



Review**Recent enhances in transmutation theory and integral transforms****Andrey Muravnik^{1,*} and Sergei Sitnik²**¹ RUDN University, Miklukho-Maklaya ul. 6, Moscow, Russia² Moscow State University of Civil Engineering, 26, Yaroslavskoye Shosse, Moscow, Russia* **Correspondence:** Email: amuravnik@yandex.ru; Tel: +74959550968.

Abstract: The main content of this survey is to systematically expound the theory of transmutation operators and their applications to differential equations with singularities in coefficients. In particular, applications to differential equations with Bessel operators are considered. Also, we consider some problems for differential–difference equations. We provide a short historical background, necessary notation, definitions, and auxiliary facts. The next topics are considered in a survey style: Sonin and Poisson transmutations, Buschman–Erdélyi transmutations and some of their applications, a universal compositional method to construct transmutations, and a new class of transmutations with sharp estimates of kernels for perturbed differential equations with the Bessel operator.

Keywords: differential equations; singular solutions; transmutation theory; integral transforms; special functions; difference equations; Gauss hypergeometric functions; Bessel operators; Fox–Wright generalized hypergeometric functions; Sturm–Liouville problems; fractional integrals and derivatives

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1. Introduction

Over the last several decades, interest in singular and degenerate elliptic boundary-value problems has risen; due to application needs. This direction originates from the fundamental Keldysh work [1] finding (on examples of second-order equations with power-like coefficients) and the main properties of the posing of boundary-value conditions for such equations. Further, strongly degenerating boundary-value problems arise in the theory of singular points of solutions (including regular ones) of elliptic equations (see, e. g., [2–4] and reference therein).

Functional–differential equations form a special class (differential–difference equations or equations with retarded or deviated independent variables are their particular cases). Their theory originated in the works of Myshkis, Hale, G. Kamenskii, and El’sgol’ts (see [5–8]). Substantial results in

this direction were obtained by Skubachevskii, Rossovskii, and Muravnik; we cite only [8, 9]. It is important that even prototype problems for functional–differential equations might possess compactly-supported solutions (such as Rvachev atomic functions); this phenomenon is possible not only for differential but also for difference equations. Also, functional–differential equations include the important class of equations with Dunkl operators related to Bessel differential operators, appearing at the intersection of group theory and the theory of group symmetries (Coxeter groups), differential equations, integral transforms, quantum physics, and crystallography. Also, the class of functional–differential equations includes problems with involute (Carleman) translations (see [10]).

We select a special class of singular partial differential equations such that its typical representative is the following B -elliptic equation with Bessel operators with respect to each variable:

$$\sum_{k=1}^n B_{\nu, x_k} u(x_1, \dots, x_n) = f. \quad (1.1)$$

B -hyperbolic and B -parabolic equations are considered in the same way. This convenient terminology was introduced in [11]. The study of this class of equations is originated in works of Euler, Poisson, and Darboux. Then it was continued in the Weinstein generalized axially symmetric potential theory (see [12], in works of Aldashev, Bers, I. Egorov, Zhitomirskii, Kapilevich, Kilbas, Kudryavtsev, Lizorkin, Sh. Karimov, E. Karimov, Marichev, Matiychuk, L. Mikhailov, Olevskii, Pul'kin, M. M. Smirnov, S. Tersenov, Khasanov, Kan Cher Khe, Yanushauskas, and others. The importance of equations from these classes is due to their usage in applications as well; they are often applied to problems related to the theory of axially symmetric potential, Euler–Poisson–Darboux equations and many others.

The most comprehensive investigation of equations with Bessel operators was performed by Kipriyanov and his disciples L. A. Ivanov, A. Ryzhkov, Aziev, Arkhipov, Baydakov, Bogachev, Brodskii, Vinogradova, V. Zaytsev, Zasorin, G. Kagan, Katrakhova, Kipriyanova, Kononenko, Klyuchantsev, A. Kulikov, Larin, Leizin, Lyapin, Lyakhov, Muravnik, I. Polovinkin, A. Sazonov, Sitnik, Ukrainskii, Shatskii, E. Shishkina, and Yaroslavl'tseva; the main results in this direction are presented in [11].

For the much more detailed presentation of results of two talented disciples of I. A. Kipriyanov, V. V. Katrakhov and V. Z. Meshkov, including their short biographical information, see [13–15]. In these references there are above many more considerations of the new boundary–value problem of V. V. Katrakhov for solutions of differential equations with essential singularities, and solutions by V. Z. Meshkov of the famous Landis problem on the strict growth estimates for solutions of the Schrödinger equation with bounded potential.

To describe classes of solutions of the corresponding equations, Kipriyanov introduced and investigated function spaces named in his honor afterwards.

In all above the mentioned questions, the essential role is played by transmutation theory.

Definition 1.1. Let (A, B) be a pair of operators. A nonzero operator T is called a *transmutation operator* if the relation

$$T A = B T \quad (1.2)$$

holds.

Also, relation (1.2) is called the *intertwining property*; in this case, we say that the transmutation operator T *intertwines* the operators A and B . To convert (1.2) into a strict definition, one has to specify function spaces (sets) such that the operators A , B , and T act in them. Usually, the invertibility and continuity are included in the definition of transmutation operators; however, these requirements are not compulsory. In particular implementations, A and B are frequently (but not compulsory) differential operators, while T is a linear integral operator; all three act in standard spaces.

It is clear that the notion of transmutation operators is a generalization of the linear-algebra notion of the similarity of matrices. Note that no efficient methods to verify the similarity of two finite matrices exist because there are no efficient methods (apart from direct computations) to verify whether their Jordan forms coincide each other.

We briefly present the prototype scheme illustrating the usage of transmutation operators to obtain binding relations between solutions of the perturbed and unperturbed equation provided that the intertwining property is proved for them. For example, suppose that rather complex operator A is studied, while the needed properties are already known for a prototype (simpler) operator B . Then, if a transmutation operator of kind (1.2) exists, then it is frequently possible to extend properties of the prototype operator B to A . This is a prototype scheme of a typical usage of transmutation operators in particular problems, explained in a few words.

In particular, if an equation $Au = f$ is considered, then applying to it the transmutation operator T with the intertwining property (1.2), we obtain an equation of the kind $Bv = g$, where $v = Tu$ and $g = Tf$. Therefore, if the equation with the operator B is simpler and relations for its solutions are already known, then we obtain the representations for solutions of the first equation as well: $u = T^{-1}v$. Certainly, in the framework of the explained approach, the existence of the inverse transmutation operator is needed and it has to act in the corresponding spaces; moreover, an explicit representation of this inverse operator is needed to obtain explicit representations of solutions. This is a simple example of applications of the technique of transmutation operators in the theory of differential equations (both ordinary and partial ones). Note that if a pair of linear transmutation operators exists, then they provide the relation between solutions and representations of solutions both for homogeneous and inhomogeneous equations (as well as for equations with a spectral parameter).

For the basics of transmutations, we refer to the classical monographs [16–25] and some recent books [13–15], edited collections [26, 27] and papers [28, 29].

The necessity of the theory of transmutation operators is confirmed by its numerous applications. Methods of transmutation operators are especially important for the theory of differential equations of various types. Many fundamental results for various classes of differential equations are proved by means of methods of transmutation operators.

Nowadays, the theory of transmutation operators, as with functional differential equations, are separate areas of mathematics, located at the intersection of differential, integral, and integrodifferential equations, functional analysis, function theory, complex analysis, theory of special functions and fractional differential and integral calculus, harmonic analysis, theory of optimal control, inverse problems, and scattering problems.

2. Special functions and integral transforms

2.1. Special functions

Here, we provide brief definitions and explanations for special functions used in this survey. We follow classic monographs [30–33], and also definitions and notations in [13–15].

The following definition via the Euler second-kind integral is used:

$$\Gamma(z) = \int_0^{\infty} y^{z-1} e^{-y} dy, \quad \Re z > 0; \quad (2.1)$$

this integral converges for all complex z such that $\Re z > 0$.

The Bessel functions are defined as solutions of the Bessel differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0,$$

where the order α is an arbitrary complex number.

First-kind Bessel functions denoted by $J_\alpha(x)$ are solutions finite at the origin for integer or nonnegative α . These functions can be defined by means of the Taylor expansion around the origin or more general power series (if α is not integer):

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

If α is not integer, then the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent. Hence, they are solutions of the equation. If α is integer, then the following relation holds:

$$J_{-\alpha}(x) = (-1)^\alpha J_\alpha(x).$$

This means that the functions are linearly dependent (in this case). Then the second solution of the equation is the second-kind Bessel function, i.e., the *Neumann function*, which is the solution $N_\alpha(x)$ of the Bessel equation, tending to infinity as $x \rightarrow 0$. This function is related to $J_\alpha(x)$ as follows:

$$N_\alpha(x) = \frac{J_\alpha(x) \cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

(if α is integer, then the limit with respect to α , computed, e.g., by means of L'Hôpital's rule, is taken). A linear combination of the first-kind and second-kind Bessel function is a complete solution of the Bessel equation:

$$y(x) = C_1 J_\alpha(x) + C_2 N_\alpha(x).$$

The notation $N_\alpha(x) = Y_\alpha(x)$ is frequently used as well.

Modified Bessel functions or Bessel functions of imaginary independent variables are functions

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

and the *MacDonald function* is

$$K_\nu(x) = \frac{\pi}{2 \sin(\pi\nu)} [I_{-\nu}(x) - I_\nu(x)], \nu \notin \mathbb{Z}.$$

If $\nu \in \mathbb{Z}$, then the MacDonald function is computed by means of the limit passage with respect to the index (the L'Hôpital's rule is applied).

The Gauss hypergeometric function is defined inside the disk $\{|z| < 1\}$ as the sum of the hypergeometric series

$$\begin{aligned} {}_2F_1(a, b; c; z) &= F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad \Re c > \Re b > 0 \quad (2.2) \end{aligned}$$

(see [30, p. 373, relation 15.3.1]). For $|z| \geq 1$, it is defined as the analytic continuation of this series. In relation (2.2), the parameters a, b , and c and the variable z are allowed to be complex, c is different from each nonpositive integer, and $(a)_k$ is the Pochhammer symbol.

The hypergeometric series (2.2) converges only in the unit disk of the complex plane. This results in the necessity to construct the analytic continuation of the hypergeometric function to the whole complex plane. One way to do this is to use the Euler integral representation

$${}_2F_1(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt,$$

$$0 < \Re \beta < \Re \gamma, \quad |\arg(1-z)| < \pi;$$

the right-hand side is defined under the specified conditions providing the convergence of the integral.

It is important that various special and elementary functions can be obtained from the hypergeometric function if we assign particular values to parameters and transform the independent variable.

The Legendre functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are generalizations of Legendre polynomials to noninteger degrees (see [31]).

The Legendre functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are the solutions of the general Legendre equation

$$(1-x^2)y'' - 2xy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-x^2} \right] y = 0,$$

where the complex numbers λ and μ are called the degree and order (respectively) of the Legendre function. For complex values of the parameters and the independent variable, these functions can be determined as follows:

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left[\frac{1+z}{1-z} \right]^{\mu/2} {}_2F_1(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}), \quad |1-z| < 2,$$

and

$$Q_\lambda^\mu(z) = \frac{\sqrt{\pi} \Gamma(\lambda + \mu + 1)}{2^{\lambda+1} \Gamma(\lambda + \frac{3}{2})} \frac{e^{i\mu\pi} (z^2 - 1)^{\mu/2}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda + \mu + 1}{2}, \frac{\lambda + \mu + 2}{2}; \lambda + \frac{3}{2}; \frac{1}{z^2}\right), \quad |z| > 1,$$

where Γ is the gamma-function and ${}_2F_1$ is the hypergeometric function. Also, we use the direct values of the functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ on the cut $x \in [-1; 1]$, denoted by $\mathbb{P}_\nu^\mu(x)$ and $\mathbb{Q}_\nu^\mu(x)$.

Legendre functions are special cases of the Gauss hypergeometric function. Therefore, numerous series expansions, integral representations, and relations for the continuation as well as expressions via elementary functions for special values of parameters are known for them.

Legendre polynomials are Legendre functions of order $\mu = 0$ for nonnegative integers $\lambda = n$.

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is the entire function (in $z \in \mathbb{C}$) of order $1/\alpha$ defined by the power series

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \alpha, \beta \in \mathbb{C}, \Re \alpha > 0, \Re \beta > 0. \quad (2.3)$$

Lately, functions of the hypergeometric type (such as Mittag-Leffler functions, Wright functions, and Fox functions) are broadly applied in problems of probability theory and mathematical statistics. In particular important distribution functions, probability densities, and their characteristics are expressed via them.

Wright functions, or more exactly Fox–Wright functions, intermediately generalize Mittag-Leffler functions and are defined by series similar to (2.3); the difference is that the fraction for the common term of the series might contain an arbitrary finite amount of gamma-functions both in the numerator and denominator.

Let $p, q \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $p^2 + q^2 \neq 0$, $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$ ($\alpha_i, \beta_j \neq 0$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$). Then the Fox–Wright function is defined by the power series

$${}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \quad z \in \mathbb{C}.$$

The general definition of the Meyer G -function is given by the following integral in the complex plane:

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds.$$

This is an integral of the so-called Mellin–Barnes type. It is well defined under the following assumptions: $0 \leq m' \leq q$, $0 \leq n \leq p$, $m, n, p, q \in \mathbb{Z}$, and $a_k - b_j \neq 1, 2, 3, \dots$ for $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Under these assumptions, there are no coinciding poles $\Gamma(b_j - s)$, $j = 1, 2, \dots, m$, and $\Gamma(1 - a_k + s)$, $k = 1, 2, \dots, n$, $z \neq 0$. Moreover, once the circuit over the region with poles is over, the contour is continued up and down along the same vertical line.

In all cases, the Meyer G -function is representable as a finite sum of generalized hypergeometric functions and satisfies a differential equation with polynomial coefficients (it can be presented explicitly). Meyer functions are used, e.g., to represent the Obreshkov transform.

The H -function was introduced by Fox as integrals in 1961 and as series much earlier. It is defined by means of the Mellin–Barnes integral:

$$H(x) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, A_1) & (a_2, A_2) & \dots & (a_p, A_p) \\ (b_1, B_1) & (b_2, B_2) & \dots & (b_q, B_q) \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\left(\prod_{j=1}^m \Gamma(b_j + B_j s) \right) \left(\prod_{j=1}^n \Gamma(1 - a_j - A_j s) \right)}{\left(\prod_{j=m+1}^q \Gamma(1 - b_j - B_j s) \right) \left(\prod_{j=n+1}^p \Gamma(a_j + A_j s) \right)} z^{-s} ds,$$

where each empty product is assigned to be equal to one, $m, n, p, q \in \mathbb{N}_0$, $0 \leq n \leq p$, $1 \leq m \leq q$, $A_i, B_j \in \mathbb{R}_+$, a_i and b_j are either real or complex, $i = 1, \dots, p$, $j = 1, \dots, q$, L is a suitable contour separating the poles

$$\zeta_{j\nu} = -\left(\frac{b_j + \nu}{B_j} \right), \quad j = 1, \dots, m, \quad \nu = 0, 1, 2, \dots,$$

of the gamma-function $\Gamma(b_j + sB_j)$ from the poles

$$\omega_{\lambda k} = \left(\frac{1 - a_\lambda + k}{A_\lambda} \right), \quad \lambda = 1, \dots, n, \quad k = 0, 1, 2, \dots,$$

of the gamma-function $\Gamma(1 - a_\lambda - sA_\lambda)$ such that

$$A_\lambda(b_j + \nu) \neq B_j(a_\lambda - k - 1), \quad j = 1, \dots, m, \quad \lambda = 1, \dots, n, \quad \nu, k = 0, 1, 2, \dots,$$

and, once the circuit over the region with poles is over, the integration contour is continued up and down along the same vertical line.

Fox functions are applied to construct the Green function for the fractional-diffusion equation; the Stankovich transform is used for this purpose.

Multivariable hypergeometric functions are used as well as they are important for the theory of degenerating and singular differential equations.

2.2. Integral transforms

The *Fourier transform*, *sine transform*, *cosine transform*, and *Hankel transform* have the following forms, respectively:

$$(Ff)(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-ity) f(y) dy,$$

$$(F_c f)(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(ty) f(y) dy,$$

$$(F_s f)(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(ty) f(y) dy.$$

They are unitary in $L_2(0, \infty)$ and coincide with their inverse transforms.

The Hankel (Fourier–Bessel) transform has the form

$$(H_\nu f)(t) = \frac{1}{t^\nu} \int_0^\infty J_\nu(ty) f(y) dy$$

or

$$(H_\nu f)(\xi) = \int_0^\infty j_{\frac{\nu-1}{2}}(x\xi) f(x) x^\nu dx.$$

Due to the property that $j_\nu(0) = 1$, it is more convenient to use the Hankel transform with the function $j_{\frac{\nu-1}{2}}(x\xi)$ in the kernel.

The Mellin transform $f(x)$ is the function $g(s)$ defined by the relation

$$g(s) = Mf(s) = \int_0^\infty x^{s-1} f(x) dx. \quad (2.4)$$

Define the Mellin convolution as follows:

$$(f_1 * f_2)(x) = \int_0^\infty f_1\left(\frac{x}{y}\right) f_2(y) \frac{dy}{y}. \quad (2.5)$$

In Mellin images, the convolution operator with kernel K acts as a multiplier:

$$MAf(s) = \int_0^\infty K\left(\frac{x}{y}\right) f(y) \frac{dy}{y} = MK * f(s) = m_A(s)Mf(s), \quad (2.6)$$

$$m_A(s) = MK(s).$$

Theorem 2.1. (see [13, 14]) *Let the convolution operator A acts according to relation (2.6) as a multiplier in Mellin images. Then, the following boundedness conditions for the direct and inverse operators and relations for their norms are satisfied:*

(a) *It admits an extension till an operator bounded in $L_2(0, \infty)$ if and only if*

$$\sup_{\xi \in \mathbb{R}} \left| m_A \left(i\xi + \frac{1}{2} \right) \right| = M_2 < \infty, \quad (2.7)$$

where $\|A\|_{L_2} = M_2$.

(b) *Under the additional condition that the kernel K is nonnegative, it admits an extension till an operator bounded in $L_p(0, \infty)$, $p > 1$, if and only if*

$$\sup_{\xi \in \mathbb{R}} \left| m_A \left(i\xi + \frac{1}{p} \right) \right| = M_p < \infty, \quad (2.8)$$

where $\|A\|_{L_p} = M_p$.

(c) The inverse operator A^{-1} acts according to relation (2.6) as well and its multiplier is $\frac{1}{m_A}$. It admits an extension till an operator bounded in $L_2(0, \infty)$ if and only if

$$\inf_{\xi \in \mathbb{R}} \left| m_A \left(i\xi + \frac{1}{2} \right) \right| = m_2 > 0, \quad (2.9)$$

where $\|A^{-1}\|_{L_2} = \frac{1}{m_2}$.

(d) Let the operators A and A^{-1} be defined and bounded in $L_2(0, \infty)$. Then, they are unitary if and only if the relation

$$\left| m_A \left(i\xi + \frac{1}{2} \right) \right| = 1 \quad (2.10)$$

holds for almost all ξ .

The last theorem summarizes results of many mathematicians: Schur, Hardy, Littlewood, Polya, Kober, Mikhlin, and Hörmander.

The following important achievement of 1970s is related to the Mellin transform: The known Slater theorem allowing one to explicitly restore the majority of Mellin originals via their images is completely proved and adopted for the computing of integrals (a simple algorithm of hypergeometric functions is used for that. Formally, this result can be easily obtained by means of the general Mellin–Barnes inversion relation (via residues). However, quite complicated and careful work is required to justify it strictly; one has to investigate the asymptotic behavior of hypergeometric functions near residues and at infinity, and a substantial diversity is typical for such a behavior. This work is only started by Slater, while it was completed by Oleg Marichev. The Slater–Marichev theorem yields a universal and powerful method to compute integrals. This method allows one to solve various problems from the theory of partial differential equations; also, it is the base of the forefront technologies of the symbol integrating of the *MATHEMATICA* package.

The Slater theorem reads as follows. Let

$$\Gamma \left[\begin{matrix} a_1, & a_2, & \dots, & a_A \\ b_1, & b_2, & \dots, & b_B \end{matrix} \right] = \Gamma[(a), (b)] = \frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_B)}, \quad (2.11)$$

where each empty product is assigned to be equal to one,

$$(a) + s = a_1 + s, a_2 + s, \dots, a_A + s,$$

$$(b)' - b_k = b_1 - b_k, \dots, b_{k-1} - b_k, b_{k+1} - b_k, \dots, b_B - b_k, \quad (2.12)$$

$$\begin{aligned} \Sigma_A(z) = \sum_{j=1}^A z^{a_j} \Gamma \left[\begin{matrix} (a)' - a_j, & (b) + a_j \\ (c) - a_j & (d) + a_j \end{matrix} \right] \\ \times_{B+C} F_{A+D-1} \left(\begin{matrix} (b) + a_j, & 1 + a_j - (c); & (-1)^{C-A} z \\ 1 + a_j - (a)', & (d) + a_j \end{matrix} \right), \end{aligned} \quad (2.13)$$

$$\Sigma_B(1/z) = \sum_{k=1}^B z^{-b_k} \Gamma \left[\begin{matrix} (b)' - b_k, (a) + b_k \\ (d) - b_k (c) + b_k \end{matrix} \right] \\ \times_{A+D} F_{B+C-1} \left(\begin{matrix} (a) + b_k, & 1 + a_k - (d); & \frac{(-1)^{D-B}}{z} \\ 1 + b_k - (b)', & (c) + b_k \end{matrix} \right), \quad (2.14)$$

and

$$|\arg z| < \pi.$$

If the series converge, then $\Sigma_A(z)$ and $\Sigma_B(1/z)$ are functions of the hypergeometric type and they pass into each other if we change places of the A -dimensional complex vector $(a) = a_1, a_2, \dots, a_A$ and the analogous B -dimensional vector (b) , change places of the C -dimensional vector (c) and the D -dimensional vector (d) , and replace z with $1/z$. These functions analytically depend on the complex parameters (a) , (b) , (c) , and (d) and on the variable z . If the vectors (a) (or (b)) have coinciding parameters or parameters such that the difference between them are integers, then the vectors $(a)' - a_j$ ($(b)' - b_k$) contain nonzero or negative integer components and, due to the property $\Gamma(-n) = \infty$, $n = 0, 1, 2, \dots$, the functions $\Sigma_A(z)$ and $\Sigma_B(1/z)$ might have ambiguous points of the kind $\infty - \infty$. In such logarithmic cases, the values of $\Sigma_A(z)$ and $\Sigma_B(1/z)$ are understood as the corresponding limits of the “regular” functions $\Sigma_A(z)$ and $\Sigma_B(1/z)$ as their parameters continuously tend to the considered special values.

Note that if the restriction $|\arg z| < \pi$ is not imposed, then $\Sigma_A(z)$ and $\Sigma_B(1/z)$ are not guaranteed to be one-valued functions.

Theorem (Slater). *Let*

$$K^*(s) = \Gamma \left[\begin{matrix} (a) + s & (b) - s \\ (c) + s, & (d) - s \end{matrix} \right], \quad (2.15)$$

where (a) , (b) , (c) , and (d) have A , B , C , and D components a_j , b_k , c_l , and d_m , respectively. Let the following two groups of conditions be satisfied:

$$-\Re a_j < \Re s < \Re b_k, \quad j = 1, 2, \dots, A, \quad k = 1, 2, \dots, B, \quad (2.16)$$

and

$$\begin{cases} A + B > C + D, \\ A + B = C + D, & \Re s(A + D - B - C) < -\Re v, \\ A = C, B = D, & \Re v < 0, \end{cases} \quad (2.17)$$

where

$$v = \sum_{j=1}^A a_j + \sum_{k=1}^B b_k - \sum_{l=1}^C c_l - \sum_{m=1}^D d_m.$$

Then, the following relations hold for the values of s :

$$K^*(s) = \begin{cases} \int_0^\infty x^{s-1} \Sigma_A(x) dx, & A + D > B + C, \\ \int_0^1 x^{s-1} \Sigma_A(x) dx + \int_1^\infty x^{s-1} \Sigma_B(1/x) dx, & A + D = B + C, \\ \int_0^\infty x^{s-1} \Sigma_B(1/x) dx, & A + D < B + C, \end{cases} \quad (2.18)$$

and $\Sigma_A(1) = \Sigma_B(1)$ provided that $A + D = B + C$, $\Re \nu + C - A + 1 < 0$, and $A \geq C$.

Corollary. If conditions (2.16)-(2.17) are satisfied, then the preimage of the function

$$K^*(s) = \Gamma \left[\begin{matrix} (a) + s & (b) - s \\ (c) + s, & (d) - s \end{matrix} \right],$$

is a function $K(x)$ of the hypergeometric type such that it is equal to one of the following expressions:

$$K(x) = \begin{cases} \Sigma_A(x), & x > 0, A + D > B + C, \\ \Sigma_A(x), & 0 < x < 1, A + D = B + C, \\ \Sigma_B(1/x), & x > 1, A + D = B + C, \\ \Sigma_B(1/x), & x > 0, A + D < B + C, \end{cases} \quad (2.19)$$

where $K(1) = \Sigma_A(1) - \Sigma_B(1)$, provided that $A + D = B + C$, $\Re \nu + C - A + 1 < 0$, and $A \geq C$.

2.3. Various forms of fractional integrodifferentiating

Operators of fractional integrodifferentiating are important for various areas of contemporary mathematics. Consider fractional Riemann–Liouville integrals and derivatives.

Definition 2.1. Let $\varphi(x) \in L_1(a, b)$. Then, the integrals

$$(I_{a+}^\alpha \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \quad x > a, \quad (2.20)$$

and

$$(I_{b-}^\alpha \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}} dt, \quad x < b, \quad (2.21)$$

where $\alpha > 0$, are respectively called the *left-side* and *right-side fractional Riemann–Liouville integrals* of order α .

For a function $f(x)$, $x \in [a, b]$, the expressions

$$(D_{a+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x \frac{f(t) dt}{(x-t)^{\alpha-n+1}}, \quad (2.22)$$

and

$$(D_{b-}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_x^b \frac{f(t) dt}{(t-x)^{\alpha-n+1}}, \quad (2.23)$$

where $n = [\alpha] + 1$, $\alpha > 0$, are respectively called the *left-side* and *right-side fractional Riemann–Liouville derivatives* of order α .

In special cases, the Riemann–Liouville operators most important for applications are defined (for positive values of α) as follows:

$$I_{0+,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \quad \text{and} \quad I_{-,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t) dt.$$

For other values of α , they are defined by means of the analytic continuation (regularization).

Note that there are numerous variants of fractional integrodifferentiating operators: Those of Gerasimov–Caputo introduced by Gerasimov in 1948 and rediscovered by Caputo in 1968, Marchaud, Weyl, Riesz, Erdélyi–Kober, Hadamard, Gel’fond–Leontiev (the double title is proposed by Korobeinik), Dzhrbashyan–Nersesyan, etc.

Now, consider Erdélyi–Kober fractional integrals and derivatives.

For $\alpha > 0$, *Erdélyi–Kober operators* are defined by the following relations:

$$I_{0+;2,y}^{\alpha} f = \frac{2}{\Gamma(\alpha)} x^{-2(\alpha+y)} \int_0^x (x^2 - t^2)^{\alpha-1} t^{2y+1} f(t) dt, \quad (2.24)$$

$$I_{-;2,y}^{\alpha} f = \frac{2}{\Gamma(\alpha)} x^{2y} \int_x^{\infty} (t^2 - x^2)^{\alpha-1} t^{2(1-\alpha-y)-1} f(t) dt. \quad (2.25)$$

For $\alpha > -n$, $n \in \mathbb{N}$, they are defined by the following relations:

$$I_{0+;2,y}^{\alpha} f = x^{-2(\alpha+y)} \left(\frac{d}{dx^2} \right)^n x^{2(\alpha+y+n)} I_{0+;2,y}^{\alpha+n} f, \quad (2.26)$$

$$I_{-;2,y}^{\alpha} f = x^{2y} \left(-\frac{d}{dx^2} \right)^n x^{2(\alpha-y)} I_{-;2,y-n}^{\alpha+n} f. \quad (2.27)$$

For other values of α , they are defined by means of the analytic continuation (similarly to fractional Liouville integrodifferentiating operators).

Now, consider the most general form of such operators, i.e., fractional integrals with respect to arbitrary functions.

The fractional integral with respect to an arbitrary function $g(x)$ is defined as follows:

$$I_{0+,g}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) dt, \quad (2.28)$$

$$I_{-,g}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (g(t) - g(x))^{\alpha-1} g'(t) f(t) dt, \quad (2.29)$$

where $\Re \alpha > 0$. For the remaining values of α , the relations are easily extended. Fractional integrals (2.24) are obtained if we assign $g(x) = x$ in (2.28), the Erdélyi–Kober integrals (2.24) are obtained if we assign $g(x) = x^2$, and the Hadamard integrals are obtained if we assign $g(x) = \ln x$.

The relation to transmutation operators is as follows: Up to factors, the Sonin–Poisson–Delsarte operators are the Erdélyi–Kober operators, i.e., the fractional powers

$$\left(\frac{d}{dg(x)}\right)^{-\alpha} = \left(\frac{d}{2xdx}\right)^{-\alpha}, \quad g(x) = x^2.$$

Therefore, the main properties of these transmutation operators can be obtained from the theory of fractional integrodifferentiating operators instead of to invent them again (as it is frequently done). A. Dzhrbashyan attracted our attention to the fact that fractional integrating operators with respect to the function (2.28) are special cases of more general operators introduced and studied by M. Dzhrbashyan.

3. Main classes of differential equations with Bessel operators and related transmutation operators

For differential equations with Bessel operators

$$B_\nu u(x) = \frac{d^2 u}{dx^2} + \frac{2\nu + 1}{x} \frac{du}{dx}, \quad (3.1)$$

we use the following terminology due to Kipriyanov [11].

Equations with Bessel operators of the kind

$$\sum_{k=1}^n B_{\nu, x_k} u(x_1, \dots, x_n) = f(t, x) \quad (3.2)$$

are called a *B-elliptic* equations. Sometimes, they are called *Laplace–Bessel equations*.

B-hyperbolic equations are equations with Bessel operators of the kind

$$B_{\nu, t} u(t, x_1, \dots, x_n) = \sum_{k=1}^n B_{\nu, x_k} u(t, x_1, \dots, x_n) + f(t, x). \quad (3.3)$$

If the spatial variable is unique, then we obtain the *Euler–Poisson–Darboux equation*.

B-parabolic equations are equations with Bessel operators of the kind

$$\frac{\partial}{\partial t} u(t, x_1, \dots, x_n) = \sum_{k=1}^n B_{\nu, x_k} u(t, x_1, \dots, x_n) + f(t, x). \quad (3.4)$$

The same terms are preserved for incomplete equations, where one or several Bessel operators are reduced to the second derivatives and spectral parameters are added to the equations. We list several known classes of differential equations with Bessel operators without any claims for this list to be complete; only main references are provided.

B-elliptic equations of the kind

$$\sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial u}{\partial x_i} \right) = f(x) \quad (3.5)$$

form a type including the Weinstein equation of the generalized axially symmetric potential theory (see [12]); equations of this class are profoundly and comprehensively studied in works of Kipriyanov and his school (see [11] and the above information).

The generalized Euler–Poisson–Darboux equations are given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\nu}{t} \frac{\partial u}{\partial t} = \sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial u}{\partial x_i} \right), \quad (3.6)$$

$$u = u(x_1, \dots, x_n, t), t > 0, x_i > 0, \nu, i = 1, \dots, n.$$

The Euler–Poisson–Darboux equation with the spectral potential is

$$\frac{\partial^2 u}{\partial t^2} + \frac{\nu}{t} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \pm \lambda^2 u, \lambda \in \mathbb{R} \quad (3.7)$$

and the generalized Euler–Poisson–Darboux equation with the spectral potential is

$$\frac{\partial^2 u}{\partial t^2} + \frac{\nu}{t} \frac{\partial u}{\partial t} = \sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial u}{\partial x_i} \right) - \lambda^2 u, \lambda \in \mathbb{R}. \quad (3.8)$$

Further, generalized wave equations with variable potentials of the kind

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2} + p(x)u \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} + q(t)u = \frac{\partial^2 u}{\partial x^2} + p(x)u, \quad (3.9)$$

are generalized to B -hyperbolic equations with variable potentials of the kind

$$B_{\nu,t}u = B_{\nu,x}u + p(x)u \quad \text{and} \quad B_{\nu,t}u + q(t)u = B_{\nu,x}u + p(x)u, \quad (3.10)$$

their multidimensional analogs

$$B_{\nu,t}u + q(t)u = \sum_{k=1}^n B_{\nu,x_k}u + p(x_k)u, \quad (3.11)$$

and the following B -ultrahyperbolic equations with variable potentials:

$$\sum_{j=1}^m (B_{\nu,t_j}u + q(t_j)u) = \sum_{k=1}^n (B_{\nu,x_k}u + p(x_k)u). \quad (3.12)$$

Usually, these equations are solved (explicitly or implicitly) by means of transmutation operators for Sturm–Liouville operators or perturbed Bessel operators.

Numerous applications of differential equations with Bessel operators are provided in cited papers as well. Thus, this class of equations is important both for the theory of partial differential equations and for practical applications.

The following useful Darboux–Weinstein relation (the term is introduced by J. L. Lions) can be verified intermediately:

$$B_\nu \left(y^{-2\nu} f(y) \right) = y^{-2\nu} B_\nu f(y); \quad (3.13)$$

this easily reduces the case where $\Re \nu < 0$ to the case where $\Re \nu > 0$. This relation helps to reduce problems connected with Bessel operators with negative parameter to ones with positive one.

Define the class of transmutation operators intertwining the Bessel differential operator with the second derivative:

$$T(B_\nu)f = (D^2)Tf, \quad B_\nu = D^2 + \frac{2\nu+1}{x}D, \quad D^2 = \frac{d^2}{dx^2}, \quad \nu \in \mathbb{C}. \quad (3.14)$$

To construct transmutation operators, one can establish relations between solutions of differential equations. Solutions of equations of the kind $B_\nu f = \lambda f$ are Bessel functions, while solutions of the equation $D^2 f = \lambda f$ are trigonometric or exponential functions. Therefore, transmutation operators of the kind (3.14) grow from the following relations (Poisson and Sonin):

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})2^{\nu-1}x^\nu} \int_0^x (x^2 - t^2)^{\nu-\frac{1}{2}} \cos(t) dt, \quad \Re \nu > \frac{1}{2}, \quad (3.15)$$

and

$$J_\nu(x) = \frac{2^{\nu+1}x^\nu}{\sqrt{\pi}\Gamma(\frac{1}{2} - \nu)} \int_x^\infty (t^2 - x^2)^{-\nu-\frac{1}{2}} \sin(t) dt, \quad -\frac{1}{2} < \Re \nu < \frac{1}{2}. \quad (3.16)$$

Integral (3.15) has been studied since 1769 (Euler). For $\nu = 0$, its value was computed by Parseval in 1805. For integer values of ν , relation (3.15) was obtained by Plana in 1821. For half-integer values of ν , Poisson derived it in 1823. For the general case, relation (3.15) was established by Lommel in 1868. Relation (3.16) was derived by Sonin in 1880.

Definition 3.1. The *Poisson transmutation operator* is the expression

$$P_\nu f = \frac{1}{\Gamma(\nu+1)2^\nu x^{2\nu}} \int_0^x (x^2 - t^2)^{\nu-\frac{1}{2}} f(t) dt, \quad \Re \nu > -\frac{1}{2}. \quad (3.17)$$

The *Sonin transmutation operator* is the expression

$$S_\nu f = \frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2} - \nu)} \frac{d}{dx} \int_0^x (x^2 - t^2)^{-\nu-\frac{1}{2}} t^{2\nu+1} f(t) dt, \quad \Re \nu < \frac{1}{2}. \quad (3.18)$$

Operators (3.17)-(3.18) act as transmutation operators according to the relations

$$S_\nu B_\nu = D^2 S_\nu \quad \text{and} \quad P_\nu D^2 = B_\nu P_\nu. \quad (3.19)$$

They can be defined for all complex values of ν .

The idea to study operators of kind (3.17)-(3.18) is attributed to Liouville. Sonin was the first to use them in the framework of the theory of Bessel functions. Delsarte was the first to use them as transmutation operators. Further, Delsarte ideas are used to continue the investigation of these operators (Delsarte and Lions). That is why we call (3.17)-(3.18) *Sonin–Poisson–Delsarte transmutation operators*.

It is possible to say that Sonin–Poisson–Delsarte operators (3.17)-(3.18) are the most famous objects of the theory of transmutation operators; hundreds of papers are devoted to them, their applications, and their generalizations.

Definition 3.2. The *generalized translation operator* is the solution $u(x, y) = T_x^\nu f(x)$ of the problem

$$\begin{aligned} (B_\nu)_y u(x, y) &= \left(\frac{\partial^2}{\partial y^2} + \frac{2\nu + 1}{y} \frac{\partial}{\partial y} \right) u(x, y) = \frac{\partial^2}{\partial x^2} u(x, y), \\ u(x, 0) &= f(x), \quad u_y(x, 0) = 0. \end{aligned} \quad (3.20)$$

To justify this term, we note that, in the special case $\nu = -\frac{1}{2}$, the generalized translation operator is reduced to the operator

$$T_x^\nu f(x) = \frac{1}{2} (f(x + y) + f(x - y)).$$

For the generalized translation operator defined by (3.20), the following explicit relation is obtained by Delsarte:

$$T_x^\nu f(x) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi f(\sqrt{x^2 + y^2 - 2xy \cos t}) \sin^{2\nu} t \, dt. \quad (3.21)$$

Arbitrary pairs of differential (or even arbitrary) operators can be considered in (3.20) as well. For example, the following definition yields the classical translation:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \quad u(x, 0) = f(x), \quad T_x^\nu f(x) = f(x + y).$$

However, from the viewpoint of applications to solutions of partial differential equations with singularities at coefficients, investigated in the present monograph, the specified Sonin–Poisson–Delsarte operators possess a number of disadvantages preventing their applying in various important cases. The disadvantages are as follows. First, Sonin–Poisson–Delsarte operators introduced above are transmutation operators only on the set of even functions; therefore, it is impossible to consider functions with singularities at the origin. Second, they do not preserve the compactness of the support and the property to rapidly decrease at infinity. Third, they change the smoothness of transmuted functions.

This results in the necessity to introduce and study other classes of transmutation operators for differential equations with Bessel operators.

Fractional powers of Bessel operators

Note that there exists a theory of fractional powers of Bessel operators and their applications to fractional-order differential equations. This approach allows one to avoid the implicit definition of fractional powers of Bessel operators in images of the Hankel integral transform; instead, they are explicitly defined as a particular integral operator with special functions in the kernels (see [13, 14]).

The main definitions and properties of fractional powers of Bessel operators are as follows.

We consider real powers of the singular Bessel differential operator

$$B_\nu = D^2 + \frac{\nu}{x} D, \quad \nu \geq 0 \quad (3.22)$$

on the real semiaxis $(0, \infty)$.

Definition 3.3. Let $f(x) \in C^{2k}(0, b]$. Under the assumption that $f^{(i)}(b) = 0, 0 \leq i \leq 2k - 1, k \in \mathbb{N}$, define the *right-hand Bessel fractional integrating operator* by the relation

$$\begin{aligned} (B_{b-}^{\nu,k}f)(x) &= \frac{1}{\Gamma(2k)} \int_x^b \left(\frac{y^2 - x^2}{2y} \right)^{2k-1} {}_2F_1\left(k + \frac{\nu-1}{2}, k; 2k; 1 - \frac{x^2}{y^2}\right) f(y) dy \\ &= \frac{\sqrt{\pi}}{2^{2k-1}\Gamma(k)} \int_x^b (y^2 - x^2)^{k-\frac{1}{2}} \left(\frac{y}{x} \right)^{\frac{\nu}{2}} P_{\frac{\nu}{2}-1}^{\frac{1}{2}-k} \left(\frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} \right) \right) f(y) dy. \end{aligned}$$

Definition 3.4. Under the assumption that $f^{(i)}(a) = 0, 0 \leq i \leq 2k - 1, k \in \mathbb{N}$, define the *right-hand Bessel fractional integrating operator* by the relation

$$\begin{aligned} (B_{a+}^{\nu,k}f)(x) &= \frac{1}{\Gamma(2k)} \int_a^x \left(\frac{x^2 - y^2}{2x} \right)^{2k-1} {}_2F_1\left(k + \frac{\nu-1}{2}, k; 2k; 1 - \frac{y^2}{x^2}\right) f(y) dy \\ &= \frac{\sqrt{\pi}}{2^{2k-1}\Gamma(k)} \int_a^x (x^2 - y^2)^{k-\frac{1}{2}} \left(\frac{x}{y} \right)^{\frac{\nu}{2}} P_{\frac{\nu}{2}-1}^{\frac{1}{2}-k} \left(\frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} \right) \right) f(y) dy, \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function and $P_{\nu}^{\mu}(z)$ is the Legendre function.

It is useful to express Bessel fractional integrals via Legendre functions; it simplifies the original definition because the Gauss hypergeometric function depends on three parameters, while the Legendre function depends on two.

4. Theory of Buschman–Erdélyi transmutation operators

In this chapter, the theory of Buschman–Erdélyi transmutation operators are systematically explained. Special cases of integral operators of this class are considered in the previous chapter.

Note, that, following the tradition of the theory of transmutation operators and the corresponding literature, we frequently use the term “operators” in cases where the term “differential expressions” is more exact. In main theorems, the function spaces of their validity are provided. For results containing explicit relations, if no particular function classes are mentioned, then we assume that we mean compactly supported on the positive semiaxis functions (infinitely differentiable functions different from zero on an interval (a, b) , where $a > 0$ and $b < \infty$).

Buschman–Erdélyi integral transmutation operators

The considered class of transmutation operators is very important: under a suitable choice of parameters, it simultaneously generalizes the Sonin–Poisson–Delsarte transmutation operator and their adjoints, Riemann–Liouville and Erdélyi–Kober fractional integrodifferentiating operators, and Meler–Fock integral transforms.

Let us define first-kind Buschman–Erdélyi operators.

Definition 4.1. The following integral operators are called the *first-kind Buschman–Erdélyi operators*:

$$B_{0+}^{\nu,\mu} f = \int_0^x (x^2 - t^2)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{x}{t}\right) f(t) dt, \quad (4.1)$$

$$E_{0+}^{\nu,\mu} f = \int_0^x (x^2 - t^2)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu\left(\frac{t}{x}\right) f(t) dt, \quad (4.2)$$

$$B_-^{\nu,\mu} f = \int_x^\infty (t^2 - x^2)^{-\frac{\mu}{2}} P_\nu^\mu\left(\frac{t}{x}\right) f(t) dt, \quad (4.3)$$

$$E_-^{\nu,\mu} f = \int_x^\infty (t^2 - x^2)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu\left(\frac{x}{t}\right) f(t) dt. \quad (4.4)$$

Here, $P_\nu^\mu(z)$ is the first-kind Legendre function (see [31]), $\mathbb{P}_\nu^\mu(z)$ is the same function on the cut $-1 \leq t \leq 1$, and $f(x)$ is a locally summable function satisfying growth restrictions as $x \rightarrow 0$ and $x \rightarrow \infty$. The parameters μ and ν are complex numbers such that $\Re \mu < 1$ and $\Re \nu \geq -1/2$.

In the same way, it is possible to define the second-kind Buschman–Erdélyi operators with Legendre functions of the second kind as kernels (see [13–15]).

Now, we present results for Buschman–Erdélyi transmutation operators and their applications to differential equations with singularities at coefficients.

In what follows, we deal with the semiaxis. Therefore, L_2 denotes the space $L_2(0, \infty)$ and $L_{2,k}$ denotes the weight space $L_{2,k}(0, \infty)$.

First, we extend (4.1)–(4.4) to the case where $\mu = 1$ (this case is important, but it was not investigated earlier).

Definition 4.2. For $\mu = 1$, the *Buschman–Erdélyi operators of zero-order smoothness* are introduced by the relations

$$B_{0+}^{\nu,1} f = {}_1S_{0+}^\nu f = \frac{d}{dx} \int_0^x P_\nu\left(\frac{x}{t}\right) f(t) dt, \quad (4.5)$$

$$E_{0+}^{\nu,1} f = {}_1P_-^\nu f = \int_0^x P_\nu\left(\frac{t}{x}\right) \frac{df(t)}{dt} dt, \quad (4.6)$$

$$B_-^{\nu,1} f = {}_1S_-^\nu f = \int_x^\infty P_\nu\left(\frac{t}{x}\right) \left(-\frac{df(t)}{dt}\right) dt, \quad (4.7)$$

$$E_-^{\nu,1} f = {}_1P_{0+}^\nu f = \left(-\frac{d}{dx}\right) \int_x^\infty P_\nu\left(\frac{x}{t}\right) f(t) dt, \quad (4.8)$$

where $P_\nu(z) = P_\nu^0(z)$ is a Legendre function.

Under obvious additional conditions on the functions from (4.5)–(4.8), these relations can be differentiated under the integral sign or integrated by parts.

Theorem 4.1. *On suitable functions, the Buschman–Erdélyi can be factorized via fractional Riemann–Liouville and Buschman–Erdélyi integrals of zero-order smoothness as follows:*

$$B_{0+}^{\nu,\mu} f = I_{0+}^{1-\mu} {}_1S_{0+}^{\nu} f, \quad B_{-}^{\nu,\mu} f = {}_1P_{-}^{\nu} I_{-}^{1-\mu} f \quad (4.9)$$

and

$$E_{0+}^{\nu,\mu} f = {}_1P_{0+}^{\nu} I_{0+}^{1-\mu} f, \quad E_{-}^{\nu,\mu} f = I_{-}^{1-\mu} {}_1S_{-}^{\nu} f. \quad (4.10)$$

Definition 4.3. The number $\rho = 1 - \Re\mu$ is called the *smoothness order* of the Buschman–Erdélyi operators defined by (4.5)–(4.8).

Theorem 4.2. *The Buschman–Erdélyi operators given by (4.1)–(4.4) are defined if $\Re\mu < 1$ or $\mu \in \mathbb{N}$ and the following assumptions are satisfied:*

(a) *for the operator $B_{0+}^{\nu,\mu}$, we have*

$$\int_0^x \sqrt{y} |f(y) \ln y| dy < \infty$$

provided that $\nu = -\frac{1}{2}$ and $\mu \neq \frac{1}{2}$, while we have

$$\int_0^x y^{-\Re\nu} |f(y)| dy < \infty$$

for all other cases;

(b) *for the operator $E_{0+}^{\nu,\mu}$, no additional assumptions are imposed;*

(c) *for the operator $E_{-}^{\nu,\mu}$, we have*

$$\int_x^\infty y^{-\Re\nu} |f(y)| dy < \infty;$$

(d) *for the operator $B_{-}^{\nu,\mu}$, we have*

$$\int_x^\infty y^{-\frac{1}{2}-\Re\nu} |\ln y \cdot f(y)| dy < \infty$$

provided that $\nu = -\frac{1}{2}$ and $\mu \neq \frac{1}{2}$, while we have

$$\int_x^\infty y^{\Re(\nu-\mu)} |f(y)| dy < \infty$$

for all other cases.

Apart from the Bessel operator, consider the differential operator

$$L_\nu = D^2 - \frac{\nu(\nu+1)}{x^2} = \left(\frac{d}{dx} - \frac{\nu}{x}\right)\left(\frac{d}{dx} + \frac{\nu}{x}\right) = \left(\frac{d}{dx} + \frac{\nu+1}{x}\right)\left(\frac{d}{dx} - \frac{\nu+1}{x}\right); \quad (4.11)$$

for $\nu \in \mathbb{N}$, this is the angular momentum operator from quantum mechanics. They are linked by the following easy verifiable relations.

Let a pair $\{X_\nu, Y_\nu\}$ of transmutation operators intertwine L_ν and the second derivative, i.e.,

$$X_\nu L_\nu = D^2 X_\nu \text{ and } Y_\nu D^2 = L_\nu Y_\nu. \quad (4.12)$$

Introduce the new pair of transmutation operators as follows:

$$S_\nu = X_{\nu-1/2} x^{\nu+1/2} \text{ and } P_\nu = x^{-(\nu+1/2)} Y_{\nu-1/2}. \quad (4.13)$$

This pair intertwines the Bessel operator and the second derivative:

$$S_\nu B_\nu = D^2 S_\nu \text{ and } P_\nu D^2 = B_\nu P_\nu. \quad (4.14)$$

Conversely, the specified relations can be used to pass from transmutation operators for the Bessel operator to transmutation operators for the angular momentum operator. Namely, if a pair $\{S_\nu, P_\nu\}$ of transmutation operators possessing property (4.14) is given, then the pair of transmutation operators defined by the relations

$$X_\nu = S_{\nu+1/2} x^{-(\nu+1)} \text{ and } Y_\nu = x^{\nu+1} P_{\nu+1/2} \quad (4.15)$$

satisfy relations (4.12).

Transmutation operators intertwining the second derivative with the angular momentum operator have the following advantage compared with transmutation operators intertwining the second derivative with the Bessel operator: Under suitable conditions, they are bounded in a same space instead of a pair of different spaces. Transmutation operators acting according to relations (4.12) are still called Sonin-type and Poisson-type transmutation operators, respectively.

Theorem 4.3. *If $\Re \mu \leq 1$, then the operator $B_{0+}^{\nu, \mu}$ is a Sonin-type transmutation operator satisfying relation (4.12) on functions $f(x) \in \Phi(B_{0+}^{\nu, \mu})$.*

Now, consider various factorizations of Buschman–Erdélyi operators via Erdélyi–Kober operators and fractional Riemann–Liouville integrals (see definitions in [34]).

Theorem 4.4. *The following relations factorizing Buschman–Erdélyi operators via fractional integrodifferentiating operators and Erdélyi–Kober operators hold:*

$$B_{0+}^{\nu, \mu} = I_{0+}^{\nu+1-\mu} I_{0+; 2, \nu+\frac{1}{2}}^{-(\nu+1)} \left(\frac{2}{x}\right)^{\nu+1}, \quad (4.16)$$

$$E_{0+}^{\nu, \mu} = \left(\frac{x}{2}\right)^{\nu+1} I_{0+; 2, -\frac{1}{2}}^{\nu+1} I_{0+}^{-(\nu+\mu)}, \quad (4.17)$$

$$B_{-}^{\nu, \mu} = \left(\frac{2}{x}\right)^{\nu+1} I_{-; 2, \nu+1}^{-(\nu+1)} I_{-}^{\nu-\mu+2}, \quad (4.18)$$

and

$$E_{-}^{\nu, \mu} = I_{-}^{-(\nu+\mu)} I_{-; 2, 0}^{\nu+1} \left(\frac{x}{2}\right)^{\nu+1}. \quad (4.19)$$

Theorem 4.5.

(1) Buschman–Erdélyi operators of the zero-order smoothness act according to (2.6), i.e., as multipliers in Mellin images. The following relations hold for their symbols:

$$\begin{aligned} m_{1S_{0+}^{\nu}}(s) &= \frac{\Gamma(-\frac{s}{2} + \frac{\nu}{2} + 1)\Gamma(-\frac{s}{2} - \frac{\nu}{2} + \frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{s}{2})\Gamma(1 - \frac{s}{2})} \\ &= \frac{2^{-s}}{\sqrt{\pi}} \frac{\Gamma(-\frac{s}{2} - \frac{\nu}{2} + \frac{1}{2})\Gamma(-\frac{s}{2} + \frac{\nu}{2} + 1)}{\Gamma(1 - s)}, \Re s < \min(2 + \Re \nu, 1 - \Re \nu), \end{aligned} \quad (4.20)$$

$$m_{1P_{0+}^{\nu}}(s) = \frac{\Gamma(\frac{1}{2} - \frac{s}{2})\Gamma(1 - \frac{s}{2})}{\Gamma(-\frac{s}{2} + \frac{\nu}{2} + 1)\Gamma(-\frac{s}{2} - \frac{\nu}{2} + \frac{1}{2})}, \Re s < 1, \quad (4.21)$$

$$m_{1P_{-}^{\nu}}(s) = \frac{\Gamma(\frac{s}{2} + \frac{\nu}{2} + 1)\Gamma(\frac{s}{2} - \frac{\nu}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2} + \frac{1}{2})}, \Re s > \max(\Re \nu, -1 - \Re \nu), \quad (4.22)$$

and

$$m_{1S_{-}^{\nu}}(s) = \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2} + \frac{1}{2})}{\Gamma(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{2})\Gamma(\frac{s}{2} - \frac{\nu}{2})}, \Re s > 0. \quad (4.23)$$

(2) The symbols satisfy the relations

$$m_{1P_{0+}^{\nu}}(s) = 1/m_{1S_{0+}^{\nu}}(s), \quad m_{1P_{-}^{\nu}}(s) = 1/m_{1S_{-}^{\nu}}(s) \quad (4.24)$$

and

$$m_{1P_{-}^{\nu}}(s) = m_{1S_{0+}^{\nu}}(1 - s), \quad m_{1P_{0+}^{\nu}}(s) = m_{1S_{-}^{\nu}}(1 - s). \quad (4.25)$$

(3) For L_2 -norms of Buschman–Erdélyi operators of the zero-order smoothness, the relations

$$\|_1 S_{0+}^{\nu}\| = \|_1 P_{-}^{\nu}\| = 1/\min(1, \sqrt{1 - \sin \pi \nu}) \quad (4.26)$$

and

$$\|_1 P_{0+}^{\nu}\| = \|_1 S_{-}^{\nu}\| = \max(1, \sqrt{1 - \sin \pi \nu}) \quad (4.27)$$

are valid.

(4) The norms of operators (4.5)–(4.8) are 2-periodic with respect to ν , i.e., $\|x^{\nu}\| = \|x^{\nu+2}\|$, where x^{ν} is any of the operators (4.5)–(4.8).

(5) The norms of the operators ${}_1S_{0+}^{\nu}$ and ${}_1P_{-}^{\nu}$ are not bounded in total with respect to ν , and each norm is not less than 1. If $\sin \pi \nu \leq 0$, then these norms are equal to 1. The specified operators are unbounded in L_2 if and only if $\sin \pi \nu = 1$ (or $\nu = (2k) + 1/2$, $k \in \mathbb{Z}$).

(6) The norms of the operators ${}_1P_{0+}^{\nu}$ and ${}_1S_{-}^{\nu}$ are bounded in total with respect to ν , and each norm does not exceed $\sqrt{2}$. All these operators are bounded in L_2 for all values of ν . If $\sin \pi \nu \geq 0$, then their L_2 -norm is equal to 1. The greatest value of the norm, equal to $\sqrt{2}$, is achieved if and only if $\sin \pi \nu = -1$ (or $\nu = -1/2 + (2k)$, $k \in \mathbb{Z}$).

To formulate a special case as a corollary, assume that operators (4.5)–(4.8) are defined on functions $f(x)$ such that representations for them hold (to achieve this, it suffices to assume that $xf(x) \rightarrow 0$ as $x \rightarrow 0$). Then, for $\nu = 1$, we have the relations

$${}_1P_{0+}^1 f = (I - H_1)f \text{ and } {}_1S_-^1 f = (I - H_2)f, \quad (4.28)$$

where H_1 and H_2 are the Hardy operators,

$$H_1 f = \frac{1}{x} \int_0^x f(y) dy \text{ and } H_2 f = \int_x^\infty \frac{f(y)}{y} dy \quad (4.29)$$

and I is the identity operator.

Corollary 4.1. *The operators (4.28) are unitary and mutually inverse in L_2 . They intertwine differential expressions d^2/dx^2 and $d^2/dx^2 - 2/x^2$.*

Also, one can show that operators (4.28) are Cayley transforms of symmetric operators $\pm 2i(xf(x))$ under a suitable choice of domains.

In the unitary case, Buschman–Erdélyi operators of the zero-order smoothness form a pair of Watson biorthogonal transforms, while their kernels form pairs of nonsymmetric Fourier kernels.

For the theory of integral equations, it is important to study the unitary property. In this case, the inverse operator is to be sought in the form of an integral such that its integration limits are different from the ones of the original integral.

Also, consider the case where $\nu = i\alpha - \frac{1}{2}$, $\alpha \in \mathbb{R}$, related to the Meler–Fock transform.

Theorem 4.6. *Let $\nu = i\alpha - \frac{1}{2}$, $\alpha \in \mathbb{R}$. Then, all operators (4.5)–(4.8) are bounded in L_2 and their norms satisfy the relations*

$$\|{}_1S_{0+}^{i\alpha-\frac{1}{2}}\| = \|{}_1P_-^{i\alpha-\frac{1}{2}}\| = 1.$$

5. Constructing of transmutation operators: composition methods

Each class of transmutation operators, investigated above, is constructed by its own method. This causes a necessity to develop a general scheme to construct transmutation operators. In the present chapter, we propose such a scheme. This is the factorization (composition) method based on the representation of transmutation operators by a composition of integral transforms. The composition methods provides algorithms to construct new transmutation operators. Also, it includes (as special cases) Sonin–Poisson–Delsarte, Vekua–Erdélyi–Lowndes, and Buschman–Erdélyi transmutation operators of various types, Sonin–Katrakhov and Poisson–Katrakhov unitary transmutation operators, generalized Erdélyi–Kober operators, and the classes of elliptic, hyperbolic, and parabolic transmutation operators introduced in [16–18]. In the present chapter, we generalize them, and introduce classes of B -elliptic, B -hyperbolic, and B -parabolic transmutation operators.

The composition method to construct transmutation operators is developed in [13, 14]. The initial idea close to this method is applied by Katrakhov to construct a special class of transmutation operators; these are called Sonin–Katrakhov and Poisson–Katrakhov transmutation operators.

In this section, we provide main definitions and some results that can be obtained by means of the composition method. Since the formulations of the results are quite cumbersome, we omit their proofs. In the future, it is planned to present the the composition method completely (including some examples).

The general scheme of the composition method is as follows. The input data is a pair A, B of operators of an arbitrary type and the related generalized invertible Fourier transforms F_A and F_B acting by relations

$$F_A A = g(t) F_A \quad \text{and} \quad F_B B = g(t) F_B, \quad (5.1)$$

where t is the dual variable and $g(t)$ is an arbitrary suitable function (the most obvious choice is $g(t) = -t^2$, cf. classical integral transforms).

The goal of the composition is to construct (formally) an output pair of mutually inverse transmutation operators P and S acting according to the relations

$$S = F_B^{-1} \frac{1}{w(t)} F_A \quad \text{and} \quad P = F_A^{-1} w(t) F_B, \quad (5.2)$$

where $w(t)$ is an arbitrary weight function. Then, P and S are mutually inverse transmutation operators intertwining the original operators A and B :

$$S A = B S \quad \text{and} \quad P B = A P. \quad (5.3)$$

The formal verifications is the direct substitution. The main difficulty is to compute the introduced composition in an explicit integral form and to find the corresponding domains of the operators.

The main merits of the composition method are as follows.

- The composition method is simple: Numerous transmutation operators are constructed from basic blocks (classical integral transforms) according to a simple rule.
- All transmutation operators known earlier in the explicit form can be constructed by means of the (unified) composition method.
- Using the (unified) composition method, one can construct various new transmutation operators.
- The composition method provides a simple possibility to obtain inverse transforms in the same composition form.
- The composition method provides a possibility to derive estimates of norms of direct and inverse transmutation operators, using known estimates of norms for classical integral transforms in various function spaces.
- The composition method provides a possibility to obtain explicit relations binding solutions of the intertwined differential equations.

The composition method has a disadvantage as well: Usually, the action of basic blocks (integral transforms) is known in classical spaces, but, applying them to differential equations or differential operators for particular problems, we need estimates and the action on functions of other classes, e.g., classes of functions vanishing at the origin and/or infinity. In such cases, one can start, applying the composition method to obtain an explicit form of the transmutation operator, and to extend it to desired spaces afterwards.

We emphasize that relations of kind (5.1)-(5.2) are new neither for the theory of integral transforms nor for applications to differential equations. However, the composition method is new for applications in the theory of transmutation operators.

Selecting the classical Fourier transform in other problems for integral transforms and related differential equations, we see that relations (5.2) of the composition method lead to the class of pseudodifferential operators with symbols $w(t)$ and $\frac{1}{w(t)}$. Selecting the classical Fourier transform and the weight function $w(t) = (\pm it)^{-s}$ for the operators $A = B = D^2$ and $F_A = F_B = F$, we obtain the fractional Riemann–Liouville integral on the whole real line. Selecting the weight function $w(t) = |t|^{-s}$, we obtain Riesz potentials. Once $w(t) = (1 + t^2)^{-s}$ is selected, relations (5.2) yield Bessel potentials. For $w(t) = (1 \pm it)^{-s}$, we obtain modified Bessel potentials (see [34]).

If we select the classical Hankel integral transform and the weight functions

$$A = B = B_\nu, \quad F_A = F_B = H_\nu, \quad g(t) = -t^2, \quad \text{and} \quad w(t) = j_\nu(st), \quad (5.4)$$

then we obtain the Delsarte generalized translation operators (see [21, 23, 24, 35]); recall that the Bessel operator is denoted by B_ν , the Hankel transform is denoted by H_ν , and the normalized (“small”) Bessel function is denoted by $j_\nu(\cdot)$. In the general case, selecting data $A = B, F_A = F_B$ and arbitrary weight functions $g(t)$ and $w(t)$ for the composition method, we obtain transmutation operators commuting with the operator A (cf. the generalized translation operator commuting with the Bessel operator).

Applying the composition method instead of classical approaches, one can construct various explicit integral forms of fractional powers of the Bessel operator. Neither semigroup-theory methods nor spectral methods are able to provide such constructions.

Applying the composition method, one can obtain relations binding solutions of perturbed and unperturbed differential equations linked by transmutation operators. For example, applying the composition method to equations with Bessel operators of the kind

$$\sum_{k=1}^n A_k \left(\frac{\partial^2 u}{\partial x_k^2} + \frac{\nu_k}{x_k} \frac{\partial u}{\partial x_k} \right) \pm \lambda^2 u = 0, \quad (5.5)$$

$$x_k > 0, A_k = \text{const}, \nu_k = \text{const}, \lambda = \text{const},$$

we find binding relations for solutions of the unperturbed equation

$$\sum_{k=1}^n A_k \frac{\partial^2 v}{\partial x_k^2} \pm \lambda^2 v = 0. \quad (5.6)$$

In particular, this class of relations includes binding relations for solutions of equations with Bessel operators with different parameters; they are called translation operators with respect to the parameter (see [13, 14]). A similar idea called the *subordination principle* is used in the theory of fractional-order equations. In fact, it consists of the constructing of transmutation operators performing a translation with respect to the order of the equation; in particular, this allows one to express solutions of fractional-order equations via solutions of classical integer-order equations.

Thus, one can conclude that the composition method is efficient for the constructing of transmutation operators, it is related to a number of other known methods and problems, all known explicit representations for transmutation operators are obtained by means of it, and numerous new

classes of transmutation operators can be constructed by means of it as well. The algorithm of the composition method is applied in the following three stages (steps).

- Step 1. For the given operator pair A, B and related pair F_A, F_B of generalized Fourier transforms, determine and compute the pair P, S of mutually inverse transmutation operators by means of the main relations of the method, given by (5.1)-(5.2).
- Step 2. Find exact conditions and determine function classes such that the transmutation operators constructed on Step 1 satisfy the intertwining properties given by (5.3).
- Step 3. On the corresponding function spaces, apply the well-defined transmutation operators in Steps 1-2 to solve problems for differential equations, e.g., to establish correspondence relations between solutions of perturbed and unperturbed differential equations.

Further, the composition method is used to construct hyperbolic, elliptic, and parabolic (in the Carroll sense) transmutation operators, generalized Erdélyi–Kober operators, etc.

We use the classification of transmutation operators, proposed by Carroll; it is related to the type of the partial differential equation such that the kernel of the said operator satisfies it. In the same way, we introduce and use natural generalizations of these notions such as B -elliptic, B -hyperbolic, and B -parabolic transmutation operators. To do that, we use the Kipriyanov definitions of the corresponding classes of singular partial differential equations. For example, in these terms, the classical Sonin–Poisson–Delsarte intertwining operators are B -hyperbolic.

5.1. B -hyperbolic transmutation operators

Translation operators with respect to the parameter of the Bessel operator, satisfying the relation

$$TB_\nu = B_\mu T \quad (5.7)$$

are called B -hyperbolic transmutation operators. Such operators are sought in the factorized form

$$T_{\nu, \mu}^{(\varphi)} = F_\mu^{-1} (\varphi(t) F_\nu). \quad (5.8)$$

If $\nu = -\frac{1}{2}$ or $\mu = -\frac{1}{2}$, then such operators are reduced to already studied ones. Assume that $\varphi(t) = t^\alpha$ and $T^{(\varphi)} = T^{(\alpha)}$.

Theorem 5.1. *If*

$$-2 - 2\Re\mu < \Re\alpha < \Re(\nu - \mu),$$

then the integral representation

$$\begin{aligned} (T_{\nu, \mu}^{(\alpha)} f)(x) = & \frac{C_1}{x^{2\mu+\alpha+2}} \int_0^x y^{2\nu+1} {}_2F_1\left(\mu + \frac{\alpha}{2} + 1, \frac{\alpha}{2} + 1; \nu + 1; \frac{y^2}{x^2}\right) f(y) dy \\ & + C_2 \int_x^\infty y^{-2\mu+2\nu-\alpha-1} {}_2F_1\left(\mu + \frac{\alpha}{2} + 1, \mu - \nu + \frac{\alpha}{2} + 1; \mu + 1; \frac{x^2}{y^2}\right) f(y) dy, \end{aligned} \quad (5.9)$$

where

$$C_1 = \frac{2^{\mu-\nu+\alpha+1}\Gamma\left(\mu + \frac{\alpha}{2} + 1\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma(\nu + 1)}, C_2 = \frac{2^{\mu-\nu+\alpha+1}\Gamma\left(\mu + \frac{\alpha}{2} + 1\right)}{\Gamma\left(\nu - \mu - \frac{\alpha}{2}\right)\Gamma(\mu + 1)},$$

and ${}_2F_1$ is the Gauss hypergeometric function, holds.

Consider several special cases of operator (5.9).

(a) Let $\alpha = -1 - 2\mu + \nu$, $\Re \nu > -1$, and $\Re \mu > -1$. Then,

$$\begin{aligned} (T_{\nu,\mu}^{(-1-2\mu+\nu)} f)(x) &= 2^{-\mu} \frac{1}{x^\mu} \int_x^\infty y^\nu (y^2 - x^2)^{\frac{\mu}{2}} P_{\frac{\nu}{2}-\frac{1}{2}}^{-\mu} \left(1 - 2\frac{x^2}{y^2}\right) f(y) dy \\ &\quad + 2^{1-\mu} e^{i\mu\pi} \frac{\Gamma\left(\mu + \frac{\nu}{2} + \frac{1}{2}\right)}{\Gamma\left(\mu - \frac{\nu}{2} + \frac{1}{2}\right)} \frac{1}{x^\mu} \int_0^x y^\nu (x^2 - y^2)^{\frac{\mu}{2}} Q_{\frac{\nu}{2}-\frac{1}{2}}^\mu \left(2\frac{x^2}{y^2} - 1\right) f(y) dy. \end{aligned}$$

(b) Let $\alpha = 0$ and $-1 < \Re \mu < \Re \nu$. In this case, we obtain the remarkable operator of “descending” with respect to the parameter, depending neither on the initial nor on the final values of the parameters ν and μ , but only on the value $\gamma = \nu - \mu$ of the “descending:”

$$(T_{\nu,\mu}^{(0)} f)(x) = \frac{2^{1-(\nu-\mu)}}{\Gamma(\nu-\mu)} \int_x^\infty y(y^2 - x^2)^{\nu-\mu-1} f(y) dy.$$

The only difference between this integral and the fractional integral I_{-,x^2} with respect to the function $g(x) = x^2$ is a constant factor. This form of the operator is discovered by Erdélyi; it is a special case of the Erdélyi–Kober operators or Lowndes operators.

(c) Let $\alpha = 2\nu$ and $-1 < \Re(\nu + \mu) < 0$. Then,

$$\begin{aligned} (T_{\nu,\mu}^{(2\nu)} f)(x) &= \frac{\sin(\pi\mu)}{\pi} 2^{\mu+\nu+1} \Gamma(\nu + \mu + 1) \int_x^\infty y^{2\nu+1} (y^2 - x^2)^{-\mu-\nu-1} f(y) dy \\ &\quad - \frac{\sin(\pi\nu)}{\pi} 2^{\mu+\nu+1} \Gamma(\nu + \mu + 1) \int_0^x y^{2\nu+1} (x^2 - y^2)^{-\mu-\nu-1} f(y) dy. \end{aligned}$$

(d) If $\mu = \nu$ and $-2\Re \nu - 2 < \Re \alpha < 0$, then we obtain the following family of operators commutative with B_ν :

$$(T_{\nu,\nu}^{(\alpha)} f)(x) = \frac{2^{\alpha+2}}{\sqrt{\pi}} \frac{e^{-i\frac{\pi\alpha}{2}}}{\Gamma\left(-\frac{1}{2} - \frac{\alpha}{2}\right)} \frac{1}{x^{\nu+\frac{1}{2}}} \int_0^\infty y^{\nu+\frac{3}{2}} |x^2 - y^2|^{-\frac{\alpha}{2}-1} Q_{\nu-\frac{1}{2}}^{\frac{\alpha}{2}+1} \left(\frac{x^2 + y^2}{2xy}\right) f(y) dy.$$

(e) If $\mu = -\nu$ and $2\Re \nu - 2 < \Re \alpha < \Re 2\nu$, then

$$(T_{\nu,-\nu}^{(\alpha)} f)(x) = 2^{-2\nu+\alpha+1} \frac{\Gamma\left(-\nu + \frac{\alpha}{2} + 1\right)}{\Gamma\left(2\nu - \frac{\alpha}{2}\right)} x^\nu \int_0^\infty |y^2 - x^2|^{\nu-\frac{\alpha}{2}-1} y^{\nu+1} P_{\nu-\frac{\alpha}{2}-1}^\nu \left(\frac{x^2 + y^2}{|x^2 - y^2|}\right) f(y) dy.$$

Another approach to the constructing of translation operators with respect to a parameter is as follows. Consider a Sonin-type operator constructed according to the above relations and such that $\varphi = \varphi_1$ and a Poisson-type operator such that $\nu = \mu$ and $\varphi = \varphi_2$. Then, their composition $T = P_\mu S_\nu$ is the desired operator. If the Fourier sine transform or the Fourier cosine transform are selected in both cases, then we obtain (5.8), where $\varphi = \frac{\varphi_2}{\varphi_1}$. Therefore, the problem on the boundedness of such an operator in Lebesgue spaces $L_{p,\gamma}(0, \infty)$ with power weights is reduced to the problem on the possibility to divide in the spaces of symbols. If different transforms are selected, then we arrive at the following factorization:

$$T_{\nu,\mu}^{(\varphi_1,\varphi_2)} = F_\mu^{-1} \cdot \varphi_2(t) \cdot F_{\left\{\begin{smallmatrix} s \\ c \end{smallmatrix}\right\}} \cdot F_{\left\{\begin{smallmatrix} c \\ s \end{smallmatrix}\right\}}^{-1} \frac{1}{\varphi_1(t)} F_\nu. \quad (5.10)$$

It is easy to show that the composition of the Fourier transforms is reduced to the so-called semiaxis Hilbert transforms

$$(F_s F_c f)(x) = \int_0^\infty \frac{x}{x^2 - y^2} f(y) dy \quad (5.11)$$

and

$$(F_c F_s f)(x) = \int_0^\infty \frac{y}{y^2 - x^2} f(y) dy, \quad (5.12)$$

where the integral is treated in the principal-value sense; these transforms multiplied by suitable constants are unitary in $L_2(0, \infty)$. Thus, in the considered case, operator (5.10) is factorized via two Fourier–Bessel transforms and one of the two-weight transforms

$$(A_1 f)(x) = x \varphi_2(x) \int_0^\infty \frac{f(y)}{(x^2 - y^2)} \cdot \frac{dy}{\varphi_1(x)} \quad \text{and} \quad (A_2 f)(x) = \varphi_2(x) \int_0^\infty \frac{y}{\varphi_1(x)} \cdot \frac{f(y)}{(y^2 - x^2)} dy.$$

In this case, the boundedness problem for (5.10) is reduced to the problem on two-weight estimates for the semiaxis Hilbert transforms in the corresponding spaces.

To construct other classes of B -hyperbolic transmutation operators, one can use the transform with the kernel Neumann function $Y_\nu(z)$ instead of the Fourier–Bessel transform F_ν .

Operators of translations with respect to parameters of Bessel operators have important applications in the theory of singular differential equations (see [13–15]).

5.2. B -elliptic transmutation operators

B -elliptic transmutation operators satisfy the relation

$$TB_\nu = -D^2 T. \quad (5.13)$$

This unusual class of transmutation operators establishes a relation between solutions of B -elliptic and B -hyperbolic differential equations, expressing their solutions via each other.

To construct such operators, in the previous factorizations, we replace the Fourier sine transform and cosine transform with the Laplace transform or change the Fourier–Bessel transform with one of

the following transforms with the Macdonald function and Neumann function:

$$(K_\nu f)(t) = \frac{1}{t^\nu} \int_0^\infty y^{\nu+1} K_\nu(ty) f(y) dy \quad (5.14)$$

and

$$(Y_\nu f)(t) = \frac{1}{t^\nu} \int_0^\infty y^{\nu+1} Y_\nu(ty) f(y) dy. \quad (5.15)$$

Also, it is possible to use the transform with the Struwe function in the kernel.

Theorem 5.2. *Let $|\Re \nu| + \Re(\alpha + \nu) < 1$. Then,*

$$(A_\nu^\alpha f)(x) = F_c^{-1} t^{-\alpha} K_\nu f$$

is a B -elliptic operator satisfying relation (5.13). Then the integral representation

$$\begin{aligned} (A_\nu^\alpha f)(x) &= \frac{\pi \Gamma(1 - \alpha)}{4 \sin \frac{\pi}{2}(1 - \alpha - 2\nu)} \int_0^\infty y^{\nu+1} (x^2 + y^2)^{\frac{\alpha+\nu-1}{2}} \\ &\times \left[P_{-\alpha-\nu}^{-\nu} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + P_{-\alpha-\nu}^{-\nu} \left(-\frac{x}{\sqrt{x^2 + y^2}} \right) \right] f(y) dy. \end{aligned}$$

holds.

Define an operator satisfying relation (5.13) as follows:

$$(C_\nu^\alpha f)(x) = L(t^{-\alpha} F_\nu f),$$

where L is the Laplace transform.

Theorem 5.3. *Let $\Re \alpha < 1$. Then, the integral representation*

$$(A_\nu^\alpha f)(x) = \Gamma(1 - \alpha) \int_0^\infty y^{\nu+1} (x^2 + y^2)^{\frac{\alpha+\nu-1}{2}} P_{-\alpha-\nu}^{-\nu} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) f(y) dy$$

holds.

For $|\Re \nu| + \Re(\alpha + \nu) < 1$, a similar representation holds for the operator L as well:

$$L(t^{-\alpha} Y_\nu f)(x) = -\frac{2}{\pi} \Gamma(1 - \alpha) \int_0^\infty y^{\nu+1} (x^2 + y^2)^{\frac{\alpha+\nu-1}{2}} Q_{-\alpha-\nu}^{-\nu} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) f(y) dy.$$

Also, similar expressions are derived for a broader class of B -elliptic transmutation operators intertwining B_ν and $(-B_\mu)$.

Consider the above basic operators A_ν^α and C_ν^α for $\nu = \pm \frac{1}{2}$. These operators are defined by the relations

$$(A^\beta f)(x) = \left(F^{-1} \begin{Bmatrix} s \\ c \end{Bmatrix} t^\beta L \right) f \quad \text{and} \quad (C^\beta f)(x) = \left(L t^{-\beta} F \begin{Bmatrix} s \\ c \end{Bmatrix} \right) f,$$

where L is the Laplace transform. They intertwine D^2 and $-D^2$. In this case, it is easier to compute the kernels of the integral operators directly. This leads to the relation

$$(C^\beta f)(x) = \sqrt{\frac{2}{\pi}} \Gamma(1-\beta) \int_0^\infty \frac{f(y)}{(x^2+y^2)^{\frac{1-\beta}{2}}} \left\{ \frac{\sin}{\cos} \right\} \left[(1-\beta) \operatorname{atan} \frac{y}{x} \right] dy,$$

where $\Re \beta < 1 + \delta$ and $\delta = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, and

$$(A^\beta f)(x) = \sqrt{\frac{2}{\pi}} \Gamma(1+\beta) \int_0^\infty \frac{f(y)}{(x^2+y^2)^{\frac{1+\beta}{2}}} \left\{ \frac{\sin}{\cos} \right\} \left[(1+\beta) \operatorname{atan} \frac{x}{y} \right] dy,$$

where $\Re \beta > -\delta - 1$ and δ is defined above. In particular, for $\beta = 0$, we obtain the following pair of transmutation operators, related to the half-space Poisson integrals:

$$(C^0 f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{y f(y)}{x^2 + y^2} dy, \quad (A^0 f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x f(y)}{x^2 + y^2} dy.$$

These operators and the operators C^β and A^β for special positive integer values of β are constructed in [16].

5.3. B -parabolic transmutation operators

This unusual class of transmutation operators allows one to express solutions of parabolic equations via solutions of hyperbolic ones and vice versa.

We introduce the integral transforms

$$(F'_c f)(x) = (F_c f)(\sqrt{x}), \quad (F'_s f)(x) = (F_s f)(\sqrt{x}), \quad \text{and} \quad (Pf)(x) = \left(L \varphi(t) F' \begin{Bmatrix} s \\ c \end{Bmatrix} \right) (x).$$

Then, on compactly supported functions, the operator P intertwines the second and first derivative as follows:

$$PD^2 f = DPf.$$

Thus, this operator is parabolic in the Carroll sense.

5.4. Lowndes-type operators of translations with respect to the spectral parameter

Operators of this type arise once solutions of Helmholtz equations are expressed via harmonic functions. Their study was initiated by Vekua and Erdélyi and was continued by Lowndes. This is the reason to call them the Vekua–Erdélyi–Lowndes transmutation operators; similar operators are applied to resolve singular differential equations (see, e. g., [13–15]).

Consider the operator

$$T_1 = F_\nu^{-1} (\varphi(t) F'_\mu),$$

where the following transform with the parameter λ is introduced:

$$(F'_\mu f)(t) = \frac{1}{t^\nu} \int_0^\infty y^{\nu+1} J_\nu(y \sqrt{t^2 + \lambda^2}) f(y) dy.$$

The operator T_1 satisfies the relation

$$T_1 B_\mu = (B_\nu - \lambda^2) T_1. \quad (5.16)$$

Theorem 5.4. *If $-1 < \Re \nu < \Re \mu$ and $\varphi(x) = x^\mu (x^2 + \lambda^2)^{-\frac{\mu}{2}}$, then the integral representation*

$$(T_1 f)(x) = \lambda^{1+\nu-\mu} \int_0^\infty y(y^2 - x^2)^{\frac{\mu-\nu-1}{2}} J_{\nu-\mu-1}(\lambda \sqrt{y^2 - x^2}) f(y) dy$$

holds.

A number of other operators obtained in the factorized form is provided below.

(a) Let $\nu = 1$ and $\varphi(x) = x^{\mu-2} (x^2 + \lambda^2)^{\frac{\mu}{2}}$.

Then, the operator $T_2 = F_1^{-1} \varphi(t) F'_\mu$ satisfying the relation

$$T_2 B_\mu = (B_1 - \lambda^2) T_2$$

for $\Re \lambda > 0$ and $\Re \mu > -1$ can be represented as follows:

$$(T_2 f)(x) = \frac{1}{x^2 \lambda^2} \int_0^\infty y^{\nu+1} J_\mu(\lambda y) f(y) dy - \frac{1}{x^2 \lambda^2} \int_x^\infty y(y^2 - x^2)^{\frac{\mu}{2}} J_\mu(\lambda \sqrt{y^2 - x^2}) f(y) dy.$$

(b) Let $\varphi(x) = x^\mu (x^2 + \lambda^2)^2$, $\Re \lambda > 0$, $T_3 = F_\nu^{-1} \varphi F'_\mu$, and $-1 < \Re \nu < -\Re \mu$. Then,

$$\begin{aligned} (T_3 f)(x) &= \frac{2 \sin(\pi \mu)}{\pi} \lambda^{\mu+\nu+1} \int_0^x y^{2\mu+1} (x^2 - y^2)^{-\frac{\mu+\nu+1}{2}} K_{\mu+\nu+1}(\lambda \sqrt{x^2 - y^2}) f(y) dy \\ &+ \lambda^{\nu+\mu+1} \int_x^\infty y^{2\mu+1} (y^2 - x^2)^{-\frac{\mu+\nu+1}{2}} \left[\sin(\pi \nu) Y_{\mu+\nu+1}(\lambda \sqrt{y^2 - x^2}) - \cos(\pi \nu) J_{\nu+\mu+1}(\lambda \sqrt{y^2 - x^2}) \right] f(y) dy. \end{aligned}$$

(c) Let $\varphi(x) = x^{\mu-1} / (x^2 + \lambda^2)(x^2 + \frac{\lambda^2}{2})^{\frac{\mu}{2}}$.

Assign $T_4 = Y_\nu^{-1} \varphi F'_\mu$ and assume that $\Re \lambda > 0$ and $-\frac{1}{2} < \Re \nu < 3 + \Re \mu$. Then, the following relation holds:

$$(T_4 f)(x) = -\frac{\lambda^{\nu-\mu-1}}{2^{-\frac{\mu}{2}}} \cdot \frac{K_\nu(\lambda x)}{x^\nu} \int_0^\infty y^{\mu+1} J_\mu\left(\frac{\lambda y}{\sqrt{2}}\right) f(y) dy.$$

The provided examples demonstrate that it is important to have a freedom to select the function φ .

Operators T satisfying the relation

$$TB_\mu = (B_\mu + \lambda^2)T \quad (5.17)$$

are constructed in the same way. For example, there exists a function φ such that the operator

$$T_5 = (F'_\mu)^{-1} \varphi F_\nu$$

is its Lowndes operator:

$$(T_4 f)(x) = \frac{\lambda^{\nu-\mu+1}}{x^{2\mu}} \int_0^\infty y^{2\nu+1} (x^2 - y^2)^{\frac{\mu-\nu-1}{2}} J_{\mu-\nu-1}(\lambda \sqrt{y^2 - x^2}) f(y) dy.$$

The operators contained in relations (5.16)-(5.17) are linear. Therefore, the said relations can be represented as follows:

$$T_1(B_\mu + \lambda^2) = B_\nu T \text{ and } T(B_\nu - \lambda^2) = B_\mu T.$$

In all operators, the substitution $\lambda \rightarrow i\lambda$ can be justified. For the most general transmutation operators, the relation

$$T(B_\nu + \alpha) = (B_\mu + \beta)T \quad (5.18)$$

is equivalent for the relation pair

$$T(B_\nu + \alpha - \beta) = B_\mu T, \quad (B_\mu + \beta - \alpha)T = TB_\nu$$

considered above.

Operators satisfying the previous relations can be obtained intermediately as well. Lowndes-type B -elliptic operators satisfying the relation

$$T(B_\nu + \lambda) = (-B_\mu + \beta)T$$

are considered in the same way.

Selecting the values $\nu = \mu = -\frac{1}{2}$ of the parameters, we obtain operators intertwining D^2 and $D^2 \pm \lambda^2$.

Also, the composition method can be applied if the quadratic (fractional) Fourier or Hankel transform is taken as the integral transform. According to the general scheme of the composition method, we can construct transmutation operators intertwining the differential operators D^2 and

$$\left(\sin^2 \frac{\alpha}{2} L_\nu - \frac{1}{2} i \sin \alpha (XD + DX) - X^2 \cos^2 \frac{\alpha}{2} \right),$$

where

$$L_\nu = -\frac{1}{4}D^2 - \frac{\nu^2 - 1/4}{x^2} + \frac{1}{4}x^2 - \frac{\nu + 1}{2}$$

with arbitrary parameters α and ν . To do that, one has to assign

$$A = \left(\sin^2 \frac{\alpha}{2} L_\nu - \frac{1}{2} i \sin \alpha (XD + DX) - X^2 \cos^2 \frac{\alpha}{2} \right), \quad B = D^2,$$

$$F(A) = H_\nu^\alpha, F(B) = F_c, \text{ and } g(t) = -t^2$$

(in our terms), where H_ν^α is the quadratic Fourier–Fresnel transform and F_c is the Fourier cosine-transform. By X , we denote multiplication on variable x .

Also, the composition method can be successfully applied to the constructing of fractional powers of the Bessel operator. Using this method, solutions of numerous integrodifferential equations can be obtained (see [13, 14]).

6. Applications of transmutation operator method to perturbed Bessel equations with variable potentials

Consider the problem to construct an integral relation for solutions of the differential equation

$$B_\alpha g(x) - q(x)g(x) = \lambda^2 g(x) \quad (6.1)$$

with a given asymptotic behavior, where B_α is the Bessel operator; in this section, it is convenient to define it in the form

$$B_\alpha g = g''(x) + \frac{2\alpha}{x}g'(x), \quad \alpha > 0. \quad (6.2)$$

Note that various forms to represent constants in the Bessel operator are used throughout the monograph. This depends on the problem to be solved and is caused by easier expressions of the obtained results. No misunderstanding is expected because problems considered at different sections of the monograph are disjoint.

The above problem is solved by means of the transmutation operator method. It suffices to construct a pair of mutually inverse transmutation operators such that the first one

$$S_\alpha h(x) = h(x) + \int_x^\infty S(x, t)h(t) dt \quad (6.3)$$

intertwines the operators $B_\alpha - q(x)$ and B_α by means of the relation

$$S_\alpha(B_\alpha - q(x))h = B_\alpha S_\alpha h, \quad (6.4)$$

while second one is inverse to the first one, is constructed as an integral operator with kernel $P(x, t)$,

$$P_\alpha h(x) = h(x) + \int_x^\infty P(x, t)h(t) dt, \quad (6.5)$$

and acts according to the relation

$$P_\alpha B_\alpha h = (B_\alpha - q(x))P_\alpha h$$

on functions $h \in C^2(0, \infty)$.

As a result, on solutions of Eq (6.1), the function $S_\alpha u = v$ is expressed via solutions of the unperturbed equation, which is Eq (6.1) without the potential term. In fact, it is expressed via Bessel functions and the function $u = P_\alpha v$ satisfies the original perturbed equation given by (6.1). The integral representation of the solution, given by (6.5), describes the kernel $P(x, t)$ explicitly. This technique reflects one of the main utilities of transmutation operators: They express complicated differential equations via simpler ones (see above). Note that, since linear transmutation operators are used, it follows that the same pair of mutually inverse transmutation operators yields representations both for solutions of Eq (6.1) with a spectral parameter and for solutions of the (simpler) case of the homogeneous equation

$$B_\alpha h(x) - q(x)h(x) = 0.$$

If we have to represent solutions of the perturbed equation, i.e., Eq (6.1), then constructing of the direct transmutation operator can be skipped; one can immediately pass to the constructing of an integral representation of kind (6.5) for the desired solution.

In [36], an original technique to solve such problems is developed. Using it, one can consider singular potentials satisfying the most exact estimate at the origin: $|q(x)| \leq cx^{-3/2+\varepsilon}$, $\varepsilon > 0$ where α is an integer. This technique based on the application of generalized Paley–Wiener theorems is broadly developed and recognized.

In many problems of mathematics and physics, one has to consider strongly singular potentials, e.g., potentials admitting arbitrary power singularities at the origin. In the present work, we formulate results on integral representations of solutions of equations with such singular potentials. The only requirement for the potential is to be majorized by a function summable at infinity. In particular, the class of admissible potentials includes the singular potential $q = x^{-2}$, the strongly singular potential with the power singularity $q = x^{-2-\varepsilon}$, $\varepsilon > 0$, Ukawa-type potentials $q = e^{-\alpha x}/x$, Bargmann and Batman–Chadan potentials and a number of others. No additional assumptions (such as a fast oscillation at the origin or a constant sign) are imposed on the function $q(x)$, which allows one to use the same method to study both attracting and repelling potentials.

Special-kind transmutation operators constructed in the present work differ from those known previously. Earlier, only the cases where the main integral equation for the kernel of the transmutation operator has the same integration limits (either $[0; a]$ or $[a; \infty]$ in both integrals). Here, we show that different limits can be considered. This provides a possibility to cover a broader class of potentials with singularities at the origin. Also, we improve the classical Levitan scheme of [35], expressing the Green function (used in the proofs) via the Legendre function (instead of more general Gauss hypergeometric function) depending on fewer parameters; this allows one to avoid undetermined constants in estimates of earlier papers.

The resolving of the main integral equation for kernels of transmutation operators

We introduce new variables and functions as follows:

$$\xi = \frac{t+x}{2}, \quad \eta = \frac{t-x}{2}, \quad \xi \geq \eta > 0,$$

$$K(x, t) = \left(\frac{x}{t}\right)^\alpha P(x, t), \text{ and } u(\xi, \eta) = K(\xi - \eta, \xi + \eta). \quad (6.6)$$

We introduce the notation $\nu = \alpha - 1$. Thus, to justify representation (6.5) for solutions of Eq (6.1), it suffices to define the function $u(\xi, \eta)$. It is known that if there exists a twice continuously differentiable solution $u(\xi, \eta)$ of the integral equation

$$u(\xi, \eta) = -\frac{1}{2} \int_{\xi}^{\infty} R_{\nu}(s, 0; \xi, \eta) q(s) ds - \int_{\xi}^{\infty} ds \int_0^{\eta} q(s + \tau) R_{\nu}(s, \tau; \xi, \eta) u(s, \tau) d\tau$$

under the assumption that $0 < \tau < \eta < \xi < s$, then the desired function $P(x, t)$ is defined by relations (6.6) via this solution $u(\xi, \eta)$. Here, $R_{\nu} = R_{\alpha-1}$ is the Riemann function arising under the resolving of a Goursat problem for the singular hyperbolic equation

$$\frac{\partial^2 u(\xi, \eta)}{\partial \xi \partial \eta} + \frac{4\alpha(\alpha - 1)\xi\eta}{(\xi^2 - \eta^2)^2} u(\xi, \eta) = q(\xi + \eta)u(\xi, \eta).$$

This function is known in its explicit form (see [35]): It is expressed via the Gauss hypergeometric function ${}_2F_1$ as follows:

$$R_{\nu} = \left(\frac{s^2 - \eta^2}{s^2 - \tau^2} \cdot \frac{\xi^2 - \tau^2}{\xi^2 - \eta^2} \right)^{\nu} {}_2F_1 \left(-\nu, -\nu; 1; \frac{s^2 - \xi^2}{s^2 - \eta^2} \cdot \frac{\eta^2 - \tau^2}{\xi^2 - \tau^2} \right). \quad (6.7)$$

This expression may be simplified: It is proved that, in the considered case, the Riemann function is expressed via the Legendre function as follows:

$$R_{\nu}(s, \tau, \xi, \eta) = P_{\nu} \left(\frac{1 + A}{1 - A} \right), \quad A = \frac{\eta^2 - \tau^2}{\xi^2 - \tau^2} \cdot \frac{s^2 - \xi^2}{s^2 - \eta^2}. \quad (6.8)$$

The main content of this section is the following assertion.

Theorem 6.1. *Let $q(r) \in C^1(0, \infty)$ and the conditions*

$$|q(s + \tau)| \leq |p(s)|, \quad \forall s, \forall \tau, \quad 0 < \tau < s \quad \text{and} \quad \int_{\xi}^{\infty} |p(t)| dt < \infty \quad \forall \xi > 0 \quad (6.9)$$

be satisfied. Then there exists an integral representation of kind (6.5) such that its kernel satisfies the estimate

$$\begin{aligned} |P(r, t)| &\leq \left(\frac{t}{r}\right)^{\alpha} \frac{1}{2} \int_{\frac{t+r}{2}}^{\infty} P_{\alpha-1} \left(\frac{y^2(t^2 + r^2) - (t^2 - r^2)}{2try^2} \right) |p(y)| dy \\ &\times \exp \left[\left(\frac{t-r}{2} \right) \frac{1}{2} \int_{\frac{t+r}{2}}^{\infty} P_{\alpha-1} \left(\frac{y^2(t^2 + r^2) - (t^2 - r^2)}{2try^2} \right) |p(y)| dy \right]. \end{aligned}$$

The kernel $P(x, t)$ of the transmutation operator and the solution of Eq (6.1) are twice continuously differentiable on $(0, \infty)$ with respect to their independent variables.

The following classes of potentials satisfy conditions (6.9). If $|q(x)|$ monotonously decreases, then one can assign $p(x) = |q(x)|$. If a potential has an arbitrary singularity at the origin, increases for $0 < x < M$ (e.g., the Coulomb potential $q = -\frac{1}{x}$), and is truncated by zero at infinity, i.e., $q(x) = 0$ for $x > M$, one can assign $p(x) = |q(M)|$ for $x < M$ and $p(x) = 0$ for $x \geq M$. Potentials with the estimate $q(x + \tau) \leq c|q(x)| = |p(x)|$ satisfy condition (6.9) as well.

In particular, the above conditions are satisfied by the following potentials arising in applications: strongly singular potentials with power singularities of the kind $q(x) = x^{-2-\varepsilon}$, various Bargmann potentials

$$q_1(x) = -\frac{e^{-ax}}{(1 + \beta e^{-ax})^2}, \quad q_2(x) = \frac{c_2}{(1 + c_3x)^2}, \quad \text{and} \quad q_3(x) = \frac{c_4}{ch^2(c_5x)},$$

and the Ukawa potentials

$$q_4(x) = -\frac{e^{-ax}}{x} \quad \text{and} \quad q_5(x) = \int_x^\infty e^{-at} dc(t).$$

Remark 6.1. In fact, the explicit form of the Riemann function, given by (6.8), is not required to prove the above theorem. Only the existence of the Riemann function, its positivity, and a special monotonicity property are used. These facts are rather general. Therefore, the obtained results can be extended to a quite broad class of differential equations.

The estimate from Theorem 6.1 for general-kind potentials can be transformed into a rougher (though more visible) one.

Theorem 6.2. *Let the assumptions of Theorem 6.1 be satisfied. Then, the kernel $P(x, t)$ of the transmutation operator satisfies the estimate*

$$|P(x, t)| \leq \frac{1}{2} \left(\frac{t}{x} \right)^\alpha P_{\alpha-1} \left(\frac{t^2 + x^2}{2tx} \right) \int_x^\infty |p(y)| dy \exp \left[\frac{1}{2} \left(\frac{t-x}{2} \right) P_{\alpha-1} \left(\frac{t^2 + x^2}{2tx} \right) \int_x^\infty |p(y)| dy \right].$$

Note that an exponential singularity as $x \rightarrow 0$ is admitted for the kernel of the integral representation.

For the class of potentials with power singularities of the kind

$$q(x) = x^{-(2\beta+1)}, \quad \beta > 0, \tag{6.10}$$

the obtained estimates can be simplified such that the precision is preserved. The restriction for β is caused by the summability condition at infinity.

Theorem 6.3. *For potentials of kind (6.10), Theorem 6.1 holds with the estimate*

$$|P(x, t)| \leq \left(\frac{t}{x} \right)^\alpha \frac{\Gamma(\beta) 4^{\beta-1}}{(t^2 - x^2)^\beta} \cdot P_{\alpha-1}^{-\beta} \left(\frac{t^2 + x^2}{2tx} \right) \exp \left[\left(\frac{t-x}{x} \right) \frac{\Gamma(\beta) 4^{\beta-1}}{(t^2 - x^2)^\beta} P_{\alpha-1}^{-\beta} \left(\frac{t^2 + x^2}{2tx} \right) \right],$$

where $P_\nu^\mu(\cdot)$ is the Legendre function, the value of β is determined by (6.10), and the value of α is determined by (6.2).

This estimate is obtained by means of a long chain of computations using the well-known Slater–Marichev theorem; it helps to compute the needed integrals (in terms of hypergeometric functions) once they are reduced to the Mellin convolution.

Also, the obtained estimate can be simplified and expressed via elementary functions for potentials of the kind $q(x) = x^{-(2\beta+1)}$, where the parameters are bound by the relation $\beta = \alpha - 1$.

Corollary 6.1. *If $\beta = \alpha - 1$, then the estimate from Theorem 6.3 takes the form*

$$|P(x, t)| \leq \left(\frac{t}{x}\right)^{\beta+1} \frac{2^{\beta-2}}{\beta} \left[\frac{t^2 + x^2}{2tx}\right]^\beta \exp \left[\left(\frac{t-x}{2}\right) \frac{2^{\beta-2}}{\beta} \left[\frac{t^2 + x^2}{2tx}\right]^\beta \right] \\ = \frac{1}{4\beta} \frac{1}{x^{2\beta+1}} (t^2 + x^2)^\beta \exp \left[\frac{2^{\beta-2}}{\beta} \left(\frac{t-x}{2}\right) \left(\frac{t^2 + x^2}{2tx}\right)^\beta \right]. \quad (6.11)$$

If $\alpha = 0$ in relations (6.1)–(6.5), then Theorem 6.1 is reduced to the known estimates of the kernel of the integral representation of Jost solutions for the Sturm–Liouville equation.

7. Applications of Fourier–Bessel transform to B -parabolic functional-differential equations

Full proofs of the results presented below are provided in [37].

7.1. Singular integrodifferential equations

In what follows, assuming that $\nu > -\frac{1}{2}$, we use the notation

$$B_y \stackrel{\text{def}}{=} \frac{1}{y^{2\nu+1}} \frac{\partial}{\partial y} \left(y^{2\nu+1} \frac{\partial}{\partial y} \right) = \frac{\partial^2}{\partial y^2} + \frac{2\nu+1}{y} \frac{\partial}{\partial y}, \\ T_y^h f(y) \stackrel{\text{def}}{=} \frac{\Gamma(\nu+1)}{\sqrt{\pi} \Gamma(\nu+\frac{1}{2})} \int_0^\pi f(\sqrt{y^2 + h^2 - 2yh \cos \theta}) \sin^{2\nu} \theta d\theta,$$

and

$$j_\nu(y) \stackrel{\text{def}}{=} \frac{2^\nu \Gamma(\nu+1)}{y^\nu} J_\nu(y)$$

to define the Bessel operator with respect to y , the corresponding generalized translation operator, and the corresponding (uniformly) normalized Bessel function of the first kind respectively.

Assuming that u_0 and f are continuous and bounded, while $s \in \mathbb{N}$ and $a_1, \dots, a_s, h_1, \dots, h_s$ are real constants, we investigate the following problem:

$$\frac{\partial u}{\partial t} - B_x u - \sum_{k=1}^s a_k T_x^{h_k} u = f(x, t), \quad x > 0, t > 0, \quad (7.1)$$

$$\frac{\partial u}{\partial x} \Big|_{x=0} = 0, \quad t > 0, \quad (7.2)$$

$$u \Big|_{t=0} = u_0(x), \quad x > 0. \quad (7.3)$$

Note that, in general, any solution of problem (7.1)–(7.3) is defined only in the quarter $\mathbb{R}_+^1 \times \mathbb{R}_+^1$ of the plane, while, to apply the generalized translation operator, we need it to be defined for negative values of the variable x as well. To provide that, we use the even with respect to x extension of the solution; the specified extension exists by virtue of the *evenness condition* (7.2). In other words, we can consider problem (7.1)–(7.3) in the whole half-plane $\mathbb{R}^1 \times \mathbb{R}_+^1$, replacing condition (7.2) by the evenness (with respect to the variable x) requirement imposed on the solution. For *differential B-parabolic* equations, such problems are well defined (see, e.g., [38–44] and references therein).

Using the fact that both the Bessel operator and the generalized translation operator act as multipliers in Fourier–Bessel (Hankel) images, one can explicitly construct integral representations of classical solutions of the above problem. The following assertion is valid.

Theorem 7.1. *Let u_0 be continuous and bounded in $[0, +\infty)$ and f be continuous and bounded in $[0, +\infty) \times (0, +\infty)$. Then, the function*

$$\frac{1}{4^{\nu}\Gamma^2(\nu+1)} \left[\int_0^\infty \xi^{2\nu+1} \mathcal{E}(\xi, t) T_x^\xi u_0(x) d\xi + \int_0^t \int_0^\infty \xi^{2\nu+1} \mathcal{E}(\xi, t) T_x^\xi f(x, t-\tau) d\xi d\tau \right]$$

is a classical solution of problem (7.1)–(7.3), where

$$\mathcal{E}(x, t) \stackrel{\text{def}}{=} \mathcal{E}_{a,h}(x, t) = \int_0^\infty \xi^{2\nu+1} e^{-t \left[\xi^2 - \sum_{k=1}^s a_k j_\nu(h_k \xi) \right]} j_\nu(x\xi) d\xi. \quad (7.4)$$

Uniqueness classes and the long-time behavior of solutions are provided below for more general cases.

7.2. Multidimensional case

Equations of kind (7.1) contain only one spatial (independent) variable. This variable is special, which means that the investigated equation is singular with respect to it. The above technique and approach can be used to generalize it substantially: One can consider equations containing both nonspecial spatial variables (the second-derivative operator and classical translation operators act with respect to them) and special spatial variables (Bessel operators and generalized translation operators act with respect to them). Moreover, the amount of spatial variables of both kinds can be arbitrary. Thus, in the cylinder $\mathbb{R}_+^{m+n} \times (0, \infty)$, where

$$\mathbb{R}_+^{m+n} = \left\{ (x, y) \mid x \in \mathbb{R}^m, y_1 > 0, \dots, y_n > 0 \right\},$$

the following problem is investigated:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^m \left[\frac{\partial^2 u}{\partial x_i^2} + \sum_{s=1}^{m_i} a_{is} u(x + h_{is}, y, t) \right] - \sum_{l=1}^n \left(B_{k_l, y_l} u + \sum_{r=1}^{n_l} b_{lr} T_{y_l}^{g_{lr}} u \right) = f(x, y, t), \quad (7.5)$$

$$\left. \frac{\partial u}{\partial y_l} \right|_{y_l=0} = 0 \quad (l = \overline{1, n}), \quad t > 0, \quad (7.6)$$

$$u|_{t=0} = u_0(x, y), \quad (x, y) \in \overline{\mathbb{R}_+^{m+n}}, \quad (7.7)$$

where h_{is} are vectors parallel to the i th coordinate axis of the space $\mathbb{R}^m$, $i \in \overline{1, m}$, for any s , and a_{is} , b_{lr} , and g_{lr} are real constants for all values of their indices, while the generalized translation operator is defined as the superposition of the one-dimensional operators: $T_y^h = T_{y_1}^{h_1} \cdots T_{y_n}^{h_n}$. The functions u_0 , f , $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}$, and $\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$ are assumed to be continuous and bounded, and the function f satisfies condition (7.6).

As above, problem (7.5)–(7.7) can be considered in the whole subspace $\mathbb{R}^{m+n} \times (0, +\infty)$ if condition (7.6) is replaced by the requirement that the function u is even with respect to each variable y_l .

The following assertions are valid.

Theorem 7.2. *Let u_0 be a continuous and bounded in $\overline{\mathbb{R}_+^{m+n}}$ function. Let f be a continuous and bounded in $\overline{\mathbb{R}_+^{m+n}} \times (0, +\infty)$ function such that it satisfies condition (7.6) and the functions $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}$ and $\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$ are continuous and bounded in $\overline{\mathbb{R}_+^{m+n}} \times (0, +\infty)$. Then the function*

$$u(x, y, t) = \frac{2^{n-m-|k|}}{\pi^m \prod_{l=1}^n \Gamma^2\left(\frac{k_l+1}{2}\right)} \left[\int_{\overline{\mathbb{R}_+^{m+n}}} \prod_{l=1}^n \eta_l^{k_l} \mathcal{E}(\xi, \eta, t) T_y^\eta u_0(x - \xi, y) d\xi d\eta \right. \\ \left. + \int_0^t \int_{\overline{\mathbb{R}_+^{m+n}}} \prod_{l=1}^n \eta_l^{k_l} \mathcal{E}(\xi, \eta, \tau) T_y^\eta f(x - \xi, y, t - \tau) d\xi d\eta d\tau \right], \quad (7.8)$$

where

$$\mathcal{E}_1(x, t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^m} e^{-t\left(|\xi|^2 - \sum_{i=1}^m \sum_{s=1}^{m_i} a_{is} \cos h_{is} \cdot \xi\right)} \cos\left(x \cdot \xi + t \sum_{i=1}^m \sum_{s=1}^{m_i} a_{is} \sin h_{is} \cdot \xi\right) d\xi, \quad (7.9)$$

$$\mathcal{E}_2(y, t) \stackrel{\text{def}}{=} \prod_{l=1}^n \int_0^\infty \eta_l^{k_l} e^{-t\left[\eta_l^2 - \sum_{r=1}^{n_l} b_{lr} j_{\nu_l}(g_{lr} \eta_l)\right]} j_{\nu_l}(y_l \eta_l) d\eta_l, \quad (7.10)$$

and

$$\mathcal{E}(x, y, t) \stackrel{\text{def}}{=} \mathcal{E}_1(x, y, t) \mathcal{E}_2(x, y, t), \quad (7.11)$$

satisfies problem (7.5)–(7.7) in the classical sense.

Theorem 7.3. *Under the assumptions of Theorem 7.2, function (7.8) is bounded in $\overline{\mathbb{R}_+^{m+n}} \times [0, T]$ for any positive T and problem (7.5)–(7.7) has no other solutions bounded in $\overline{\mathbb{R}_+^{m+n}} \times [0, T]$ for any positive T .*

Remark 7.1. The requirement of the boundedness of the function f and its derivatives can be weakened: It can be replaced by the requirement of their boundedness in any layer $\overline{\mathbb{R}_+^{m+n}} \times [0, T]$.

Thus, in the general case of several nonspecial spatial variables $x_1, \dots, x_m$ and several special spatial variables $y_1, \dots, y_l$, the Poissonian kernel $\mathcal{E}$ is modified (compared with the prototype integrodifferential case considered in Section 7.1) from (7.4) to (7.11).

7.3. Long-time behavior of solutions

To investigate the long-time behavior of the solution of problem (7.5)–(7.7), we introduce the operator $\mathcal{L}$ acting as follows:

$$\mathcal{L}u \stackrel{\text{def}}{=} \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} + \sum_{l=1}^n B_{k_l, y_l} u + \sum_{a_{is} < 0} a_{is} u(x + h_{is}, y, t) + \sum_{b_{lr} < 0} b_{lr} T_{y_l}^{g_{lr}} u.$$

Further, without loss of generality, we redenote the vector h_{is} by $(\underbrace{0, \dots, 0}_{i-1 \text{ times}}, h_{is}, 0, \dots, 0)$ (note that here h_{is} is a scalar parameter).

Now, denote the operator $\left(\sum_{a_{is} < 0} a_{is} + \sum_{b_{lr} < 0} b_{lr} \right) I - \mathcal{L}$ by R and consider the real part of its symbol:

$$\text{Re}R(\xi, \eta) = \sum_{a_{is} < 0} a_{is} + \sum_{b_{lr} < 0} b_{lr} + |\xi|^2 + |\eta|^2 - \sum_{a_{is} < 0} a_{is} \cos h_{is} \xi_i - \sum_{b_{lr} < 0} b_{lr} j_{y_l} (g_{lr} \eta_l)$$

(cf. [8, §8]). We say that $R(\xi, \eta)$ is *positive definite* if there exists a positive C such that $\text{Re}R(\xi, \eta) \geq C(|\xi|^2 + |\eta|^2)$ for any (ξ, η) from $\overline{\mathbb{R}_+^{m+n}}$.

Together with Eq (7.5), consider the equation

$$\frac{\partial w}{\partial t} = \sum_{i=1}^m \frac{\partial^2 w}{\partial x_i^2} + \sum_{l=1}^n B_{k_l, y_l} w, \quad (x, y) \in \overline{\mathbb{R}_+^{m+n}}, \quad t > 0, \quad (7.12)$$

and the initial-value condition

$$w|_{t=0} = w_0(x, y), \quad (x, y) \in \overline{\mathbb{R}_+^{m+n}}, \quad (7.13)$$

where w_0 is continuous and bounded.

It is known from [38] (see also [39, 44]) that problem (7.6), (7.12), and (7.13) has a unique classical bounded solution $w(x, y, t)$.

The following assertion is valid.

Theorem 7.4. *Let $f(x, y) \equiv 0$ and $R(\xi, \eta)$ be positive definite. Then, the following limit relation is valid for any (x, y) from $\overline{\mathbb{R}_+^{m+n}}$:*

$$e^{-t \left(\sum_{i=1}^m \sum_{s=1}^{m_i} a_{is} + \sum_{l=1}^n \sum_{r=1}^{n_l} b_{lr} \right)} u(x, y, t) - w \left(\frac{x_1 + q_1 t}{p_1}, \dots, \frac{x_m + q_m t}{p_m}, \frac{y_1}{p_{m+1}}, \dots, \frac{y_n}{p_{m+n}}, t \right) \xrightarrow{t \rightarrow \infty} 0, \quad (7.14)$$

where

$$p_i = \sqrt{1 + \frac{1}{2} \sum_{s=1}^{m_i} a_{is} h_{is}^2}, \quad q_i = \sum_{s=1}^{m_i} a_{is} h_{is}, \quad i = \overline{1, m},$$

$$p_{m+l} = \sqrt{1 + \frac{1}{2(k_l + 1)} \sum_{r=1}^{n_l} b_{lr} g_{lr}^2}, \quad l = \overline{1, n},$$

and

$$w_0(x, y) = u_0(p_1 x_1, \dots, p_m x_m, p_{m+1} y_1, \dots, p_{m+n} y_n).$$

8. Conclusions

This survey is devoted to recent enhances in transmutation theory and integral transforms, and also functional differential equations. Methods of differential and functional equations, function theory and special functions are applied to many related problems. The survey is almost equally consists of known basic and classical results together with author's original contribution.

Author contributions

Author contributions are divided equally in compiling material and editing. More exact S.M. is mostly responsible for Sections 1, 3–5 and A.M. for Sections 2, 6, 7. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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