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*Research article***Strategy consensus of large-scale networked evolutionary games under profile constraints via network aggregation****Huixin Kan, Shihua Fu\*, Rui Wang, and Yifan Zhao**

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**Abstract:** Analyzing large-scale networked evolutionary games (NEGs) with arbitrary network structures remains challenging due to complex network dynamics. Strategy consensus requires all players to adopt an identical strategy over long-term evolution. Different from existing studies, this paper investigated the strategy consensus of large-scale NEGs under profile constraints, where all profile trajectories must avoid a predefined restricted-profile set. Using network aggregation, we decomposed the original NEG into multiple smaller-size control subgames. We derived a reachability criterion for each subgame under local profile constraints and proposed a trajectory-matching condition for coupled subgames. We further discussed the expansion of restricted-profile sets through profile projection and derived a sufficient condition for global strategy consensus under profile constraints starting from an arbitrary initial profile. Examples were given to verify the effectiveness of the proposed method.

**Keywords:** large-scale networked evolutionary games; network aggregation; strategy consensus; profile constraint; semi-tensor product

**Mathematics Subject Classification:** 91A22, 93A15, 93Dxx, 15A30

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**1. Introduction**

Networked evolutionary games (NEGs) provide an effective framework for characterizing the strategy evolution of interacting players over complex networks. In an NEG, players are represented by nodes, and their interactions are characterized by edges. Each player updates its strategy according to a prescribed strategy updating rule by utilizing strategy and payoff information obtained from its neighborhood. Accordingly, the collective evolutionary behavior of an NEG is jointly determined by the network topology, the underlying game types, and the strategy updating rule. Owing to their ability to characterize strategic interactions and behavioral adaptation among interconnected

individuals, NEGs have been widely applied to the investigation of collective behavior in social systems [1], biological populations [2], social–ecological systems [3], and engineering systems [4, 5].

The semi-tensor product (STP) of matrices was originally developed as an efficient algebraic tool for handling high-dimensional arrays [6]. It provides an algebraic framework for the modeling and analysis of NEGs [7, 8]. By mapping each strategy to a vector and formulating strategy-updating rules as logical functions, the dynamics of strategy profiles in an NEG can be transformed into an algebraic state space representation. Based on this framework, extensive studies on NEGs have been investigated, including the existence of equilibria [9, 10], stability degree analysis of profiles [11, 12], strategy optimization [13, 14], and engineering applications [15, 16]. The STP has also been widely used in the modeling and analysis of Boolean networks. Typical research topics include observability [17], topological structure [18, 19], and stability [20, 21]. Besides, it has found applications in finite-state machines [22, 23], and shift registers [24, 25]. Beyond the scope of logical systems, the STP solves conventional quaternion matrix equations [26, 27] and addresses split-quaternion matrix problems [28, 29]. Related methods are further extended to octonion matrix equations [30]. In addition, STP-based quaternion matrix approaches have been applied to image-processing tasks such as image restoration and strict image authentication [31–33].

Strategy consensus is an important collective behavior of NEGs, which requires all players to eventually adopt an identical strategy. Using the STP method, various strategy consensus problems of NEGs have been investigated [34, 35]. In practical systems, strategy evolution is often subject to various constraints. Inviolable constraints in autonomous games were considered in [36], while local and coupled nonlinear convex constraints were studied in nonlinear multi-coalition games in [37]. Similar constraint requirements also arise in NEGs, where certain strategy profiles may be inadmissible. For instance, in a traffic network, if the number of drivers choosing the same route exceeds a prescribed congestion threshold, the corresponding strategy profile can be regarded as restricted. Furthermore, reference [38] demonstrated that an NEG with bankruptcy risks could ultimately be formulated as a game model subject to profile constraints. These cases show that only reaching consensus is not enough, all profiles along the evolution must stay away from an undesirable set. Robust consensus of NEGs with attackers and forbidden profiles was investigated in [39], where robust constrained reachability was introduced and a state-feedback control procedure was developed. Nevertheless, due to the impact of computational complexity, most existing results concerning strategy consensus are established for networks of small scale.

A major difficulty in extending the strategy consensus for small-scale networks to large-scale NEGs lies in the exponential growth of the strategy profile space. For an NEG consisting of  $n$  players, each having  $k$  strategies, the system contains  $k^n$  possible strategy profiles. Consequently, the analysis of the system becomes computationally prohibitive as the number of players increases, which limits the direct application of STP-based methods to large-scale NEGs. To alleviate this difficulty, network aggregation [40–42] and other structure-based approaches [43, 44] have been developed for the efficient analysis and control of large-scale logical networks. Drawing on these aggregation methods, Li et al. decomposed a large-scale networked evolutionary matrix game into several small-scale control subgames [45]. Instead of solving global consensus directly, they only need to find valid local strategies for each subgame under a matchability condition. Although reference [45] investigated strategy consensus in large-scale network evolutionary games, it failed to take restricted-profile constraints into consideration. Furthermore, the target of strategy consensus studied therein

corresponded merely to a unique consensus profile. However, each player in the game may have multiple alternative strategies, and in practical task execution, agents are often required to achieve strategy consensus under diverse scenarios [46]. Under such circumstances, the conclusions in [45] cannot characterize these practical situations.

Motivated by the above discussions, this paper addresses the strategy consensus problem for large-scale NEG with a restricted-profile set from a given initial strategy profile via network aggregation. We first partition the large-scale NEG into multiple small control subgames and project the global restricted-profile set onto each subgame to obtain local restricted-profile constraints. For each subgame, admissible local trajectories that reach compatible consensus profiles without entering the associated local restricted-profile set are identified. Since some players serve as state players in one subgame and act as input players for its connected subgames, a trajectory-matching condition is introduced to characterize the compatibility of local trajectories. Based on this condition, a sufficient condition is derived for global strategy consensus that avoids constrained profiles throughout evolution. The main contributions of this paper are summarized as follows:

- (1) The strategy consensus problems for large-scale NEG with a constrained set from a given initial strategy profile are investigated. The strategy consensus problem of NEG is transformed into the set stability problem of large-scale logical networks. The work in [45] corresponds to a special case where the undesirable set is empty and the target is a single prescribed strategy consensus profile. When the target is a strategy consensus set, the analysis becomes more challenging because possible strategy consensus cycles within the target set need to be identified, instead of only checking whether a prescribed strategy profile is a fixed point.
- (2) A trajectory-matching condition is established to judge the compatibility of local admissible trajectories among subgames. Based on this condition, a sufficient condition is derived for the large-scale NEG to achieve strategy consensus while avoiding the prescribed undesirable set. Moreover, we find that the projected local restrictions may induce an enlarged restricted-profile set in the global space. Consequently, under profile constraints, the local admissible sets obtained after aggregation cannot, in general, be directly combined to recover the admissible set of the original NEG.

The remainder of this paper is organized as follows. Section 2 introduces the notation and preliminaries used throughout the paper. Section 3 presents the system model and network aggregation. Section 4 contains our main results, with illustrative examples presented in Section 5. Section 6 concludes the paper.

## 2. Preliminaries

In this section, we give some necessary preliminaries, which will be used throughout this paper. We first list some notations as

- $\mathbb{Z}_+$  denotes the set of all positive integers.
- $\mathbb{R}_{m \times n}$  is the set of  $m \times n$  real matrices.
- $\mathcal{D}_k := \left\{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$  and  $\mathcal{D}_k^n := \underbrace{\mathcal{D}_k \times \cdots \times \mathcal{D}_k}_n$ .

- $\Delta_n$  represents  $\{\delta_n^i \mid i = 1, 2, \dots, n\}$ , where  $\delta_n^i$  is the  $i$ -th column of the identity matrix  $I_n$ .
- An  $m \times n$  matrix  $L$  is called a logical matrix if  $L = [\delta_m^{i_1} \delta_m^{i_2} \dots \delta_m^{i_n}]$  and we briefly denote  $L$  by  $L = \delta_m[i_1 \ i_2 \ \dots \ i_n]$ .
- $\text{Col}_i(L)$  is the  $i$ -th column of matrix  $L$ .  $\text{Col}(L)$  is the set of columns of  $L$ .
- $\mathcal{L}_{m \times n}$  denotes the set of  $m \times n$  logical matrices.
- $(L)_{i,j}$  denotes the element on the  $(i, j)$  entry of matrix  $L$ .
- $\mathcal{B}_{m \times n}$  represents the set of  $m \times n$  Boolean matrices.
- Let  $a, b \in \mathcal{B}$ . The Boolean addition is defined by  $a +_{\mathcal{B}} b := a \vee b = \begin{cases} 0, & a = b = 0, \\ 1, & \text{otherwise.} \end{cases}$
- $A +_{\mathcal{B}} B$ : the Boolean addition of  $A, B \in \mathcal{B}_{m \times n}$ , where  $(A +_{\mathcal{B}} B)_{i,j} = (A)_{i,j} \vee (B)_{i,j}$ .
- $\sum_{\mathcal{B} \ i=1}^n A_i = A_1 +_{\mathcal{B}} A_2 +_{\mathcal{B}} \dots +_{\mathcal{B}} A_n$ , where  $A_i \in \mathcal{B}_{m \times n}, i \in \{1, 2, \dots, n\}$ .

**Definition 2.1.** ([6]) Let  $A \in \mathbb{R}_{m \times n}$ ,  $B \in \mathbb{R}_{p \times q}$ . The STP of  $A$  and  $B$  is defined as  $A \ltimes B = (A \otimes I_{\frac{s}{n}})(B \otimes I_{\frac{s}{p}})$ , where  $s = \text{lcm}(n, p)$  and  $\otimes$  is the Kronecker product.

**Remark 2.2.** It is worth noting that STP degenerates into the conventional matrix product when  $n = p$ , and it inherits all properties of the conventional matrix product. Thus, we omit the symbol “ $\ltimes$ ” in this article.

**Definition 2.3.** ([6]) Let  $M \in \mathbb{R}_{m \times s}$  and  $N \in \mathbb{R}_{n \times s}$ . Define the Khatri-Rao product of  $M$  and  $N$ , denoted by  $M * N$ , as  $M * N = [\text{Col}_1(M)\text{Col}_1(N) \ \text{Col}_2(M)\text{Col}_2(N) \ \dots \ \text{Col}_s(M)\text{Col}_s(N)] \in \mathbb{R}_{mn \times s}$ .

**Lemma 2.4.** ([6]) Let  $f(s_1, s_2, \dots, s_n) : \mathcal{D}_k^n \rightarrow \mathcal{D}_k$  be a logical function. For each  $s_i \in \mathcal{D}_k$ , let  $x_i \in \Delta_k$  denote the vector form of  $s_i, i = 1, 2, \dots, n$ . Then there exists a unique matrix  $M_f$ , such that the vector form of  $f$  can be presented as

$$f(s_1, s_2, \dots, s_n) = M_f \ltimes_{i=1}^n x_i,$$

where  $M_f \in \mathcal{L}_{k \times k^n}$  is the structural matrix of  $f$  and  $i = 1, 2, \dots, n$ .

### 3. Problem formulation

#### 3.1. The algebraic form of NEGs

**Definition 3.1.** [7] A finite game, denoted by  $G = (N, S, C)$ , consists of three fundamental ingredients:

- (1)  $N = \{1, 2, \dots, n\}$  is the set of players;
- (2)  $S = S_1 \times S_2 \times \dots \times S_n$  is the set of strategy profiles of all players, and  $S_i = \{1, 2, \dots, k_i\}$  is the set of strategies of player  $i, i \in N$ ;
- (3)  $C = \{c_1, c_2, \dots, c_n\}$  is the set of payoff functions, where  $c_i : S \rightarrow \mathbb{R}$  is the payoff function of player  $i$ .

We consider repeated games over networks. Let  $S = \{1, 2, \dots, k\}$  be the common strategy set of all players. At each time, player  $i$  only interacts with its neighbors, and its aggregate payoff  $p_i : S^{|N_i|+1} \rightarrow \mathbb{R}$  is

$$p_i = \sum_{j \in N_i} p_{ij}(s_i, s_j), \quad s_i, s_j \in S, \quad (3.1)$$

where  $p_{ij} : S \times S \rightarrow \mathbb{R}$  denotes the payoff of player  $i$  playing with its neighbor  $j$ .

Each player updates its strategy according to a specific strategy updating rule. In this paper, the myopic best-response updating rule [42] is adopted. Let  $s_i(t)$  be the strategy of player  $i$  at time  $t$ . We can express the strategy choice of player  $i$  at time  $t + 1$  as

$$s_i(t + 1) \in Q_i(t) := \arg \max_{s_i \in S} p_i(s_i, s_j(t), j \in N_i). \quad (3.2)$$

Player  $i$  may possess more than one best response, that is,  $|Q_i(t)| > 1$ . In this case, we set  $s_i(t + 1) = \min\{s \mid s \in Q_i(t)\}$ .

We denote each strategy  $j \in S$  by the standard vector  $\delta_k^j$ . Let  $x_i(t)$  be the vector form of  $s_i(t)$ , so  $x_i(t) \in \Delta_k$  for  $i = 1, 2, \dots, n$ . The strategy profile  $(s_1(t), s_2(t), \dots, s_n(t))$  uniquely corresponds to a  $k^n$ -dimensional vector defined as  $x(t) := \times_{i=1}^n x_i(t) \in \Delta_{k^n}$ .

Above that, the large-scale NEG can be respresented as the algebraic form

$$x(t + 1) = Fx(t), \quad (3.3)$$

where  $F$  is the state transition matrix. For more details on the algebraic representation and the construction of  $F$ , see [7].

The initial strategy profile is written as  $x(0)$ , which is equivalent to  $x_0 \in \Delta_{k^n}$ . Denote player  $i$ 's strategy and the overall NEG profile under algebraic form (3.3) at time  $t$  starting from  $x_0$  by  $x_i(t; x_0)$  and  $x(t; x_0)$ , respectively. The strategy consensus definition for NEGs is then presented below. Denote the restricted-profile set of NEG (3.3) as

$$\Omega = \{\delta_{k^n}^{\sigma_1}, \delta_{k^n}^{\sigma_2}, \dots, \delta_{k^n}^{\sigma_r}\} \subseteq \Delta_{k^n}.$$

The admissible initial set is  $\Theta = \{\delta_{k^n}^{\alpha_1}, \delta_{k^n}^{\alpha_2}, \dots, \delta_{k^n}^{\alpha_d}\} \in \Delta_{k^n} \setminus \Omega$ , where  $d + r = k^n$ . The global strategy consensus set is denoted by  $\Phi := \{(\delta_k^i)^n \mid i = 1, 2, \dots, k\}$ . Under the restricted-profile constraint, the admissible global strategy consensus set is defined as  $\Phi_\Omega := \Phi \setminus \Omega$ .

**Definition 3.2.** An NEG with algebraic form (3.3) is said to reach a strategy consensus while avoiding the undesirable set  $\Omega \subset \Delta_{k^n}$  starting from  $x_0 \in \Theta$ , if there exists a positive integer  $T > 0$  such that:

- (i)  $x(t; x_0) \in \Phi_\Omega, \quad \forall t \geq T;$
- (ii) for all  $t \geq 0$ , the trajectory satisfies  $x(t; x_0) \notin \Omega$ .

In the following, we give the algebraic expression of NEG (3.3) with the restricted-profile set  $\Omega$  by constructing a Boolean matrix  $I_\Omega \in \mathcal{B}_{k^n \times k^n}$ .

Define a Boolean matrix  $I_\Omega \in \mathcal{B}_{k^n \times k^n}$  related to  $\Omega$  as follows:

$$\text{Col}_i(I_\Omega) = \begin{cases} 0_{k^n}, & \delta_{k^n}^i \in \Omega, \\ \text{Col}_i(I_{k^n}), & \delta_{k^n}^i \notin \Omega. \end{cases} \quad (3.4)$$

We can denote the state transition matrix under the constrained set as

$$F_{\Omega} = I_{\Omega} F I_{\Omega}.$$

For any initial strategy profile  $x_0 \notin \Omega$ , if  $x(1) = F_{\Omega} x_0 = 0_{k^n}$ , then  $F x_0 \in \Omega$ . Therefore, we construct a new algebraic expression of NEG (3.3) under constrained set  $\Omega$  as

$$\mathcal{X}(t+1) = F_{\Omega} \mathcal{X}(t), \quad x_0 \in \Theta, \quad (3.5)$$

where  $\mathcal{X}(t) \in \mathcal{B}_{k^n \times 1}$  is the state that belongs to  $\Theta \cup \{0_{k^n}\}$ .

With the constrained algebraic form defined above, we now derive the following result, which serves as the key criterion for verifying strategy consensus of the NEG under the constrained set.

### 3.2. Network aggregation for large-scale NEGs

Consider a large-scale NEG with a constrained set, where  $V$  denotes the node set representing all players. Partition  $V$  into the following  $\omega$  blocks:

$$V = V_1 \cup V_2 \cdots \cup V_{\omega}, \quad (3.6)$$

where  $V_i := \{x_i^1, \dots, x_i^{n_i}\} \subseteq V$  and  $x_i^j$  denotes the  $j$ th player in the  $i$ th subgame.  $V_i \neq \emptyset$  and  $V_i \cap V_j = \emptyset$  for all  $i, j \in \{1, \dots, \omega\}$  with  $i \neq j$ .

For each block  $V_i$ , denote the sets of input players and output players of  $V_i$  by

$$U_i := \left( \bigcup_{j \in V_i} \mathcal{N}_j \right) \setminus V_i := \{u_i^1, \dots, u_i^{q_i}\}$$

and

$$Y_i := \left( \bigcup_{j \in V \setminus V_i} \mathcal{N}_j \right) \cap V_i = \{y_i^1, \dots, y_i^{p_i}\}.$$

For the subsequent analysis, we consider the NEG and all its aggregation blocks to be connected. It follows that  $U_i, Y_i \neq \emptyset$  for  $i = 1, 2, \dots, \omega$ .

Each block  $V_i$  forms a subgame, denoted by  $\Lambda_i$ . Let  $x_i^j(t)$  denote the strategy of  $x_i^j$  at time  $t$ . According to (3.2), the dynamics of  $\Lambda_i$  can be expressed as

$$x_i^j(t+1) = f_i^j(x_i^1(t), \dots, x_i^{q_i}(t), u_i^1(t), \dots, u_i^{p_i}(t)), \quad (3.7)$$

where  $j = 1, 2, \dots, n_i, i = 1, 2, \dots, \omega$ .

Using the STP method, the algebraic form of (3.7) can be given by

$$x_{\Lambda_i}(t+1) = F_i u_{\Lambda_i}(t) x_{\Lambda_i}(t), \quad (3.8)$$

where  $x_{\Lambda_i}(t) = \bowtie_{j=1}^{n_i} x_i^j(t)$ ,  $u_{\Lambda_i}(t) = \bowtie_{j=1}^{q_i} u_i^j(t)$ , and  $F_i \in \mathcal{L}_{k^{n_i} \times k^{n_i+q_i}}$ .

It follows from the aggregation framework that the large-scale NEG is partitioned into several small-scale control subgames  $\Lambda_i$ , where  $V_i$  is the set of players in subgame  $\Lambda_i$ . The aggregation partition is generally not unique. Smaller subgames reduce the dimension of the local transition matrices and thus

lower the computational complexity of each subgame. However, a finer partition may increase the connections among subgames and the number of input players, which makes trajectory matching more complicated. Therefore, smaller subgames do not necessarily lead to a more efficient analysis.

To describe the relation between the global and local profiles, we introduce the following natural projection.

Let  $A = \{a_1, a_2, \dots, a_v\}$  be an arbitrary ordered set of players, and let  $B = \{b_1, b_2, \dots, b_r\}$  be an ordered subset of  $A$ , where the ordering of  $B$  is inherited from  $A$ . The natural projection  $P_{A,B} : \Delta_{k^{|A|}} \rightarrow \Delta_{k^r}$  is defined by

$$P_{A,B}(x_A(t)) = x_B(t) = \times_{l=1}^r x_{b_l}(t).$$

For illustration, let  $A = \{1, 2, 3, 4\}$  be the set of four players, each with two strategies, and let  $B = \{1, 3, 4\} \subseteq A$  be a subset of  $A$ . For  $x_A(t) = \delta_{16}^9 = \delta_2^2 \times \delta_2^1 \times \delta_2^1 \times \delta_2^1$ , we have  $P_{A,B}(\delta_{16}^9) = \delta_2^2 \times \delta_2^1 \times \delta_2^1 = \delta_8^5 = x_B(t)$ .

For any set  $\Upsilon \subseteq \Delta_{k^{|A|}}$ , the same notation is used for its elementwise projection, namely,

$$P_{A,B}(\Upsilon) = \{P_{A,B}(x_A) : x_A \in \Upsilon\}.$$

For the global player set  $V$  and player set  $V_i$  of subgame  $\Lambda_i$ , the corresponding state and input strategy profiles are given by

$$x_{\Lambda_i}(t) = P_{V,V_i}(x(t)), \quad u_{\Lambda_i}(t) = P_{V,U_i}(x(t)).$$

For the undesirable set  $\Omega \subseteq \Delta_{k^n}$ , the projected undesirable set of subgame  $\Lambda_i$  is

$$\Omega_i = P_{V,V_i}(\Omega).$$

Accordingly, the admissible state set of subgame  $\Lambda_i$  is defined as

$$\Theta_i := \Delta_{k^{n_i}} \setminus \Omega_i.$$

We next formulate and characterize the reachability of its projected consensus profile. By the projection mapping defined above, we denote

$$x_{\Lambda_i,0} := P_{V,V_i}(x_0), \quad u_{\Lambda_i,0} := P_{V,U_i}(x_0).$$

Define the subgame  $\Lambda_i$ 's restriction matrix  $I_{\Omega_i} \in \mathcal{B}_{k^{n_i} \times k^{n_i}}$  by

$$\text{Col}_l(I_{\Omega_i}) = \begin{cases} 0_{k^{n_i}}, & \delta_{k^{n_i}}^l \in \Omega_i, \\ \text{Col}_l(I_{k^{n_i}}), & \delta_{k^{n_i}}^l \notin \Omega_i. \end{cases} \quad (3.9)$$

For each subgame  $\Lambda_i$ , the constrained state transition matrix is

$$F_{i,\Omega_i} = I_{\Omega_i} F_i (I_{k^{q_i}} \otimes I_{\Omega_i}), \quad (3.10)$$

where  $F_{i,\Omega_i} \in \mathcal{B}_{k^{n_i} \times k^{n_i+q_i}}$ .

Thus, the constrained dynamics of subgame  $\Lambda_i$  can be rewritten as

$$X_{\Lambda_i}(t+1) = F_{i,\Omega_i} u_{\Lambda_i}(t) X_{\Lambda_i}(t), t \geq 0, \quad (3.11)$$

where  $x_0 \in \Theta$ ,  $\mathcal{X}_{\Lambda_i}(0) = P_{V,V_i}(x_0)$ , and  $\mathcal{X}_{\Lambda_i}(t) \in \Theta_i \cup \{\mathbf{0}_{k^{n_i}}\}$ .

Decompose matrix  $F_{i,\Omega_i}$  with respect to the input profile  $u_i = \delta_{k^{q_i}}^\mu$ ,  $\mu = 1, 2, \dots, k^{q_i}$ , and we have

$$F_{i,\Omega_i} = [F_{i,\Omega_i}^1, F_{i,\Omega_i}^2, \dots, F_{i,\Omega_i}^{k^{q_i}}],$$

where each block  $F_{i,\Omega_i}^\mu \in \mathcal{B}_{k^{n_i} \times k^{n_i}}$  represents the constrained transition under the  $\mu$ -th input profile.

Define the aggregated transition matrix

$$M_{i,\Omega_i} = \sum_{\mathcal{B}} \sum_{\mu=1}^{k^{q_i}} F_{i,\Omega_i}^\mu.$$

If  $(M_{i,\Omega_i})_{r,s} > 0$ , then there exists at least one input profile driving the local state from  $\delta_{k^{n_i}}^s$  to  $\delta_{k^{n_i}}^r$ , while both of these two profiles do not belong to set  $\Omega_i$ .

More generally, the  $T$ -step reachability matrix is  $M_{i,\Omega_i}^T$ . For any integer  $T > 0$ , if

$$(M_{i,\Omega_i}^T)_{r,s} > 0, \quad (3.12)$$

then there exists an admissible  $T$ -step profile sequence from  $\delta_{k^{n_i}}^s$  to  $\delta_{k^{n_i}}^r$  that avoids the set  $\Omega_i$ .

**Example 3.3.** Consider an NEG whose aggregated network structure is shown in Figure 1.  $S = \{1, 2\}$  is the strategy set of the NEG and  $\Omega = \{\bar{x}_1, \bar{x}_2\}$  is the undesirable set, where  $\bar{x}_1 = (1, 2, 1, 2, 1, 2, 2, 2, 1)$ ,  $\bar{x}_2 = (2, 1, 2, 1, 2, 1, 1, 2, 1)$ . Set the initial strategy profile  $x_0 = (2, 1, 1, 2, 2, 2, 2, 2, 2)$ . Every player in the network employs the same payoff matrix  $A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ .

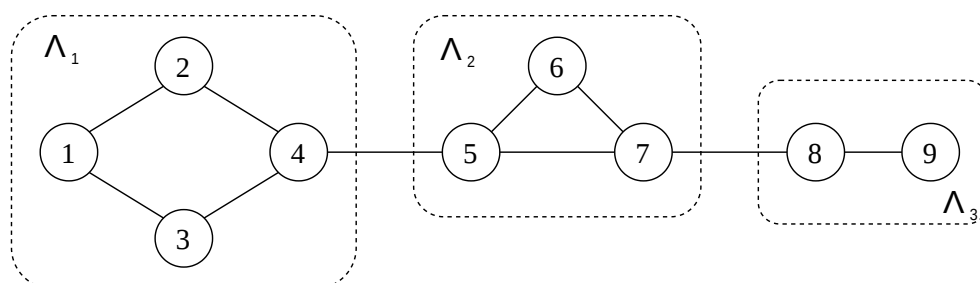


**Figure 1.** Network aggregation of the nine-player NEG.

According to Figure 1, we have

$$\begin{aligned} V_1 &= \{1, 2, 3, 4\}, U_1 = \{5\}, Y_1 = \{4\}, \\ V_2 &= \{5, 6, 7\}, U_2 = \{4, 8\}, Y_2 = \{5, 7\}, \\ V_3 &= \{8, 9\}, U_3 = \{7\}, Y_3 = \{8\}. \end{aligned}$$

By virtue of the projection mapping, we derive the restricted-profile sets corresponding to subgames  $\Lambda_1$ ,  $\Lambda_2$ , and  $\Lambda_3$  as

$$\Omega_1 = \{(1, 2, 1, 2), (2, 1, 2, 1)\}, \Omega_2 = \{(1, 2, 2), (2, 1, 1)\}, \Omega_3 = \{(2, 1)\}.$$

The initial strategy profile for each subgame is given by

$$x_{\Lambda_1,0} = (2, 1, 1, 2), \quad x_{\Lambda_2,0} = (2, 2, 2), \quad x_{\Lambda_3,0} = (2, 2).$$

We denote the strategies 1 and 2 as  $\delta_2^1, \delta_2^2$ , respectively. The global strategy consensus set is  $\Phi = \{(\delta_2^1)^9, (\delta_2^2)^9\}$ . Since neither consensus profile belongs to  $\Omega$ , one has  $\Phi_\Omega = \Phi$ . Now, we consider the consensus profile  $(\delta_2^1)^9$ . So the corresponding local consensus profiles of subgames  $\Lambda_i, i = 1, 2, 3$ , are  $(\delta_2^1)^4, (\delta_2^1)^3, (\delta_2^1)^2$ .

Evidently,  $x_{\Lambda_1}(t) = x_1(t)x_2(t)x_3(t)x_4(t)$  and  $u_{\Lambda_1}(t) = x_5(t)$ . The dynamics of subgame  $\Lambda_1$  with the undesirable set  $\Omega_1$  is given as

$$X_{\Lambda_1}(t+1) = F_{1,\Omega_1} u_{\Lambda_1}(t) X_{\Lambda_1}(t),$$

where  $F_{1,\Omega_1} = \delta_{16}[1, 1, 1, 1, 0, 9, 9, 1, 7, 0, 7, 1, 7, 9, 15, 1, 1, 1, 1, 1, 0, 10, 10, 1, 7, 0, 7, 1, 7, 10, 16]$ .

Similarly, we can derive the algebraic forms and corresponding structure matrices of subgames  $\Lambda_2$  and  $\Lambda_3$ .  $F_{2,\Omega_2} = \delta_8[1, 1, 1, 0, 0, 3, 1, 3, 1, 1, 1, 0, 0, 3, 2, 4, 1, 1, 1, 0, 0, 3, 1, 7, 1, 1, 1, 0, 0, 3, 2, 8]$  and  $F_{3,\Omega_3} = \delta_4[1, 1, 0, 2, 1, 0, 0, 4]$ .

Subsequently, we compute the 3-step reachability matrix for each subgame. We further obtain the following results:

$$\begin{aligned} (M_{1,\Omega_1}^3)_{1,10} &> 0, \\ (M_{2,\Omega_2}^3)_{1,8} &> 0, \\ (M_{3,\Omega_3}^3)_{1,4} &> 0. \end{aligned}$$

Thus, all three subgames can independently reach their respective consensus targets at the common time  $t = 3$ , while avoiding the corresponding restricted-profile sets.

Nevertheless, the reachability of subgames may not guarantee compatibility among their local evolutions. To demonstrate this point, consider the evolutionary processes of subgames  $\Lambda_1$  and  $\Lambda_2$ .

$$\Lambda_1 : \quad \delta_{16}^{10} \xrightarrow{\delta_2^2} \delta_{16}^7 \xrightarrow{\delta_2^1} \delta_{16}^9 \xrightarrow{\delta_2^1} \delta_{16}^1.$$

$$\Lambda_2 : \quad \delta_8^8 \xrightarrow{\delta_4^4} \delta_8^7 \xrightarrow{\delta_4^3} \delta_8^2 \xrightarrow{\delta_4^1} \delta_8^1.$$

The trajectory of  $\Lambda_1$  is generated under the inputs  $u_{\Lambda_1}(0) = \delta_2^2, u_{\Lambda_1}(1) = \delta_2^1, u_{\Lambda_1}(2) = \delta_2^1$ . Player 5 serves as both an output player of  $\Lambda_2$  and an input player of  $\Lambda_1$ . However, from the trajectory of  $\Lambda_2$ , the strategies of player 5 satisfy  $x_5(0) = 2 \sim \delta_2^2, x_5(1) = 2 \sim \delta_2^2, x_5(2) = 1 \sim \delta_2^1$ . For the profile trajectories of the two subgames to match, it is necessary that  $u_{\Lambda_1}(0) = x_5(0), u_{\Lambda_1}(1) = x_5(1), u_{\Lambda_1}(2) = x_5(2)$ . However, at time  $t = 1$ ,  $\Lambda_1$  needs an input value 1, while  $\Lambda_2$  produces  $x_5(1) = 2$ . Thus, the mismatch occurs at step  $t = 1$ . Therefore, the two profile trajectories do not match.

This example illustrates that local reachability of all control subgames at a common time does not suffice to ensure global reachability. An extra constraint is needed to guarantee matching between the input sequences of each subgame and the state trajectories produced by its adjacent subgames.

#### 4. Matching condition and global strategy consensus

We first define a vector for the  $i$ th subgame  $\Lambda_i$ , which stands for the input vector containing all inputs received by  $\Lambda_i$  from its connected subgames at time  $t$ .

For any  $i \neq j$ , define

$$C_{ji} := Y_j \cap U_i.$$

The set  $C_{ji}$  consists of the players that belong to subgame  $\Lambda_j$  and simultaneously serve as input players of subgame  $\Lambda_i$ .

If subgame  $\Lambda_i$  does not interact with  $\Lambda_j$ , i.e.,  $C_{ji} = \emptyset$ , then we set

$$P_{V_j, C_{ji}}(X_{\Lambda_j}(t)) := 1, \quad t \geq 0.$$

This assignment introduces no extra factors into the subsequent product and facilitates computation.

Accordingly, the input profile received by subgame  $\Lambda_i$  at time  $t$  is determined by

$$u_{\Lambda_i}(t) = \prod_{j \neq i} P_{V_j, C_{ji}}(X_{\Lambda_j}(t)), \quad i, j = 1, \dots, \omega. \quad (4.1)$$

When the subgames are connected,  $u_{\Lambda_i}(t)$  is no longer free but is determined by the strategy profiles of its neighboring subgames.

Then, we establish a group of iterative formulas taking the subsequent form

$$\begin{cases} X_{\Lambda_1}(t+1) = F_{1, \Omega_1} u_{\Lambda_1}(t) X_{\Lambda_1}(t), \\ X_{\Lambda_2}(t+1) = F_{2, \Omega_2} u_{\Lambda_2}(t) X_{\Lambda_2}(t), \\ \vdots \\ X_{\Lambda_\omega}(t+1) = F_{\omega, \Omega_\omega} u_{\Lambda_\omega}(t) X_{\Lambda_\omega}(t). \end{cases} \quad (4.2)$$

For subgame  $\Lambda_i$  (3.11), the admissible strategy consensus set is defined as  $\Phi_{i, \Omega_i} := \Phi_i \setminus \Omega_i$ , where  $\Phi_i = \{(\delta_k^1)^{n_i}, (\delta_k^2)^{n_i}, \dots, (\delta_k^k)^{n_i}\}$  is the strategy consensus set of subgame  $\Lambda_i$ .

Based on the above iterative dynamics and the admissible strategy consensus set, we next introduce cyclic trajectories for subgame  $\Lambda_i$ .

**Definition 4.1.** Consider the subgame  $\Lambda_i$ ,  $i \in \{1, 2, \dots, \omega\}$ .

(i) Let  $\eta$  be a positive integer and the state trajectory

$$(\delta_k^{j_1})^{n_i} \rightarrow (\delta_k^{j_2})^{n_i} \rightarrow \dots \rightarrow (\delta_k^{j_\eta})^{n_i} \rightarrow (\delta_k^{j_1})^{n_i}$$

is called a strategy consensus cycle under the control sequence  $u_{\Lambda_i}(0), u_{\Lambda_i}(1), \dots, u_{\Lambda_i}(\eta-1)$ , if

$$(\delta_k^{j_{t+1}})^{n_i} = F_{i, \Omega_i} u_{\Lambda_i}(t) (\delta_k^{j_t})^{n_i}, \quad \forall t \in \{1, 2, \dots, \eta\},$$

where  $\delta_k^{j_{\eta+1}} := \delta_k^{j_1}$ , and all states of the trajectory are pairwise distinct.

(ii) The cycle

$$(\delta_k^{j_1})^{n_i} \rightarrow (\delta_k^{j_2})^{n_i} \rightarrow \dots \rightarrow (\delta_k^{j_\eta})^{n_i} \rightarrow (\delta_k^{j_1})^{n_i}$$

is called a control-state matched strategy consensus cycle if

$$u_{\Lambda_i}(t) = (\delta_k^{j_{t+1}})^{q_i}, \quad t \in \{1, 2, \dots, \eta-1\}.$$

To find the control-state matched strategy consensus cycles of subgame  $\Lambda_i$ , we construct several matrices restricted to the admissible strategy consensus set  $\Phi_{i,\Omega_i}$ .

Construct the transition matrix  $F_{i,\Phi_{i,\Omega_i}}$  restricted to  $\Phi_{i,\Omega_i}$  by

$$\text{Col}_j(F_{i,\Phi_{i,\Omega_i}}) = \begin{cases} \text{Col}_j(F_{i,\Omega_i}), & \delta_{k^{n_i}}^j \in \Phi_{i,\Omega_i}, \\ \mathbf{0}_{k^{n_i}}, & \text{otherwise.} \end{cases}$$

Then decompose

$$F_{i,\Phi_{i,\Omega_i}} = [F_{i,\Phi_{i,\Omega_i}}^1, F_{i,\Phi_{i,\Omega_i}}^2, \dots, F_{i,\Phi_{i,\Omega_i}}^{k_{q_i}}],$$

and further define the one-step controllability matrix

$$M_{i,\Phi_{i,\Omega_i}} = \sum_{\mathcal{B}}^{k_{q_i}} F_{i,\Phi_{i,\Omega_i}}^\mu.$$

Using the above transition matrix  $M_{i,\Phi_{i,\Omega_i}}$ , we compute

$$C_{i,\Phi_{i,\Omega_i}} = \sum_{\mathcal{B}}^k (M_{i,\Phi_{i,\Omega_i}})^j.$$

The upper bound  $k$  is the number of candidate strategies for each player. Any path longer than  $k$  must contain repeated consensus strategy profiles and therefore yields no new reachability relations. Then, we determine the set

$$\mathcal{R}_i = \{\delta_{k^{n_i}}^l \mid (C_{i,\Phi_{i,\Omega_i}})_{l,l} = 1\}.$$

Every  $\delta_{k^{n_i}}^l \in \mathcal{R}_i$  can return to itself within at most  $k$  steps while remaining in  $\Phi_{i,\Omega_i}$ .

Thus, every profile in  $\mathcal{R}_i$  lies on at least one strategy-consensus cycle contained in  $\Phi_{i,\Omega_i}$ . The following proposition further provides a condition under which such a profile is on a control-state matched strategy consensus cycle.

**Proposition 4.2.** *For subgame  $\Lambda_i$ , there exists a control-state matched strategy consensus cycle starting from  $(\delta_k^l)^{n_i} \in \mathcal{R}_i$ , if there exists a positive integer  $d_l \leq k$  such that*

$$\begin{aligned} (\delta_k^{l_1})^{n_i} &= F_{i,\Phi_{i,\Omega_i}}(\delta_k^l)^{q_i+n_i}, \\ (\delta_k^{l_2})^{n_i} &= F_{i,\Phi_{i,\Omega_i}}(\delta_k^{l_1})^{q_i+n_i}, \\ &\vdots \\ (\delta_k^l)^{n_i} &= F_{i,\Phi_{i,\Omega_i}}(\delta_k^{l_{d_l-1}})^{q_i+n_i}. \end{aligned}$$

Moreover, the control-state matched strategy consensus cycle is  $(\delta_k^l)^{n_i} \rightarrow (\delta_k^{l_1})^{n_i} \rightarrow \dots \rightarrow (\delta_k^{l_{d_l-1}})^{n_i} \rightarrow (\delta_k^l)^{n_i}$ .

*Proof.* By the first equality, we have

$$\begin{aligned} (\delta_k^{l_1})^{n_i} &= F_{i,\Phi_{i,\Omega_i}}(\delta_k^l)^{q_i+n_i} \\ &= F_{i,\Phi_{i,\Omega_i}}(\delta_k^l)^{q_i}(\delta_k^l)^{n_i}. \end{aligned}$$

The equality indicates that when  $x_{\Lambda_i}(t) = (\delta_k^l)^{n_i}$  and  $u_{\Lambda_i} = (\delta_k^l)^{q_i}$ , subgame  $\Lambda_i$  evolves to profile  $(\delta_k^l)^{n_i}$  at the next time. Both the state players and the input players adopt the same strategy  $\delta_k^l$ , so the control-state matching condition is satisfied.

Similarly, from the second equality, we have

$$(\delta_k^{l_2})^{n_i} = F_{i, \Phi_{i, \Omega_i}}(\delta_k^{l_1})^{q_i} (\delta_k^{l_1})^{n_i},$$

which yields the matched transition  $(\delta_k^{l_1})^{n_i} \rightarrow (\delta_k^{l_2})^{n_i}$ .

Repeating the same argument for the remaining equalities, we obtain that the profile of subgame  $\Lambda_i$  can evolve along the following trajectory:

$$(\delta_k^l)^{n_i} \rightarrow (\delta_k^{l_1})^{n_i} \rightarrow \dots \rightarrow (\delta_k^{l_{d_l-1}})^{n_i} \rightarrow (\delta_k^l)^{n_i}.$$

Since the control-state matching condition is satisfied for each transition, the trajectory is a control-state matched strategy consensus cycle.  $\square$

**Remark 4.3.** For each  $(\delta_k^l)^{n_i} \in \mathcal{R}_i$ , it is sufficient to start with the matched control input  $u_{\Lambda_i}(0) = (\delta_k^l)^{q_i}$  and follow the control-state matching rule thereafter. There is no need to enumerate all possible control sequences. For each profile  $(\delta_k^l)^{n_i} \in \mathcal{R}_i$ , the control-state matching condition fixes the control input uniquely at each time instant. Note that  $|\mathcal{R}_i| \leq k$ . Hence, at most  $|\mathcal{R}_i|$  candidate trajectories need to be examined.

By computing the control-state matching strategy consensus cycles for each subgame, we can verify whether all subgames admit control-state matched strategy consensus cycles with a common period. To realize global strategy consensus, we further require every subgame to achieve internal strategy consensus at a common time instant. This yields the following theorem.

**Theorem 4.4.** Consider the large-scale NEG under the aggregation framework (3.6) with the restricted-profile set  $\Omega$ . The NEG achieves strategy consensus starting from  $x_0$  if the following three conditions hold:

- (i) There exists a nonnegative integer  $\eta \in \{1, 2, \dots, k\}$  such that for each subgame  $\Lambda_i$ ,  $i \in \{1, 2, \dots, \omega\}$ , there exists a control-state matched strategy consensus cycle of length  $\eta$  as

$$(\delta_k^{j_1})^{n_i} \rightarrow (\delta_k^{j_2})^{n_i} \rightarrow \dots \rightarrow (\delta_k^{j_\eta})^{n_i} \rightarrow (\delta_k^{j_1})^{n_i}.$$

- (ii) There exist a common positive integer  $T \geq 0$  and a common index  $\beta \in \{1, 2, \dots, \eta\}$  such that

$$\mathcal{X}_{\Lambda_i}(T; x_{\Lambda_i, 0}) = (\delta_k^{j_\beta})^{n_i}, \quad i = 1, 2, \dots, \omega.$$

- (iii) The trajectories of  $\Lambda_i$ ,  $i = \{1, 2, \dots, \omega\}$ , satisfy the iterative formulas

$$\begin{cases} x_{\Lambda_i}(0) = P_{V, V_i}(x_0), \\ u_{\Lambda_i}(t) = \times_{j \neq i} P_{V_j, C_{j_i}}(\mathcal{X}_{\Lambda_j}(t)), \\ \mathcal{X}_{\Lambda_i}(t+1) = F_{i, \Omega_i} u_{\Lambda_i}(t) \mathcal{X}_{\Lambda_i}(t), \end{cases} \quad t \in \{0, 1, \dots, T-1\}. \quad (4.3)$$

*Proof.* If condition (i) holds, each subgame  $\Lambda_i$  admits a control-state matched strategy consensus cycle with the common period  $\eta$  of the form

$$(\delta_k^{j_1})^{n_i} \rightarrow (\delta_k^{j_2})^{n_i} \rightarrow \cdots \rightarrow (\delta_k^{j_\eta})^{n_i}.$$

Since there are at most  $k$  distinct strategy consensus states and the cycle contains distinct states, we have  $\eta \in \{1, 2, \dots, k\}$ .

If condition (ii) holds, that is, there exists a common time  $T \geq 0$  and a common index  $\beta \in \{1, \dots, \eta\}$  such that

$$\mathcal{X}_{\Lambda_i}(T; x_{\Lambda_i,0}) = (\delta_k^{j_\beta})^{n_i}, \quad i = 1, \dots, \omega,$$

then, at time  $T$ , all subgames adopt the identical strategy  $\delta_k^{j_\beta}$ , which appears on the control-state matched cycle of length  $\eta$  we have obtained. Since  $V_1, \dots, V_\omega$  form a partition of the player set  $V$ , the global profile at time  $T$  satisfies

$$\begin{aligned} x(T; x_0) &= \times_{i=1}^\omega \mathcal{X}_{\Lambda_i}(T; x_{\Lambda_i,0}) \\ &= \times_{i=1}^\omega (\delta_k^{j_\beta})^{n_i} \\ &= (\delta_k^{j_\beta})^n \in \Phi. \end{aligned}$$

Thus, the overall NEG reaches a strategy consensus profile at time  $T$ . Since profile  $(\delta_k^{j_\beta})^{n_i}$  is on the control-state matched strategy consensus cycle, it can be seen that the control input satisfies

$$\begin{aligned} u_{\Lambda_i}(T) &= \times_{j \neq i} P_{V_j, C_{ji}}(\mathcal{X}_{\Lambda_j}(T; x_{\Lambda_j,0})) \\ &= (\delta_k^{j_\beta})^{q_i} \end{aligned}$$

at time  $T$ . Then

$$\begin{aligned} \mathcal{X}_{\Lambda_i}(T+1; x_{\Lambda_i,0}) &= F_{i, \Omega_i} u_{\Lambda_i}(T) \mathcal{X}_{\Lambda_i}(T; x_{\Lambda_i,0}) \\ &= F_{i, \Omega_i} (\delta_k^{j_\beta})^{q_i} (\delta_k^{j_\beta})^{n_i} \\ &= (\delta_k^{j_{\beta+1}})^{n_i}, \end{aligned}$$

where  $j_{\beta+1} \in \{1, 2, \dots, \eta\}$ ,  $i = 1, \dots, \omega$ .

Now consider any  $t > T$ , and suppose there exists  $j_\ell, \ell \in \{1, 2, \dots, \eta\}$  such that

$$\mathcal{X}_{\Lambda_i}(t; x_{\Lambda_i,0}) = (\delta_k^{j_\ell})^{n_i}, \quad i = 1, \dots, \omega.$$

Then

$$\begin{aligned} u_{\Lambda_i}(t) &= \times_{j \neq i} P_{V_j, C_{ji}}((\delta_k^{j_\ell})^{n_j}) \\ &= (\delta_k^{j_\ell})^{q_i} \end{aligned}$$

and

$$\begin{aligned} \mathcal{X}_{\Lambda_i}(t+1; x_{\Lambda_i,0}) &= F_{i, \Omega_i} (\delta_k^{j_\ell})^{q_i} (\delta_k^{j_\ell})^{n_i} \\ &= (\delta_k^{j_{\ell+1}})^{n_i}, \quad i = 1, \dots, \omega, \end{aligned}$$

where  $j_{\eta+1} := j_1$ .

Since the control-state matching strategy consensus cycle is a cycle within the set  $\Phi_\Omega$ , it follows that

$$\begin{aligned} x(t+1; x_0) &= \bowtie_{i=1}^{\omega} (\delta_k^{j_{i+1}})^{n_i} \\ &= (\delta_k^{j_{\ell+1}})^n \in \Phi_\Omega. \end{aligned}$$

That is,  $x(t; x_0) \in \Phi_\Omega, \forall t \geq T$ .

If condition (iii) holds, the trajectories of each subgame  $\Lambda_i$  satisfy the iterative formulas (4.3). At time  $t = 0$ , for  $i = 1, \dots, \omega$ , we have

$$\begin{aligned} \mathcal{X}_{\Lambda_i}(0) &= P_{V, V_i}(x_0), \\ u_{\Lambda_i}(0) &= \bowtie_{j \neq i} P_{V_j, C_{ji}}(\mathcal{X}_{\Lambda_j}(0)). \end{aligned}$$

Hence, the input for each subgame is determined by the corresponding outputs from its neighboring subgames, and

$$\mathcal{X}_{\Lambda_i}(1) = F_{i, \Omega_i} u_{\Lambda_i}(0) \mathcal{X}_{\Lambda_i}(0), \quad i = 1, \dots, \omega,$$

which means the matched local transitions from  $t = 0$  to  $t = 1$ .

Suppose the local trajectories are matched at time  $t = a$ , where  $0 \leq a < T$ . According to iterative formulas, we have

$$u_{\Lambda_i}(a) = \bowtie_{j \neq i} P_{V_j, C_{ji}}(\mathcal{X}_{\Lambda_j}(a))$$

and

$$\mathcal{X}_{\Lambda_i}(a+1) = F_{i, \Omega_i} u_{\Lambda_i}(a) \mathcal{X}_{\Lambda_i}(a).$$

Thus, the transitions from time  $t = a$  to  $t = a+1$  are also matched. Moreover, since  $V = V_1 \cup \dots \cup V_\omega$  is a partition of the player set, the strategy profile of the large-scale NEG can be obtained by combining the trajectories of all subgames as

$$x(t; x_0) = \bowtie_{i=1}^{\omega} \mathcal{X}_{\Lambda_i}(t; x_{\Lambda_i, 0}).$$

According to (4.3), the input profile of each subgame is determined by the states of its neighboring subgames. Therefore, the matched trajectories of all subgames are mutually compatible and realize the evolution of the original NEG.  $\square$

**Remark 4.5.** *The three conditions form a sequential mechanism ensuring strategy consensus of the large-scale NEG initiated from  $x_0$  while avoiding the restricted set  $\Omega$ . Condition (i) requires that each subgame admits a control-state matched consistent strategy cycle, and all cycles share the same period, cycle strategies, and evolution order. This ensures that the global system can maintain consistency persistently after entering the consensus state. Condition (ii) requires that all subgames reach the same strategy on their respective cycles at the same time instant  $T$ . Condition (iii) requires that the input of each subgame matches the states generated by other subgames, so that the local trajectories prior to reaching the consensus state can form a global trajectory avoiding  $\Omega$ . Collectively, these conditions constitute the complete mechanism of the sufficient criterion for achieving strategy consensus without entering  $\Omega$ .*

To further illustrate the computational advantage of network aggregation, we compare the computational costs of directly analyzing the original NEG and using network aggregation. The comparison is summarized in Table 1.

**Table 1.** Complexity comparison between the original network and the aggregated network.

|                          | Original network | Network aggregation                                                    |
|--------------------------|------------------|------------------------------------------------------------------------|
| Maximum state space size | $k^n$            | $k^\varsigma$ , $\varsigma = \max\{n_1, n_2, \dots, n_\omega\}$        |
| Time complexity          | $O(k^{3n})$      | $O\left(\sum_{i=1}^\omega k^{3n_i} + \sum_{i=1}^\omega k^{q_i}\right)$ |

Subgame  $\Lambda_i$  has  $k^{q_i}$  possible input profiles. Therefore, the computational cost associated with input matching over all  $\omega$  subgames is  $O(\sum_{i=1}^\omega k^{q_i})$ . Hence, the overall computational complexity after aggregation is  $O\left(\sum_{i=1}^\omega k^{3n_i} + \sum_{i=1}^\omega k^{q_i}\right)$ . When the network is partitioned such that  $n_i, q_i \ll n$ , the exponential terms involved in the aggregated analysis are much smaller than those in the direct analysis. Therefore, the aggregation method can significantly reduce the computational complexity.

**Remark 4.6.** *The projection of the set of restricted profiles onto subgames may produce an enlarged set of restricted profiles. After obtaining local sets of restricted profiles for each subgame, their union expands the original global set of restricted profiles  $\Omega$ . We denote this enlarged set as  $\overline{\Omega}$ , where*

$$\overline{\Omega} := \bigcup_{i=1}^\omega \{x \in \Delta_{k^n} \mid P_{V,V_i}(x) \in \Omega_i\}.$$

Clearly,  $\Omega \subseteq \overline{\Omega}$ . Since  $\Omega \subseteq \overline{\Omega}$ , we get  $\Delta_{k^n} \setminus \overline{\Omega} \subseteq \Delta_{k^n} \setminus \Omega$ . Thus, trajectories admissible under  $\Omega$  may not be realizable under  $\overline{\Omega} \setminus \Omega$ . For this reason, the aggregation-based conditions in Theorem 4.4 are conservative and only provide a sufficient condition for the strategy consensus of the original NEG. Precisely, a finer partition induces projections onto smaller player sets, which may cause more globally admissible profiles to be excluded by the resulting local restricted sets, leading to a larger  $\overline{\Omega}$  and increased conservatism.

**Example 4.7.** *Consider an NEG with three players, each having two strategies  $\delta_2^1$  and  $\delta_2^2$ . Let the global restricted profile set be  $\Omega = \{\delta_8^1\}$ . Partition players into  $V_1 = \{1, 2\}$  and  $V_2 = \{3\}$ . Projecting  $\Omega$  onto the two subgames yields*

$$\Omega_1 = P_{V,V_1}(\Omega) = \{\delta_4^1\}, \quad \Omega_2 = P_{V,V_2}(\Omega) = \{\delta_2^1\}.$$

*The subgame aggregation gives the enlarged restricted set*

$$\overline{\Omega} = \{\delta_4^1 \ltimes \delta_2^1, \delta_4^1 \ltimes \delta_2^2\} \cup \{\delta_4^2 \ltimes \delta_2^1, \delta_4^3 \ltimes \delta_2^1, \delta_4^4 \ltimes \delta_2^1\} = \{\delta_8^1, \delta_8^2, \delta_8^3, \delta_8^5, \delta_8^7\}.$$

*We see  $\Omega \subsetneq \overline{\Omega}$ .*

## 5. An illustrative example

**Example 5.1.** *Consider an NEG with 20 players and strategy set  $S = \{1, 2, 3, 4\}$ , where players update their strategies via the myopic best-response rule. Figure 2 illustrates the aggregation structure. Each*

$$A_2 = \begin{bmatrix} 0 & 4 & 3 & 3 \\ 4 & 1 & 0 & 0 \\ 3 & 0 & -10 & -9 \\ 3 & 0 & -9 & -11 \end{bmatrix}.$$

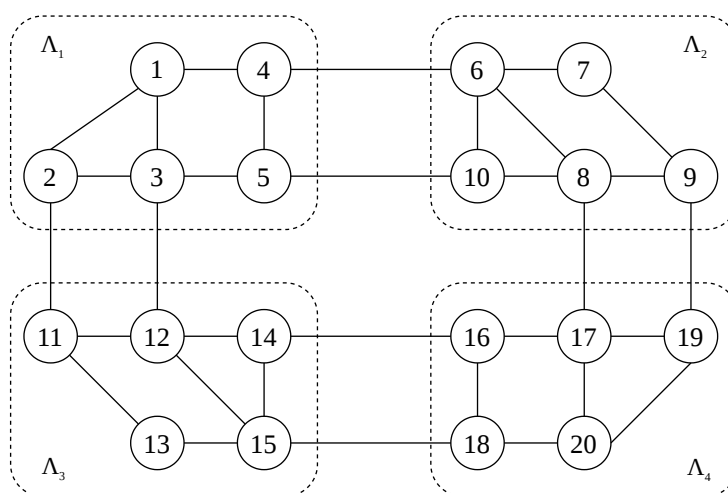
$$\begin{aligned}\bar{x}_1 &= (1, 2, 2, 3, 4, 2, 2, 2, 3, 4, 1, 2, 2, 3, 4, 1, 2, 2, 2, 4), \\ \bar{x}_2 &= (4, 3, 2, 1, 2, 2, 4, 1, 3, 2, 3, 2, 4, 2, 1, 2, 1, 3, 4, 2), \\ \bar{x}_3 &= (2, 4, 3, 2, 1, 3, 1, 2, 4, 2, 2, 3, 1, 2, 4, 4, 2, 1, 3, 2).\end{aligned}$$


**Figure 2.** Network aggregation of the twenty-player NEG.

According to Figure 2, the NEG is divided into subgames  $\Lambda_i, i = 1, 2, 3, 4$ . We denote strategies 1, 2, 3, 4 by  $\delta_4^1, \delta_4^2, \delta_4^3, \delta_4^4$ , respectively. First, we have

$$\begin{aligned} V_1 &= \{1, 2, 3, 4, 5\}, U_1 = \{6, 10, 11, 12\}, Y_1 = \{2, 3, 4, 5\}, \\ V_2 &= \{6, 7, 8, 9, 10\}, U_2 = \{4, 5, 17, 19\}, Y_2 = \{6, 8, 9, 10\}, \\ V_3 &= \{11, 12, 13, 14, 15\}, U_3 = \{2, 3, 16, 18\}, Y_3 = \{11, 12, 14, 15\}, \\ V_4 &= \{16, 17, 18, 19, 20\}, U_4 = \{8, 9, 14, 15\}, Y_4 = \{16, 17, 18, 19\}. \end{aligned}$$

Then, the nonempty matching sets are given by

$$C_{12} = \{4, 5\}, C_{21} = \{6, 10\}, C_{13} = \{2, 3\}, C_{31} = \{11, 12\},$$

$$C_{24} = \{8, 9\}, C_{42} = \{17, 19\}, C_{34} = \{14, 15\}, C_{43} = \{16, 18\}.$$

By the projection mapping, we get the undesirable sets  $\Omega_i, i = 1, 2, 3, 4$ .

$$\begin{aligned}\Omega_1 &= \{(1, 2, 2, 3, 4), (4, 3, 2, 1, 2), (2, 4, 3, 2, 1)\} \sim \{\delta_{45}^{92}, \delta_{45}^{914}, \delta_{45}^{485}\}, \\ \Omega_2 &= \{(2, 2, 2, 3, 4), (4, 2, 1, 3, 2), (3, 1, 2, 4, 2)\} \sim \{\delta_{45}^{348}, \delta_{45}^{458}, \delta_{45}^{542}\}, \\ \Omega_3 &= \{(1, 2, 2, 3, 4), (3, 2, 4, 2, 1), (2, 3, 1, 2, 4)\} \sim \{\delta_{45}^{92}, \delta_{45}^{629}, \delta_{45}^{392}\}, \\ \Omega_4 &= \{(1, 2, 2, 2, 4), (2, 1, 3, 4, 2), (4, 2, 1, 3, 2)\} \sim \{\delta_{45}^{88}, \delta_{45}^{302}, \delta_{45}^{842}\}.\end{aligned}$$

Accordingly, the Boolean matrices  $I_{\Omega_i} \in \mathcal{B}_{4^5 \times 4^5}$  related to  $\Omega_i$  are given as follows:

$$\begin{aligned}\text{Col}_l(I_{\Omega_1}) &= 0, l \in \{92, 914, 485\}, & \text{Col}_l(I_{\Omega_2}) &= 0, l \in \{348, 458, 542\}, \\ \text{Col}_l(I_{\Omega_3}) &= 0, l \in \{92, 629, 392\}, & \text{Col}_l(I_{\Omega_4}) &= 0, l \in \{88, 302, 842\}.\end{aligned}$$

The constrained structure matrices are given by

$$F_{i, \Omega_i} = I_{\Omega_i} F_i (I_{4^{q_i}} \otimes I_{\Omega_i}), \quad i = 1, 2, 3, 4.$$

Since  $n_i = 5$  and  $q_i = 4$ , each matrix  $F_{i, \Omega_i} \in \mathcal{B}_{4^{n_i} \times 4^{n_i+q_i}}$  contains  $4^9$  columns. Hence, its expanded form is omitted for brevity.

For each subgame  $\Lambda_i$ , the admissible local strategy consensus set is

$$\Phi_{i, \Omega_i} = \{(\delta_4^1)^5, (\delta_4^2)^5, (\delta_4^3)^5, (\delta_4^4)^5\}.$$

To identify the strategy consensus profiles belonging to a cycle, we first restrict the constrained dynamics of each subgame  $\Lambda_i$  to the strategy consensus set  $\Phi_{i, \Omega_i}$ , and obtain the transition matrix  $F_{i, \Phi_{i, \Omega_i}}$ . Using  $F_{i, \Phi_{i, \Omega_i}}$ , the one-step reachability matrix  $M_{i, \Phi_{i, \Omega_i}}$  is constructed via a Boolean sum over all valid control inputs. We further calculate

$$C_{i, \Phi_{i, \Omega_i}} = \sum_{\mathcal{B}}^4 (M_{i, \Phi_{i, \Omega_i}})^j, \quad i = 1, 2, 3, 4.$$

Inspection over the diagonal entries of  $C_{i, \Phi_{i, \Omega_i}}$  yields the strategy consensus profiles that return to themselves after a finite sequence of valid transitions. Direct calculation gives

$$\mathcal{R}_i = \{(\delta_4^1)^5, (\delta_4^2)^5\} = \{\delta_{45}^1, \delta_{45}^{342}\}, \quad i = 1, 2, 3, 4.$$

Thus, for each subgame, the profiles  $(\delta_4^1)^5$  and  $(\delta_4^2)^5$  belong to a strategy consensus cycle. Moreover, according to Proposition 4.2, since  $q_i = 4$ , direct computation yields

$$(\delta_4^2)^5 = F_{i, \Phi_{i, \Omega_i}}(\delta_4^1)^{4+5}, \quad (\delta_4^1)^5 = F_{i, \Phi_{i, \Omega_i}}(\delta_4^2)^{4+5}, \quad i = 1, 2, 3, 4.$$

Hence, each subgame possesses the control-state matched strategy consensus cycle

$$(\delta_4^1)^5 \rightarrow (\delta_4^2)^5 \rightarrow (\delta_4^1)^5,$$

with the common period  $\eta = 2$ . Therefore, condition (i) of Theorem 4.4 is satisfied.

To verify conditions (ii) and (iii), consider the valid initial profile  $x_0 = (1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$ . The corresponding initial profiles of the four subgames are

$$x_{\Lambda_1, 0} = (1, 1, 2, 2, 2), \quad x_{\Lambda_i, 0} = (2, 2, 2, 2, 2), \quad i = 2, 3, 4.$$

According to the iterative formule (4.3), at  $t = 0$ , the input profiles of the four subgames are determined by their connected subgames as

$$\begin{aligned} u_{\Lambda_1}(0) &= P_{V_2, C_{21}}(x_{\Lambda_2,0}) \ltimes P_{V_3, C_{31}}(x_{\Lambda_3,0}) \\ &= \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \sim (2, 2, 2, 2), \\ u_{\Lambda_2}(0) &= P_{V_1, C_{12}}(x_{\Lambda_1,0}) \ltimes P_{V_4, C_{42}}(x_{\Lambda_4,0}) \\ &= \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \sim (2, 2, 2, 2), \\ u_{\Lambda_3}(0) &= P_{V_1, C_{13}}(x_{\Lambda_1,0}) \ltimes P_{V_4, C_{43}}(x_{\Lambda_4,0}) \\ &= \delta_4^1 \ltimes \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \sim (1, 2, 2, 2), \\ u_{\Lambda_4}(0) &= P_{V_2, C_{24}}(x_{\Lambda_2,0}) \ltimes P_{V_3, C_{34}}(x_{\Lambda_3,0}) \\ &= \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \ltimes \delta_4^2 \sim (2, 2, 2, 2). \end{aligned}$$

For  $t = 1$ , we have

$$\begin{cases} \mathcal{X}_{\Lambda_1}(1; x_{\Lambda_1,0}) = F_{1, \Omega_1} u_{\Lambda_1}(0) x_{\Lambda_1,0} = \delta_{4^5}^5 \sim (1, 1, 2, 1, 1), \\ \mathcal{X}_{\Lambda_i}(1; x_{\Lambda_i,0}) = F_{i, \Omega_i} u_{\Lambda_i}(0) x_{\Lambda_i,0} = \delta_{4^5}^5 \sim (1, 1, 1, 1, 1), \quad i = 2, 3, 4. \end{cases}$$

The input of subgames  $\Lambda_i$ ,  $i = 1, 2, 3, 4$ , are given as

$$\begin{aligned} u_{\Lambda_1}(1) &= P_{V_2, C_{21}}(\mathcal{X}_{\Lambda_2}(1; x_{\Lambda_2,0})) \ltimes P_{V_3, C_{31}}(\mathcal{X}_{\Lambda_3}(1; x_{\Lambda_3,0})) \\ &= \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^1 \sim (1, 1, 1, 1), \\ u_{\Lambda_2}(1) &= P_{V_1, C_{12}}(\mathcal{X}_{\Lambda_1}(1; x_{\Lambda_1,0})) \ltimes P_{V_4, C_{42}}(\mathcal{X}_{\Lambda_4}(1; x_{\Lambda_4,0})) \\ &= \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^1 \sim (1, 1, 1, 1), \\ u_{\Lambda_3}(1) &= P_{V_1, C_{13}}(\mathcal{X}_{\Lambda_1}(1; x_{\Lambda_1,0})) \ltimes P_{V_4, C_{43}}(\mathcal{X}_{\Lambda_4}(1; x_{\Lambda_4,0})) \\ &= \delta_4^1 \ltimes \delta_4^2 \ltimes \delta_4^1 \ltimes \delta_4^1 \sim (1, 2, 1, 1), \\ u_{\Lambda_4}(1) &= P_{V_2, C_{24}}(\mathcal{X}_{\Lambda_2}(1; x_{\Lambda_2,0})) \ltimes P_{V_3, C_{34}}(\mathcal{X}_{\Lambda_3}(1; x_{\Lambda_3,0})) \\ &= \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^1 \sim (1, 1, 1, 1). \end{aligned}$$

For  $t = 2$ , we have

$$\begin{aligned} \mathcal{X}_{\Lambda_i}(2; x_{\Lambda_i,0}) &= F_{i, \Omega_i} u_{\Lambda_i}(1) \mathcal{X}_{\Lambda_i}(1; x_{\Lambda_i,0}) \\ &= (\delta_4^2)^5 \sim (2, 2, 2, 2, 2), \quad i = 1, 2, 3, 4. \end{aligned}$$

Hence, at time  $t = 2$ , every player within each subgame adopts strategy 2 on its corresponding control-state matched strategy consensus cycle. Therefore, condition (ii) is satisfied.

Consequently, each subgame achieves strategy consensus within itself at  $t = 2$ , and the global strategy profile is

$$x(2; x_0) = (\delta_4^2)^{20}.$$

Afterward, the subgames follow their control-state matched strategy consensus cycles, which yield

$$(\delta_4^2)^{20} \rightarrow (\delta_4^1)^{20} \rightarrow (\delta_4^2)^{20} \rightarrow (\delta_4^1)^{20} \rightarrow \dots$$

Therefore, by Theorem 4.4, the NEG achieves strategy consensus starting from  $x_0$ .

If the original NEG in this example is analyzed directly, the strategy-consensus analysis involves  $4^{20}$  strategy profiles, and the corresponding computational term is  $4^{60}$ . Such a large-scale calculation is difficult to perform directly. Under the aggregation shown in Figure. 2, the NEG is divided into four subgames, each containing five state players and four input players. Thus, each subgame has only  $4^5 = 1024$  state profiles and  $4^4 = 256$  possible input profiles. The corresponding computational term is  $(4 \cdot 4^{15} + 4 \cdot 4^4) \ll 4^{60}$ . Therefore, the aggregation method greatly reduces the computational burden for the considered large-scale NEG.

## 6. Conclusions

We have investigated constrained strategy consensus problems for large-scale NEG, where all state trajectories must stay away from an undesirable set. Network aggregation has been utilized to decompose the original game into smaller subgames and reduce computational complexity. The definition of control-state matched strategy consensus cycles has been given, and a proposition for checking their existence has been presented. A sufficient condition for achieving global strategy consensus starting from a given initial profile has been further derived by analyzing reachability properties and trajectory matching among subgames. Due to the expansion of the undesirable set under aggregation, only sufficient conditions have been obtained in this paper. In future work, we will investigate strategy consensus reachability for globally feasible initial profiles and explore necessary and sufficient conditions to overcome this limitation.

## Author contributions

Huixin Kan and Shihua Fu: Methodology, Formal analysis, Investigation, Resources, Writing—original draft preparation, Conceptualization, Validation; Rui Wang and Yifan Zhao: Writing—review and editing, Supervision. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

## References

1. A. Basak, S. Sengupta, Evolution of cooperation in multichannel games on multiplex networks, *PLOS Comput. Biol.*, **20** (2024), e1012678. <https://doi.org/10.1371/journal.pcbi.1012678>
2. Y. Kuo, O. Carja, Evolutionary graph theory beyond pairwise interactions: Higher-order network motifs shape times to fixation in structured populations, *PLOS Comput. Biol.*, **20** (2024), e1011905. <https://doi.org/10.1371/journal.pcbi.1011905>
3. B. Allen, G. Lippner, Y. Chen, B. Fotouhi, N. Momeni, S. Yau, et al., Evolutionary dynamics on any population structure, *Nature*, **544** (2017), 227–230. <https://doi.org/10.1038/nature21723>
4. Y. Hu, Z. Wang, X. Li, Impact of policies on electric vehicle diffusion: An evolutionary game of small world network analysis, *J. Clean. Prod.*, **265** (2020), 121703. <https://doi.org/10.1016/j.jclepro.2020.121703>
5. Y. Jia, Y. Li, J. Feng, Networked evolutionary game-based demand response via feedback controls, *IEEE Trans. Autom. Sci. Eng.*, **21** (2024), 1638–1646. <https://doi.org/10.1109/TASE.2023.3249769>
6. D. Cheng, H. Qi, Z. Li, *Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach*, London: Springer, 2011. <https://doi.org/10.1007/978-0-85729-097-7>
7. D. Cheng, F. He, H. Qi, T. Xu, Modeling, analysis and control of networked evolutionary games, *IEEE Trans. Autom. Control*, **60** (2015), 2402–2415. <https://doi.org/10.1109/TAC.2015.2404471>
8. X. Yang, Z. Geng, H. Li, Matrix-based method for the analysis and control of networked evolutionary games: A survey, *Games*, **14** (2023), 22. <https://doi.org/10.3390/g14020022>
9. S. Fu, J. Wang, Z. Wang, C. D. Vecchio, J. Zhao, Equilibrium existence and convergence of congestion games with stochastic disturbances, *Sci. China Inf. Sci.*, **69** (2026), 140209. <https://doi.org/10.1007/s11432-025-4798-y>
10. L. Yu, S. Fu, J. Pan, J. E. Feng, Z. Wang, Weighted Nash equilibrium of incomplete-profile networked evolutionary games with multiple payoffs, *Asian J. Control*, **26** (2024), 1472–1482. <https://doi.org/10.1002/asjc.3274>
11. D. Wang, S. Fu, J. Wang, Y. Li, Z. Wang, The stability degree analysis of profiles for networked evolutionary games with switching topology, *Nonlinear Anal. Hybrid Syst.*, **50** (2023), 101396. <https://doi.org/10.1016/j.nahs.2023.101396>
12. Z. Wang, S. Fu, J. Wang, X. Zhao, Stable degree analysis for profile of networked evolutionary games with disturbances, *Int. J. Control Autom. Syst.*, **22** (2024), 2064–2073. <https://doi.org/10.1007/s12555-023-0314-x>
13. Y. Tang, L. Li, J. Lu, Modeling and optimization for networked evolutionary games with player exit mechanism: semi-tensor product of matrices method, *Phys. A*, **590** (2022), 126710. <https://doi.org/10.1016/j.physa.2021.126710>
14. S. Fu, L. Li, J. Feng, Strategy optimization of controlled evolutionary games on a two-layer coupled network using Lebesgue sampling, *Nonlinear Anal. Hybrid Syst.*, **56** (2025), 101570. <https://doi.org/10.1016/j.nahs.2024.101570>

15. K. Jiang, J. Wang, Y. Yan, C. Li, Y. Wu, Inertia-based strategy updating rule design for congestion games and its application to NFV networks, *Inf. Sci.*, **755** (2026), 123817. <https://doi.org/10.1016/j.ins.2026.123817>
16. Q. Zhang, J. Wu, J. Xie, G. Wang, Y. Huang, Networked evolutionary game-based energy trading strategy for smart grid with time-varying delays, *IEEE Trans. Ind. Inform.*, **21** (2025), 2659–2668. <https://doi.org/10.1109/TII.2024.3514139>
17. Y. Sun, S. Fu, L. Xia, J. Xu, Minimal observability of switching Boolean networks, *Math. Methods Appl. Sci.*, **48** (2025), 3291–3303. <https://doi.org/10.1002/mma.10485>
18. L. Deng, S. Fu, X. Li, J. Wang, A matrix approach to the analysis and computation of robust cycles for Boolean control networks with disturbances, *J. Appl. Math. Comput.*, **70** (2024), 4827–4850. <http://doi.org/10.1007/s12190-024-02158-5>
19. S. Fu, D. Cheng, J. Feng, J. Zhao, Matrix expression of finite Boolean-type algebras, *Appl. Math. Comput.*, **395** (2021), 125880. <https://doi.org/10.1016/j.amc.2020.125880>
20. C. Huang, W. Xiong, J. Lu, D. Huang, Asymptotic stability of delayed Boolean networks with random data dropouts, *IEEE Trans. Neural Netw. Learn. Syst.*, **35** (2024), 18886–18891. <https://doi.org/10.1109/TNNLS.2023.3301220>
21. J. Wang, W. Liu, S. Fu, J. Xia, On robust set stability and set stabilization of probabilistic Boolean control networks, *Appl. Math. Comput.*, **422** (2022), 126992. <https://doi.org/10.1016/j.amc.2022.126992>
22. Y. Yan, D. Cheng, J. E. Feng, H. Li, J. Yue, Survey on applications of algebraic state space theory of logical systems to finite state machines, *Sci. China Inf. Sci.*, **66** (2023), 111201. <https://doi.org/10.1007/s11432-022-3538-4>
23. C. Dong, Y. Yan, H. Li, J. Yue, Semi-tensor product approach to controllability, reachability, and stabilizability of extended finite state machines, *Front. Inf. Technol. Electron. Eng.*, **25** (2024), 1370–1377. <https://doi.org/10.1631/fitee.2300578>
24. S. Wang, S. Fu, J. Wang, X. Zhao, Nonsingularity of grain-like cascade feedback shift registers subject to fault attacks over finite fields, *Nonlinear Anal. Hybrid Syst.*, **63** (2027), 101803. <https://doi.org/10.1016/j.nahs.2026.101803>
25. H. Li, Z. Liu, W. Li, Nonsingularity of grain-like cascade feedback shift registers subject to fault attacks, *Sci. China Inf. Sci.*, **67** (2024), 192203. <https://doi.org/10.1007/s11432-023-4044-8>
26. W. Ding, Y. Li, D. Wang, A real method for solving quaternion matrix equation  $X - A\hat{X}B = C$  based on semi-tensor product of matrices, *Adv. Appl. Clifford Algebras*, **31** (2021), 4–17. <http://doi.org/10.1007/s00006-021-01180-1>
27. X. Fan, Y. Li, J. Sun, J. Zhao, Solving quaternion linear system  $AXB = E$  based on semi-tensor product of quaternion matrices, *Banach J. Math. Anal.*, **17** (2023), 25. <https://doi.org/10.1007/s43037-023-00251-8>
28. Z. Zhou, Z. Meng, Y. Zhang, Y. Song, F. Zhang, Real algorithms for special least squares solutions of the complex split quaternion matrix equations  $AX = B$  and  $AXC = B$ , *Numer. Algor.*, **102** (2026), 505–530. <http://doi.org/10.1007/s11075-025-02138-1>

29. A. Wei, Y. Li, W. Ding, J. Zhao, A new structure-preserving algorithm based on the semi-tensor product of matrices for split quaternion matrix LDU decomposition and its applications, *J. Comput. Appl. Math.*, **451** (2024), 116072. <https://doi.org/10.1016/j.cam.2024.116072>
30. X. Liu, Y. Li, W. Ding, R. Tao, A real method for solving octonion matrix equation  $AXB = C$  based on semi-tensor product of matrices, *Adv. Appl. Clifford Algebras*, **34** (2024), 12. <https://doi.org/10.1007/s00006-024-01316-z>
31. X. Fan, Y. Li, W. Ding, J. Zhao, Semi-tensor product of quaternion matrices and its application, *Math. Methods Appl. Sci.*, **46** (2023), 6450–6462. <https://doi.org/10.1002/mma.8916>
32. W. Ding, Y. Li, Z. Liu, R. Tao, M. Zhang, Algebraic method for LU decomposition in commutative quaternion based on semi-tensor product of matrices and application to strict image authentication, *Math. Methods Appl. Sci.*, **47** (2024), 6036–6050. <https://doi.org/10.1002/mma.9905>
33. M. Zhang, Y. Li, J. Sun, W. Ding, The  $\mathcal{L}_C$ -structure-preserving algorithms of quaternion  $LDL^H$  decomposition and Cholesky decomposition, *Adv. Appl. Clifford Algebras*, **33** (2023), 57. <http://doi.org/10.1007/s00006-023-01298-4>
34. Z. Wang, S. Fu, J. Wang, X. Zhao, Strategy consensus of networked evolutionary games with time invariant delays, *Dyn. Games Appl.*, **14** (2024), 981–996. <https://doi.org/10.1007/s13235-023-00522-x>
35. F. Sun, B. Wang, J. Feng, Approximate consensus of networked evolutionary matrix game: A finite-cost event-triggered mechanism, *Nonlinear Anal. Hybrid Syst.*, **61** (2026), 101701. <https://doi.org/10.1016/j.nahs.2026.101701>
36. Z. Deng, Distributed algorithm design for autonomous games with inviolable constraints and its application to smart grids, *IEEE Trans. Autom. Control*, Early access, 2026. <http://doi.org/10.1109/TAC.2026.3724126>
37. Z. Deng, X. Xie, Distributed strategy for autonomous nonlinear multi-coalition games of high-order players over unconnected intra-coalition communications, *Automatica*, **185** (2026), 112749. <https://doi.org/10.1016/j.automatica.2025.112749>
38. L. Xia, J. Wang, S. Fu, Y. Gao, Control design to minimize the number of bankrupt players for networked evolutionary games with bankruptcy mechanism, *AIMS Math.*, **9** (2024), 35702–35720. <https://doi.org/10.3934/math.20241694>
39. Y. Li, X. Ding, H. Li, Robust consensus of networked evolutionary games with attackers and forbidden profiles, *Entropy*, **20** (2018), 15. <https://doi.org/10.3390/e20010015>
40. Y. Zhao, B. Ghosh, D. Cheng, Control of large-scale Boolean networks via network aggregation, *IEEE Trans. Neural Netw. Learn. Syst.*, **27** (2016), 1527–1536. <https://doi.org/10.1109/TNNLS.2015.2442593>
41. W. Liu, S. Fu, J. Wang, R. D. Leone, J. Xia, Stability of large-scale probabilistic Boolean networks via network aggregation, *Neural Netw.*, **194** (2026), 108108. <https://doi.org/10.1016/j.neunet.2025.108108>
42. S. Wang, H. Li, Aggregation method to reachability and optimal control of large-size Boolean control networks, *Sci. China Inf. Sci.*, **66** (2023), 179202. <https://doi.org/10.1007/s11432-021-3388-y>

43. J. Zhong, Q. Pan, W. Xu, B. Chen, A framework of pinning control for nonperiodical stable behaviors of large-scale asynchronous Boolean networks, *IEEE Trans. Autom. Control*, **69** (2024), 5711–5726. <https://doi.org/10.1109/TAC.2024.3351553>
44. H. Li, X. Pang, A polynomial-time criterion for stability of large-scale switched conjunctive Boolean networks, *Automatica*, **159** (2024), 111340. <https://doi.org/10.1016/j.automatica.2023.111340>
45. H. Li, S. Wang, W. Li, Aggregation method to strategy consensus of large-size networked evolutionary matrix games, *IEEE Trans. Autom. Control*, **69** (2024), 3975–3981. <https://doi.org/10.1109/TAC.2023.3340558>
46. M. Jiang, S. Yu, Y. Yan, Y. Zhao, Strategy consensus of evolutionary games on networks with degree-based pinning and incentive control, *IEEE Trans. Comput. Soc. Syst.*, **13** (2026), 601–611. <https://doi.org/10.1109/TCSS.2025.3589373>



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