
6. Y.-W. Chen, M. Larbani, Two-person zero-sum game approach for fuzzy multiple attribute decision making problems, *Fuzzy Set. Syst.*, **157** (2006), 34–51. <https://doi.org/10.1016/j.fss.2005.06.004>

7. J. H. Park, I. Y. Park, Y. C. Kwun, X. Tan, Extension of the topsis method for decision making problems under interval-valued intuitionistic fuzzy environment, *Appl. Math. Model.*, **35** (2011), 2544–2556. <https://doi.org/10.1016/j.apm.2010.11.025>
8. A. Al-Barakati, P. Rani, Assessment of healthcare waste treatment methods using an interval-valued intuitionistic fuzzy double normalization-based multiple aggregation approach, *Environ. Dev. Sustain.*, **26** (2024), 19397–19424. <https://doi.org/10.1007/s10668-023-03154-8>
9. L. Bian, C. Che, Enhanced combined techniques for interval-valued intuitionistic fuzzy multiple-attribute group decision-making and applications to quality evaluation of large-scale multi-view 3d reconstruction, *IEEE Access*, **11** (2023), 120502–120515. <https://doi.org/10.1109/ACCESS.2023.3327310>
10. N. Chai, W. Zhou, Z. Jiang, Sustainable supplier selection using an intuitionistic and interval-valued fuzzy mcdm approach based on cumulative prospect theory, *Inform. Sciences*, **626** (2023), 710–737. <https://doi.org/10.1016/j.ins.2023.01.070>
11. J. Xu, J. Y. Dong, S. P. Wan, J. Gao, Multiple attribute decision making with triangular intuitionistic fuzzy numbers based on zero-sum game approach, *Iran. J. Fuzzy Syst.*, **16** (2019), 97–112. <https://doi.org/10.22111/ijfs.2019.4648>
12. A. Singh, A. Gupta, A. Mehra, Matrix games with 2-tuple linguistic information, *Ann. Oper. Res.*, **287** (2020), 895–910. <https://doi.org/10.1007/s10479-018-2810-6>
13. S. Kumar, M. Kumar, A game theoretic approach to solve multiple group decision making problems with interval-valued intuitionistic fuzzy decision matrices, *Int. J. Manag. Sci. Eng. Manag.*, **16** (2021), 34–42. <https://doi.org/10.1080/17509653.2020.1852125>
14. Z. Wang, K. W. Li, J. Xu, A mathematical programming approach to multi-attribute decision making with interval-valued intuitionistic fuzzy assessment information, *Expert Syst. Appl.*, **38** (2011), 12462–12469. <https://doi.org/10.1016/j.eswa.2011.04.027>
15. S.-P. Wan, D.-F. Li, Fuzzy mathematical programming approach to heterogeneous multiattribute decision-making with interval-valued intuitionistic fuzzy truth degrees, *Inform. Sciences*, **325** (2015), 484–503. <https://doi.org/10.1016/j.ins.2015.07.014>
16. H. Garg, A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems, *Appl. Soft Comput.*, **38** (2016), 988–999. <https://doi.org/10.1016/j.asoc.2015.10.040>
17. Y. Yang, Z.-S. Chen, Y.-L. Li, H.-X. Lv, Commentary on “a new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems”, *Appl. Soft Comput.*, **49** (2016), 611–615. <https://doi.org/10.1016/j.asoc.2016.08.050>
18. Z.-Y. Bai, An interval-valued intuitionistic fuzzy topsis method based on an improved score function, *Sci. World J.*, **2013** (2013), 879089. <https://doi.org/10.1155/2013/879089>
19. S.-M. Chen, Z.-C. Huang, Multiattribute decision making based on interval-valued intuitionistic fuzzy values and linear programming methodology, *Inform. Sciences*, **381** (2017), 341–351. <https://doi.org/10.1016/j.ins.2016.11.010>
20. C. Y. Wang, S. M. Chen, Multiple attribute decision-making based on interval-valued intuitionistic fuzzy sets, linear programming methodology, and the extended topsis method, *Inform. Sciences*, **397–398** (2017), 155–167. <https://doi.org/10.1016/j.ins.2017.02.045>

21. C.-Y. Wang, S.-M. Chen, An improved multiattribute decision making method based on new score function of interval-valued intuitionistic fuzzy values and linear programming methodology, *Inform. Sciences*, **411** (2017), 176–184. <https://doi.org/10.1016/j.ins.2017.05.022>
22. K. Kumar, H. Garg, Topsis method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment, *Comp. Appl. Math.*, **37** (2018), 1319–1329. <https://doi.org/10.1007/s40314-016-0402-0>
23. V. L. G. Nayagam, S. Jeevaraj, P. Dhanasekaran, An intuitionistic fuzzy multi-criteria decision-making method based on non-hesitance score for interval-valued intuitionistic fuzzy sets, *Soft Comput.*, **21** (2017), 7077–7082. <https://doi.org/10.1007/s00500-016-2249-0>
24. J. Selvaraj, A. Majumdar, A new ranking method for interval-valued intuitionistic fuzzy numbers and its application in multi-criteria decision-making, *Mathematics*, **9** (2021), 2647. <https://doi.org/10.3390/math9212647>
25. S. Kumar, S. Rani, M. Kumar, A new order function for interval-valued intuitionistic fuzzy numbers and its application in group decision making, *Fuzzy Inform. Eng.*, **13** (2021), 111–126. <https://doi.org/10.1080/16168658.2021.1936961>
26. S.-M. Chen, C.-A. Tsai, Multiattribute decision making using novel score function of interval-valued intuitionistic fuzzy values and the means and the variances of score matrices, *Inform. Sciences*, **577** (2021), 748–768. <https://doi.org/10.1016/j.ins.2021.07.055>
27. Z. Wang, F. Xiao, W. Ding, Interval-valued intuitionistic fuzzy jenson-shannon divergence and its application in multi-attribute decision making, *Appl. Intell.*, **52** (2022), 16168–16184. <https://doi.org/10.1007/s10489-022-03347-0>
28. S.-M. Chen, M.-H. Hsu, Multiple attribute decision making based on novel score function of interval-valued intuitionistic fuzzy values, score matrix, and nonlinear programming model, *Inform. Sciences*, **645** (2023), 119332. <https://doi.org/10.1016/j.ins.2023.119332>
29. J. Jana, S. K. Roy, Linguistic pythagorean hesitant fuzzy matrix game and its application in multi-criteria decision making, *Appl. Intell.*, **53** (2023), 1–22. <https://doi.org/10.1007/s10489-022-03442-2>
30. Y. Wu, M. Chen, M. Chadli, Zero-sum-game-based distributed fuzzy adaptive self-triggered control of swarm uavs under intermittent communication and dos attacks, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 5371–5384. <https://doi.org/10.1109/TFUZZ.2024.3423709>
31. S. Jiafu, D. Wang, B. Xu, F. Zhang, N. Zhang, An interval-valued intuitionistic fuzzy group decision-making method for evaluating online knowledge payment products, *Appl. Soft Comput.*, **150** (2024), 111046. <https://doi.org/10.1016/j.asoc.2023.111046>
32. S. S. Nirmalsingh, P. J. Robinson, Solving madm problems based on matrix games with a new defuzzification method for triangular intuitionistic fuzzy numbers, *International Journal for Research in Applied Science and Engineering Technology*, **12** (2024), 971–979. <https://doi.org/10.22214/ijraset.2024.64250>
33. Y. Deng, W. Wu, S. Tong, Model-free policy iteration optimal control of fuzzy systems via a two-player zero-sum game, *Int. J. Syst. Sci.*, **56** (2025), 3958–3970. <https://doi.org/10.1080/00207721.2025.2480192>
34. Y. Zeng, Z. A. Hussein, M. H. Chyad, A. Farhadi, J. Yu, H. Rahbarimagham, Integrating type-2 fuzzy logic controllers with digital twin and neural networks for advanced hydropower system management, *Sci. Rep.*, **15** (2025), 5140. <https://doi.org/10.1038/s41598-025-89866-5>

35. S. Kumar, H. Garg, A novel two-level fuzzy set theoretic approach to solve multi-objective matrix games and its applications, *Ann. Oper. Res.*, in press. <https://doi.org/10.1007/s10479-025-06524-9>
36. X. Xie, S. Sathasivam, H. Ma, An interval-valued intuitionistic fuzzy group decision-making approach based on projection measurement extended vikor for patent quality evaluation, *PLoS One*, **20** (2025), e0323019. <https://doi.org/10.1371/journal.pone.0323019>
37. X. Xie, S. Sathasivam, Research on the interval-valued intuitionistic fuzzy multi-attribute decision-making model based on topsis extended by projection measure, *International Journal of Information Technology & Decision Making*, **25** (2026), 735–763. <https://doi.org/10.1142/S021962202550066>
38. X. Wu, Q. Liu, L. Liu, M.-S. Yang, X. Zhang, New jensen–shannon divergence measures for intuitionistic fuzzy sets with the construction of a parametric intuitionistic fuzzy topsis, *Complex Intell. Syst.*, **11** (2025), 134. <https://doi.org/10.1007/s40747-024-01761-0>
39. Ö. N. Bilişik, N. H. Duman, E. Taş, A novel interval-valued intuitionistic fuzzy critic-topsis methodology: an application for transportation mode selection problem for a glass production company, *Expert Syst. Appl.*, **235** (2024), 121134. <https://doi.org/10.1016/j.eswa.2023.121134>
40. Y. Qin, S. R. M. Hashim, J. Sulaiman, A new distance measure and corresponding topsis method for interval-valued intuitionistic fuzzy sets in multi-attribute decision-making, *AIMS Math.*, **8** (2023), 26459–26483. <https://doi.org/10.3934/math.20231351>
41. M. Arshi, A. Hadi-Vencheh, A. Aazami, T. Hamlehvar, The multiple attribute group decision-making problems with interval-valued intuitionistic fuzzy numbers: a linear programming approach, *Journal of Optimization in Industrial Engineering*, **18** (2025), 151–161. <https://doi.org/10.71720/joie.2025.1196756>
42. G.-F. Yu, D.-F. Li, A novel intuitionistic fuzzy goal programming method for heterogeneous madm with application to regional green manufacturing level evaluation under multi-source information, *Comput. Ind. Eng.*, **174** (2022), 108796. <https://doi.org/10.1016/j.cie.2022.108796>
43. W. Wang, X. Liu, An extended linmap method for multi-attribute group decision making under interval-valued intuitionistic fuzzy environment, *Procedia Computer Science*, **17** (2013), 490–497. <https://doi.org/10.1016/j.procs.2013.05.063>
44. D.-F. Li, G.-H. Chen, Z.-G. Huang, Linear programming method for multiattribute group decision making using if sets, *Inform. Sciences*, **180** (2010), 1591–1609. <https://doi.org/10.1016/j.ins.2010.01.017>
45. S. Kumar, R. Tyagi, A novel score function for picture fuzzy numbers and its based entropy method to multiple attribute decision-making, In: *Soft computing for problem solving*, Singapore: Springer, 2023, 719–729. https://doi.org/10.1007/978-981-19-6525-8_55
46. J. Zheng, M. Dong, Interval-valued intuitionistic fuzzy multi-attribute decision-making based on entropy and bidirectional projection, *Int. J. Comput. Intell. Syst.*, **18** (2025), 39. <https://doi.org/10.1007/s44196-025-00763-1>
47. J. Wang, Q. Cai, G. Wei, N. Liao, An extended edas approach based on cumulative prospect theory for multiple attributes group decision making with interval-valued intuitionistic fuzzy information, *Informatica*, **35** (2024), 421–452. <https://doi.org/10.15388/24-INFOR547>
48. C. Suo, X. Li, Y. Li, Distance-based knowledge measure and entropy for interval-valued intuitionistic fuzzy sets, *Mathematics*, **11** (2023), 3468. <https://doi.org/10.3390/math11163468>

49. C. Yue, R. Huang, D. Towey, Z. Xian, G. Wu, An entropy-based group decision-making approach for software quality evaluation, *Expert Syst. Appl.*, **238** (2024), 121979. <https://doi.org/10.1016/j.eswa.2023.121979>
50. S. Fu, Y.-Z. Xiao, H.-J. Zhou, Interval-valued intuitionistic fuzzy multi-attribute group decision-making method considering risk preference of decision-makers and its application, *Sci. Rep.*, **12** (2022), 11597. <https://doi.org/10.1038/s41598-022-15815-1>
51. A. Ohlan, Novel entropy and distance measures for interval-valued intuitionistic fuzzy sets with application in multi-criteria group decision-making, *Int. J. Gen. Syst.*, **51** (2022), 413–440. <https://doi.org/10.1080/03081079.2022.2036138>
52. S. Kumar, R. Tyagi, A vikor approach based on connection numbers for linguistic intuitionistic fuzzy decision-making: applications to coal mine safety and cyclone disaster management, *Transactions on Fuzzy Sets and Systems*, **6** (2027), 138–156. <https://doi.org/10.71602/tfss.2027.1214680>
53. C. Yue, Picture fuzzy normalized projection and extended vikor approach to software reliability assessment, *Appl. Soft Comput.*, **88** (2020), 106056. <https://doi.org/10.1016/j.asoc.2019.106056>
54. R. Baki, B. Ecer, A. Aktas, A decision framework for supplier selection in digital supply chains of e-commerce platforms using interval-valued intuitionistic fuzzy vikor methodology, *J. Theor. Appl. Electron. Commer. Res.*, **20** (2025), 23. <https://doi.org/10.3390/jtaer20010023>
55. Y. Huang, M. Tang, Enhanced exptodim framework based on vikor for interval-valued intuitionistic fuzzy multiple-attribute group decision-making and applications to for college english blended teaching quality evaluation, *Soft Comput.*, **29** (2025), 4559–4570. <https://doi.org/10.1007/s00500-025-10778-7>
56. F. Dammak, L. Baccour, A. M. Alimi, A new ranking method for topsis and vikor under interval valued intuitionistic fuzzy sets and possibility measures, *Journal of Intelligent & Fuzzy Systems*, **38** (2020), 4459–4469. <https://doi.org/10.3233/JIFS-191223>
57. X. Xie, S. Sathasivam, H. Ma, Z. Yue, A novel group decision-making method based on extended vikor and interval-valued intuitionistic fuzzy, *IEEE Access*, **12** (2024), 153330–153345. <https://doi.org/10.1109/ACCESS.2024.3471871>
58. H. A. Dağıstanlı, An interval-valued intuitionistic fuzzy VIKOR approach for R&D project selection in defense industry investment decisions, *Journal of Soft Computing and Decision Analytics*, **2** (2024), 1–13. <https://doi.org/10.31181/jsda21202428>
59. S. Kumar, S. R. Mondal, R. Tyagi, A novel p-norm-based ranking algorithm for multiple-attribute decision making using interval-valued intuitionistic fuzzy sets and its applications, *Axioms*, **14** (2025), 722. <https://doi.org/10.3390/axioms14100722>
60. S. Kumar, H. Garg, R. Tyagi, Two decision-making algorithms for interval-valued intuitionistic fuzzy information with generalized score function and its application to prioritizing sustainable cryptocurrencies, *Applied and Computational Mathematics*, **25** (1) (2026), 139–169. <https://doi.org/10.30546/1683-6154.25.1.2026.139>
61. S. Kumar, R. Tyagi, Set pair analysis based algorithm for ivif decision-making with mean and variance, In: *Optimization techniques and applications*, Singapore: Springer, 2026, 199–211. https://doi.org/10.1007/978-981-95-5044-9_15

62. S.-M. Chen, C.-Y. Hsieh, X.-Y. Zou, Multiattribute decision making system based on interval-valued intuitionistic fuzzy weights of attributes, *Int. J. Pattern Recogn. Artif. Intell.*, **40** (2026), 2659002. <https://doi.org/10.1142/S0218001426590020>
63. K. Patra, S. Sen, S. K. Mondal, An improved similarity measure for interval-valued intuitionistic fuzzy numbers and its application to multi-attribute decision-making problem, *Mathematics*, **14** (2026), 374. <https://doi.org/10.3390/math14020374>
64. Z. S. Xu, Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision-making, *Control and Decision*, **22** (2007), 215–219.
65. G. W. Wei, X. R. Wang, Some geometric aggregation operators on interval-valued intuitionistic fuzzy sets and their application to group decision-making, In: *2007 International conference on computational intelligence and security (CIS 2007)*, Harbin, China, 2007, 495–499. <https://doi.org/10.1109/CIS.2007.84>
66. R. Yao, H. Guo, Multiattribute group decision making based on novel score function considering both the angle and fuzzy information under interval-valued intuitionistic fuzzy environment, *Inform. Sciences*, **644** (2023), 119233. <https://doi.org/10.1016/j.ins.2023.119233>
67. D.-F. Li, Mathematical-programming approach to matrix games with payoffs represented by atanassov's interval-valued intuitionistic fuzzy sets, *IEEE Trans. Fuzzy Syst.*, **18** (2010), 1112–1128. <https://doi.org/10.1109/TFUZZ.2010.2065812>
68. Z. S. Xu, Intuitionistic fuzzy aggregation operators, *IEEE Trans. Fuzzy Syst.*, **15** (2007), 1179–1187. <https://doi.org/10.1109/TFUZZ.2006.890678>
69. S.-M. Chen, J.-M. Tan, Handling multicriteria fuzzy decision-making problems based on vague set theory, *Fuzzy Set. Syst.*, **67** (1994), 163–172. [https://doi.org/10.1016/0165-0114\(94\)90084-1](https://doi.org/10.1016/0165-0114(94)90084-1)
70. D. H. Hong, C.-H. Choi, Multicriteria fuzzy decision-making problems based on vague set theory, *Fuzzy Set. Syst.*, **114** (2000), 103–113. [https://doi.org/10.1016/S0165-0114\(98\)00271-1](https://doi.org/10.1016/S0165-0114(98)00271-1)
71. J. H. Park, K. M. Lim, J. Park, Y. C. Kwun, Distances between interval-valued intuitionistic fuzzy sets, *J. Phys.: Conf. Ser.*, **96** (2008), 012089. <https://doi.org/10.1088/1742-6596/96/1/012089>
72. Z. S. Xu, J. Chen, On geometric aggregation over interval-valued intuitionistic fuzzy information, In: *Fourth international conference on fuzzy systems and knowledge discovery (FSKD 2007)*, Haikou, China, 2007, 466–471. <https://doi.org/10.1109/FSKD.2007.427>
73. J. Ye, Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment, *Expert Syst. Appl.*, **36** (2009), 6899–6902. <https://doi.org/10.1016/j.eswa.2008.08.042>
74. V. L. G. Nayagam, S. Muralikrishnan, G. Sivaraman, Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets, *Expert Syst. Appl.*, **38** (2011), 1464–1467. <https://doi.org/10.1016/j.eswa.2010.07.055>
75. R. D. Luce, H. Raiffa, *Games and decisions: introduction and critical survey*, New York: John Wiley & Sons, 1957.
76. Z. Xu, An overview of methods for determining owa weights, *Int. J. Intell. Syst.*, **20** (2005) 843–865. <https://doi.org/10.1002/int.20097>
77. R. R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision-making, *IEEE Trans. Syst. Man Cyber.*, **18** (1988), 183–190. <https://doi.org/10.1109/21.87068>

78. S. Cherif, N. Baklouti, H. Hagra, A. M. Alimi, Novel intuitionistic-based interval type-2 fuzzy similarity measures with application to clustering, *IEEE Trans. Fuzzy Syst.*, **30** (2022), 1260–1271. <https://doi.org/10.1109/TFUZZ.2021.3057697>

Appendix A

Proof. For a fixed IFN $\alpha = \langle \xi, \zeta \rangle$, the proposed GSF S_{α}^{new} is a function of λ_1 only. Suppose that $S_{\alpha}^{new} = f(\lambda_1)$.

That is,

$$f(\lambda_1) = (\xi - \zeta) - (\xi^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi - \zeta),$$

where $\lambda_2 (\neq \lambda_1)$ such that $\lambda_1 \lambda_2 = 1$.

For $0 < \lambda_1 \leq \lambda'_1 < 1$, we have

$$\xi^{\lambda_1} \geq \xi^{\lambda'_1}.$$

For a given λ_1 , there is a number λ_2 such that $\lambda_2 = \frac{1}{\lambda_1} (\neq \lambda_1)$. Similarly, there is a number λ'_2 such that $\lambda'_2 = \frac{1}{\lambda'_1} (\neq \lambda'_1)$. Therefore the inequality $\lambda_1 \leq \lambda'_1$ is transformed into the form $\lambda'_2 \leq \lambda_2$.

Now, we have

$$\zeta^{\lambda_2} \leq \zeta^{\lambda'_2}, \text{ if } \lambda'_2 \leq \lambda_2.$$

For $\lambda_1 \leq \lambda'_1$ and for a given α , we get

$$\xi^{\lambda_1} \geq \xi^{\lambda'_1}, -\zeta^{\lambda_2} \geq -\zeta^{\lambda'_2}. \quad (8.1)$$

The inequalities in Eq (8.1) can be combined as

$$\xi^{\lambda_1} - \zeta^{\lambda_2} \geq \xi^{\lambda'_1} - \zeta^{\lambda'_2}.$$

This gives

$$(\xi - \zeta) - (\xi^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi - \zeta) \leq (\xi - \zeta) - (\xi^{\lambda'_1} - \zeta^{\lambda'_2})(1 - \xi - \zeta).$$

That is,

$$f(\lambda_1) \leq f(\lambda'_1), \quad \lambda_1 \leq \lambda'_1.$$

It shows that the proposed GSF S_{α}^{new} is a monotonically increasing function with respect to $\lambda_1 \in (0, 1)$. $\square$

Appendix B

Proof. For two IFNs, we have $\xi_1 < \xi_2$ and $\zeta_1 > \zeta_2$, which implies that

$$\xi_1 - \zeta_1 < \xi_2 - \zeta_2. \quad (8.2)$$

For $0 < \lambda_1 < 1$, the following inequalities come out

$$\xi_1 \leq \xi_1^{\lambda_1} \text{ and } \zeta_1 \geq \zeta_1^{\lambda_2}, \quad (8.3)$$

$$\xi_2 \leq \xi_2^{\lambda_1} \text{ and } \zeta_2 \geq \zeta_2^{\lambda_2}, \quad (8.4)$$

where $\lambda_1 \lambda_2 = 1$.

From the inequalities given in (8.3) and (8.4), we get the following inequalities:

$$\xi_1 - \zeta_1 \leq \xi_1^{\lambda_1} - \zeta_1^{\lambda_2} \text{ and } \xi_2 - \zeta_2 \leq \xi_2^{\lambda_1} - \zeta_2^{\lambda_2}. \quad (8.5)$$

Now consider a set H_1 of the values of $S_{\alpha_1}^{new}$ for $0 < \lambda_1 < 1$, i.e.,

$$H_1 = \{(\xi_1 - \zeta_1) - (\xi_1^{\lambda_1} - \zeta_1^{\lambda_2})(1 - \xi_1 - \zeta_1) \mid 0 < \lambda_1 < 1\}.$$

Using (8.5), we get $-(\xi_1^{\lambda_1} - \zeta_1^{\lambda_2})(1 - \xi_1 - \zeta_1) \leq (\xi_1 - \zeta_1)$. This gives that every element of the set H_1 is less than or equal to the value $(\xi_1 - \zeta_1)$.

Similarly, every element of a set

$$H_2 = \{(\xi_2 - \zeta_2) - (\xi_2^{\lambda_1} - \zeta_2^{\lambda_2})(1 - \xi_2 - \zeta_2) \mid 0 < \lambda_1 < 1\},$$

is less than or equal to the value $(\xi_2 - \zeta_2)$.

By Theorem 3.1, the functions $S_{\alpha_1}^{new}$ and $S_{\alpha_2}^{new}$ are non-decreasing functions with respect to $\lambda_1 \in (0, 1)$. This result and Inequality (8.2) imply that any element of the set H_1 is less than to the corresponding element of set H_2 with respect to the value of λ_1 . It implies that $S_{\alpha_1}^{new} < S_{\alpha_2}^{new}$. $\square$

Appendix C

Proof. Let $\alpha_1 = \langle \xi_1, \zeta_1 \rangle$ and $\alpha_2 = \langle \xi_2, \zeta_2 \rangle$ be two IFNs where $\xi_1 \leq \xi_2$ and $\zeta_1 = \zeta_2 = \zeta$. For these IFNs, we assume that

$$S_{\alpha_1}^{new} = (\xi_1 - \zeta) - (\xi_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi_1 - \zeta) = f(\xi_1),$$

and

$$S_{\alpha_2}^{new} = (\xi_2 - \zeta) - (\xi_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi_2 - \zeta) = f(\xi_2),$$

where λ_1 and $\lambda_2 (\neq \lambda_1)$ are parameters with $\lambda_1 \lambda_2 = 1$.

The inequality $\xi_1 \leq \xi_2$ gives the result $\xi_1^{\lambda_1} \leq \xi_2^{\lambda_1}$, for $\lambda_1 \in (0, 1)$.

This gives

$$(\xi_1^{\lambda_1} - \zeta^{\lambda_2}) \leq (\xi_2^{\lambda_1} - \zeta^{\lambda_2}). \quad (8.6)$$

By utilizing Inequality $(1 - \xi_1 - \zeta) \geq (1 - \xi_2 - \zeta)$, the inequality (8.6) leads to the following result:

$$-(\xi_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi_1 - \zeta) \leq -(\xi_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi_2 - \zeta). \quad (8.7)$$

By using $\xi_1 - \zeta \leq \xi_2 - \zeta$, Eq (8.7) becomes

$$(\xi_1 - \zeta) - (\xi_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi_1 - \zeta) \leq (\xi_2 - \zeta) - (\xi_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi_2 - \zeta).$$

Therefore, $f(\xi_1) \leq f(\xi_2)$ whenever $\xi_1 \leq \xi_2$.

It shows that the proposed GSF S_{α}^{new} is a monotonically increasing function with respect to ξ .

In similar way, it can be proved that the proposed GSF S_{α}^{new} is a monotonically decreasing function with respect to ζ . $\square$

Appendix D

Proof. For an arbitrary IFN $\alpha = \langle \xi, \zeta \rangle$, the proposed GSF is given by

$$S_{\alpha}^{new} = (\xi - \zeta) - (\xi^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi - \zeta),$$

where $\lambda_1 \lambda_2 = 1$ and $\lambda_1 \in (0, 1)$.

Theorem 3.3 found that for a given value of ζ , the function S_{α}^{new} is a monotonically increasing with respect to ξ . Hence, for any $\zeta \in [0, 1]$, we have

$$0 \leq \xi \leq 1 \Rightarrow S_{\langle 0, \zeta \rangle}^{new} \leq S_{\alpha}^{new} \leq S_{\langle 1, \zeta \rangle}^{new}. \quad (8.8)$$

Furthermore, Theorem 3.3 also derived that for a given value of ξ , S_{α}^{new} is a monotonically decreasing function with respect to ζ . Hence

$$S_{\langle 1, \zeta \rangle}^{new} \leq S_{\langle 1, 0 \rangle}^{new}, \text{ if } 0 \leq \zeta \text{ and } S_{\langle 0, 1 \rangle}^{new} \leq S_{\langle 0, \zeta \rangle}^{new}, \text{ if } \zeta \leq 1. \quad (8.9)$$

Finally, the inequalities (8.8) and (8.9) give

$$S_{\langle 0, 1 \rangle}^{new} \leq S_{\langle 0, \zeta \rangle}^{new} \leq S_{\alpha}^{new} \leq S_{\langle 1, \zeta \rangle}^{new} \leq S_{\langle 1, 0 \rangle}^{new}.$$

It is notable that $S_{\langle 1, 0 \rangle}^{new} = 1$ and $S_{\langle 0, 1 \rangle}^{new} = -1$ by using the proposed GSF $S_{\alpha}^{new} = (\xi - \zeta) - (\xi^{\lambda_1} - \zeta^{\lambda_2})(1 - \xi - \zeta)$ where $\alpha = \langle \xi, \zeta \rangle$. Hence $-1 \leq S_{\alpha}^{new} \leq 1$. $\square$

Appendix E

Proof. For a fixed IVIFN $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$, the proposed GSF $S_{\tilde{\alpha}}^{new}$ is a function of λ_1 only. Suppose that $S_{\tilde{\alpha}}^{new} = f(\lambda_1)$.

That is, we have

$$f(\lambda_1) = \frac{1}{2} \left((\Omega - \vartheta) + (\Theta - \varrho) - (\Omega^{\lambda_1} - \vartheta^{\lambda_2})(1 - \Omega - \vartheta) - (\Theta^{\lambda_1} - \varrho^{\lambda_2})(1 - \Theta - \varrho) \right),$$

where $\lambda_2 (\neq \lambda_1)$ such that $\lambda_1 \lambda_2 = 1$.

For $0 < \lambda_1 \leq \lambda'_1 < 1$, we have

$$\Omega^{\lambda_1} \geq \Omega^{\lambda'_1}, \Theta^{\lambda_1} \geq \Theta^{\lambda'_1}.$$

For a given λ_1 there is a number λ_2 such that $\lambda_2 = \frac{1}{\lambda_1} (\neq \lambda_1)$. Similarly, there is a number λ'_2 such that

$\lambda'_2 = \frac{1}{\lambda'_1} (\neq \lambda'_1)$. Therefore, the inequality $\lambda_1 \leq \lambda'_1$ is transformed into the form $\lambda'_2 \leq \lambda_2$.

Now, we have

$$\vartheta^{\lambda_2} \leq \vartheta^{\lambda'_2}, \varrho^{\lambda_2} \leq \varrho^{\lambda'_2}, \text{ if } \lambda'_2 \leq \lambda_2.$$

For $\lambda_1 \leq \lambda'_1$ and for a given $\tilde{\alpha}$, we get

$$\Omega^{\lambda_1} - \vartheta^{\lambda_2} \geq \Omega^{\lambda'_1} - \vartheta^{\lambda'_2}, \Theta^{\lambda_1} - \varrho^{\lambda_2} \geq \Theta^{\lambda'_1} - \varrho^{\lambda'_2}. \quad (8.10)$$

Using the inequalities in (8.10), we obtain

$$-(\Omega^{\lambda_1} - \vartheta^{\lambda_2})(1 - \Omega - \vartheta) \leq -(\Omega^{\lambda'_1} - \vartheta^{\lambda'_2})(1 - \Omega - \vartheta), \quad (8.11)$$

$$-(\Theta^{\lambda_1} - \varrho^{\lambda_2})(1 - \Theta - \varrho) \leq -(\Theta^{\lambda'_1} - \varrho^{\lambda'_2})(1 - \Theta - \varrho), \quad (8.12)$$

if $\lambda_1 \leq \lambda'_1$ with $1 - \Omega - \vartheta > 0$ and $1 - \Theta - \varrho > 0$.

The results in (8.11) and (8.12) obtained the following inequality if $\lambda_1 \leq \lambda'_1$

$$\begin{aligned} & \frac{1}{2} \left((\Omega - \vartheta) + (\Theta - \varrho) - (\Omega^{\lambda_1} - \vartheta^{\lambda_2})(1 - \Omega - \vartheta) - (\Theta^{\lambda_1} - \varrho^{\lambda_2})(1 - \Theta - \varrho) \right) \\ & \leq \frac{1}{2} \left((\Omega - \vartheta) + (\Theta - \varrho) - (\Omega^{\lambda'_1} - \vartheta^{\lambda'_2})(1 - \Omega - \vartheta) - (\Theta^{\lambda'_1} - \varrho^{\lambda'_2})(1 - \Theta - \varrho) \right). \end{aligned}$$

This gives

$$f(\lambda_1) \leq f(\lambda'_1), \quad \lambda_1 \leq \lambda'_1.$$

It shows that the proposed GSF $S_{\tilde{\alpha}}^{new}$ is a monotonically increasing function with respect to $\lambda_1 \in (0, 1)$. $\square$

Appendix F

Proof. Let $\tilde{\alpha}_1 = \langle [\Omega_1, \Theta_1], [\vartheta_1, \varrho_1] \rangle$ and $\tilde{\alpha}_2 = \langle [\Omega_2, \Theta_2], [\vartheta_2, \varrho_2] \rangle$ be two IVIFNs where

$$\Omega_1 \leq \Omega_2, \Theta_1 \leq \Theta_2 \text{ and } \vartheta_1 = \varrho_1 = \vartheta_2 = \varrho_2 = \zeta.$$

For these IVIFNs, assuming that

$$\begin{aligned} S_{\tilde{\alpha}_1}^{new} &= \frac{1}{2} \left((\Omega_1 - \zeta) + (\Theta_1 - \zeta) - (\Omega_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \Omega_1 - \zeta) \right. \\ & \quad \left. - (\Theta_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \Theta_1 - \zeta) \right) = f(\Omega_1, \Theta_1). \end{aligned}$$

and

$$\begin{aligned} S_{\tilde{\alpha}_2}^{new} &= \frac{1}{2} \left((\Omega_2 - \zeta) + (\Theta_2 - \zeta) - (\Omega_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \Omega_2 - \zeta) \right. \\ & \quad \left. - (\Theta_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \Theta_2 - \zeta) \right) = f(\Omega_2, \Theta_2). \end{aligned}$$

where λ_1 and $\lambda_2 (\neq \lambda_1)$ are parameters with $\lambda_1 \lambda_2 = 1$.

The inequalities

$$\Omega_1 \leq \Omega_2, \Theta_1 \leq \Theta_2 \text{ imply } \Omega_1^{\lambda_1} \leq \Omega_2^{\lambda_1}, \Theta_1^{\lambda_1} \leq \Theta_2^{\lambda_1}, \text{ for } \lambda_1 \in (0, 1).$$

This gives

$$(\Omega_1^{\lambda_1} - \zeta^{\lambda_2}) \leq (\Omega_2^{\lambda_1} - \zeta^{\lambda_2}) \text{ and } (\Theta_1^{\lambda_1} - \zeta^{\lambda_2}) \leq (\Theta_2^{\lambda_1} - \zeta^{\lambda_2}).$$

Thus, we get

$$\begin{aligned} & -(\Omega_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \Omega_1 - \zeta) - (\Theta_1^{\lambda_1} - \zeta^{\lambda_2})(1 - \Theta_1 - \zeta) \\ & \leq -(\Omega_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \Omega_2 - \zeta) - (\Theta_2^{\lambda_1} - \zeta^{\lambda_2})(1 - \Theta_2 - \zeta). \end{aligned} \quad (8.13)$$

We also have

$$(\Omega_1 - \zeta) + (\Theta_1 - \zeta) \leq (\Omega_2 - \zeta) + (\Theta_2 - \zeta). \quad (8.14)$$

Adding (8.13) and (8.14), we get

$$f(\Omega_1, \Theta_1) \leq f(\Omega_2, \Theta_2) \text{ whenever } \Omega_1 \leq \Omega_2, \Theta_1 \leq \Theta_2.$$

It shows that the proposed GSF $S_{\tilde{\alpha}}^{new}$ is a monotonically increasing function with respect to Ω and Θ .

In similar way, it can be proved that the proposed GSF $S_{\tilde{\alpha}}^{new}$ is a monotonically decreasing function with respect to ϑ and ϱ . $\square$

Appendix G

Proof. For an arbitrary IVIFN $\tilde{\alpha} = \langle [\Omega, \Theta], [\vartheta, \varrho] \rangle$, the proposed GSF is given by

$$S_{\tilde{\alpha}}^{new} = \frac{1}{2} \left((\Omega - \vartheta) + (\Theta - \varrho) - (\Omega^{\lambda_1} - \vartheta^{\lambda_2})(1 - \Omega - \vartheta) - (\Theta^{\lambda_1} - \varrho^{\lambda_2})(1 - \Theta - \varrho) \right),$$

where $\lambda_1 \lambda_2 = 1$ and $\lambda_1 \in (0, 1)$.

By Theorem 3.7, for a given value of ϑ and ϱ , $S_{\tilde{\alpha}}^{new}$ is a monotonically increasing function with respect to Ω and Θ . Consequently, for $0 \leq \Omega \leq \Theta \leq 1$, we get

$$S_{\langle [0,0], [\vartheta, \varrho] \rangle}^{new} \leq S_{\tilde{\alpha}}^{new} \leq S_{\langle [1,1], [\vartheta, \varrho] \rangle}^{new}. \quad (8.15)$$

By Theorem 3.7, for a given value of Ω and Θ , $S_{\tilde{\alpha}}^{new}$ is a monotonically decreasing function with respect to ϑ and ϱ . Thus, we get

$$S_{\langle [1,1], [\vartheta, \varrho] \rangle}^{new} \leq S_{\langle [1,1], [0,0] \rangle}^{new}, \text{ if } 0 \leq \vartheta \leq \varrho \text{ and } S_{\langle [0,0], [1,1] \rangle}^{new} \leq S_{\langle [0,0], [\vartheta, \varrho] \rangle}^{new}, \text{ if } \vartheta \leq \varrho \leq 1. \quad (8.16)$$

Finally, the results of (8.15) and (8.16) lead to the following inequality:

$$S_{\langle [0,0], [1,1] \rangle}^{new} \leq S_{\langle [0,0], [\vartheta, \varrho] \rangle}^{new} \leq S_{\tilde{\alpha}}^{new} \leq S_{\langle [1,1], [\vartheta, \varrho] \rangle}^{new} \leq S_{\langle [1,1], [0,0] \rangle}^{new}.$$

Hence $-1 \leq S_{\tilde{\alpha}}^{new} \leq 1$. $\square$



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