



Research article

A theoretical development on information measures of n -polygonal interval-value fuzzy sets

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Abstract: n -polygonal interval-valued fuzzy sets are a very effective means for describing fuzzy information. How to handle the uncertain information is the key issue. In this paper, we generalized the recent results on the entropy and similarity measures of n -polygonal interval-valued fuzzy sets based on abstract functions that meet specific conditions. Our main contribution is two formulas that derive to create various new entropies and similarity measures by the use of given functions. Meanwhile, we utilized the similarity measure function to construct the cross-entropy function, which can be understood as a dual relationship from the mathematical structure. Besides, we also proved that entropy and similarity measures for n -polygonal interval-valued fuzzy sets can be transformed by each other in a general way. Finally, a multi-attribute decision-making (MADM) method in the light of the newly proposed entropy and cross-entropy were designed to solve practical problems. The experimental results demonstrate the effectiveness of the novel method in solving practical problems in decision-making.

Keywords: n -polygonal interval-valued fuzzy set; entropy; similarity measure; cross-entropy

Mathematics Subject Classification: 90B50, 91A35

1. Introduction

With the advancement of science and technology, unknown factors and uncertainty in decision-making progress continue to increase, and fuzzy sets are particularly important in handling uncertain information. Since the concept of fuzzy theory was proposed, the continuous deepening of research has revealed some practical limitations, resulting in a series of extension forms of fuzzy sets being derived, such as interval-valued fuzzy sets [1], intuitionistic fuzzy sets [2], and hesitant fuzzy sets [3], etc. With the application of fuzzy sets in different fields, it is not difficult to see that fuzzy sets are a mathematical method that can more accurately express the evaluation information made by decision-makers than a certain number. Therefore, many researchers use fuzzy sets to handle multi-attribute decision-making

problems in different fields [4–7].

Although fuzzy sets can fully represent the uncertainty of individual descriptions of a semantic concept, different individuals may have a different understanding of the same semantic concept. Therefore, it is difficult for fuzzy sets to qualitatively describe evaluation information in the decision-making process. To address this issue, Zadeh [8] proposed another extension form of fuzzy sets—type-2 fuzzy sets, which are characterized by including primary and secondary membership functions. Obviously, the enormous computational complexity of type-2 fuzzy sets limits their development in theory and application. Therefore, some researchers began to develop a new tool to describe fuzzy information more succinctly. The concept of interval type-2 fuzzy sets (IT2FSs) was proposed by Mendel [9]. Due to the fact that the membership degree of IT2FSs is represented in interval form, semantic uncertainty can be a better measure. Consequently, IT2FSs are widely applied in various fields [10–13]. Subsequently, the interval type-2 trapezoidal fuzzy set (IT2tFS) was defined, which not only has more computational advantages than the IT2FS, but is usually more suitable for describing the fuzzy number of decision information. This laid a solid foundation for other researchers to propose multi-attribute decision-making (MADM) methods. Jin and Zhao et al. [14] proposed logarithmic information entropy, logarithmic similarity, and logarithmic cross-entropy on IT2tFSs in view of the logarithmic function to measure the fuzziness and similarity of decision information. A new MADM method with the weighted generalized Heronian mean operator of IT2tFSs was established by Dehghani et al. [15]. Besides, Pan [16] provided the entropy of IT2tFSs and constructed a new MADM method based on entropy and regret theory.

It is worth noting that the MADM method proposed according to the extended form of fuzzy sets for information measurement has gradually become the mainstream trend. Information measurement is an important research hotspot in the field of decision-making. A novel approach to construct similarity for intuitionistic fuzzy sets was proposed by Song et al. [17]. Guan and Liu [18] developed comprehensive hesitant fuzzy entropy based on the two features of fuzziness and unclarity. Some researchers are also dedicated to studying the information measures of interval-valued fuzzy sets. Xie et al. [19] generalized recent results on the entropy of interval-valued fuzzy sets based on the intuitionistic distance and its relationship with the similarity measure. Based on aggregation operators, measures of embedding and similarity measures for interval-valued fuzzy sets were proposed in [20, 21]. As is well known, interval-valued fuzzy sets are composed of two fuzzy sets, where the complexity of their operations can be imagined. Therefore, in 2021, Suo and Li [22] proposed the concept of n -polygonal interval-valued fuzzy sets (nPIVFSs), which not only reduce the complexity of traditional interval-valued fuzzy-set arithmetic operations, but also achieve the closure and linearity of operations. In addition, it was verified that fuzzy sets are more complete and delicate in characterizing fuzzy information compared to triangular interval-valued fuzzy sets by a specific case. Now, there are only a few studies constructing information measures on nPIVFSs, while most research focuses on constructing information measures for interval-valued fuzzy sets or triangular interval-valued fuzzy sets. Hence, it is essential to propose a novel method in accordance with nPIVFSs to solve MADM problems with less computational complexity and higher flexibility.

Compared with the existing studies, the following research gaps motivate this work.

(G1) Lack of information measures on nPIVFSs. Existing entropy, similarity, and cross-entropy models are developed for interval-valued fuzzy sets [19–21], triangular interval-valued fuzzy sets, and IT2FSs [13–16], whereas no information measure has been constructed for nPIVFSs, although

nPIVFSs enjoy closed and linear operations and can represent fuzzy information at an arbitrary resolution through the parameter n .

(G2) The absence of a unified generating mechanism. In most existing studies, the entropy and the similarity measure are designed separately and each takes a fixed concrete form. Consequently, it is unclear which axioms a candidate function must satisfy, how different measures are related, and whether a similarity measure can be converted into an entropy or a cross-entropy.

(G3) No tunable discrimination. The existing fixed-form measures contain no adjustable parameter, so their ability to separate close fuzzy evaluations cannot be calibrated by the decision-maker.

(G4) Computational burden of interval arithmetic. Building information measures on general interval-valued or interval type-2 fuzzy sets usually involves interval extension or α -cut-based computations, which are not closed and become costly as the problem scale grows.

The framework developed in this paper addresses G1–G4 simultaneously: It defines the first entropy, similarity, and cross-entropy on nPIVFSs (G1), generates all three measures from elementary functions f and g satisfying mild axioms and proves their mutual transformation (G2), introduces the adjustable parameter α (G3), and exploits the element-wise, closed, and linear operations of nPIVFSs (G4).

Based on the above analysis, the main contributions of this paper are summarized as follows:

(1) We develop the first axiomatic framework of entropy, the similarity measure, and cross-entropy for nPIVFSs. Any type-1 function f yields an entropy via Eq (10), and any type-2 function g yields a similarity measure via Eq (16).

(2) We prove that the three measures are mutually transformable, which shows that the entropy of an nPIVFS equals the cross-entropy between the set and its complement, thereby unifying the three concepts in one mathematical structure rather than treating them as independent definitions.

(3) The framework is flexible and tunable. Selecting $f = f_2$ or $g = g_3$ with $n = 1$ recovers the logarithmic entropy and logarithmic similarity measure of interval type-2 trapezoidal fuzzy sets proposed in [14], while the parameter α of f_4 endows the entropy with a controllable discrimination ability.

(4) We propose an entropy-weight-based MADM method with the proposed measures and validate it on the global supplier-selection case of [13]. The obtained ranking coincides with the state-of-the-art methods [13–16] while providing a larger, tunable separation among alternatives.

The rest of this paper is organized as follows: Section 2 introduces some concepts relevant to nPIVFSs, and provides intersection, union, complement, and number multiplication operations for nPIVFSs. In Section 3, the axiomatic definitions of three information measures are presented, followed by the construction of three information-measure functions, and it is verified that the newly proposed entropy is effective by some numerical examples. In Section 4, we explore the transformation relationship between information measures and provide a detailed proof. In Section 5, we propose a new method to deal with MADM problems with information measures, and demonstrate the practicality and rationality of the presented method by analyzing a case of an actual supplier selection and comparing it with existing methods. Finally, the entire manuscript is wrapped up in Section 6 with conclusions.

In order to ensure that the topic of this paper is adequately introduced to the reader, Table 1 explains all abbreviations and symbols.

Table 1. Symbol explanation.

Abbreviation	Explanation
IVFS	Interval-valued fuzzy set
n PIVFS	n -polygonal interval-valued fuzzy set
n PIVFSs	n -polygonal interval-valued fuzzy sets
n PIVFS(X)	The set of all n PIVFSs on X
Symbol	Explanation
$\tilde{A}^\xi$	ξ -component of n -polygonal interval-valued fuzzy set $\tilde{A}$
$H(\tilde{A}^L), H(\tilde{A}^U)$	Heights of lower and upper membership functions
ξ	Index for lower (L)/upper (U) membership components, $\xi \in \{L, U\}$
a_i^ξ	Coordinate points of the ξ -component for $\tilde{A}$
a_{2n+2}^ξ	Coordinate parameter corresponding to membership height
k_i^ξ, K^ξ	Coordinate and height parameters for the union operation
p_i^ξ, P^ξ	Coordinate and height parameters for the intersection operation
$E(\cdot)$	Entropy measure for n -polygonal interval-valued fuzzy sets
$S(\cdot, \cdot)$	Similarity measure between two n -polygonal interval-valued fuzzy sets
$C(\cdot, \cdot)$	Cross-entropy measure
$T(A_l)$	Closeness degree of alternative A_l
$C(n), D(n)$	Difference metrics for ranking discrimination analysis
α	Adjustable shape parameter for entropy function f_4
A	Set of alternatives in multi-attribute decision-making
C	Set of attributes in multi-attribute decision-making
w	Attribute weight vector
λ	Decision-maker weight vector
D^k	Individual decision matrix from the k -th decision-maker
D	Aggregated overall decision matrix
G^+	Positive ideal solution
G^-	Negative ideal solution
E_t	Aggregated entropy of the t -th attribute
w_t	Attribute weight calculated by the entropy weight method

2. Preliminaries

In this section, we briefly review some concepts and representations of n -polygonal interval-valued fuzzy sets.

Definition 2.1. ([22]) Let $\tilde{A}$ be an IVFS on X . If there are two $2n + 2$ -tuples ordered real numbers $(a_0^L, a_1^L, \dots, a_n^L, a_{n+1}^L, \dots, a_{2n}^L, a_{2n+1}^L)$ and $(a_0^U, a_1^U, \dots, a_n^U, a_{n+1}^U, \dots, a_{2n}^U, a_{2n+1}^U)$ ($a_i^\xi \in [0, 1], i = 0, 1, \dots, 2n + 1, \xi = L, U$), and two real numbers $H(\tilde{A}^L), H(\tilde{A}^U) \in [0, 1]$, such that the lower and upper membership functions of $\tilde{A}$ have the following forms, respectively, then we call $\tilde{A}$ an n -polygonal interval-valued fuzzy set (n PIVFS).

$$\widetilde{A}^{\xi}(x) = \begin{cases} \frac{(i-1)H(\widetilde{A}^{\xi})}{n} + \frac{H(\widetilde{A}^{\xi})(x - a_{i-1}^{\xi})}{n(a_i^{\xi} - a_{i-1}^{\xi})}, & x \in [a_{i-1}^{\xi}, a_i^{\xi}], i = 1, 2, \dots, n, \xi = L, U, \\ H(\widetilde{A}^{\xi}), & x \in [a_n^{\xi}, a_{n+1}^{\xi}], \xi = L, U, \\ \frac{(i-1)H(\widetilde{A}^{\xi})}{n} + \frac{H(\widetilde{A}^{\xi})(a_{i-1}^{\xi} - x)}{n(a_{i-1}^{\xi} - a_i^{\xi})}, & x \in [a_{i-1}^{\xi}, a_i^{\xi}], i = n+2, \dots, 2n+1, \xi = L, U, \\ 0, & \text{Otherwise.} \end{cases} \quad (2.1)$$

In this case, $\widetilde{A}$ can be written in the following form

$$\widetilde{A} = \left[\left(a_0^L, a_1^L, \dots, a_n^L, a_{n+1}^L, \dots, a_{2n}^L, a_{2n+1}^L; H(\widetilde{A}^L) \right), \right. \\ \left. \left(a_0^U, a_1^U, \dots, a_n^U, a_{n+1}^U, \dots, a_{2n}^U, a_{2n+1}^U; H(\widetilde{A}^U) \right) \right]. \quad (2.2)$$

Interestingly, $nPIVFS$ $\widetilde{A}$ degenerates into an interval-valued trapezoidal fuzzy set when $n = 1$. $nPIVFS$ s achieve the closure and linearity of operations, which effectively reduce the computational complexity of traditional interval-valued fuzzy sets.

For the convenience of later description, assume that $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ are the $nPIVFS$ s, they can be denoted as follows:

$$\widetilde{A} = \left[\left(a_0^L, a_1^L, \dots, a_n^L, a_{n+1}^L, \dots, a_{2n}^L, a_{2n+1}^L; H(\widetilde{A}^L) \right), \right. \\ \left. \left(a_0^U, a_1^U, \dots, a_n^U, a_{n+1}^U, \dots, a_{2n}^U, a_{2n+1}^U; H(\widetilde{A}^U) \right) \right], \quad (2.3)$$

$$\widetilde{B} = \left[\left(b_0^L, b_1^L, \dots, b_n^L, b_{n+1}^L, \dots, b_{2n}^L, b_{2n+1}^L; H(\widetilde{B}^L) \right), \right. \\ \left. \left(b_0^U, b_1^U, \dots, b_n^U, b_{n+1}^U, \dots, b_{2n}^U, b_{2n+1}^U; H(\widetilde{B}^U) \right) \right], \quad (2.4)$$

$$\widetilde{C} = \left[\left(c_0^L, c_1^L, \dots, c_n^L, c_{n+1}^L, \dots, c_{2n}^L, c_{2n+1}^L; H(\widetilde{C}^L) \right), \right. \\ \left. \left(c_0^U, c_1^U, \dots, c_n^U, c_{n+1}^U, \dots, c_{2n}^U, c_{2n+1}^U; H(\widetilde{C}^U) \right) \right]. \quad (2.5)$$

Definition 2.2. Let $\widetilde{A}, \widetilde{B}$ be $nPIVFS$ s and the complement of $\widetilde{A}$ be $\widetilde{A}^c$. Then the complement, union, and intersection of $\widetilde{A}$ and $\widetilde{B}$ can be expressed as

$$\widetilde{A}^c = \left[\left(a_0^{cL}, a_1^{cL}, \dots, a_n^{cL}, a_{n+1}^{cL}, \dots, a_{2n}^{cL}, a_{2n+1}^{cL}; H(\widetilde{A}^{cL}) \right), \right. \\ \left. \left(a_0^{cU}, a_1^{cU}, \dots, a_n^{cU}, a_{n+1}^{cU}, \dots, a_{2n}^{cU}, a_{2n+1}^{cU}; H(\widetilde{A}^{cU}) \right) \right], \quad (2.6)$$

$$\widetilde{A} \cup \widetilde{B} = \left[\left(k_0^L, k_1^L, \dots, k_n^L, k_{n+1}^L, \dots, k_{2n}^L, k_{2n+1}^L; K^L \right), \right. \\ \left. \left(k_0^U, k_1^U, \dots, k_n^U, k_{n+1}^U, \dots, k_{2n}^U, k_{2n+1}^U; K^U \right) \right], \quad (2.7)$$

$$\begin{aligned}\widetilde{A} \cap \widetilde{B} = & \left[(p_0^L, p_1^L, \dots, p_n^L, p_{n+1}^L, \dots, p_{2n}^L, p_{2n+1}^L; P^L), \right. \\ & \left. (p_0^U, p_1^U, \dots, p_n^U, p_{n+1}^U, \dots, p_{2n}^U, p_{2n+1}^U; P^U) \right],\end{aligned}\quad (2.8)$$

where $a_i^{c\xi} = 1 - a_{2n+1-i}^\xi$, $H(\widetilde{A}^{cU}) = 1 - H(\widetilde{A}^L)$, $H(\widetilde{A}^{cL}) = 1 - H(\widetilde{A}^U)$, $k_i^\xi = \max\{a_i^\xi, b_i^\xi\}$, $p_i^\xi = \min\{a_i^\xi, b_i^\xi\}$, $K^\xi = \max\{H(\widetilde{A}^\xi), H(\widetilde{B}^\xi)\}$, $P^\xi = \min\{H(\widetilde{A}^\xi), H(\widetilde{B}^\xi)\}$, $0 \leq a_i^\xi, b_i^\xi \leq 1$, $\xi = L, U$, and $i = 0, 1, 2, \dots, 2n+1$.

Definition 2.3. Assume that $\widetilde{A}$ is an $nPIVFS$ and $0 < k \leq 1$. Then multiplication of $\widetilde{A}$ and k can be defined as follows:

$$\begin{aligned}k\widetilde{A} = & \left[(ka_0^L, ka_1^L, \dots, ka_n^L, ka_{n+1}^L, \dots, ka_{2n}^L, ka_{2n+1}^L; H(\widetilde{A}^L)), \right. \\ & \left. (ka_0^U, ka_1^U, \dots, ka_n^U, ka_{n+1}^U, \dots, ka_{2n}^U, ka_{2n+1}^U; H(\widetilde{A}^U)) \right].\end{aligned}\quad (2.9)$$

As a matter of convenience, we utilize $nPIVFS(X)$ to denote the set of all the $nPIVFS$ s on X .

Several special $nPIVFS$ s are used frequently below, which are abbreviated as

$$\begin{aligned}\widetilde{A} &= [\underbrace{(0, 0, \dots, 0)}_{2n+2}; 0), \underbrace{(0, 0, \dots, 0)}_{2n+2}; 0] = \widetilde{0}, \\ \widetilde{A} &= [\underbrace{(1, 1, \dots, 1)}_{2n+2}; 1), \underbrace{(1, 1, \dots, 1)}_{2n+2}; 1] = \widetilde{1}, \\ \widetilde{A} &= [\underbrace{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})}_{2n+2}; \frac{1}{2}), \underbrace{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})}_{2n+2}; \frac{1}{2}] = \frac{\widetilde{1}}{2}.\end{aligned}$$

3. The functional expression of information measures for $nPIVFS$ s

In the third section, we mainly introduce the axiomatic definitions of information measures on $nPIVFS$ s, such as entropy, the similarity measure, and cross-entropy. In addition, the corresponding measure functions are also proposed.

3.1. Entropy for $nPIVFS$ s

First, we state the axiomatic definition of entropy for $nPIVFS$ s and construct an entropy function, followed by an investigation of the entropy formula for $nPIVFS$ s.

Definition 3.1. Let $\widetilde{A} \in nPIVFS(X)$, and then the function $E : nPIVFS(X) \rightarrow [0, 1]$ is called an entropy for $nPIVFS$ s if it satisfies the following four axiomatic conditions:

(E1) $E(\widetilde{A}) = 0$ iff $\widetilde{A} = \widetilde{0}$ or $\widetilde{A} = \widetilde{1}$;

(E2) $E(\widetilde{A}) = 1$ iff $\widetilde{A} = \frac{\widetilde{1}}{2}$;

(E3) $E(\widetilde{A}) = E(\widetilde{A}^c)$;

(E4) $E(\widetilde{A}) \leq E(\widetilde{B})$ if $\widetilde{A}$ is fuzzy less than $\widetilde{B}$, i.e., $a_i^\xi \leq b_i^\xi \leq \frac{1}{2}$, $H(\widetilde{A}^\xi) \leq \frac{1}{2}$ or $a_i^\xi \geq b_i^\xi \geq \frac{1}{2}$, $H(\widetilde{A}^\xi) \geq \frac{1}{2}$, $\xi = L, U$, $i = 0, 1, 2, \dots, 2n+1$.

Definition 3.2. The mapping $f : [0, 1] \rightarrow [0, 1]$ is a type-1 function, if it satisfies the following requirements:

- (i) $f(0) = f(1) = 0$;
- (ii) $f(\frac{1}{2}) = 1$;
- (iii) f monotonically increases on $[0, \frac{1}{2}]$ and monotonically decreases on $[\frac{1}{2}, 1]$.

Theorem 3.3. Let $f : [0, 1] \rightarrow [0, 1]$ be a type-1 function, and then for any $\tilde{A} \in nPIVFS(X)$,

$$E(\tilde{A}) = \frac{1}{4n+6} \sum_{\xi=L,U} \sum_{i=0}^{2n+2} \frac{f(1-a_i^\xi) + f(a_i^\xi)}{2} \quad (3.1)$$

is an entropy for $nPIVFS$.

Proof. Assuming that $\tilde{A}$ is defined by Eq (2.2), entropy $E(\tilde{A})$ should satisfy the four axiomatic conditions represented in Definition 3.1. Meanwhile, we use a_{2n+2}^ξ to denote $H(\tilde{A}^\xi)$ for convenience.

(E1) $E(\tilde{A}) = 0$ if and only if $f(1-a_i^\xi) + f(a_i^\xi) = 0$, that is, $f(1-a_i^\xi) = f(a_i^\xi) = 0$, then $a_i^\xi = 0$ or 1 for all $i = 0, 1, 2, \dots, 2n+2, \xi = L, U$. It follows that $\tilde{A} = \tilde{0}$ or $\tilde{A} = \tilde{1}$.

On the other hand, we can suppose the other way around that $\tilde{A} = \tilde{0}$ or $\tilde{A} = \tilde{1}$. Then we have $a_i^\xi = 0$ or 1 for all $i = 0, 1, 2, \dots, 2n+2, \xi = L, U$, which can conclude that $f(a_i^\xi) = 0$, thus, $E(\tilde{A}) = 0$.

(E2) Assume that $E(\tilde{A}) = 1$. Since $f(1-a_i^\xi) + f(a_i^\xi) = 2$ for all $i = 1, 2, \dots, 2n+2, \xi = L, U$, then,

$$\frac{f(1-a_i^\xi) + f(a_i^\xi)}{2} = 1,$$

that is,

$$f(1-a_i^\xi) + f(a_i^\xi) = 2.$$

According to Definition 3.2, we can obtain $a_i^\xi = \frac{1}{2}$ ($i = 0, 1, 2, \dots, 2n+2, \xi = L, U$). Therefore, we get $E(\tilde{A}) = 1$ iff $\tilde{A} = \tilde{\frac{1}{2}}$.

(E3) $E(\tilde{A}) = E(\tilde{A}^c)$ is clearly valid.

(E4) Supposed that $0 \leq a_i^\xi \leq b_i^\xi \leq \frac{1}{2}$, and then

$$\begin{aligned} & \frac{f(1-a_i^\xi) + f(a_i^\xi)}{2} - \frac{f(1-b_i^\xi) + f(b_i^\xi)}{2} \\ &= \frac{1}{2} \{ [f(1-a_i^\xi) - f(1-b_i^\xi)] + [f(a_i^\xi) - f(b_i^\xi)] \}, \end{aligned}$$

since $0 \leq a_i^\xi \leq b_i^\xi \leq \frac{1}{2}$. Then $1 \geq 1-a_i^\xi \geq 1-b_i^\xi \geq 0.5$. In addition, $f(x)$ is a monotonically increasing function for $x \in [0, 0.5]$ and is a monotonically decreasing function for $x \in [0.5, 1]$. Obviously, we can see that

$$f(1-a_i^\xi) - f(1-b_i^\xi) \leq 0, \quad f(a_i^\xi) - f(b_i^\xi) \leq 0,$$

and therefore,

$$\frac{f(1-a_i^\xi) + f(a_i^\xi)}{2} - \frac{f(1-b_i^\xi) + f(b_i^\xi)}{2} \leq 0.$$

It also follows that $E(\widetilde{A}) \leq E(\widetilde{B})$. By the same token, if $1 \geq a_i^\xi \geq b_i^\xi \geq \frac{1}{2}$, we can also speculate that $E(\widetilde{A}) \leq E(\widetilde{B})$.

We remark that by Theorem 3.3, one can construct different formulas to calculate the entropy on nPIVFSs using different functions $f : [0, 1] \rightarrow [0, 1]$ that satisfy conditions (i)–(iii) in Definition 3.2, such as

- (1) $f_1(x) = -4(x - \frac{1}{2})^2 + 1$;
- (2) $f_2(x) = -\frac{1}{\ln 2}[x \ln x + (1 - x) \ln(1 - x)]$;
- (3) $f_3(x) = \sin(\pi x)$;
- (4) $f_4(x) = 1 - |2x - 1|^\alpha, (\alpha \in (0, 1))$.

Different entropies can be constructed from different functions f . Let us see some examples.

Let $f : [0, 1] \rightarrow [0, 1]$ be given by $f(x) = f_1(x)$ for nPIVFSs, and then for any $\widetilde{A} \in nPIVFS(X)$,

$$E_1(\widetilde{A}) = \frac{1}{4n+6} \sum_{\xi=L,U} \left[\sum_{i=0}^{2n+1} \left(1 - 4(a_i^\xi - \frac{1}{2})^2 \right) + 1 - 4(H(\widetilde{A}^\xi) - \frac{1}{2})^2 \right] \quad (3.2)$$

is the entropy on $nPIVFS(X)$.

Suppose that $f : [0, 1] \rightarrow [0, 1]$ is given by $f(x) = f_2(x)$ for nPIVFSs, and then for any $\widetilde{A} \in nPIVFS(X)$,

$$E_2(\widetilde{A}) = -\frac{1}{(4n+6)\ln 2} \sum_{\xi=L,U} \left[\sum_{i=0}^{2n+1} \left(a_i^\xi \ln a_i^\xi + (1 - a_i^\xi) \ln(1 - a_i^\xi) \right) + H(\widetilde{A}^\xi) \ln H(\widetilde{A}^\xi) + (1 - H(\widetilde{A}^\xi)) \ln(1 - H(\widetilde{A}^\xi)) \right] \quad (3.3)$$

is the entropy on $nPIVFS(X)$.

Specifically, when $n = 1$ in Eq (2.2), that is, $\widetilde{A}$ is an interval-valued trapezoidal fuzzy set, then Eq (3.3) degenerates into the following form:

$$E_2(\widetilde{A})' = -\frac{1}{10\ln 2} \sum_{\xi=L,U} \left[\sum_{i=1}^4 \left(a_i^\xi \ln a_i^\xi + (1 - a_i^\xi) \ln(1 - a_i^\xi) \right) + H(\widetilde{A}^\xi) \ln H(\widetilde{A}^\xi) + (1 - H(\widetilde{A}^\xi)) \ln(1 - H(\widetilde{A}^\xi)) \right]. \quad (3.4)$$

It is easy to see that the aforementioned entropy of Eq (3.4) is introduced in Ref. [14]. In other words, the logarithmic entropy proposed in Ref. [14] is a special form of entropy that we develop in Eq (3.1).

Assume that $f : [0, 1] \rightarrow [0, 1]$ is introduced by $f(x) = f_3(x)$ for nPIVFSs, and then for any $\widetilde{A} \in nPIVFS(X)$,

$$E_3(\widetilde{A}) = -\frac{1}{4n+6} \sum_{\xi=L,U} \left[\sum_{i=0}^{2n+1} \left(\sin a_i^\xi \pi \right) + \sin H(\widetilde{A}^\xi) \pi \right] \quad (3.5)$$

is the entropy on $nPIVFS(X)$.

Provided that the mapping $f : [0, 1] \rightarrow [0, 1]$ is designated as $f(x) = f_4(x)$ for nPIVFSs, then for any $\tilde{A} \in nPIVFS(X)$,

$$E_4(\tilde{A}) = \frac{1}{4n+6} \sum_{\xi=L,U} \left[\sum_{i=0}^{2n+1} (1 - |2a_i^\xi - 1|^\alpha) + 1 - |2H(\tilde{A}^\xi) - 1|^\alpha \right] \quad (3.6)$$

is the entropy on $nPIVFS(X)$.

Theorem 3.4. For any $\tilde{A}, \tilde{B} \in nPIVFS(X)$, either $a_i^\xi \leq b_i^\xi$ or $a_i^\xi \geq b_i^\xi$, then the following property is valid:

$$E(\tilde{A} \cap \tilde{B}) + E(\tilde{A} \cup \tilde{B}) = E(\tilde{A}) + E(\tilde{B}).$$

Proof. If $a_i^\xi \leq b_i^\xi$, then $\tilde{A} \cup \tilde{B} = \tilde{B}$ and $\tilde{A} \cap \tilde{B} = \tilde{A}$, thus, $E(\tilde{A} \cap \tilde{B}) + E(\tilde{A} \cup \tilde{B}) = E(\tilde{A}) + E(\tilde{B})$. Another case can be proved in the same way. $\square$

Corollary 3.5. For any $\tilde{A}, \tilde{B} \in nPIVFS(X)$ and any type-1 function f ,

$$E_f(\tilde{A} \cap \tilde{B}) + E_f(\tilde{A} \cup \tilde{B}) = E_f(\tilde{A}) + E_f(\tilde{B}).$$

Corollary 3.6. Let $\tilde{A}, \tilde{B} \in nPIVFS(X)$. If

$$\left| a_i^\xi - \frac{1}{2} \right| \geq \left| b_i^\xi - \frac{1}{2} \right|,$$

then $E_f(\tilde{A}) \leq E_f(\tilde{B})$ for every type-1 function f , where $\xi \in \{L, U\}$, $i = 0, 1, \dots, 2n+2$.

Note 1: In particular, condition (E4) of Definition 3.1 can be relaxed to this form, which does not require the two sets to be comparable on one side of $1/2$.

Corollary 3.7. Let $\tilde{A}, \tilde{B} \in nPIVFS(X)$, and let f be the mapping given in Definition 3.2, which is L_f -Lipschitz continuous on $[0, 1]$. If for any $\varepsilon > 0$, $|a_i^\xi - b_i^\xi| \leq \varepsilon$, then $|E_f(\tilde{A}) - E_f(\tilde{B})| \leq L_f \varepsilon$.

Below, it is explained that the newly proposed entropy is effective through some specific examples. For convenience, we assume that the domain X is a single point set, i.e., $X = \{x\}$. For comparison with entropy in [14], we take $n = 1$.

Example 3.8. Let $X = \{x\}$, and $\tilde{A} = [(0.2, 0.3, 0.4, 0.5; 0.9), (0.1, 0.3, 0.4, 0.5; 1)]$ and $\tilde{B} = [(0.7, 0.8, 0.9, 0.9; 0.9), (0.6, 0.8, 0.9, 0.9; 1)]$ are two nPIVFSs defined on X .

We calculate $E_1(\tilde{A}) = 0.469$, $E_3(\tilde{A}) = 0.670$, $E_4(\tilde{A}) = 0.428(\alpha = 0.5)$; $E_1(\tilde{B}) = 0.449$, $E_3(\tilde{B}) = 0.484$, $E_4(\tilde{B}) = 0.190(\alpha = 0.5)$. It can be concluded that E_1 , E_3 , and E_4 can effectively distinguish the fuzziness of information in nPIVFSs $\tilde{A}$ and $\tilde{B}$. However, E_2 fails when calculating $\tilde{A}$ and $\tilde{B}$.

Example 3.9. Let $X = \{x\}$, and $\tilde{A} = [(0.2, 0.3, 0.4, 0.5; 0.9), (0.1, 0.3, 0.4, 0.5; 0.9)]$ and $\tilde{B} = [(0.7, 0.8, 0.9, 0.9; 0.9), (0.6, 0.8, 0.9, 0.9; 0.9)]$ are two nPIVFSs on X . After calculation, it is obtained that

$$\begin{aligned} E_1(\tilde{A}) &= 0.532, E_2(\tilde{A}) = 0.683, E_3(\tilde{A}) = 0.745, E_4(\tilde{A}) = 0.438(\alpha = 0.5), \\ E_1(\tilde{B}) &= 0.500, E_2(\tilde{B}) = 0.348, E_3(\tilde{B}) = 0.480, E_4(\tilde{B}) = 0.200(\alpha = 0.5). \end{aligned}$$

E_1 , E_2 , E_3 , and E_4 can effectively distinguish nPIVFSs $\tilde{A}$ and $\tilde{B}$. On the other hand, E_1 , E_2 , E_3 , and E_4 are special cases of E defined in Theorem 3.3, which further confirms the rationality of the newly proposed entropy for nPIVFSs.

When the adjustment parameter $\alpha = 0.5$, the values of $E_4(\widetilde{A})$ and $E_4(\widetilde{B})$ calculated by entropy E_4 differ greatly, which can clearly distinguish the fuzziness degree of $\widetilde{A}$ and $\widetilde{B}$. It can be seen that the entropy measure with adjustable parameter α has superior discrimination performance. Introducing the adjustment parameter can strengthen the ability of entropy to identify the information fuzziness of nPIVFSs, and the parameter-adjustable form possesses better practicability.

3.2. Similarity measure and cross-entropy for nPIVFSs

In this section, we present the axiomatic definition of the similarity measure for nPIVFSs, and then develop functional expressions of the similarity measure and cross-entropy for nPIVFSs.

Definition 3.10. Let $\widetilde{A}, \widetilde{B} \in nPIVFS(X)$, and then the function $S : nPIVFS(X) \times nPIVFS(X) \rightarrow [0, 1]$ is called a similarity measure for nPIVFSs, if S satisfies the following four conditions:

- (S1) $S(\widetilde{A}, \widetilde{B}) = 0$ iff $\widetilde{A} = \widetilde{0}, \widetilde{B} = \widetilde{1}$ or $\widetilde{A} = \widetilde{1}, \widetilde{B} = \widetilde{0}$;
- (S2) $S(\widetilde{A}, \widetilde{B}) = 1$ iff $a_i^\xi = b_i^\xi, i = 0, 1, 2, \dots, 2n+2, \xi = L, U$;
- (S3) $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$;
- (S4) $S(\widetilde{A}, \widetilde{C}) \leq S(\widetilde{A}, \widetilde{B}), S(\widetilde{A}, \widetilde{C}) \leq S(\widetilde{B}, \widetilde{C})$, if $a_i^\xi \leq b_i^\xi \leq c_i^\xi$ or $a_i^\xi \geq b_i^\xi \geq c_i^\xi, i = 0, 1, 2, \dots, 2n+2, \xi = L, U$.

Definition 3.11. Let the mapping $g : [0, 1] \rightarrow [0, 1]$ be a strictly monotone decreasing type-2 function, and it satisfies the following requirements:

- (i) $g(1) = 0$;
- (ii) $g(0) = 1$.

Theorem 3.12. A mapping $S : nPIVFS(X) \times nPIVFS(X) \rightarrow [0, 1]$ is a similarity measure for nPIVFSs based on the aforementioned type-2 function in Definition 3.11, which can be defined as

$$S(\widetilde{A}, \widetilde{B}) = \frac{1}{4n+6} \sum_{\xi=L,U} \sum_{i=0}^{2n+2} g(|a_i^\xi - b_i^\xi|). \quad (3.7)$$

Proof. As long as Eq (3.7) satisfies all four axiomatic properties above, it can be used as a reasonable metric formula of the similarity measure on nPIVFSs.

- (S1) Obviously, $g(|a_i^\xi - b_i^\xi|) \geq 0$, and then for $a_i^\xi, b_i^\xi \in [0, 1], i = 0, 1, 2, \dots, 2n+2, \xi = L, U$,

$$S(\widetilde{A}, \widetilde{B}) = 0 \Leftrightarrow g(|a_i^\xi - b_i^\xi|) = 0 \Leftrightarrow |a_i^\xi - b_i^\xi| = 1.$$

Then it can obtain that

$$a_i^\xi = 1, b_i^\xi = 0 \text{ or } a_i^\xi = 0, b_i^\xi = 1,$$

which indicates that $\widetilde{A} = \widetilde{0}, \widetilde{B} = \widetilde{1}$ or $\widetilde{A} = \widetilde{1}, \widetilde{B} = \widetilde{0}$.

On the other hand, if $a_i^\xi = 1, b_i^\xi = 0$ or $a_i^\xi = 0, b_i^\xi = 1$ for all $i = 0, 1, 2, \dots, 2n+2, \xi = L, U$, then $|a_i^\xi - b_i^\xi| = 1$, and thus, $g(|a_i^\xi - b_i^\xi|) = 0$. It follows that $S(\widetilde{A}, \widetilde{B}) = 0$.

(S2) According to the above analysis, when $i = 0, 1, \dots, 2n+2, \xi = L, U$, in the same way, we have that

$$S(\widetilde{A}, \widetilde{B}) = 1 \Leftrightarrow g(|a_i^\xi - b_i^\xi|) = 1 \Leftrightarrow a_i^\xi - b_i^\xi = 0 \Leftrightarrow a_i^\xi = b_i^\xi.$$

- (S3) It is easy to see that $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$.

(S4) Assuming that $0 \leq a_i^\xi \leq b_i^\xi \leq c_i^\xi \leq 1, \xi = L, U, i = 0, 1, 2, \dots, 2n + 2$,

$$-1 \leq a_i^\xi - c_i^\xi \leq b_i^\xi - c_i^\xi \leq 0, \quad -1 \leq a_i^\xi - c_i^\xi \leq a_i^\xi - b_i^\xi \leq 0,$$

it can be derived that

$$0 \leq |b_i^\xi - c_i^\xi| \leq |a_i^\xi - c_i^\xi| \leq 1, \quad 0 \leq |a_i^\xi - b_i^\xi| \leq |a_i^\xi - c_i^\xi| \leq 1.$$

Notice the monotonically increasing property of $g(x)$ for $x \in [0, 1]$, and therefore,

$$g(|b_i^\xi - c_i^\xi|) \geq g(|a_i^\xi - c_i^\xi|); \quad g(|a_i^\xi - b_i^\xi|) \geq g(|a_i^\xi - c_i^\xi|).$$

We can deduce that

$$S(\widetilde{A}, \widetilde{C}) \leq S(\widetilde{A}, \widetilde{B}), S(\widetilde{A}, \widetilde{C}) \leq S(\widetilde{B}, \widetilde{C}).$$

On the other hand, if $1 \geq a_i^\xi \geq b_i^\xi \geq c_i^\xi \geq 0$, through the same method, we can also speculate that

$$S(\widetilde{A}, \widetilde{C}) \leq S(\widetilde{A}, \widetilde{B}), S(\widetilde{A}, \widetilde{C}) \leq S(\widetilde{B}, \widetilde{C}).$$

Note 2: An nPIVFS is stored as two ordered $(2n + 2)$ -tuples together with two heights. The operations in Eqs (6)–(9) are component-wise and require $O(n)$ elementary operations; the entropy, similarity, and cross-entropy in Eqs (10), (16), and (19) are averages of $4n + 6$ bounded terms and hence also require $O(n)$ operations. Consequently, for q alternatives, m attributes, and T decision-makers, the MADM procedure of Section 5 runs in $O(Tqmn)$ time and $O(qmn)$ memory, i.e., the cost grows linearly with the number of polygonal vertices.

Next, the following functions $g : [0, 1] \rightarrow [0, 1]$ satisfying conditions (i)–(iii) in Definition 3.11 are introduced:

$$(1) g_1(x) = 1 - x; \quad (2) g(x) = 1 - xe^{x-1}; \quad (3) g_2(x) = \frac{1-x}{1+x};$$

$$(4) g_3(x) = -\frac{1}{\ln 2} \left[\frac{x+1}{2} \ln \frac{x+1}{2} + \left(1 - \frac{x+1}{2}\right) \ln \left(1 - \frac{x+1}{2}\right) \right].$$

Various similarity measures are constructed by selecting different functions g . For example, when $g(x) = g_3(x)$ and $n = 1$ are applied to Theorem 3.12, then the similarity measure of nPIVFSs can be obtained as follows:

$$\begin{aligned} S(\widetilde{A}, \widetilde{B}) = & -\frac{1}{10 \ln 2} \sum_{\xi=L,U} \sum_{i=0}^4 \left[\frac{|a_i^\xi - b_i^\xi| + 1}{2} \ln \frac{|a_i^\xi - b_i^\xi| + 1}{2} \right. \\ & \left. + \left(1 - \frac{|a_i^\xi - b_i^\xi| + 1}{2}\right) \ln \left(1 - \frac{|a_i^\xi - b_i^\xi| + 1}{2}\right) \right]. \end{aligned} \quad (3.8)$$

Apparently, the aforementioned similarity measure is Eq (7) in Ref. [14]. On another level, Ref. [14] is covered by the newly proposed similarity measure in Theorem 3.12.

Let $g : [0, 1] \rightarrow [0, 1]$ be given by $g(x) = g_1(x)$, and then for any $\widetilde{A}, \widetilde{B} \in nPIVFS(X)$, the similarity measure is

$$S(\widetilde{A}, \widetilde{B}) = \frac{1}{4n+6} \sum_{\xi=L,U} \left[\sum_{i=0}^{2n+1} (1 - |a_i^\xi - b_i^\xi|) + 1 - |H(\widetilde{A}^\xi) - H(\widetilde{B}^\xi)| \right]. \quad (3.9)$$

Subsequently, we explore the cross-entropy in view of the general expression of the similarity measure for nPIVFSs. First, we will elaborate on the axiomatic definition of the cross-entropy of nPIVFSs, and then provide a functional representation of the cross-entropy, exploring its effectiveness as an effective measure.

Definition 3.13. Let $\tilde{A}, \tilde{B} \in nPIVFS(X)$, and the mapping $C : nPIVFS(X) \times nPIVFS(X) \rightarrow [0, 1]$ is called a cross-entropy for nPIVFSs, which meets the following conditions:

(C1) $C(\tilde{A}, \tilde{B}) \geq 0$;

(C2) $C(\tilde{A}, \tilde{B}) = 0 \Leftrightarrow a_i^\xi = b_i^\xi$ for all $i = 0, 1, 2, \dots, 2n + 2, \xi = L, U$.

Theorem 3.14. Let $\tilde{A}, \tilde{B} \in nPIVFS(X)$, and then the cross-entropy can be constructed as

$$C(\tilde{A}, \tilde{B}) = 1 - S(\tilde{A}, \tilde{B}). \quad (3.10)$$

Proof. Please refer to Theorem 3.12 for this proof.

From Theorem 3.14, it can be seen that the similarity measure and cross-entropy here are in a dual relationship.

3.3. Comparison with representative information measures

To position the proposed measures within the literature, this subsection compares them with representative entropy, similarity, and cross-entropy models from the five aspects suggested in the Introduction: flexibility, computational complexity, parameterization capability, discrimination ability, and applicability to decision-making. Table 2 summarizes the comparison and Table 3 lists the explicit relations by which existing measures are covered by the proposed framework.

(1) Flexibility. Existing measures are fixed-form: [14] fixes the logarithmic function, [16] fixes a particular entropy, and [19, 21] construct measures from distances or aggregation operators. By contrast, Eqs (10) and (16) are generator-based: any f with $f(0) = f(1) = 0$, $f(1/2) = 1$, increasing on $[0, 1/2]$ and decreasing on $[1/2, 1]$, yields a valid entropy, and any decreasing g with $g(0) = 1$, $g(1) = 0$ yields a valid similarity measure. New measures can therefore be produced by choosing f or g according to the semantic requirement, without re-proving the axioms.

(2) Parameterization capability and discrimination ability. The proposed family contains a one-parameter entropy E_4 generated by $f_4(x) = 1 - |2x - 1|^\alpha$, $\alpha \in (0, 1)$. Since $|2x - 1| < 1$ for every $x \neq 1/2$, a smaller α makes the entropy more concentrated around the most fuzzy point $1/2$ and hence it is more sensitive to deviations from it. The decision-maker can thus calibrate the discrimination power. The numerical examples confirm this advantage: In Example 3.5, the separation between two nPIVFSs produced by E_4 ($\alpha = 0.5$) is $0.428 - 0.190 = 0.238$, whereas the quadratic entropy E_1 yields only $0.469 - 0.449 = 0.020$ and the sine entropy E_3 yields $0.670 - 0.484 = 0.186$. Moreover, Example 3.5 shows that a fixed logarithmic measure of the type used in [14] may fail to separate the two sets, while the parameter-adjustable E_4 still succeeds.

(3) Computational complexity. An nPIVFS is encoded by two ordered $(2n + 2)$ -tuples and two heights; the complement, union, intersection and scalar multiplication act component-wise and therefore cost $O(n)$. The entropy, similarity and cross-entropy are averages over $O(n)$ bounded terms and also cost $O(n)$. In contrast, measures on general interval-valued or interval type-2 fuzzy sets usually require interval extension or α -cut computations whose results are not closed in a simple finite

representation. The price of the higher resolution is only a linear growth in n , which can be selected by the user.

(4) Applicability to decision-making. Because all operations and measures are element-wise and explicit, the proposed framework leads to a fully numerical MADM procedure: Attribute weights are derived from the entropy via Eq (23), and alternatives are ranked by the closeness coefficient built on the cross-entropy. Applied to the supplier-selection case of [13], the proposed method produces the ranking $A_2 > A_1 > A_3$ for every $n = 1, \dots, 6$ and every $\alpha = 0.1, \dots, 0.9$, which coincides with the results of [13–16] while offering an adjustable separation among alternatives.

Overall, the distinctive features of the proposed measures are that they are defined on nPIVFSs, generated by elementary functions rather than fixed ad hoc formulas, mutually transformable, parameter-adjustable, and computationally explicit. These features are not shared by the existing models listed in Table 2.

Table 2. Comparison of proposed information measures with representative models.

Aspect	Jin et al. [14]	Wang & Liu [13] / Pan [16]	Xie et al. [19] / Pekala et al. [21]	This paper
Fuzzy structure	IT2tFS	IT2FS / IT2tFS	IVFS	nPIVFS ($n = 1$ reduces to IT2tFS)
Measures	Logarithmic entropy, similarity, and cross-entropy	Entropy	Distance/aggregation-based entropy and similarity	Unified entropy, similarity, and cross-entropy
Generating mechanism	Fixed logarithmic function	Fixed	Distance/aggregation operator	Any admissible f or g
Tunable parameters	None	None	None	α (and free choice of f, g, n)
Computational complexity	Low (trapezoidal structure)	Interval extension	Interval arithmetic	$O(n)$ element-wise, closed, and linear
Discrimination ability	Fixed; may fail to distinguish	–	–	Adjustable; separation 0.238 in Example 3.5
Relation among E, S, C	Independently defined	Independently defined	Partially discussed	Unified via $E = g(C(A, A^c))$
MADM applicability	Yes	Yes	Mainly theoretical	Entropy weights plus cross-entropy closeness

Table 3. Existing information measures as special cases of the proposed framework.

Existing measure	Corresponding form in this paper	Conditions
Logarithmic entropy of [14] on IT2tFS	E_2 , Eqs (12)–(13)	$n = 1, f = f_2$
Similarity measure Eq (7) of [14]	Eq (17)	$n = 1, g = g_3$
Classical quadratic / sine entropy	E_1 / E_3 , Eqs (11) and (14)	$f = f_1 / f = f_3$
General entropy / similarity	Eq (10) / Eq (16)	Any admissible f / g

4. Relationships among the information measures

In this section, we will further consider the potential interrelationships among the functional expressions of information measures on nPIVFSs. Meanwhile, the transformation relationship of three Eqs (3.1), (3.7), and (3.10) is discussed.

Theorem 4.1. Let $\tilde{A} \in nPIVFS(X)$, and then based on $S(\tilde{A}, \tilde{A}^c)$ mentioned in Eq (3.7) and g defined in Definition 3.11, the entropy is established as

$$E(\tilde{A}) = 1 - g(S(\tilde{A}, \tilde{A}^c)). \quad (4.1)$$

Proof. It is sufficient to verify that Eq (4.1) meets all the axiomatic properties mentioned in Definition 3.1.

(E1) From Definition 3.1, and we have

$$E(\tilde{A}) = 0 \Leftrightarrow S(\tilde{A}, \tilde{A}^c) = 0 \Leftrightarrow g(|a_i^\xi - a_i^{c\xi}|) = 0 \Leftrightarrow |a_i^\xi - a_i^{c\xi}| = 1,$$

we can calculate that $a_i^\xi = 0, a_i^{c\xi} = 1$ or $a_i^\xi = 1, a_i^{c\xi} = 0$. Thus, $\tilde{A} = \tilde{0}$ or $\tilde{A} = \tilde{1}$.

(E2) From Definitions 3.1 and 3.10, we can see that

$$\widetilde{A} = \frac{1}{2} \Leftrightarrow E(\widetilde{A}) = 1 \Leftrightarrow \widetilde{A}^c = \frac{1}{2} \Leftrightarrow S(\widetilde{A}, \widetilde{A}^c) = 1.$$

(E3) We know

$$(a_i^{c\xi})^c = (1 - a_{2n+1-i}^\xi)^c = a_i^\xi,$$

and we can deduce that $S(\widetilde{A}, \widetilde{A}^c) = S(\widetilde{A}^c, \widetilde{A})$. Therefore,

$$E(\widetilde{A}^c) = 1 - g(S(\widetilde{A}^c, (\widetilde{A}^c)^c)) = 1 - g(S(\widetilde{A}^c, \widetilde{A})) = 1 - g(S(\widetilde{A}, \widetilde{A}^c)) = E(\widetilde{A}).$$

(E4) Suppose that $0 \leq a_i^\xi \leq b_i^\xi \leq 0.5$, and it is easy to get $0.5 \leq b_i^{c\xi} \leq a_i^{c\xi} \leq 1$ for all $i = 0, 1, 2, \dots, 2n+2, \xi = L, U$. It can be derived that

$$|a_i^\xi - a_i^{c\xi}| \geq |b_i^\xi - b_i^{c\xi}|,$$

in terms of the monotonically decreasing property of $g(x)$ for $x \in [0, 1]$. Therefore, $g(|a_i^\xi - a_i^{c\xi}|) \leq g(|b_i^\xi - b_i^{c\xi}|)$. It follows that $S((\widetilde{A}, \widetilde{A}^c)) \leq S((\widetilde{B}, \widetilde{B}^c))$. Thus,

$$1 - g(S((\widetilde{A}, \widetilde{A}^c))) \leq 1 - g(S((\widetilde{B}, \widetilde{B}^c))),$$

and then $E(\widetilde{A}) \leq E(\widetilde{B})$.

Specially, if $g(x) = 1 - x$, $x \in [0, 1]$, then $E(\widetilde{A}) = S(\widetilde{A}, \widetilde{A}^c)$. It can be concluded that the model was proposed in Ref. [14].

Corollary 4.2. Assume that $\widetilde{A} \in nPIVFS(X)$, and we have

$$E(\widetilde{A}) = g(1 - S(\widetilde{A}, \widetilde{A}^c)). \quad (4.2)$$

In terms of Corollary 4.2 and Theorem 4.1, Theorem 4.3 is revealed.

Theorem 4.3. Let $\widetilde{A} \in nPIVFS(X)$, and then the entropy can be defined as

$$E(\widetilde{A}) = g(C(\widetilde{A}, \widetilde{A}^c)). \quad (4.3)$$

5. The application of nPIVFSs' information measures

With the development of society, innovation, and uncertainty of dynamic change, and the difficulty of decision-making management significantly intensifies. The degree of uncertainty is mainly influenced by the increasing objective information and the complexity of subjective evaluation information by decision-makers. Therefore, information measures for nPIVFSs to calculate fuzzy information play an indispensable role. Now, we investigate a new approach with information measures for nPIVFSs to analyze and evaluate alternatives.

5.1. A MADM method with n PIVFS information measures

Assume that $A = \{A_1, A_2, \dots, A_q\}$ is a set of alternatives and $C = \{c_1, c_2, \dots, c_m\}$ is an attribute class. Let $w = \{w_1, w_2, \dots, w_m\}^T$ be the weight of all attributes, which are unknown. Suppose that there are T decision-makers $DK = \{d_1, d_2, \dots, d_T\}$, and the weight vector of the decision-makers is $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_T\}^T$. Without loss of generality, the evaluation value of an alternative A_t ($l = 1, 2, \dots, q$) with respect to the attribute c_t ($t = 1, 2, \dots, m$) provided by the decision-makers d_k ($k = 1, 2, \dots, T$) is in the form of

$$d_{lt}^k = [(a_0^L(ltk), a_1^L(ltk), \dots, a_n^L(ltk), \dots, a_{2n+1}^L(ltk), H(s_{ltk})^L, \\ (a_0^U(ltk), a_1^U(ltk), \dots, a_n^U(ltk), \dots, a_{2n+1}^U(ltk), H(s_{ltk})^U)].$$

In addition, in terms of the transformation criteria of given linguistic variables, a polygonal interval-valued fuzzy decision matrix can be obtained. The details are as follows:

$$D^k = \begin{matrix} & c_1 & c_2 & \cdots & c_m \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{matrix} & \begin{pmatrix} d_{11}^k & d_{12}^k & \cdots & d_{1m}^k \\ d_{21}^k & d_{22}^k & \cdots & d_{2m}^k \\ \vdots & \vdots & \cdots & \vdots \\ d_{q1}^k & d_{q2}^k & \cdots & d_{qm}^k \end{pmatrix} \end{matrix},$$

where $k = 1, 2, \dots, T$.

Next, we aggregate the individual matrices $D^k = (d_{lt}^k)_{q \times m}$ into an overall decision matrix $D = (d_{lt})_{q \times m}$, where $d_{lt} = \oplus_{k=1}^T (\lambda_k d_{lt}^k)$ refers to [13].

Further, the criteria weight, which is unknown, is to be calculated as

$$w_t = \frac{1 - E_t}{\sum_{t=1}^m (1 - E_t)}, \quad (5.1)$$

where $E_t = \frac{1}{q} \sum_{l=1}^q E(d_{lt})$ and $E(d_{lt})$ follows from Eq (3.3), $t = 1, 2, \dots, m$.

Then the weighted decision matrix is constructed by considering the importance of attributes in the decision-making process:

$$\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ d_{q1} & d_{q2} & \cdots & d_{qm} \end{pmatrix} \begin{pmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & w_m \end{pmatrix} = \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \cdots & \tilde{d}_{1m} \\ \tilde{d}_{21} & \tilde{d}_{22} & \cdots & \tilde{d}_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{q1} & \tilde{d}_{q2} & \cdots & \tilde{d}_{qm} \end{pmatrix}. \quad (5.2)$$

In the decision-making process, when reflecting the gap between evaluation options, it is usually necessary to first select the positive ideal and negative ideal options in the system composed of all options. The positive ideal and negative ideal solutions are expressed as

$$G^+ = (G_1^+, G_2^+, \dots, G_m^+),$$

$$G^- = (G_1^-, G_2^-, \dots, G_m^-),$$

where G_t^+ and G_t^- ($t = 1, 2, \dots, m$) can be defined as follows:

$$\begin{cases} G_t^+ = [(\max_{1 \leq l \leq q} \tilde{a}_0^L(lt), \max_{1 \leq l \leq q} \tilde{a}_1^L(lt), \dots, \max_{1 \leq l \leq q} \tilde{a}_n^L(lt), \dots, \max_{1 \leq l \leq q} \tilde{a}_{2n+1}^L(lt)), \\ (\max_{1 \leq l \leq q} \tilde{a}_0^U(lt), \max_{1 \leq l \leq q} \tilde{a}_1^U(lt), \dots, \max_{1 \leq l \leq q} \tilde{a}_n^U(lt), \dots, \max_{1 \leq l \leq q} \tilde{a}_{2n+1}^U(lt))]; \\ G_t^- = [(\min_{1 \leq l \leq q} \tilde{a}_0^L(lt), \min_{1 \leq l \leq q} \tilde{a}_1^L(lt), \dots, \min_{1 \leq l \leq q} \tilde{a}_n^L(lt), \dots, \min_{1 \leq l \leq q} \tilde{a}_{2n+1}^L(lt)), \\ (\min_{1 \leq l \leq q} \tilde{a}_0^U(lt), \min_{1 \leq l \leq q} \tilde{a}_1^U(lt), \dots, \min_{1 \leq l \leq q} \tilde{a}_n^U(lt), \dots, \min_{1 \leq l \leq q} \tilde{a}_{2n+1}^U(lt))]. \end{cases} \quad (5.3)$$

According to the attribute weight vector $w = \{w_1, w_2, \dots, w_m\}^T$ determined by Eq (5.1), the cross-entropy between the alternative A_l and the positive ideal and negative ideal alternatives (G^+, G^-) can be calculated by the following formulas:

$$C^+(A_l) = \sum_{t=1}^m w_t C(d_{lt}, G_t^+), \quad (5.4)$$

$$C^-(A_l) = \sum_{t=1}^m w_t C(d_{lt}, G_t^-), \quad (5.5)$$

where $C(\cdot, \cdot)$ follows from Eqs (3.9) and (3.10), $l = 1, 2, \dots, q$.

Finally, the closeness degree of the alternative A_l ($l = 1, 2, \dots, q$) to the ideal alternative is then expressed as:

$$T(A_l) = \frac{C^+(A_l)}{C^+(A_l) + C^-(A_l)}. \quad (5.6)$$

Based on the size of $T(A_l)$, the ranking of solutions can be obtained. The larger the $T(A_l)$, the better the A_l , $l = 1, 2, \dots, q$.

Note 3: Let q , m , T , and n denote the numbers of alternatives, attributes, decision-makers, and polygonal vertices, respectively. The aggregation $d_{lt} = \oplus_{k=1}^T (\lambda_k d_{lt}^k)$ costs $O(Tqmn)$; the entropy weights in Eq (23), the weighted matrix in Eq (24), the ideal solutions in Eq (25), and the cross-entropies in Eqs (26)–(27) each cost $O(qmn)$; the closeness degrees and the final ranking cost $O(qm)$ and $O(q \log q)$, respectively. Consequently the whole procedure runs in $O(Tqmn)$ time and $O(qmn)$ memory, i.e., the cost is linear in each of q , m , T , and n . When $n = 1$, the model reduces to the interval type-2 trapezoidal case with ten scalars per set, which is the cheapest configuration.

5.2. Examples to validate the novel entropy

We take the case proposed by Wang and Liu [13] to illustrate the practicality of the proposed method. Besides, by selecting different n values for horizontal comparison, it is verified that nPIVFSs depict information more accurately as n values continue to increase. n denotes the polygon order of an n -polygonal interval-valued fuzzy set (nPIVFS). When $n = 1$, nPIVFS degenerates into the interval-valued trapezoidal type-2 fuzzy set reported in existing literature. As n increases, nPIVFS can characterize fuzzy information with higher granularity, accompanied by higher computational complexity. In this paper, we set $n = 1, 2, \dots, 6$ to compare model performances under different description granularities, and also verify that the existing models are special cases of our proposed framework. Table 4 and Table A1 in the Appendix provide the linguistic terms and their corresponding nPIVFSs when $n = 1, 2, \dots, 6$, respectively (the main text only provides the corresponding results for $n = 1, 2, 3$, please refer to the Appendix for $n = 4, 5, 6$).

Table 4. Linguistic variables for the ratings and ordered representations.

Linguistic variables	Corresponding ordered representations $n = 1$
Very Low (VL)	$[(0.00, 0.00, 0.00, 0.05; 0.90), (0.00, 0.00, 0.00, 0.10; 1.00)]$
Low (L)	$[(0.05, 0.10, 0.10, 0.20; 0.90), (0.00, 0.10, 0.10, 0.30; 1.00)]$
Medium Low (ML)	$[(0.20, 0.30, 0.30, 0.40; 0.90), (0.10, 0.30, 0.30, 0.50; 1.00)]$
Medium (M)	$[(0.40, 0.50, 0.50, 0.60; 0.90), (0.30, 0.50, 0.50, 0.70; 1.00)]$
Medium High (MH)	$[(0.60, 0.70, 0.70, 0.80; 0.90), (0.50, 0.70, 0.70, 0.90; 1.00)]$
High (H)	$[(0.80, 0.90, 0.90, 0.95; 0.90), (0.70, 0.90, 0.90, 1.00; 1.00)]$
Very High (VH)	$[(0.95, 1.00, 1.00, 1.00; 1.00), (0.90, 1.00, 1.00, 1.00; 1.00)]$
Linguistic variables	Corresponding ordered representations $n = 2$
Very Low (VL)	$[(0.00, 0.00, 0.00, 0.00, 0.05; 0.90), (0.00, 0.00, 0.00, 0.00, 0.10; 1.00)]$
Low (L)	$[(0.05, 0.10, 0.20, 0.20, 0.20; 0.90), (0.00, 0.10, 0.10, 0.25, 0.30; 1.00)]$
Medium Low (ML)	$[(0.20, 0.25, 0.30, 0.40, 0.40; 0.90), (0.10, 0.20, 0.25, 0.38, 0.40; 1.00)]$
Medium (M)	$[(0.40, 0.45, 0.48, 0.60, 0.60; 0.90), (0.30, 0.35, 0.40, 0.60, 0.70; 1.00)]$
Medium High (MH)	$[(0.60, 0.62, 0.65, 0.78, 0.80; 0.90), (0.50, 0.60, 0.63, 0.75, 0.90; 1.00)]$
High (H)	$[(0.70, 0.75, 0.78, 1.00, 1.00; 0.90), (0.55, 0.58, 0.72, 1.00, 1.00; 1.00)]$
Very High (VH)	$[(0.95, 1.00, 1.00, 1.00, 1.00; 1.00), (0.90, 1.00, 1.00, 1.00, 1.00; 1.00)]$
Linguistic variables	Corresponding ordered representations $n = 3$
Very Low (VL)	$[(0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 0.05; 0.90), (0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 0.10; 1.00)]$
Low (L)	$[(0.05, 0.10, 0.11, 0.15, 0.18, 0.20, 0.20, 0.20; 0.90), (0.00, 0.10, 0.10, 0.15, 0.18, 0.20, 0.25, 0.30; 1.00)]$
Medium Low (ML)	$[(0.20, 0.25, 0.30, 0.30, 0.35, 0.38, 0.40, 0.40; 0.90), (0.10, 0.20, 0.25, 0.30, 0.32, 0.35, 0.38, 0.40; 1.00)]$
Medium (M)	$[(0.40, 0.45, 0.48, 0.50, 0.50, 0.58, 0.60, 0.60; 0.90), (0.30, 0.35, 0.40, 0.48, 0.50, 0.50, 0.60, 0.70; 1.00)]$
Medium High (MH)	$[(0.60, 0.62, 0.65, 0.70, 0.70, 0.75, 0.78, 0.80; 0.90), (0.50, 0.60, 0.63, 0.70, 0.70, 0.72, 0.75, 0.90; 1.00)]$
High (H)	$[(0.70, 0.75, 0.78, 0.90, 1.00, 1.00, 1.00, 1.00; 0.90), (0.55, 0.58, 0.72, 0.90, 1.00, 1.00, 1.00, 1.00; 1.00)]$
Very High (VH)	$[(0.95, 1.00, 1.00, 1.00, 1.00, 1.00, 1.00, 1.00; 1.00), (0.90, 1.00, 1.00, 1.00, 1.00, 1.00, 1.00, 1.00; 1.00)]$

We assume that a manufacturing company wants to consider the best global supplier for one of the most critical components used in the assembly process. Now, we will choose from the final global suppliers A_l ($l = 1, 2, 3$). In order to ensure fairness and rationality in the selection, the company's personnel department dispatches three decision-makers d_k ($k = 1, 2, 3$), whose weight vector is $\lambda = \{0.3, 0.45, 0.25\}^T$. Decision makers, based on their own experience, provide evaluation values for alternatives A_l from four attributes: c_1 : product quality, c_2 : risk factor, c_3 : quality service, c_4 : profile of supplier.

First, we establish the primal $nPIVFS$ decision matrix, as follows:

$$D^1 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} d_{11}^1 & d_{12}^1 & d_{13}^1 & d_{14}^1 \\ d_{21}^1 & d_{22}^1 & d_{23}^1 & d_{24}^1 \\ d_{31}^1 & d_{32}^1 & d_{33}^1 & d_{34}^1 \end{pmatrix} \end{matrix} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} MH & H & VH & VH \\ H & MH & H & H \\ VH & VH & M & H \end{pmatrix} \end{matrix},$$

$$D^2 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} d_{11}^2 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ d_{21}^2 & d_{22}^2 & d_{23}^2 & d_{24}^2 \\ d_{31}^2 & d_{32}^2 & d_{33}^2 & d_{34}^2 \end{pmatrix} \end{matrix} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} H & VH & H & H \\ MH & H & VH & VH \\ H & VH & MH & VH \end{pmatrix} \end{matrix},$$

$$D^3 = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} d_{11}^3 & d_{12}^3 & d_{13}^3 & d_{14}^3 \\ d_{21}^3 & d_{22}^3 & d_{23}^3 & d_{24}^3 \\ d_{31}^3 & d_{32}^3 & d_{33}^3 & d_{34}^3 \end{pmatrix} \end{matrix} = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} MH & H & H & H \\ H & VH & VH & H \\ MH & H & MH & VH \end{pmatrix} \end{matrix}.$$

Second, we aggregate the decision matrices of all decision-makers to obtain an overall decision matrix $D = (d_{ij})_{3 \times 4}$, and when $n = 1, 2, \dots, 6$, the results are listed in Tables 5 and 6 and Tables A2–A4 in the Appendix.

$$D = \begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \end{pmatrix} \end{matrix},$$

where $d_{lt} = \oplus_{k=1}^3 (\lambda_k d_{lt}^k)$ refers to Ref. [13], $l = 1, 2, 3$; $t = 1, 2, 3, 4$.

Table 5. Aggregate $nPIVFS$ decision matrix $n = 1, 2$.

Attributes	$A_1(n = 1)$	$A_1(n = 2)$
c_1	[(0.69,0.71,0.85,0.87;0.90),(0.59,0.65,0.85,0.95;1.00)]	[(0.69,0.71,0.74,0.85,0.86,0.87;0.90),(0.59,0.65,0.68,0.85,0.87,0.95;1.00)]
c_2	[(0.03,0.04,0.13,0.13;0.90),(0.00,0.03,0.12,0.21;1.00)]	[(0.03,0.04,0.05,0.08,0.13,0.13;0.90),(0.00,0.03,0.04,0.12,0.16,0.21;1.00)]
c_3	[(0.85,0.88,0.97,0.97;0.90),(0.76,0.78,0.90,0.92,1.00;1.00)]	[(0.85,0.88,0.90,0.95,0.97,0.97;0.90),(0.76,0.78,0.80,0.88,0.92,1.00;1.00)]
c_4	[(0.85,0.88,0.97,0.97;0.90),(0.76,0.78,0.90,0.92,1.00;1.00)]	[(0.85,0.88,0.90,0.95,0.97,0.97;0.90),(0.76,0.78,0.80,0.86,0.92,1.00;1.00)]
Attributes	$A_2(n = 1)$	$A_2(n = 2)$
c_1	[(0.71,0.74,0.81,0.88;0.90),(0.61,0.64,0.81,0.81,1.00;1.00)]	[(0.71,0.74,0.75,0.78,0.81,0.88;0.90),(0.61,0.64,0.67,0.68,0.81,1.00;1.00)]
c_2	[(0.08,0.08,0.14,0.22;0.90),(0.03,0.06,0.14,0.30;1.00)]	[(0.08,0.08,0.08,0.14,0.16,0.22;0.90),(0.03,0.06,0.07,0.09,0.14,0.30;1.00)]
c_3	[(0.91,0.92,0.97,0.99;0.90),(0.84,0.85,0.97,1.00;1.00)]	[(0.91,0.92,0.93,0.95,0.97,0.99;0.90),(0.84,0.85,0.88,0.90,0.97,1.00;1.00)]
c_4	[(0.87,0.89,0.95,0.97;0.90),(0.79,0.81,0.95,1.00;1.00)]	[(0.87,0.89,0.90,0.93,0.95,0.97;0.90),(0.79,0.81,0.82,0.85,0.95,1.00;1.00)]
Attributes	$A_3(n = 1)$	$A_3(n = 2)$
c_1	[(0.80,0.81,0.88,0.93;0.90),(0.71,0.74,0.88,0.98;1.00)]	[(0.80,0.81,0.82,0.85,0.88,0.93;0.90),(0.71,0.74,0.75,0.82,0.88,0.98;1.00)]
c_2	[(0.01,0.01,0.03,0.09;0.90),(0.00,0.01,0.03,0.15;1.00)]	[(0.01,0.01,0.01,0.02,0.03,0.09;0.90),(0.00,0.01,0.01,0.02,0.03,0.15;1.00)]
c_3	[(0.54,0.55,0.64,0.84;0.90),(0.44,0.45,0.64,0.84;1.00)]	[(0.54,0.55,0.56,0.57,0.64,0.84;0.90),(0.44,0.45,0.50,0.55,0.64,0.84;1.00)]
c_4	[(0.90,0.91,0.97,0.99;0.90),(0.84,0.85,0.97,1.00;1.00)]	[(0.90,0.91,0.92,0.95,0.97,0.99;0.90),(0.84,0.85,0.86,0.88,0.97,1.00;1.00)]

Table 6. Aggregate $nPIVFS$ decision matrix $n = 3$.

Attributes	A_1
c_1	[(0.69,0.71,0.74,0.76,0.79,0.82,0.85,0.87;0.90),(0.59,0.65,0.68,0.76,0.79,0.79,0.85,0.95;1.00)]
c_2	[(0.03,0.04,0.05,0.06,0.06,0.11,0.13,0.13;0.90),(0.00,0.03,0.04,0.06,0.06,0.10,0.12,0.21;1.00)]
c_3	[(0.85,0.88,0.90,0.90,0.93,0.95,0.97,0.97;0.90),(0.76,0.78,0.80,0.82,0.84,0.90,0.92,1.00;1.00)]
c_4	[(0.85,0.88,0.90,0.90,0.93,0.95,0.97,0.97;0.90),(0.76,0.78,0.80,0.82,0.84,0.90,0.92,1.00;1.00)]
Attributes	A_2
c_1	[(0.71,0.74,0.75,0.75,0.78,0.81,0.81,0.88;0.90),(0.61,0.64,0.67,0.75,0.78,0.81,0.81,1.00;1.00)]
c_2	[(0.08,0.08,0.08,0.08,0.09,0.14,0.14,0.22;0.90),(0.03,0.06,0.07,0.08,0.10,0.14,0.14,0.30;1.00)]
c_3	[(0.91,0.92,0.93,0.93,0.95,0.97,0.97,0.99;0.90),(0.84,0.85,0.88,0.91,0.94,0.97,0.97,1.00;1.00)]
c_4	[(0.87,0.89,0.90,0.92,0.93,0.95,0.95,0.97;0.90),(0.79,0.81,0.82,0.83,0.90,0.95,0.95,1.00;1.00)]
Attributes	A_3
c_1	[(0.80,0.81,0.82,0.84,0.87,0.88,0.88,0.93;0.90),(0.71,0.74,0.75,0.80,0.81,0.88,0.88,0.98;1.00)]
c_2	[(0.01,0.01,0.01,0.02,0.02,0.03,0.03,0.09;0.90),(0.00,0.01,0.02,0.03,0.03,0.03,0.03,0.15;1.00)]
c_3	[(0.54,0.55,0.57,0.61,0.62,0.64,0.64,0.84;0.90),(0.44,0.45,0.55,0.56,0.60,0.64,0.64,0.84;1.00)]
c_4	[(0.90,0.91,0.92,0.94,0.95,0.97,0.97,0.99;0.90),(0.84,0.85,0.88,0.91,0.93,0.97,0.97,1.00;1.00)]

Then based on Eq (3.6) with the step size of α as 0.1 and Eq (5.1), the value of α controls the discrimination capability of entropy for fuzzy information. We adopt $\alpha = 0.1, 0.2, \dots, 0.9$ for parameter sensitivity tests, so as to search for the parameter with optimal discrimination performance. When $n = 1, 2, \dots, 6$, we can substitute the numerical value of $E(d_{it})_n^\alpha$ to calculate the results (the specific date can be found in the Appendix).

Therefore, we can get the aggregate entropy and attribute weights when $n = 1, 2, \dots, 6; \alpha = 0.1, 0.2, \dots, 0.9$ (the specific date can be found in Appendix), where E_t^n and w^n are the entropy and attribute values when $n = 1, 2, \dots, 6$, respectively, $t = 1, 2, \dots, m$.

The weighted decision matrix is obtained according to Eq (5.2), and the specific results are listed in Tables A5–A10 in the Appendix. It is easy to obtain positive and negative ideal solutions from Tables A11–A16 in the Appendix.

Then based on Eqs (5.4) and (5.5), we can get the results (the specific date can be found in the Appendix).

Finally, we derive the closeness of alternatives based on Eq (5.6), and results are calculated (the specific date can be found in Appendix).

Obviously, by analysis and calculation, we can obtain $T_n(A_2) > T_n(A_1) > T_n(A_3)$ for all $n = 1, 2, \dots, 6$, so the most suitable global supplier is A_2 .

Comparing and analyzing our method with other methods in the same case reveals the following advantages:

(a) Compared with the other methods in Refs. [13–16], it is verified that our proposed method is rational and feasible. We calculated that the ranking of alternatives is consistent with the results of the

other four literatures, please refer to Table 7.

Table 7. Comparison of different methods.

Methods	Ranking order	Alternatives order
The proposed method	$T_n(A_2) \geq T_n(A_1) \geq T_n(A_3) (n = 1, 2, \dots, 6)$	$A_2 \succ A_1 \succ A_3$
Ali et al. [15]	$Rank(A_2) \geq Rank(A_1) \geq Rank(A_3)$	$A_2 \succ A_1 \succ A_3$
Wang et al. [13]	$Rank(A_2) \geq Rank(A_1) \geq Rank(A_3)$	$A_2 \succ A_1 \succ A_3$
Pan [16]	$PU(A_2) \geq PU(A_1) \geq PU(A_3)$	$A_2 \succ A_1 \succ A_3$
Jin et al. [14]	$\widehat{T}(A_2) \geq \widehat{T}(A_1) \geq \widehat{T}(A_3)$	$A_2 \succ A_1 \succ A_3$

(b) In this paper, when the function in the total expression of information measures is taken as a logarithmic function and $n = 2$, they cover the logarithmic entropy and logarithmic similarity measure of IT2tFSs in Ref. [14]. Therefore, the newly proposed information measures have a broad range of adaptability and effectiveness.

(c) The operation of the $nPIVFS$ defined in this article is essentially $2n + 2$ numbers operations, which helps to greatly reduce the complexity of the extension operation of traditional interval-valued fuzzy sets.

(d) It can be found that the values of $C(n) = T_n(A_2) - T_n(A_1)$ and $D(n) = T_n(A_1) - T_n(A_3)$ ($n = 1, 2, \dots, 6$) tend to increase gradually with the increase of the n value (see Figure 1). The reason is that as n value increases, the $nPIVFS$ portrays fuzzy information more finely. When α takes the value of 0.5, the function $f_4(x)$ delivers optimal distinguishability for fuzzy information. For this reason, both difference metrics $C(n)$ and $D(n)$ attain their peak values under the same α setting of 0.5.

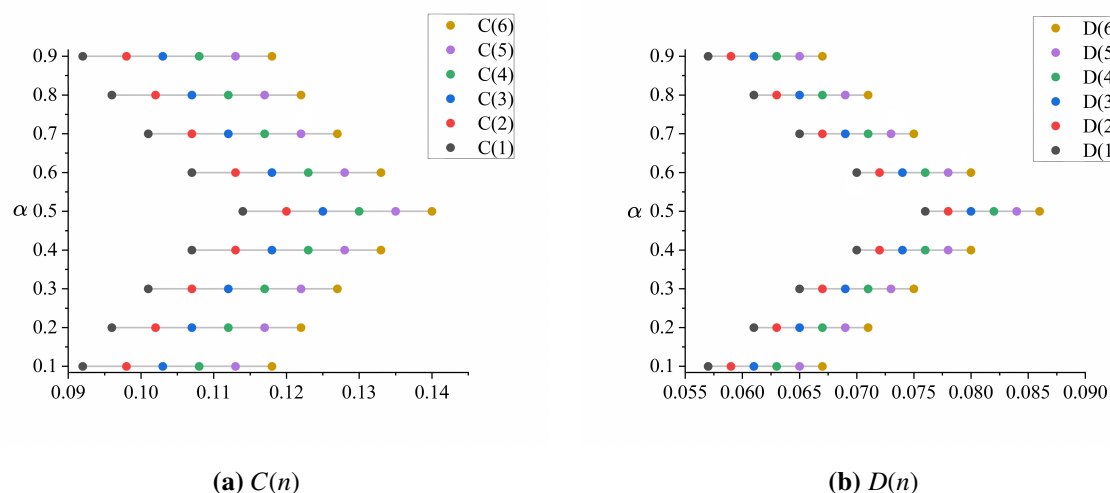


Figure 1. Difference values.

To ensure a fair and reproducible comparison, all methods considered below are applied to the same decision matrices D^k ($k = 1, 2, 3$) and the same decision-maker weight vector $\lambda = (0.30, 0.45, 0.25)$ as in Wang and Liu [13]. Since an $nPIVFS$ reduces to an interval type-2 trapezoidal fuzzy set when $n = 1$, the proposed framework and the methods of [13–16] are defined on the same information space, which

makes the comparison meaningful.

Table 8 reports the closeness degrees obtained by the proposed method together with the rankings produced by four existing entropy-based MADM methods on the same case. For the proposed method, we report $T(A_1)$, $T(A_2)$, $T(A_3)$ at $n = 3$, $\alpha = 0.5$; the same ranking is obtained for all combinations of $n = 1, \dots, 6$ and $\alpha = 0.1, \dots, 0.9$ (see Table 9).

Table 8. Quantitative comparison on the same supplier-selection case.

Method	$T(A_1)$	$T(A_2)$	$T(A_3)$	Ranking	ρ	τ
Proposed ($n = 3$, $\alpha = 0.5$)	0.455	0.512	0.410	$A_2 > A_1 > A_3$	–	–
Wang and Liu [13]	n.r.	n.r.	n.r.	$A_2 > A_1 > A_3$	1	1
Jin et al. [14]	n.r.	n.r.	n.r.	$A_2 > A_1 > A_3$	1	1
Dehghani and Titkanlue [15]	n.r.	n.r.	n.r.	$A_2 > A_1 > A_3$	1	1
Pan [16]	n.r.	n.r.	n.r.	$A_2 > A_1 > A_3$	1	1

We evaluate the stability of the proposed ranking over the whole parameter space. As reported in Table 9, the order $A_2 > A_1 > A_3$ is obtained in all $6 \times 9 = 54$ combinations of $n \in \{1, \dots, 6\}$ and $\alpha \in \{0.1, \dots, 0.9\}$, giving a ranking consistency rate of 100%. Across the 54 configurations, the closeness degrees lie in the ranges $T(A_1) \in [0.415, 0.457]$, $T(A_2) \in [0.506, 0.544]$, and $T(A_3) \in [0.349, 0.418]$; consequently, the gap between A_2 and A_1 is never smaller than 0.049, and the ranking is not sensitive to n or α .

Table 9. Ranking consistency over the parameter space.

Parameter configurations	Configs	Identical rankings	Consistency rate
$n = 1, \dots, 6$; $\alpha = 0.1, \dots, 0.9$	54	54	100%
Cross-method comparison with [13–16]	4	4	100%

Table 10 lists the closeness degrees and the separation indices $C = T(A_2) - T(A_1)$ and $D = T(A_1) - T(A_3)$ for $\alpha = 0.5$ and $n = 1, \dots, 6$. The ranking is invariant, while the separation is largest at $n = 1$ and remains essentially stable, with a mild increase for $n \geq 2$.

Table 10. Closeness degrees and separation indices for $\alpha = 0.5$.

n	$T(A_1)$	$T(A_2)$	$T(A_3)$	C	D	Ranking
1	0.428	0.537	0.354	0.109	0.074	$A_2 > A_1 > A_3$
2	0.457	0.514	0.411	0.057	0.046	$A_2 > A_1 > A_3$
3	0.455	0.512	0.410	0.057	0.045	$A_2 > A_1 > A_3$
4	0.456	0.513	0.409	0.057	0.047	$A_2 > A_1 > A_3$
5	0.455	0.515	0.408	0.060	0.047	$A_2 > A_1 > A_3$
6	0.454	0.516	0.407	0.062	0.047	$A_2 > A_1 > A_3$

The baseline case uses $\lambda = (0.30, 0.45, 0.25)$. To examine whether the result depends on the preference structure, we recompute the whole procedure for several alternative weight vectors. All scenarios yield the same ranking $A_2 > A_1 > A_3$, indicating that the proposed method is robust to variations in decision-maker preferences.

6. Conclusions

Main findings: In this article, we first define the intersection, union, complement, and multiplication operations of nPIVFSs, which take the form of two sets of $2n + 2$ number operations, thereby reducing the complexity of traditional interval-valued fuzzy set operations. In addition, we investigate the entropy, similarity measure, and cross-entropy functions of nPIVFSs in Theorem 3.3, Theorem 3.12, and Theorem 3.14. The transformation relationship between entropy, the similarity measure, and cross-entropy are proven in Theorem 4.1, Corollary 4.2, and Theorem 4.3. In detail, it is the first work to study the information measures on nPIVFSs. As a result, bringing the selected different functions into the newly established total expression of information measures will result in various information measures. In summary, a new MADM method based on entropy and cross-entropy is presented to solve MADM problems.

Limitations and weaknesses: Several limitations should be acknowledged. (i) The choice of n controls a resolution–cost trade-off. Finer polygons represent fuzzy information more precisely, but the number of stored scalars grows as $4n + 6$ and the overall cost is $O(Tqmn)$; no automatic rule for selecting n is provided. (ii) The generating function f (or g) and the parameter α must be chosen by the user; our sensitivity analysis shows that the value of α that maximizes the separation between alternatives is data-dependent, and a data-driven selection procedure is still missing. (iii) The monotonicity axioms (E4) and (S4) are based on comparison conditions; extending the measures to arbitrary, non-comparable nPIVFSs requires additional assumptions. (iv) The transformation relations are established for the class of generating functions considered here and for the constant-1/2 reference set; a complete characterization of the admissible pairs (f, g) and their inverses remains open. (v) The MADM method assumes complete and comparable evaluation matrices, homogeneous decision-makers (a common n and a single weight vector λ), and attribute weights derived solely from entropy, without incorporating subjective or reliability information. (vi) The empirical validation is limited to a single supplier-selection case; large-scale, real-world benchmarks and statistical significance tests are still needed.

Future research directions: Several directions follow naturally. (1) *Dynamic decision-making*: extending the measures to time-varying nPIVFS sequences, with dynamic entropy/similarity and online updating for real-time decision support. (2) *Group decision-making under heterogeneous information*: allowing decision-makers to use different representations and different credibility levels, together with consensus-reaching mechanisms. (3) *Hybrid uncertainty models*: combining nPIVFSs with intuitionistic, hesitant, probabilistic, or evidence-theoretic representations, and developing cross-entropy and divergence measures across hybrid structures. (4) *Machine learning and AI*: learning the generating function f/g or the parameter α from data (e.g., through differentiable entropy layers), and integrating the measures as interpretable loss functions or regularizers in neural networks and graph learning. (5) *Scalability and algorithms*: automatic selection of n , distributed/parallel and GPU implementations for large-scale MADM, and an analysis of the approximation error as a function of n . (6) *Theory*: Characterizing the class of admissible generating functions, establishing uniqueness or minimality of representations, and studying the statistical estimation of the proposed measures from finite samples.

Author contributions

Xue Zhao: Writing–review and editing, supervision, project administration; Lili Zhang: Conceptualization, methodology, software, validation, writing–original draft, writing–review and editing; Chunfeng Suo: Writing–review and editing, supervision, project administration, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in the publication of this paper.

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Appendix

(I) The values of $E(d_{lt})_n^\alpha$ ($n = 1, 2, \dots, 6; \alpha = 0.1, 0.2, \dots, 0.9$):

$$E(d_{lt})_{n=1}^{\alpha=0.1} = \begin{pmatrix} 0.058 & 0.018 & 0.026 & 0.026 \\ 0.056 & 0.028 & 0.015 & 0.020 \\ 0.035 & 0.010 & 0.120 & 0.015 \end{pmatrix}, E(d_{lt})_{n=1}^{\alpha=0.2} = \begin{pmatrix} 0.110 & 0.036 & 0.050 & 0.050 \\ 0.110 & 0.055 & 0.029 & 0.040 \\ 0.067 & 0.019 & 0.210 & 0.030 \end{pmatrix},$$

$$E(d_{lt})_{n=1}^{\alpha=0.3} = \begin{pmatrix} 0.160 & 0.053 & 0.074 & 0.074 \\ 0.150 & 0.081 & 0.043 & 0.059 \\ 0.098 & 0.028 & 0.290 & 0.044 \end{pmatrix}, E(d_{lt})_{n=1}^{\alpha=0.4} = \begin{pmatrix} 0.200 & 0.069 & 0.096 & 0.096 \\ 0.190 & 0.110 & 0.056 & 0.077 \\ 0.130 & 0.037 & 0.360 & 0.058 \end{pmatrix},$$

$$E(d_{lt})_{n=1}^{\alpha=0.5} = \begin{pmatrix} 0.240 & 0.085 & 0.120 & 0.120 \\ 0.230 & 0.130 & 0.069 & 0.095 \\ 0.160 & 0.045 & 0.410 & 0.072 \end{pmatrix}, E(d_{lt})_{n=1}^{\alpha=0.6} = \begin{pmatrix} 0.270 & 0.100 & 0.140 & 0.140 \\ 0.270 & 0.150 & 0.082 & 0.110 \\ 0.180 & 0.054 & 0.460 & 0.085 \end{pmatrix},$$

$$E(d_{lt})_{n=1}^{\alpha=0.7} = \begin{pmatrix} 0.310 & 0.120 & 0.160 & 0.160 \\ 0.300 & 0.170 & 0.095 & 0.130 \\ 0.210 & 0.062 & 0.500 & 0.098 \end{pmatrix}, E(d_{lt})_{n=1}^{\alpha=0.8} = \begin{pmatrix} 0.340 & 0.130 & 0.180 & 0.180 \\ 0.330 & 0.190 & 0.110 & 0.140 \\ 0.230 & 0.070 & 0.530 & 0.110 \end{pmatrix},$$

$$E(d_{lt})_{n=1}^{\alpha=0.9} = \begin{pmatrix} 0.360 & 0.140 & 0.200 & 0.200 \\ 0.360 & 0.210 & 0.120 & 0.160 \\ 0.250 & 0.078 & 0.560 & 0.120 \end{pmatrix}, E(d_{lt})_{n=2}^{\alpha=0.1} = \begin{pmatrix} 0.041 & 0.013 & 0.018 & 0.018 \\ 0.040 & 0.020 & 0.010 & 0.015 \\ 0.025 & 0.007 & 0.083 & 0.011 \end{pmatrix},$$

$$E(d_{lt})_{n=2}^{\alpha=0.2} = \begin{pmatrix} 0.079 & 0.026 & 0.036 & 0.036 \\ 0.076 & 0.040 & 0.021 & 0.028 \\ 0.048 & 0.013 & 0.150 & 0.021 \end{pmatrix}, E(d_{lt})_{n=2}^{\alpha=0.3} = \begin{pmatrix} 0.110 & 0.038 & 0.053 & 0.053 \\ 0.110 & 0.058 & 0.031 & 0.042 \\ 0.070 & 0.020 & 0.210 & 0.031 \end{pmatrix},$$

$$E(d_{lt})_{n=2}^{\alpha=0.4} = \begin{pmatrix} 0.140 & 0.050 & 0.069 & 0.069 \\ 0.140 & 0.075 & 0.040 & 0.055 \\ 0.091 & 0.026 & 0.260 & 0.041 \end{pmatrix}, E(d_{lt})_{n=2}^{\alpha=0.5} = \begin{pmatrix} 0.170 & 0.061 & 0.084 & 0.084 \\ 0.170 & 0.092 & 0.050 & 0.068 \\ 0.110 & 0.032 & 0.300 & 0.051 \end{pmatrix},$$

$$E(d_{lt})_{n=2}^{\alpha=0.6} = \begin{pmatrix} 0.200 & 0.072 & 0.099 & 0.099 \\ 0.190 & 0.110 & 0.059 & 0.080 \\ 0.130 & 0.039 & 0.330 & 0.061 \end{pmatrix}, E(d_{lt})_{n=2}^{\alpha=0.7} = \begin{pmatrix} 0.220 & 0.083 & 0.110 & 0.110 \\ 0.210 & 0.120 & 0.068 & 0.091 \\ 0.150 & 0.044 & 0.360 & 0.070 \end{pmatrix},$$

$$E(d_{lt})_{n=2}^{\alpha=0.8} = \begin{pmatrix} 0.240 & 0.093 & 0.130 & 0.130 \\ 0.230 & 0.140 & 0.076 & 0.100 \\ 0.160 & 0.050 & 0.380 & 0.079 \end{pmatrix}, E(d_{lt})_{n=2}^{\alpha=0.9} = \begin{pmatrix} 0.260 & 0.100 & 0.140 & 0.140 \\ 0.250 & 0.150 & 0.085 & 0.110 \\ 0.180 & 0.056 & 0.400 & 0.087 \end{pmatrix},$$

$$E(d_{lt})_{n=3}^{\alpha=0.1} = \begin{pmatrix} 0.032 & 0.010 & 0.014 & 0.014 \\ 0.031 & 0.016 & 0.008 & 0.011 \\ 0.019 & 0.005 & 0.065 & 0.008 \end{pmatrix}, E(d_{lt})_{n=3}^{\alpha=0.2} = \begin{pmatrix} 0.061 & 0.020 & 0.028 & 0.028 \\ 0.059 & 0.031 & 0.016 & 0.022 \\ 0.037 & 0.010 & 0.120 & 0.017 \end{pmatrix},$$

$$E(d_{lt})_{n=3}^{\alpha=0.3} = \begin{pmatrix} 0.087 & 0.029 & 0.041 & 0.041 \\ 0.084 & 0.045 & 0.024 & 0.033 \\ 0.055 & 0.015 & 0.160 & 0.024 \end{pmatrix}, E(d_{lt})_{n=3}^{\alpha=0.4} = \begin{pmatrix} 0.110 & 0.039 & 0.053 & 0.053 \\ 0.110 & 0.059 & 0.031 & 0.043 \\ 0.071 & 0.020 & 0.200 & 0.032 \end{pmatrix},$$

$$\begin{aligned}
E(d_{lt})_{n=3}^{\alpha=0.5} &= \begin{pmatrix} 0.130 & 0.047 & 0.065 & 0.065 \\ 0.130 & 0.072 & 0.039 & 0.053 \\ 0.086 & 0.025 & 0.230 & 0.040 \end{pmatrix}, \quad E(d_{lt})_{n=3}^{\alpha=0.6} = \begin{pmatrix} 0.150 & 0.056 & 0.077 & 0.077 \\ 0.150 & 0.084 & 0.046 & 0.062 \\ 0.100 & 0.030 & 0.260 & 0.047 \end{pmatrix}, \\
E(d_{lt})_{n=3}^{\alpha=0.7} &= \begin{pmatrix} 0.170 & 0.064 & 0.088 & 0.088 \\ 0.170 & 0.096 & 0.053 & 0.071 \\ 0.110 & 0.035 & 0.280 & 0.054 \end{pmatrix}, \quad E(d_{lt})_{n=3}^{\alpha=0.8} = \begin{pmatrix} 0.190 & 0.072 & 0.099 & 0.099 \\ 0.180 & 0.110 & 0.059 & 0.080 \\ 0.130 & 0.039 & 0.300 & 0.061 \end{pmatrix}, \\
E(d_{lt})_{n=3}^{\alpha=0.9} &= \begin{pmatrix} 0.200 & 0.080 & 0.110 & 0.110 \\ 0.200 & 0.120 & 0.066 & 0.088 \\ 0.140 & 0.043 & 0.310 & 0.068 \end{pmatrix}, \quad E(d_{lt})_{n=4}^{\alpha=0.1} = \begin{pmatrix} 0.026 & 0.008 & 0.012 & 0.012 \\ 0.025 & 0.013 & 0.007 & 0.009 \\ 0.016 & 0.004 & 0.053 & 0.007 \end{pmatrix}, \\
E(d_{lt})_{n=4}^{\alpha=0.2} &= \begin{pmatrix} 0.050 & 0.016 & 0.023 & 0.023 \\ 0.048 & 0.025 & 0.013 & 0.018 \\ 0.031 & 0.008 & 0.096 & 0.014 \end{pmatrix}, \quad E(d_{lt})_{n=4}^{\alpha=0.3} = \begin{pmatrix} 0.071 & 0.024 & 0.033 & 0.033 \\ 0.069 & 0.037 & 0.019 & 0.027 \\ 0.045 & 0.013 & 0.130 & 0.020 \end{pmatrix}, \\
E(d_{lt})_{n=4}^{\alpha=0.4} &= \begin{pmatrix} 0.091 & 0.032 & 0.044 & 0.044 \\ 0.088 & 0.048 & 0.026 & 0.035 \\ 0.058 & 0.017 & 0.160 & 0.026 \end{pmatrix}, \quad E(d_{lt})_{n=4}^{\alpha=0.5} = \begin{pmatrix} 0.110 & 0.039 & 0.054 & 0.054 \\ 0.110 & 0.059 & 0.032 & 0.043 \\ 0.071 & 0.021 & 0.190 & 0.033 \end{pmatrix}, \\
E(d_{lt})_{n=4}^{\alpha=0.6} &= \begin{pmatrix} 0.120 & 0.046 & 0.063 & 0.063 \\ 0.120 & 0.069 & 0.037 & 0.051 \\ 0.082 & 0.025 & 0.210 & 0.039 \end{pmatrix}, \quad E(d_{lt})_{n=4}^{\alpha=0.7} = \begin{pmatrix} 0.140 & 0.053 & 0.072 & 0.072 \\ 0.140 & 0.078 & 0.043 & 0.058 \\ 0.094 & 0.028 & 0.230 & 0.044 \end{pmatrix}, \\
E(d_{lt})_{n=4}^{\alpha=0.8} &= \begin{pmatrix} 0.150 & 0.059 & 0.081 & 0.081 \\ 0.150 & 0.087 & 0.048 & 0.065 \\ 0.100 & 0.032 & 0.240 & 0.050 \end{pmatrix}, \quad E(d_{lt})_{n=4}^{\alpha=0.9} = \begin{pmatrix} 0.170 & 0.066 & 0.089 & 0.089 \\ 0.160 & 0.096 & 0.054 & 0.072 \\ 0.110 & 0.036 & 0.260 & 0.056 \end{pmatrix}, \\
E(d_{lt})_{n=5}^{\alpha=0.1} &= \begin{pmatrix} 0.022 & 0.007 & 0.009 & 0.009 \\ 0.021 & 0.011 & 0.005 & 0.007 \\ 0.013 & 0.003 & 0.045 & 0.005 \end{pmatrix}, \quad E(d_{lt})_{n=5}^{\alpha=0.2} = \begin{pmatrix} 0.042 & 0.014 & 0.019 & 0.019 \\ 0.041 & 0.021 & 0.011 & 0.015 \\ 0.026 & 0.007 & 0.082 & 0.011 \end{pmatrix}, \\
E(d_{lt})_{n=5}^{\alpha=0.3} &= \begin{pmatrix} 0.060 & 0.020 & 0.028 & 0.028 \\ 0.058 & 0.031 & 0.016 & 0.023 \\ 0.038 & 0.011 & 0.110 & 0.017 \end{pmatrix}, \quad E(d_{lt})_{n=5}^{\alpha=0.4} = \begin{pmatrix} 0.077 & 0.027 & 0.037 & 0.037 \\ 0.075 & 0.041 & 0.022 & 0.030 \\ 0.049 & 0.014 & 0.140 & 0.022 \end{pmatrix}, \\
E(d_{lt})_{n=5}^{\alpha=0.5} &= \begin{pmatrix} 0.092 & 0.033 & 0.045 & 0.045 \\ 0.089 & 0.050 & 0.027 & 0.036 \\ 0.060 & 0.017 & 0.160 & 0.028 \end{pmatrix}, \quad E(d_{lt})_{n=5}^{\alpha=0.6} = \begin{pmatrix} 0.110 & 0.039 & 0.053 & 0.053 \\ 0.100 & 0.058 & 0.032 & 0.043 \\ 0.070 & 0.021 & 0.180 & 0.033 \end{pmatrix}, \\
E(d_{lt})_{n=5}^{\alpha=0.7} &= \begin{pmatrix} 0.120 & 0.045 & 0.061 & 0.061 \\ 0.110 & 0.066 & 0.036 & 0.049 \\ 0.079 & 0.024 & 0.190 & 0.038 \end{pmatrix}, \quad E(d_{lt})_{n=5}^{\alpha=0.8} = \begin{pmatrix} 0.130 & 0.050 & 0.068 & 0.068 \\ 0.130 & 0.074 & 0.041 & 0.055 \\ 0.089 & 0.027 & 0.210 & 0.042 \end{pmatrix}, \\
E(d_{lt})_{n=5}^{\alpha=0.9} &= \begin{pmatrix} 0.140 & 0.056 & 0.075 & 0.075 \\ 0.140 & 0.081 & 0.046 & 0.061 \\ 0.097 & 0.030 & 0.220 & 0.047 \end{pmatrix}, \quad E(d_{lt})_{n=6}^{\alpha=0.1} = \begin{pmatrix} 0.019 & 0.006 & 0.008 & 0.008 \\ 0.019 & 0.009 & 0.005 & 0.006 \\ 0.012 & 0.003 & 0.039 & 0.005 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
E(d_{lt})_{n=6}^{\alpha=0.2} &= \begin{pmatrix} 0.037 & 0.012 & 0.017 & 0.017 \\ 0.035 & 0.018 & 0.009 & 0.013 \\ 0.022 & 0.006 & 0.071 & 0.009 \end{pmatrix}, & E(d_{lt})_{n=6}^{\alpha=0.3} &= \begin{pmatrix} 0.052 & 0.018 & 0.025 & 0.025 \\ 0.051 & 0.027 & 0.014 & 0.020 \\ 0.033 & 0.009 & 0.097 & 0.015 \end{pmatrix}, \\
E(d_{lt})_{n=6}^{\alpha=0.4} &= \begin{pmatrix} 0.067 & 0.023 & 0.032 & 0.032 \\ 0.065 & 0.035 & 0.019 & 0.026 \\ 0.042 & 0.012 & 0.120 & 0.019 \end{pmatrix}, & E(d_{lt})_{n=6}^{\alpha=0.5} &= \begin{pmatrix} 0.080 & 0.028 & 0.039 & 0.039 \\ 0.077 & 0.043 & 0.023 & 0.032 \\ 0.052 & 0.015 & 0.140 & 0.024 \end{pmatrix}, \\
E(d_{lt})_{n=6}^{\alpha=0.6} &= \begin{pmatrix} 0.091 & 0.034 & 0.046 & 0.046 \\ 0.089 & 0.050 & 0.027 & 0.037 \\ 0.060 & 0.018 & 0.150 & 0.028 \end{pmatrix}, & E(d_{lt})_{n=6}^{\alpha=0.7} &= \begin{pmatrix} 0.100 & 0.039 & 0.053 & 0.053 \\ 0.100 & 0.057 & 0.032 & 0.043 \\ 0.069 & 0.021 & 0.170 & 0.033 \end{pmatrix}, \\
E(d_{lt})_{n=6}^{\alpha=0.8} &= \begin{pmatrix} 0.110 & 0.043 & 0.059 & 0.059 \\ 0.110 & 0.064 & 0.036 & 0.048 \\ 0.077 & 0.023 & 0.180 & 0.037 \end{pmatrix}, & E(d_{lt})_{n=6}^{\alpha=0.9} &= \begin{pmatrix} 0.120 & 0.048 & 0.065 & 0.065 \\ 0.120 & 0.071 & 0.039 & 0.053 \\ 0.084 & 0.026 & 0.190 & 0.041 \end{pmatrix}.
\end{aligned}$$

(II) Aggregate entropy and the attribute weights ($n = 1, 2, \dots, 6; \alpha = 0.1, 0.2, \dots, 0.9$):

$$\begin{aligned}
E_t^1(\alpha = 0.1) &= (0.049, 0.019, 0.052, 0.020)^T, & w^1(\alpha = 0.1) &= (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^1(\alpha = 0.2) &= (0.094, 0.037, 0.097, 0.040)^T, & w^1(\alpha = 0.2) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^1(\alpha = 0.3) &= (0.140, 0.054, 0.140, 0.059)^T, & w^1(\alpha = 0.3) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^1(\alpha = 0.4) &= (0.170, 0.071, 0.170, 0.077)^T, & w^1(\alpha = 0.4) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^1(\alpha = 0.5) &= (0.210, 0.087, 0.200, 0.095)^T, & w^1(\alpha = 0.5) &= (0.230, 0.270, 0.230, 0.270)^T; \\
E_t^1(\alpha = 0.6) &= (0.240, 0.100, 0.230, 0.110)^T, & w^1(\alpha = 0.6) &= (0.230, 0.270, 0.230, 0.270)^T; \\
E_t^1(\alpha = 0.7) &= (0.270, 0.120, 0.250, 0.130)^T, & w^1(\alpha = 0.7) &= (0.230, 0.270, 0.230, 0.270)^T; \\
E_t^1(\alpha = 0.8) &= (0.300, 0.130, 0.270, 0.140)^T, & w^1(\alpha = 0.8) &= (0.220, 0.280, 0.230, 0.270)^T; \\
E_t^1(\alpha = 0.9) &= (0.320, 0.140, 0.290, 0.160)^T, & w^1(\alpha = 0.9) &= (0.220, 0.280, 0.230, 0.270)^T; \\
E_t^2(\alpha = 0.1) &= (0.035, 0.013, 0.037, 0.015)^T, & w^2(\alpha = 0.1) &= (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^2(\alpha = 0.2) &= (0.067, 0.026, 0.069, 0.022)^T, & w^2(\alpha = 0.2) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.3) &= (0.097, 0.039, 0.097, 0.042)^T, & w^2(\alpha = 0.3) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.4) &= (0.120, 0.050, 0.120, 0.055)^T, & w^2(\alpha = 0.4) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.5) &= (0.150, 0.062, 0.140, 0.068)^T, & w^2(\alpha = 0.5) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.6) &= (0.170, 0.073, 0.160, 0.080)^T, & w^2(\alpha = 0.6) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.7) &= (0.190, 0.083, 0.180, 0.091)^T, & w^2(\alpha = 0.7) &= (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.8) &= (0.210, 0.094, 0.190, 0.100)^T, & w^2(\alpha = 0.8) &= (0.230, 0.270, 0.240, 0.260)^T; \\
E_t^2(\alpha = 0.9) &= (0.230, 0.100, 0.210, 0.110)^T, & w^2(\alpha = 0.9) &= (0.230, 0.270, 0.240, 0.270)^T; \\
E_t^3(\alpha = 0.1) &= (0.027, 0.010, 0.029, 0.011)^T, & w^3(\alpha = 0.1) &= (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^3(\alpha = 0.2) &= (0.052, 0.020, 0.054, 0.022)^T, & w^3(\alpha = 0.2) &= (0.250, 0.250, 0.250, 0.250)^T;
\end{aligned}$$

$$\begin{aligned}
E_t^3(\alpha = 0.3) &= (0.075, 0.030, 0.076, 0.033)^T, \quad w^3(\alpha = 0.3) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^3(\alpha = 0.4) &= (0.097, 0.039, 0.095, 0.043)^T, \quad w^3(\alpha = 0.4) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^3(\alpha = 0.5) &= (0.120, 0.048, 0.110, 0.053)^T, \quad w^3(\alpha = 0.5) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^3(\alpha = 0.6) &= (0.130, 0.057, 0.130, 0.062)^T, \quad w^3(\alpha = 0.6) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^3(\alpha = 0.7) &= (0.150, 0.065, 0.140, 0.071)^T, \quad w^3(\alpha = 0.7) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^3(\alpha = 0.8) &= (0.170, 0.073, 0.150, 0.080)^T, \quad w^3(\alpha = 0.8) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^3(\alpha = 0.9) &= (0.180, 0.080, 0.160, 0.088)^T, \quad w^3(\alpha = 0.9) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^4(\alpha = 0.1) &= (0.022, 0.009, 0.024, 0.009)^T, \quad w^4(\alpha = 0.1) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^4(\alpha = 0.2) &= (0.043, 0.017, 0.044, 0.018)^T, \quad w^4(\alpha = 0.2) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^4(\alpha = 0.3) &= (0.062, 0.025, 0.062, 0.027)^T, \quad w^4(\alpha = 0.3) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^4(\alpha = 0.4) &= (0.079, 0.032, 0.077, 0.035)^T, \quad w^4(\alpha = 0.4) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^4(\alpha = 0.5) &= (0.095, 0.039, 0.091, 0.043)^T, \quad w^4(\alpha = 0.5) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^4(\alpha = 0.6) &= (0.110, 0.046, 0.100, 0.051)^T, \quad w^4(\alpha = 0.6) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^4(\alpha = 0.7) &= (0.120, 0.053, 0.110, 0.058)^T, \quad w^4(\alpha = 0.7) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^4(\alpha = 0.8) &= (0.140, 0.060, 0.120, 0.064)^T, \quad w^4(\alpha = 0.8) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^4(\alpha = 0.9) &= (0.150, 0.066, 0.130, 0.070)^T, \quad w^4(\alpha = 0.9) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^5(\alpha = 0.1) &= (0.019, 0.007, 0.020, 0.008)^T, \quad w^5(\alpha = 0.1) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^5(\alpha = 0.2) &= (0.036, 0.014, 0.037, 0.015)^T, \quad w^5(\alpha = 0.2) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^5(\alpha = 0.3) &= (0.052, 0.021, 0.052, 0.023)^T, \quad w^5(\alpha = 0.3) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^5(\alpha = 0.4) &= (0.067, 0.027, 0.065, 0.030)^T, \quad w^5(\alpha = 0.4) = (0.240, 0.260, 0.250, 0.250)^T; \\
E_t^5(\alpha = 0.5) &= (0.080, 0.033, 0.077, 0.036)^T, \quad w^5(\alpha = 0.5) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^5(\alpha = 0.6) &= (0.093, 0.039, 0.087, 0.043)^T, \quad w^5(\alpha = 0.6) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^5(\alpha = 0.7) &= (0.100, 0.045, 0.097, 0.049)^T, \quad w^5(\alpha = 0.7) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^5(\alpha = 0.8) &= (0.110, 0.050, 0.100, 0.055)^T, \quad w^5(\alpha = 0.8) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^5(\alpha = 0.9) &= (0.120, 0.056, 0.110, 0.061)^T, \quad w^5(\alpha = 0.9) = (0.240, 0.260, 0.240, 0.260)^T; \\
E_t^6(\alpha = 0.1) &= (0.016, 0.006, 0.017, 0.007)^T, \quad w^6(\alpha = 0.1) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^6(\alpha = 0.2) &= (0.031, 0.012, 0.032, 0.013)^T, \quad w^6(\alpha = 0.2) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^6(\alpha = 0.3) &= (0.045, 0.018, 0.045, 0.020)^T, \quad w^6(\alpha = 0.3) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^6(\alpha = 0.4) &= (0.058, 0.024, 0.057, 0.026)^T, \quad w^6(\alpha = 0.4) = (0.250, 0.250, 0.250, 0.250)^T; \\
E_t^6(\alpha = 0.5) &= (0.070, 0.029, 0.067, 0.032)^T, \quad w^6(\alpha = 0.5) = (0.240, 0.260, 0.250, 0.250)^T; \\
E_t^6(\alpha = 0.6) &= (0.080, 0.034, 0.076, 0.037)^T, \quad w^6(\alpha = 0.6) = (0.240, 0.260, 0.240, 0.260)^T;
\end{aligned}$$

$$E_t^6(\alpha = 0.7) = (0.090, 0.039, 0.084, 0.043)^T, \quad w^6(\alpha = 0.7) = (0.240, 0.260, 0.240, 0.260)^T;$$

$$E_t^6(\alpha = 0.8) = (0.099, 0.044, 0.091, 0.048)^T, \quad w^6(\alpha = 0.8) = (0.240, 0.260, 0.240, 0.260)^T;$$

$$E_t^6(\alpha = 0.9) = (0.110, 0.048, 0.098, 0.053)^T, \quad w^6(\alpha = 0.9) = (0.240, 0.260, 0.240, 0.260)^T.$$

(III) Positive ideal solutions and negative ideal solutions ($n = 1, 2, \dots, 6; \alpha = 0.1, 0.2, \dots, 0.9$):

$$C_{n=1}^+(\alpha = 0.1) = (0.024, 0.005, 0.012), \quad C_{n=1}^-(\alpha = 0.1) = (0.007, 0.023, 0.017);$$

$$C_{n=1}^+(\alpha = 0.2) = (0.021, 0.004, 0.013), \quad C_{n=1}^-(\alpha = 0.2) = (0.006, 0.024, 0.016);$$

$$C_{n=1}^+(\alpha = 0.3) = (0.018, 0.006, 0.011), \quad C_{n=1}^-(\alpha = 0.3) = (0.008, 0.020, 0.015);$$

$$C_{n=1}^+(\alpha = 0.4) = (0.017, 0.003, 0.014), \quad C_{n=1}^-(\alpha = 0.4) = (0.005, 0.022, 0.016);$$

$$C_{n=1}^+(\alpha = 0.5) = (0.019, 0.005, 0.010), \quad C_{n=1}^-(\alpha = 0.5) = (0.007, 0.021, 0.014);$$

$$C_{n=1}^+(\alpha = 0.6) = (0.016, 0.004, 0.012), \quad C_{n=1}^-(\alpha = 0.6) = (0.006, 0.023, 0.015);$$

$$C_{n=1}^+(\alpha = 0.7) = (0.018, 0.006, 0.011), \quad C_{n=1}^-(\alpha = 0.7) = (0.004, 0.022, 0.016);$$

$$C_{n=1}^+(\alpha = 0.8) = (0.017, 0.003, 0.013), \quad C_{n=1}^-(\alpha = 0.8) = (0.007, 0.020, 0.013);$$

$$C_{n=1}^+(\alpha = 0.9) = (0.019, 0.005, 0.010), \quad C_{n=1}^-(\alpha = 0.9) = (0.005, 0.021, 0.014);$$

$$C_{n=2}^+(\alpha = 0.1) = (0.312, 0.284, 0.297), \quad C_{n=2}^-(\alpha = 0.1) = (0.281, 0.309, 0.293);$$

$$C_{n=2}^+(\alpha = 0.2) = (0.305, 0.288, 0.291), \quad C_{n=2}^-(\alpha = 0.2) = (0.286, 0.302, 0.298);$$

$$C_{n=2}^+(\alpha = 0.3) = (0.317, 0.282, 0.294), \quad C_{n=2}^-(\alpha = 0.3) = (0.283, 0.307, 0.291);$$

$$C_{n=2}^+(\alpha = 0.4) = (0.301, 0.289, 0.296), \quad C_{n=2}^-(\alpha = 0.4) = (0.289, 0.304, 0.295);$$

$$C_{n=2}^+(\alpha = 0.5) = (0.314, 0.285, 0.292), \quad C_{n=2}^-(\alpha = 0.5) = (0.280, 0.308, 0.296);$$

$$C_{n=2}^+(\alpha = 0.6) = (0.308, 0.281, 0.295), \quad C_{n=2}^-(\alpha = 0.6) = (0.287, 0.301, 0.293);$$

$$C_{n=2}^+(\alpha = 0.7) = (0.311, 0.287, 0.290), \quad C_{n=2}^-(\alpha = 0.7) = (0.284, 0.306, 0.299);$$

$$C_{n=2}^+(\alpha = 0.8) = (0.303, 0.283, 0.298), \quad C_{n=2}^-(\alpha = 0.8) = (0.288, 0.303, 0.290);$$

$$C_{n=2}^+(\alpha = 0.9) = (0.315, 0.286, 0.293), \quad C_{n=2}^-(\alpha = 0.9) = (0.282, 0.305, 0.297);$$

$$C_{n=3}^+(\alpha = 0.1) = (0.468, 0.441, 0.453), \quad C_{n=3}^-(\alpha = 0.1) = (0.445, 0.462, 0.451);$$

$$C_{n=3}^+(\alpha = 0.2) = (0.461, 0.447, 0.456), \quad C_{n=3}^-(\alpha = 0.2) = (0.449, 0.465, 0.454);$$

$$C_{n=3}^+(\alpha = 0.3) = (0.465, 0.443, 0.450), \quad C_{n=3}^-(\alpha = 0.3) = (0.442, 0.461, 0.457);$$

$$C_{n=3}^+(\alpha = 0.4) = (0.463, 0.449, 0.452), \quad C_{n=3}^-(\alpha = 0.4) = (0.447, 0.467, 0.450);$$

$$C_{n=3}^+(\alpha = 0.5) = (0.469, 0.440, 0.455), \quad C_{n=3}^-(\alpha = 0.5) = (0.444, 0.463, 0.456);$$

$$C_{n=3}^+(\alpha = 0.6) = (0.460, 0.446, 0.458), \quad C_{n=3}^-(\alpha = 0.6) = (0.448, 0.460, 0.452);$$

$$C_{n=3}^+(\alpha = 0.7) = (0.464, 0.442, 0.451), \quad C_{n=3}^-(\alpha = 0.7) = (0.441, 0.466, 0.455);$$

$$C_{n=3}^+(\alpha = 0.8) = (0.467, 0.448, 0.454), \quad C_{n=3}^-(\alpha = 0.8) = (0.446, 0.464, 0.453);$$

$$\begin{aligned}
C_{n=3}^+(\alpha = 0.9) &= (0.462, 0.444, 0.457), C_{n=3}^-(\alpha = 0.9) = (0.443, 0.468, 0.451); \\
C_{n=4}^+(\alpha = 0.1) &= (0.563, 0.541, 0.552), C_{n=4}^-(\alpha = 0.1) = (0.547, 0.568, 0.551); \\
C_{n=4}^+(\alpha = 0.2) &= (0.558, 0.545, 0.554), C_{n=4}^-(\alpha = 0.2) = (0.542, 0.563, 0.556); \\
C_{n=4}^+(\alpha = 0.3) &= (0.561, 0.540, 0.557), C_{n=4}^-(\alpha = 0.3) = (0.549, 0.565, 0.550); \\
C_{n=4}^+(\alpha = 0.4) &= (0.555, 0.546, 0.553), C_{n=4}^-(\alpha = 0.4) = (0.544, 0.561, 0.558); \\
C_{n=4}^+(\alpha = 0.5) &= (0.566, 0.543, 0.550), C_{n=4}^-(\alpha = 0.5) = (0.541, 0.567, 0.554); \\
C_{n=4}^+(\alpha = 0.6) &= (0.559, 0.548, 0.555), C_{n=4}^-(\alpha = 0.6) = (0.548, 0.560, 0.552); \\
C_{n=4}^+(\alpha = 0.7) &= (0.562, 0.542, 0.556), C_{n=4}^-(\alpha = 0.7) = (0.543, 0.564, 0.557); \\
C_{n=4}^+(\alpha = 0.8) &= (0.557, 0.544, 0.551), C_{n=4}^-(\alpha = 0.8) = (0.546, 0.562, 0.553); \\
C_{n=4}^+(\alpha = 0.9) &= (0.564, 0.547, 0.558), C_{n=4}^-(\alpha = 0.9) = (0.540, 0.566, 0.555); \\
C_{n=5}^+(\alpha = 0.1) &= (0.631, 0.612, 0.625), C_{n=5}^-(\alpha = 0.1) = (0.617, 0.634, 0.621); \\
C_{n=5}^+(\alpha = 0.2) &= (0.627, 0.615, 0.622), C_{n=5}^-(\alpha = 0.2) = (0.613, 0.630, 0.626); \\
C_{n=5}^+(\alpha = 0.3) &= (0.634, 0.611, 0.620), C_{n=5}^-(\alpha = 0.3) = (0.619, 0.636, 0.623); \\
C_{n=5}^+(\alpha = 0.4) &= (0.623, 0.618, 0.628), C_{n=5}^-(\alpha = 0.4) = (0.610, 0.632, 0.625); \\
C_{n=5}^+(\alpha = 0.5) &= (0.630, 0.614, 0.621), C_{n=5}^-(\alpha = 0.5) = (0.616, 0.635, 0.620); \\
C_{n=5}^+(\alpha = 0.6) &= (0.625, 0.619, 0.626), C_{n=5}^-(\alpha = 0.6) = (0.612, 0.631, 0.627); \\
C_{n=5}^+(\alpha = 0.7) &= (0.632, 0.610, 0.623), C_{n=5}^-(\alpha = 0.7) = (0.618, 0.633, 0.622); \\
C_{n=5}^+(\alpha = 0.8) &= (0.621, 0.616, 0.629), C_{n=5}^-(\alpha = 0.8) = (0.614, 0.637, 0.624); \\
C_{n=5}^+(\alpha = 0.9) &= (0.635, 0.613, 0.624), C_{n=5}^-(\alpha = 0.9) = (0.611, 0.630, 0.628); \\
C_{n=6}^+(\alpha = 0.1) &= (0.682, 0.661, 0.673), C_{n=6}^-(\alpha = 0.1) = (0.665, 0.687, 0.671); \\
C_{n=6}^+(\alpha = 0.2) &= (0.676, 0.664, 0.670), C_{n=6}^-(\alpha = 0.2) = (0.662, 0.683, 0.675); \\
C_{n=6}^+(\alpha = 0.3) &= (0.685, 0.660, 0.674), C_{n=6}^-(\alpha = 0.3) = (0.667, 0.681, 0.672); \\
C_{n=6}^+(\alpha = 0.4) &= (0.672, 0.666, 0.678), C_{n=6}^-(\alpha = 0.4) = (0.660, 0.686, 0.673); \\
C_{n=6}^+(\alpha = 0.5) &= (0.681, 0.663, 0.671), C_{n=6}^-(\alpha = 0.5) = (0.664, 0.684, 0.670); \\
C_{n=6}^+(\alpha = 0.6) &= (0.674, 0.668, 0.676), C_{n=6}^-(\alpha = 0.6) = (0.661, 0.682, 0.677); \\
C_{n=6}^+(\alpha = 0.7) &= (0.683, 0.662, 0.675), C_{n=6}^-(\alpha = 0.7) = (0.666, 0.680, 0.671); \\
C_{n=6}^+(\alpha = 0.8) &= (0.670, 0.665, 0.679), C_{n=6}^-(\alpha = 0.8) = (0.663, 0.688, 0.674); \\
C_{n=6}^+(\alpha = 0.9) &= (0.684, 0.661, 0.672), C_{n=6}^-(\alpha = 0.9) = (0.660, 0.685, 0.676).
\end{aligned}$$

(IV) The closeness of alternatives ($n = 1, 2, \dots, 6; \alpha = 0.1, 0.2, \dots, 0.9$):

$$T_1^{\alpha=0.1}(A_1) = 0.415, T_1^{\alpha=0.1}(A_2) = 0.542, T_1^{\alpha=0.1}(A_3) = 0.358;$$

$$\begin{aligned}
&T_1^{\alpha=0.2}(A_1) = 0.422, T_1^{\alpha=0.2}(A_2) = 0.536, T_1^{\alpha=0.2}(A_3) = 0.361; \\
&T_1^{\alpha=0.3}(A_1) = 0.419, T_1^{\alpha=0.3}(A_2) = 0.544, T_1^{\alpha=0.3}(A_3) = 0.356; \\
&T_1^{\alpha=0.4}(A_1) = 0.424, T_1^{\alpha=0.4}(A_2) = 0.532, T_1^{\alpha=0.4}(A_3) = 0.359; \\
&T_1^{\alpha=0.5}(A_1) = 0.428, T_1^{\alpha=0.5}(A_2) = 0.537, T_1^{\alpha=0.5}(A_3) = 0.354; \\
&T_1^{\alpha=0.6}(A_1) = 0.421, T_1^{\alpha=0.6}(A_2) = 0.533, T_1^{\alpha=0.6}(A_3) = 0.357; \\
&T_1^{\alpha=0.7}(A_1) = 0.426, T_1^{\alpha=0.7}(A_2) = 0.541, T_1^{\alpha=0.7}(A_3) = 0.352; \\
&T_1^{\alpha=0.8}(A_1) = 0.430, T_1^{\alpha=0.8}(A_2) = 0.530, T_1^{\alpha=0.8}(A_3) = 0.354; \\
&T_1^{\alpha=0.9}(A_1) = 0.420, T_1^{\alpha=0.9}(A_2) = 0.540, T_1^{\alpha=0.9}(A_3) = 0.349; \\
&T_2^{\alpha=0.1}(A_1) = 0.450, T_2^{\alpha=0.1}(A_2) = 0.515, T_2^{\alpha=0.1}(A_3) = 0.418; \\
&T_2^{\alpha=0.2}(A_1) = 0.453, T_2^{\alpha=0.2}(A_2) = 0.512, T_2^{\alpha=0.2}(A_3) = 0.415; \\
&T_2^{\alpha=0.3}(A_1) = 0.446, T_2^{\alpha=0.3}(A_2) = 0.518, T_2^{\alpha=0.3}(A_3) = 0.413; \\
&T_2^{\alpha=0.4}(A_1) = 0.451, T_2^{\alpha=0.4}(A_2) = 0.509, T_2^{\alpha=0.4}(A_3) = 0.417; \\
&T_2^{\alpha=0.5}(A_1) = 0.457, T_2^{\alpha=0.5}(A_2) = 0.514, T_2^{\alpha=0.5}(A_3) = 0.411; \\
&T_2^{\alpha=0.6}(A_1) = 0.444, T_2^{\alpha=0.6}(A_2) = 0.520, T_2^{\alpha=0.6}(A_3) = 0.412; \\
&T_2^{\alpha=0.7}(A_1) = 0.452, T_2^{\alpha=0.7}(A_2) = 0.506, T_2^{\alpha=0.7}(A_3) = 0.416; \\
&T_2^{\alpha=0.8}(A_1) = 0.455, T_2^{\alpha=0.8}(A_2) = 0.513, T_2^{\alpha=0.8}(A_3) = 0.414; \\
&T_2^{\alpha=0.9}(A_1) = 0.449, T_2^{\alpha=0.9}(A_2) = 0.511, T_2^{\alpha=0.9}(A_3) = 0.415; \\
&T_3^{\alpha=0.1}(A_1) = 0.448, T_3^{\alpha=0.1}(A_2) = 0.510, T_3^{\alpha=0.1}(A_3) = 0.414; \\
&T_3^{\alpha=0.2}(A_1) = 0.452, T_3^{\alpha=0.2}(A_2) = 0.515, T_3^{\alpha=0.2}(A_3) = 0.409; \\
&T_3^{\alpha=0.3}(A_1) = 0.445, T_3^{\alpha=0.3}(A_2) = 0.508, T_3^{\alpha=0.3}(A_3) = 0.411; \\
&T_3^{\alpha=0.4}(A_1) = 0.450, T_3^{\alpha=0.4}(A_2) = 0.517, T_3^{\alpha=0.4}(A_3) = 0.408; \\
&T_3^{\alpha=0.5}(A_1) = 0.455, T_3^{\alpha=0.5}(A_2) = 0.512, T_3^{\alpha=0.5}(A_3) = 0.410; \\
&T_3^{\alpha=0.6}(A_1) = 0.443, T_3^{\alpha=0.6}(A_2) = 0.516, T_3^{\alpha=0.6}(A_3) = 0.406; \\
&T_3^{\alpha=0.7}(A_1) = 0.449, T_3^{\alpha=0.7}(A_2) = 0.508, T_3^{\alpha=0.7}(A_3) = 0.412; \\
&T_3^{\alpha=0.8}(A_1) = 0.451, T_3^{\alpha=0.8}(A_2) = 0.514, T_3^{\alpha=0.8}(A_3) = 0.407; \\
&T_3^{\alpha=0.9}(A_1) = 0.446, T_3^{\alpha=0.9}(A_2) = 0.518, T_3^{\alpha=0.9}(A_3) = 0.409; \\
&T_4^{\alpha=0.1}(A_1) = 0.447, T_4^{\alpha=0.1}(A_2) = 0.511, T_4^{\alpha=0.1}(A_3) = 0.413; \\
&T_4^{\alpha=0.2}(A_1) = 0.451, T_4^{\alpha=0.2}(A_2) = 0.516, T_4^{\alpha=0.2}(A_3) = 0.408; \\
&T_4^{\alpha=0.3}(A_1) = 0.446, T_4^{\alpha=0.3}(A_2) = 0.509, T_4^{\alpha=0.3}(A_3) = 0.410; \\
&T_4^{\alpha=0.4}(A_1) = 0.450, T_4^{\alpha=0.4}(A_2) = 0.518, T_4^{\alpha=0.4}(A_3) = 0.407; \\
&T_4^{\alpha=0.5}(A_1) = 0.456, T_4^{\alpha=0.5}(A_2) = 0.513, T_4^{\alpha=0.5}(A_3) = 0.409;
\end{aligned}$$

$$\begin{aligned}
&T_4^{\alpha=0.6}(A_1) = 0.444, T_4^{\alpha=0.6}(A_2) = 0.517, T_4^{\alpha=0.6}(A_3) = 0.405; \\
&T_4^{\alpha=0.7}(A_1) = 0.448, T_4^{\alpha=0.7}(A_2) = 0.511, T_4^{\alpha=0.7}(A_3) = 0.411; \\
&T_4^{\alpha=0.8}(A_1) = 0.452, T_4^{\alpha=0.8}(A_2) = 0.515, T_4^{\alpha=0.8}(A_3) = 0.406; \\
&T_4^{\alpha=0.9}(A_1) = 0.447, T_4^{\alpha=0.9}(A_2) = 0.519, T_4^{\alpha=0.9}(A_3) = 0.408; \\
&T_5^{\alpha=0.1}(A_1) = 0.446, T_5^{\alpha=0.1}(A_2) = 0.512, T_5^{\alpha=0.1}(A_3) = 0.412; \\
&T_5^{\alpha=0.2}(A_1) = 0.450, T_5^{\alpha=0.2}(A_2) = 0.517, T_5^{\alpha=0.2}(A_3) = 0.407; \\
&T_5^{\alpha=0.3}(A_1) = 0.445, T_5^{\alpha=0.3}(A_2) = 0.510, T_5^{\alpha=0.3}(A_3) = 0.409; \\
&T_5^{\alpha=0.4}(A_1) = 0.449, T_5^{\alpha=0.4}(A_2) = 0.519, T_5^{\alpha=0.4}(A_3) = 0.406; \\
&T_5^{\alpha=0.5}(A_1) = 0.455, T_5^{\alpha=0.5}(A_2) = 0.515, T_5^{\alpha=0.5}(A_3) = 0.408; \\
&T_5^{\alpha=0.6}(A_1) = 0.443, T_5^{\alpha=0.6}(A_2) = 0.518, T_5^{\alpha=0.6}(A_3) = 0.404; \\
&T_5^{\alpha=0.7}(A_1) = 0.447, T_5^{\alpha=0.7}(A_2) = 0.512, T_5^{\alpha=0.7}(A_3) = 0.410; \\
&T_5^{\alpha=0.8}(A_1) = 0.451, T_5^{\alpha=0.8}(A_2) = 0.516, T_5^{\alpha=0.8}(A_3) = 0.405; \\
&T_5^{\alpha=0.9}(A_1) = 0.446, T_5^{\alpha=0.9}(A_2) = 0.520, T_5^{\alpha=0.9}(A_3) = 0.407; \\
&T_6^{\alpha=0.1}(A_1) = 0.445, T_6^{\alpha=0.1}(A_2) = 0.513, T_6^{\alpha=0.1}(A_3) = 0.411; \\
&T_6^{\alpha=0.2}(A_1) = 0.449, T_6^{\alpha=0.2}(A_2) = 0.518, T_6^{\alpha=0.2}(A_3) = 0.406; \\
&T_6^{\alpha=0.3}(A_1) = 0.444, T_6^{\alpha=0.3}(A_2) = 0.511, T_6^{\alpha=0.3}(A_3) = 0.408; \\
&T_6^{\alpha=0.4}(A_1) = 0.448, T_6^{\alpha=0.4}(A_2) = 0.520, T_6^{\alpha=0.4}(A_3) = 0.405; \\
&T_6^{\alpha=0.5}(A_1) = 0.454, T_6^{\alpha=0.5}(A_2) = 0.516, T_6^{\alpha=0.5}(A_3) = 0.407; \\
&T_6^{\alpha=0.6}(A_1) = 0.442, T_6^{\alpha=0.6}(A_2) = 0.519, T_6^{\alpha=0.6}(A_3) = 0.403; \\
&T_6^{\alpha=0.7}(A_1) = 0.446, T_6^{\alpha=0.7}(A_2) = 0.513, T_6^{\alpha=0.7}(A_3) = 0.409; \\
&T_6^{\alpha=0.8}(A_1) = 0.450, T_6^{\alpha=0.8}(A_2) = 0.517, T_6^{\alpha=0.8}(A_3) = 0.404; \\
&T_6^{\alpha=0.9}(A_1) = 0.445, T_6^{\alpha=0.9}(A_2) = 0.514, T_6^{\alpha=0.9}(A_3) = 0.406.
\end{aligned}$$

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Table A2. Aggregate *nPIVFS* decision matrix $n = 4$.

Attributes		A_1	
c_1		[(0.69,0.71,0.74,0.76,0.77,0.77,0.79,0.82,0.85,0.87;0.9),(0.59,0.65,0.68,0.76,0.77,0.77,0.79,0.79,0.85,0.95;1)]	
c_2		[(0.03,0.04,0.05,0.06,0.06,0.08,0.1,0.11,0.13,0.13;0.9),(0.03,0.04,0.06,0.06,0.08,0.09,0.1,0.12,0.21;1)]	
c_3		[(0.85,0.88,0.9,0.9,0.93,0.95,0.95,0.97,0.97;0.9),(0.76,0.78,0.8,0.82,0.83,0.84,0.84,0.9,0.92,1;1)]	
c_4		[(0.85,0.86,0.88,0.9,0.9,0.93,0.95,0.97,0.97;0.9),(0.76,0.78,0.8,0.82,0.84,0.88,0.9,0.92,0.95,1;1)]	
Attributes		A_2	
c_1		[(0.71,0.74,0.75,0.75,0.78,0.81,0.81,0.83,0.85,0.88;0.9),(0.61,0.64,0.67,0.75,0.78,0.81,0.81,0.81,1;1)]	
c_2		[(0.08,0.08,0.08,0.09,0.14,0.14,0.2,0.2,0.22;0.9),(0.03,0.06,0.07,0.08,0.1,0.14,0.14,0.2,0.22,0.3;1)]	
c_3		[(0.91,0.92,0.93,0.95,0.97,0.97,0.99,0.99;0.9),(0.84,0.85,0.88,0.91,0.94,0.97,0.97,0.97,1;1)]	
c_4		[(0.87,0.89,0.9,0.92,0.93,0.95,0.95,0.97,0.97;0.9),(0.79,0.81,0.82,0.83,0.9,0.95,0.95,0.95,1;1)]	
Attributes		A_3	
c_1		[(0.8,0.81,0.82,0.84,0.87,0.88,0.88,0.9,0.91,0.93;0.9),(0.71,0.74,0.75,0.8,0.81,0.88,0.88,0.9,0.98;1)]	
c_2		[(0.01,0.01,0.01,0.02,0.02,0.03,0.03,0.06,0.08,0.09;0.9),(0.01,0.02,0.03,0.03,0.06,0.09,0.09,0.15;1)]	
c_3		[(0.54,0.55,0.57,0.61,0.62,0.64,0.64,0.7,0.75,0.84;0.9),(0.44,0.45,0.55,0.56,0.6,0.64,0.68,0.75,0.84;1)]	
c_4		[(0.9,0.91,0.92,0.94,0.95,0.97,0.97,0.99,0.99;0.9),(0.84,0.85,0.88,0.91,0.93,0.97,0.97,0.97,1;1)]	

Table A3. Aggregate *nPIVFS* decision matrix $n = 5$.

Attributes	A_1
c_1	[(0.69,0.71,0.72,0.74,0.76,0.77,0.77,0.79,0.82,0.85,0.86,0.87;0.9),(0.59,0.65,0.66,0.68,0.76,0.77,0.77,0.79,0.85,0.86,0.95;1)]
c_2	[(0.03,0.04,0.05,0.06,0.06,0.08,0.1,0.11,0.12,0.13,0.13,0.13;0.9),(0.003,0.04,0.06,0.08,0.09,0.1,0.12,0.13,0.13,0.21;1)]
c_3	[(0.85,0.88,0.9,0.9,0.93,0.95,0.95,0.97,0.97,0.97,0.97;0.9),(0.76,0.78,0.8,0.82,0.83,0.84,0.9,0.92,0.95,0.97,1;1)]
c_4	[(0.85,0.86,0.88,0.9,0.9,0.93,0.95,0.96,0.97,0.97,0.97;0.9),(0.76,0.78,0.8,0.82,0.84,0.88,0.9,0.92,0.95,1;1)]
Attributes	A_2
c_1	[(0.71,0.74,0.75,0.75,0.78,0.81,0.81,0.83,0.85,0.86,0.88;0.9),(0.61,0.64,0.67,0.75,0.78,0.81,0.81,0.81,0.85,0.86,1;1)]
c_2	[(0.08,0.08,0.08,0.08,0.09,0.14,0.14,0.2,0.2,0.2,0.22;0.9),(0.03,0.06,0.07,0.08,0.1,0.14,0.2,0.2,0.2,0.22,0.3;1)]
c_3	[(0.91,0.92,0.93,0.93,0.95,0.97,0.97,0.97,0.99,0.99;0.9),(0.84,0.85,0.88,0.91,0.94,0.97,0.97,0.97,0.97,0.97,1;1)]
c_4	[(0.87,0.89,0.9,0.92,0.93,0.95,0.95,0.97,0.97,0.97;0.9),(0.79,0.81,0.82,0.83,0.9,0.95,0.95,0.95,0.95,0.95,1;1)]
Attributes	A_3
c_1	[(0.8,0.81,0.82,0.84,0.85,0.85,0.87,0.88,0.88,0.9,0.91,0.93;0.9),(0.71,0.74,0.75,0.8,0.81,0.85,0.85,0.88,0.88,0.9,0.9,0.98;1)]
c_2	[(0.01,0.01,0.01,0.02,0.02,0.03,0.03,0.06,0.08,0.09,0.09;0.9),(0.001,0.02,0.03,0.03,0.06,0.09,0.09,0.09,0.09,0.15;1)]
c_3	[(0.54,0.55,0.57,0.61,0.62,0.64,0.64,0.7,0.75,0.8,0.82,0.84;0.9),(0.44,0.45,0.55,0.56,0.6,0.64,0.64,0.68,0.75,0.78,0.8,0.84;1)]
c_4	[(0.9,0.91,0.92,0.94,0.95,0.97,0.97,0.99,0.99,0.99;0.9),(0.84,0.85,0.88,0.91,0.93,0.97,0.97,0.97,0.97,0.97,1;1)]

Table A4. Aggregate *nPIVFS* decision matrix $n = 6$.

A_1	
Attributes	
c_1	[(0.69,0.71,0.72,0.74,0.76,0.77,0.77,0.79,0.82,0.85,0.86,0.87,0.87,0.87;0.9),(0.59,0.65,0.66,0.68,0.76,0.77,0.77,0.79,0.85,0.86,0.87,0.95;1)]
c_2	[(0.03,0.04,0.05,0.06,0.06,0.08,0.1,0.11,0.12,0.13,0.13,0.13,0.13;0.9),(0.003,0.04,0.06,0.08,0.09,0.1,0.12,0.12,0.13,0.13,0.21;1)]
c_3	[(0.85,0.88,0.9,0.9,0.93,0.95,0.95,0.97,0.97,0.97,0.97,0.97,0.97;0.9),(0.76,0.78,0.8,0.82,0.83,0.84,0.84,0.9,0.92,0.95,0.97,0.97,1;1)]
c_4	[(0.85,0.86,0.88,0.9,0.9,0.93,0.95,0.96,0.97,0.97,0.97,0.97,0.97;0.9),(0.76,0.78,0.8,0.82,0.84,0.88,0.9,0.92,0.95,0.97,0.97,0.97,1;1)]
A_2	
Attributes	
c_1	[(0.71,0.74,0.75,0.75,0.77,0.78,0.78,0.81,0.81,0.83,0.85,0.86,0.88;0.9),(0.61,0.64,0.67,0.75,0.76,0.77,0.78,0.81,0.81,0.81,0.85,0.86,1;1)]
c_2	[(0.08,0.08,0.08,0.08,0.09,0.14,0.14,0.2,0.2,0.2,0.2,0.22,0.22;0.9),(0.03,0.06,0.07,0.08,0.1,0.14,0.14,0.2,0.2,0.2,0.22,0.3;1)]
c_3	[(0.91,0.92,0.93,0.93,0.95,0.97,0.97,0.97,0.99,0.99,0.99,0.99,0.99;0.9),(0.84,0.85,0.88,0.91,0.94,0.97,0.97,0.97,0.97,0.97,0.97,1;1)]
c_4	[(0.87,0.89,0.9,0.92,0.93,0.95,0.95,0.97,0.97,0.97,0.97,0.97,0.97;0.9),(0.79,0.81,0.82,0.83,0.9,0.95,0.95,0.95,0.95,0.95,0.95,1;1)]
A_3	
Attributes	
c_1	[(0.8,0.81,0.82,0.84,0.85,0.87,0.88,0.88,0.9,0.91,0.93,0.93,0.93;0.9),(0.71,0.74,0.75,0.8,0.81,0.85,0.85,0.88,0.88,0.9,0.9,0.91,0.98;1)]
c_2	[(0.01,0.01,0.01,0.02,0.02,0.03,0.03,0.06,0.08,0.09,0.09,0.09,0.09;0.9),(0.001,0.02,0.03,0.03,0.06,0.09,0.09,0.09,0.09,0.09,0.15;1)]
c_3	[(0.54,0.55,0.57,0.61,0.62,0.64,0.64,0.7,0.75,0.8,0.82,0.84,0.84,0.84;0.9),(0.44,0.45,0.55,0.56,0.6,0.64,0.64,0.68,0.75,0.78,0.8,0.83,0.84;1)]
c_4	[(0.9,0.91,0.92,0.94,0.95,0.97,0.97,0.99,0.99,0.99,0.99,0.99,0.99;0.9),(0.84,0.85,0.88,0.91,0.93,0.97,0.97,0.97,0.97,0.97,0.97,1;1)]

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Table A10. Weighted decision matrix $n = 6$.

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Table A11. Values of G_t^+ and $G_t^-(n = 1)$.

$\alpha = 0.1$	
G^+	[(0.20, 0.21, 0.22, 0.23; 0.9), (0.17, 0.18, 0.22, 0.25; 1)]
G_1^+	G_1^- [(0.17, 0.18, 0.20, 0.22; 0.9), (0.15, 0.16, 0.20, 0.23; 1)]
G_2^+	G_2^- [(0.00, 0.00, 0.01, 0.02; 0.9), (0.00, 0.00, 0.01, 0.04; 1)]
G_3^+	G_3^- [(0.22, 0.23, 0.24, 0.24; 0.9), (0.21, 0.22, 0.24, 0.25; 1)]
G_4^+	G_4^- [(0.23, 0.24, 0.25, 0.25; 0.9), (0.21, 0.22, 0.25, 0.25; 1)]
$\alpha = 0.2$	
G^+	[(0.19, 0.20, 0.21, 0.23; 0.9), (0.17, 0.18, 0.21, 0.24; 1)]
G_1^+	G_1^- [(0.17, 0.18, 0.20, 0.21; 0.9), (0.14, 0.16, 0.20, 0.23; 1)]
G_2^+	G_2^- [(0.02, 0.02, 0.04, 0.06; 0.9), (0.01, 0.02, 0.04, 0.08; 1)]
G_3^+	G_3^- [(0.22, 0.22, 0.23, 0.24; 0.9), (0.20, 0.21, 0.23, 0.24; 1)]
G_4^+	G_4^- [(0.23, 0.23, 0.25, 0.25; 0.9), (0.22, 0.22, 0.25, 0.26; 1)]
$\alpha = 0.3$	
G^+	[(0.20, 0.21, 0.23, 0.24; 0.9), (0.18, 0.19, 0.23, 0.26; 1)]
G_1^+	G_1^- [(0.18, 0.18, 0.21, 0.22; 0.9), (0.15, 0.16, 0.21, 0.24; 1)]
G_2^+	G_2^- [(0.02, 0.03, 0.04, 0.07; 0.9), (0.01, 0.02, 0.04, 0.09; 1)]
G_3^+	G_3^- [(0.23, 0.23, 0.24, 0.25; 0.9), (0.21, 0.22, 0.24, 0.26; 1)]
G_4^+	G_4^- [(0.24, 0.24, 0.26, 0.26; 0.9), (0.22, 0.23, 0.26, 0.27; 1)]
$\alpha = 0.4$	
G^+	[(0.21, 0.22, 0.24, 0.25; 0.9), (0.19, 0.20, 0.24, 0.27; 1)]
G_1^+	G_1^- [(0.19, 0.19, 0.22, 0.23; 0.9), (0.16, 0.17, 0.22, 0.25; 1)]
G_2^+	G_2^- [(0.02, 0.03, 0.04, 0.07; 0.9), (0.01, 0.02, 0.04, 0.09; 1)]
G_3^+	G_3^- [(0.24, 0.24, 0.25, 0.26; 0.9), (0.22, 0.23, 0.25, 0.27; 1)]
G_4^+	G_4^- [(0.25, 0.25, 0.27, 0.27; 0.9), (0.23, 0.24, 0.27, 0.28; 1)]
$\alpha = 0.5$	
G^+	[(0.19, 0.19, 0.20, 0.22; 0.9), (0.16, 0.17, 0.20, 0.23; 1)]
G_1^+	G_1^- [(0.16, 0.16, 0.19, 0.20; 0.9), (0.14, 0.15, 0.19, 0.22; 1)]
G_2^+	G_2^- [(0.02, 0.02, 0.04, 0.06; 0.9), (0.01, 0.02, 0.04, 0.08; 1)]
G_3^+	G_3^- [(0.21, 0.22, 0.23, 0.23; 0.9), (0.20, 0.20, 0.23, 0.23; 1)]
G_4^+	G_4^- [(0.24, 0.24, 0.26, 0.26; 0.9), (0.22, 0.23, 0.26, 0.27; 1)]
$\alpha = 0.6$	
G^+	[(0.19, 0.20, 0.21, 0.22; 0.9), (0.16, 0.17, 0.20, 0.23; 1)]
G_1^+	G_1^- [(0.16, 0.17, 0.19, 0.21; 0.9), (0.14, 0.15, 0.19, 0.22; 1)]
G_2^+	G_2^- [(0.02, 0.02, 0.04, 0.06; 0.9), (0.01, 0.02, 0.04, 0.08; 1)]
G_3^+	G_3^- [(0.21, 0.22, 0.23, 0.24; 0.9), (0.20, 0.21, 0.23, 0.24; 1)]
G_4^+	G_4^- [(0.24, 0.25, 0.26, 0.26; 0.9), (0.22, 0.23, 0.26, 0.27; 1)]
$\alpha = 0.7$	
G^+	[(0.19, 0.19, 0.21, 0.22; 0.9), (0.17, 0.17, 0.21, 0.23; 1)]
G_1^+	G_1^- [(0.16, 0.17, 0.19, 0.21; 0.9), (0.14, 0.15, 0.19, 0.22; 1)]
G_2^+	G_2^- [(0.02, 0.02, 0.04, 0.06; 0.9), (0.01, 0.02, 0.04, 0.08; 1)]
G_3^+	G_3^- [(0.22, 0.22, 0.23, 0.24; 0.9), (0.20, 0.20, 0.23, 0.24; 1)]
G_4^+	G_4^- [(0.24, 0.24, 0.26, 0.26; 0.9), (0.22, 0.22, 0.26, 0.26; 1)]
$\alpha = 0.8$	
G^+	[(0.18, 0.18, 0.20, 0.22; 0.9), (0.16, 0.16, 0.20, 0.23; 1)]
G_1^+	G_1^- [(0.16, 0.16, 0.18, 0.20; 0.9), (0.14, 0.14, 0.18, 0.22; 1)]
G_2^+	G_2^- [(0.02, 0.02, 0.04, 0.06; 0.9), (0.01, 0.02, 0.04, 0.08; 1)]
G_3^+	G_3^- [(0.21, 0.21, 0.23, 0.23; 0.9), (0.19, 0.20, 0.22, 0.23; 1)]
G_4^+	G_4^- [(0.24, 0.25, 0.26, 0.27; 0.9), (0.23, 0.23, 0.26, 0.27; 1)]
$\alpha = 0.9$	
G^+	[(0.18, 0.19, 0.20, 0.21; 0.9), (0.16, 0.17, 0.20, 0.23; 1)]
G_1^+	G_1^- [(0.15, 0.16, 0.19, 0.19; 0.9), (0.13, 0.14, 0.19, 0.21; 1)]
G_2^+	G_2^- [(0.02, 0.02, 0.04, 0.06; 0.9), (0.01, 0.02, 0.04, 0.08; 1)]
G_3^+	G_3^- [(0.21, 0.21, 0.23, 0.23; 0.9), (0.19, 0.20, 0.23, 0.24; 1)]
G_4^+	G_4^- [(0.24, 0.24, 0.26, 0.26; 0.9), (0.22, 0.23, 0.26, 0.27; 1)]

Table A12. Values of G_t^+ and $G_t^-(n = 2)$.

$\alpha = 0.1$	
G_1^+	[0.20, 0.20, 0.21, 0.22, 0.23; 0.9], (0.18, 0.18, 0.19, 0.22, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.05, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25; 0.9], (0.21, 0.21, 0.22, 0.24, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25; 0.9], (0.21, 0.21, 0.22, 0.24, 0.24, 0.25; 1) G_4^-
$\alpha = 0.2$	
G_1^+	[0.20, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.22, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.05, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25; 0.9], (0.21, 0.21, 0.22, 0.24, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.25; 0.9], (0.21, 0.21, 0.22, 0.25, 0.25, 0.26; 1) G_4^-
$\alpha = 0.3$	
G_1^+	[0.19, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.21, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25; 0.9], (0.21, 0.21, 0.22, 0.24, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.25; 0.9], (0.21, 0.21, 0.22, 0.25, 0.25, 0.26; 1) G_4^-
$\alpha = 0.4$	
G_1^+	[0.19, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.21, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25, 0.26; 0.9], (0.20, 0.20, 0.21, 0.24, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.26; 0.9], (0.22, 0.22, 0.22, 0.25, 0.25, 0.26; 1) G_4^-
$\alpha = 0.5$	
G_1^+	[0.19, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.21, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25, 0.26; 0.9], (0.20, 0.20, 0.21, 0.23, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.26; 0.9], (0.22, 0.22, 0.23, 0.25, 0.26, 0.26; 1) G_4^-
$\alpha = 0.6$	
G_1^+	[0.19, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.21, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25, 0.26; 0.9], (0.20, 0.20, 0.21, 0.23, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.26; 0.9], (0.22, 0.22, 0.23, 0.25, 0.26, 0.26; 1) G_4^-
$\alpha = 0.7$	
G_1^+	[0.19, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.21, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25, 0.26; 0.9], (0.20, 0.20, 0.21, 0.23, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.26; 0.9], (0.22, 0.22, 0.23, 0.25, 0.26, 0.26; 1) G_4^-
$\alpha = 0.8$	
G_1^+	[0.19, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.17, 0.17, 0.19, 0.21, 0.23, 0.24; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.22, 0.22, 0.23, 0.24, 0.25, 0.26; 0.9], (0.20, 0.20, 0.21, 0.23, 0.24, 0.25; 1) G_3^-
G_4^+	[0.23, 0.23, 0.24, 0.25, 0.26; 0.9], (0.22, 0.22, 0.23, 0.25, 0.26, 0.26; 1) G_4^-
$\alpha = 0.9$	
G_1^+	[0.18, 0.18, 0.19, 0.20, 0.21, 0.22, 0.23; 0.9], (0.16, 0.16, 0.18, 0.20, 0.22, 0.23; 1) G_1^-
G_2^+	[0.02, 0.02, 0.03, 0.04, 0.06, 0.06; 0.9], (0.01, 0.01, 0.02, 0.04, 0.06, 0.08; 1) G_2^-
G_3^+	[0.21, 0.21, 0.22, 0.23, 0.24, 0.25; 0.9], (0.19, 0.19, 0.20, 0.22, 0.23, 0.24; 1) G_3^-
G_4^+	[0.22, 0.22, 0.23, 0.24, 0.25, 0.26; 0.9], (0.20, 0.20, 0.21, 0.23, 0.24, 0.25; 1) G_4^-

[illegible]

Table A14. Values of G_t^+ and $G_t^-(n=4)$.

[illegible]

Table A15. Values of G_r^+ and $G_r^-(n = 5)$.

[illegible]

Table A16. Values of G^+ and G^- ($n = 6$).

[illegible]



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