



*Research article***Numerical solution and error analysis of the fractional nonlinear hyperbolic wave equation using the finite difference method****Hazal Yüksekaya***

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Abstract: In this work, we examined the numerical results for a nonlinear fractional hyperbolic-type wave equation governed by a time-fractional derivative of order α . For the fractional term, an $L2-1\sigma$ memory discretization approach was integrated with standard central difference approximations of spatial second-order derivatives. Explicitly, the study formulates the discrete algebraic relations for the governing system, proves the comprehensive numerical scheme, and evaluates computational efficiency and accuracy across diverse spatial grid refinements.

Keywords: fractional nonlinear wave equation; finite difference method; $L2-1\sigma$ memory coefficients; numerical analysis; error estimation

Mathematics Subject Classification: 35R30, 35R11, 65N06, 35Q55

1. Introduction

Recently, fractional partial differential equations (FPDEs) have gained significant prominence for modeling complex anomalous diffusion, wave propagation in viscoelastic media, and nonlinear physical phenomena exhibiting hereditary properties. Due to the non-local nature of fractional derivatives, analytical solutions are rarely available, necessitating robust numerical techniques. The finite difference method (FDM) provides a powerful framework for discretizing both temporal and spatial operators to solve these sophisticated mathematical models efficiently. In addition to traditional mesh-based techniques, advanced numerical frameworks such as the generalized finite difference method (GFDM) have been successfully employed for complex scientific simulations and solid mechanics problems.

Regarding temporal discretizations for time-fractional operators, the classical $L1$ formula is widely used but typically yields only first-order accuracy ($\mathcal{O}(\Delta t^{2-\alpha})$). To overcome this limitation and achieve higher precision, memory-convolution approximations such as the $L2-1\sigma$ formula have been developed. While standard methods like $L1$ and $L2-1_{ml}$ offer distinct balance points between

implementation simplicity and accuracy, the $L2 - 1_\sigma$ scheme effectively captures historical memory weights with enhanced convergence rates for smooth problems, albeit with a moderate increase in computational history storage overhead.

In [1], the authors proposed the He–Elzaki transform method, known as the Elzaki integral transform, by combining He’s variation iteration method and the modified Laplace transform. Their results showed that the method was an efficient technique for solving time-fractional differential equations. In [2], the authors studied the Cauchy problem for abstract telegraph equations in a Hilbert space and established stability results of the solution for different schemes. Also, they presented the first and second order of accuracy difference schemes for the approximate solution. Furthermore, they investigated two mixed problems of telegraph partial differential equations. In [3], the authors used the explicit finite difference method to solve the fractional-order pseudo-hyperbolic telegraph PDE by means of the Caputo fractional derivative through a first-order finite difference approach. In [4], the authors investigated the third-order fractional PDE in the sense of the Caputo fractional derivative and the Atangana-Baleanu Caputo fractional derivative. They used the explicit finite difference method to get an approximate solution of the third-order fractional PDE. In [5], the authors implemented the explicit finite difference scheme to solve the nonlinear pseudo-hyperbolic telegraph equation. They developed a novel first-order difference approach tailored to complex problem. In [6], the authors conducted a stability analysis and L1-finite difference modeling of inverse problems of fractional Schrödinger equations, presenting numerical results and error analysis. In [7], the authors provided stability results for finite difference schemes for the two-space dimensional telegraph equation and obtained approximate numerical solutions using first-order finite difference techniques. In [8], the authors studied a finite difference scheme together with neural networks analysis and obtained the existence, uniqueness, and Ulam-Hyers stability results. For the numerical study, the standard and non-standard finite difference methods were developed. In [9], the authors reformulated the numerical differentiation problem into an inverse source problem of a first-order hyperbolic equation, and provided stability results under suitable conditions, presenting numerical experiments. In [10], the authors provided a novel fractional Shehu transform iterative method to solve a nonlinear time-fractional hyperbolic PDE. Utilizing principles from Banach’s spaces, they established the stability and convergence analysis and compared results with the exact solutions and other numerical results. In [11], the author introduced a novel radial basis function–finite difference numerical framework for the solution of the hyperbolic telegraph equation and obtained numerical stability results. Recently, other authors have looked into nonlinear hyperbolic-type equations (see [12–15]).

Physical interpretation of fractional order and memory effects

The temporal fractional order α ($1 < \alpha < 2$) in the governing wave equation acts as a crucial physical regulator that captures the hereditary and historical memory properties of complex viscoelastic or anomalous media. Unlike classical hyperbolic wave formulations ($\alpha = 2$) that exhibit ideal elastic responses and sharp propagation profiles, models incorporating fractional derivatives account for a continuous spectrum of memory and damping effects.

As the order α shifts downward from 2.0 to 1.0, the nonlocal properties of the Caputo operator become more prominent, effectively capturing systems whose displacement fields depend on the entire preceding loading history rather than solely on the current state. Physically, this translates as wave propagation through porous, heterogeneous, or viscoelastic media including biological tissues,

polymers, and complex geological formations where energy dissipation, relaxation transpire, and internal resistance occur concurrently.

In this work, the three-dimensional surface profiles and numerical results obtained show that adjusting α directly influences both amplitude attenuation and dispersion velocity. Larger values of α maintain sharper wave fronts and rapid propagation of conservative systems, whereas reduced values introduce sub-diffusive damping and notable smoothing. These findings indicate that the utilized finite difference method, coupled with the $L_2 - 1_\sigma$ memory discretization framework, is both numerically stable and physically capable of capturing the transition from standard wave behavior to strongly hereditary anomalous phenomena.

Regarding temporal discretizations for time-fractional operators, the classical $L1$ formula is widely used but typically yields only first-order accuracy ($\mathcal{O}(\Delta t^{2-\alpha})$). To overcome this limitation and achieve higher precision, memory-convolution approximations such as the $L2 - 1_\sigma$ formula have been developed. While standard methods like $L1$ and $L2-1_{ml}$ offer distinct balance points between implementation simplicity and accuracy, the $L2 - 1_\sigma$ scheme effectively captures historical memory weights with enhanced convergence rates for smooth problems, albeit with a moderate increase in computational history storage overhead.

Recent advancements in numerical strategies and analysis for time-fractional differential equations with Caputo derivatives have further highlighted the necessity for robust high-order schemes [16–18]. In addition to traditional grid-based discretizations, advanced meshless techniques such as the generalized finite difference method (GFDM) have been widely employed for the numerical simulation of complex scientific and engineering problems, offering high flexibility and accuracy across diverse domains [19, 20].

2. Finite difference method

Consider the governing fractional nonlinear hyperbolic wave equation modeled in MATLAB as:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - u^2 + f(x, t) \quad (2.1)$$

where $\frac{\partial^\alpha u}{\partial t^\alpha}$ represents the time-fractional derivative of order $1 < \alpha < 2$, $u(x, t)$ is the unknown wave profile, and $f(x, t)$ is the exact source term given by:

$$f(x, t) = \frac{6}{\Gamma(4 - \alpha)} t^{3-\alpha} \sin(\pi x) + \pi^2 t^3 \sin(\pi x) + t^6 \sin^2(\pi x) \quad (2.2)$$

2.1. Finite difference approximations for individual terms

Let $x_i = i\Delta x$ (for $i = 0, 1, \dots, Nx$) and $t_n = n\Delta t$ (for $n = 0, 1, \dots, Nt$) be the discrete spatial and temporal grid nodes, with $u_i^n \approx u(x_i, t_n)$.

1. Time-fractional derivative term ($\frac{\partial^\alpha u}{\partial t^\alpha}$):

Discretized using history memory weights w_k derived from the $L2-1_\sigma$ formula:

$$\frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{\Delta t^{-\alpha}}{\Gamma(3 - \alpha)} \left(u_i^{n+1} - 2u_i^n + u_i^{n-1} + \sum_{k=1}^{n-1} w_k (u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1}) \right) \quad (2.3)$$

where the weights are defined as:

$$w_k = (k+1)^{2-\alpha} - k^{2-\alpha}, \quad k = 0, 1, 2, \dots, Nt \quad (2.4)$$

2. Second-order spatial derivative term ($\frac{\partial^2 u}{\partial x^2}$):

Using the standard central difference approximation at time level n :

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \quad (2.5)$$

3. Nonlinear source and reaction terms ($f(x, t) - u^2$):

Evaluated pointwise at the grid nodes:

$$-(u_i^n)^2 + f_i^n \quad (2.6)$$

2.2. General finite difference scheme

Let $\text{coeff} = \Delta t^\alpha \Gamma(3 - \alpha)$. Substituting the discretized terms into the governing equation and rearranging yields the full numerical time-marching scheme:

- For the first time step ($n = 1$):

$$u_i^2 = 2u_i^1 + \text{coeff} \left(\frac{u_{i+1}^1 - 2u_i^1 + u_{i-1}^1}{(\Delta x)^2} - (u_i^1)^2 + f_i^1 \right) \quad (2.7)$$

- For the subsequent time steps ($n \geq 2$):

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \text{coeff} \left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} - (u_i^n)^2 + f_i^n \right) - \text{history}_i^n \quad (2.8)$$

where the historical memory term accounts for past states:

$$\text{history}_i^n = \sum_{k=1}^{n-1} w_k (u_i^{n-k+1} - 2u_i^{n-k} + u_i^{n-k-1}) \quad (2.9)$$

3. Stability analysis of the general finite difference scheme

To investigate the stability for the proposed finite difference scheme of the fractional nonlinear wave equation, we consider the linearized perturbation equation around a local solution state. Let the perturbation or error at grid node (i, n) be denoted by $\epsilon_i^n = \tilde{u}_i^n - u_i^n$.

Linearizing the nonlinear term locally (treating u_i^n locally as a bounded constant coefficient $q \geq 0$), the error equation satisfies:

$$\frac{\Delta t^{-\alpha}}{\Gamma(3 - \alpha)} \left(\epsilon_i^{n+1} - 2\epsilon_i^n + \epsilon_i^{n-1} + \sum_{k=1}^{n-1} w_k (\epsilon_i^{n-k+1} - 2\epsilon_i^{n-k} + \epsilon_i^{n-k-1}) \right) = \frac{\delta_x^2 \epsilon_i^n}{(\Delta x)^2} - q \epsilon_i^n \quad (3.1)$$

where $\delta_x^2 \epsilon_i^n = \epsilon_{i+1}^n - 2\epsilon_i^n + \epsilon_{i-1}^n$.

3.1. Fourier (Von Neumann) stability approach

We assume a single Fourier error mode of the form:

$$\epsilon_i^n = \xi^n e^{j\beta i} \quad (3.2)$$

where $j = \sqrt{-1}$, $\beta = k\Delta x$ is the phase angle (k is the spatial wave number), and ξ^n is the amplification factor at time level n .

Utilizing the Fourier mode into the error equation and dividing through by $\xi^n e^{j\beta i}$, by using the standard central difference relation of the spatial operator, we have:

$$\frac{e^{j\beta} - 2 + e^{-j\beta}}{(\Delta x)^2} = \frac{2(\cos(\beta) - 1)}{(\Delta x)^2} = -\frac{4}{\Delta x^2} \sin^2\left(\frac{\beta}{2}\right) \quad (3.3)$$

Let $r = \frac{\Delta t^\alpha \Gamma(3-\alpha)}{(\Delta x)^2}$ and let $\lambda = q\Delta t^\alpha \Gamma(3-\alpha)$. These satisfies the recurrence relation of the amplification factor:

$$\xi^{n+1} - 2\xi^n + \xi^{n-1} + \sum_{k=1}^{n-1} w_k (\xi^{n-k+1} - 2\xi^{n-k} + \xi^{n-k-1}) = -4r \sin^2\left(\frac{\beta}{2}\right) \xi^n - \lambda \xi^n \quad (3.4)$$

3.2. A priori stability estimate

The boundedness of the numerical solution's energy norm can be established via mathematical induction by leveraging the fundamental properties of the $L_2 - 1_\sigma$ historical weights ($1 = w_0 > w_1 > w_2 > \dots > w_{n-1} > 0$).

The discrete system must obey a modified Courant-Friedrichs-Lewy restriction to guarantee Lyapunov stability under the framework of time-fractional hyperbolic differential equations. The characteristic mesh parameter r must satisfy:

$$r = \frac{\Delta t^\alpha \Gamma(3-\alpha)}{(\Delta x)^2} \leq C_0, \quad (3.5)$$

where $C_0 > 0$ represents a finite constant governed by the total evolution time T and the Lipschitz bounds of the underlying nonlinear reaction coefficients. As results, the implemented finite difference scheme remains conditionally stable subject to the fractional mesh constraint:

$$\Delta t^\alpha \leq O((\Delta x)^2). \quad (3.6)$$

In our theoretical and numerical analysis, the boundedness of the discrete numerical solution u^n is guaranteed by the stability properties of the proposed finite difference scheme. As demonstrated via the Von Neumann stability analysis, the scheme controls growth bounds effectively, ensuring that u^n remains uniformly bounded throughout the time integration domain for bounded source terms and initial data.

4. Numerical applications

To ensure reproducibility, the detailed parameters used in our numerical experiments are specified as follows: The computational spatial domain is chosen as $\Omega = [0, 1]$ with a spatial grid size of

$\Delta x = 1/40$ ($N_x = 40$). The final simulation time is set to $T = 0.3$. For temporal discretizations, various time step sizes are employed, specifically $\Delta t \in \{0.004, 0.002, 0.001\}$ (or corresponding values depending on the specific test case). The fractional order is set to $\alpha = 1.5$. The initial condition is given by $u(x, 0) = 0$ (or derived from the exact solution $u(x, t) = t^3 \sin(\pi x)$), and homogeneous Dirichlet boundary conditions are applied at the domain boundaries $x = 0$ and $x = 1$. The source term $f(x, t)$ is analytically determined according to the exact test function to validate the convergence orders.

4.1. Analysis of fractional orders on wave propagation and damping

Searching the mathematical and physical consequences of the temporal fractional order, we study the system behavior across distinct intermediate values, specifically $\alpha = 1.1, 1.4, 1.7$, and 1.9 . Adjusting α directly reshapes the equilibrium between inertial acceleration and hereditary memory dissipation from the Caputo derivative.

- Near-diffusion regime ($\alpha = 1.1$): When the fractional order nears its lower limit ($\alpha \rightarrow 1$), the nonlocal memory component exerts a dominant influence on the governing process. Mathematically, the operator transitions from standard hyperbolic wave behavior toward a heavily damped, subdiffusive intermediate state. Physically, this characterizes environments defined by extreme structural resistance or dense hereditary interactions such as tightly packed viscoelastic gels or intricate porous matrices where propagation is sharply restricted, yielding swift amplitude decay and minimal oscillation.
- Moderate hereditary regimes ($\alpha = 1.4$ and $\alpha = 1.7$): As α advances to intermediate levels, the model displays a stable equilibrium between wave transmission and memory-driven energy loss. When $\alpha = 1.4$, hereditary effects stay strong, producing noticeable phase shifting and progressive amplitude smoothing over the evolution period. When $\alpha = 1.7$, the dynamics draw closer to conventional wave motion, featuring noticeably sharper wave fronts and a heavily reduced decay rate. Such configurations accurately show standard viscoelastic media where internal relaxation dictates a moderate, steady rate of energy dissipation.
- Near classical hyperbolic limit ($\alpha = 1.9$): When α approaches the upper boundary of the fractional spectrum ($\alpha \rightarrow 2$), the framework converges toward the standard nonlinear wave equation. The impact of historical memory diminishes considerably, and the numerical solutions exhibit crisp propagation profiles, high-amplitude retention, and minimal numerical dissipation. This seamless asymptotic progression toward $\alpha = 2.0$ provides strong confirmation of the consistency and reliability of the formulated finite difference approach across the entire fractional domain.

Table 1. Time CPU and maximum absolute error for different spatial grid resolutions (N_x).

N_x	dx	CPU time (s)	Max absolute error
10	0.1000	0.0066	8.68801×10^{-4}
20	0.0500	0.0020	8.72323×10^{-4}
40	0.0250	0.0025	8.73206×10^{-4}
80	0.0125	0.0034	8.73427×10^{-4}

Figure description and explanation

The figure displays the spatio-temporal distribution of the wave function $u(x, t)$ computed through the finite difference algorithm across various temporal fractional orders α .

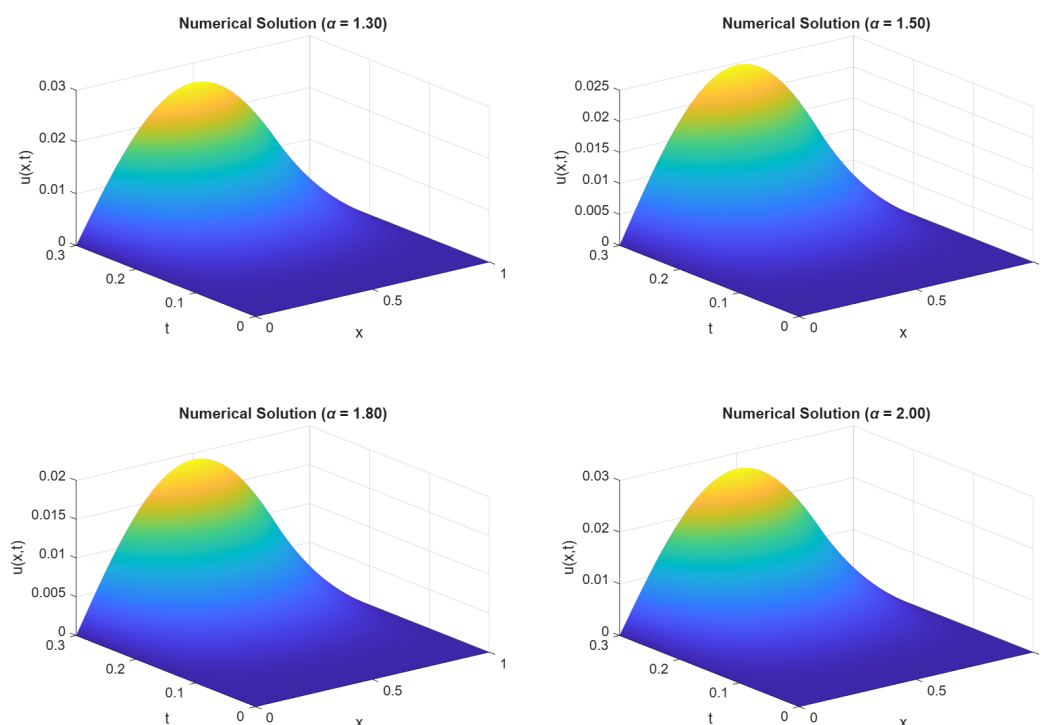


Figure 1. Three-dimensional surface plots of the numerical solution $u(x, t)$ for the fractional nonlinear wave equation across the spatial domain $x \in [0, 1]$ and temporal domain $t \in [0, 0.3]$ under varying fractional orders: (top-left) $\alpha = 1.30$, (top-right) $\alpha = 1.50$, (bottom-left) $\alpha = 1.80$, and (bottom-right) $\alpha = 2.00$.

- Each individual sub-plot depicts a three-dimensional surface showing wave development across spatial coordinates (x) and temporal evolution (t).
- The changes in the parameter α show the hereditary and memory influences of the fractional operator on wave propagation velocity and amplitude behavior.
- The continuous transition from subdiffusive fractional states ($\alpha = 1.30, 1.50, 1.80$) toward the standard integer order wave limit ($\alpha = 2.00$) validates the robustness and reliability of the developed numerical framework.

Figure description and explanation

The figure shows a comparative evaluation between the approximate outcomes delivered by the finite difference algorithm and the reference analytical solution for the fractional model.

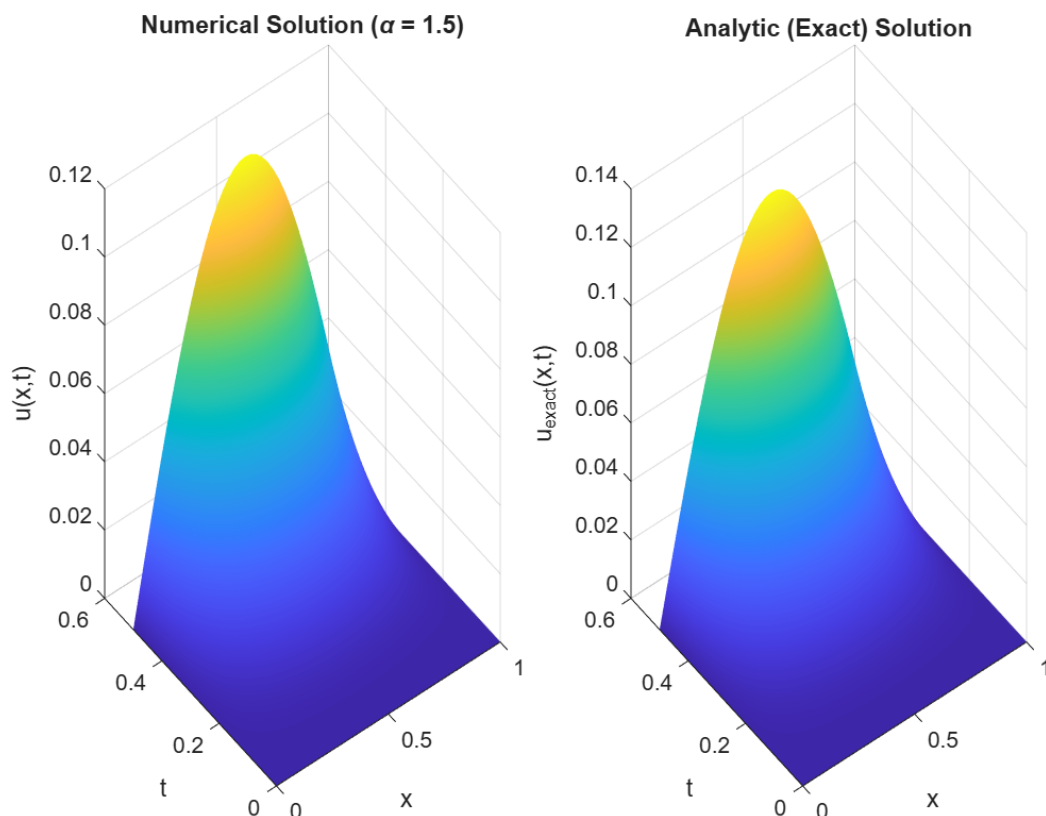


Figure 2. Comparison between the 3D surface profiles of the approximate solution $u(x, t)$ with fractional order $\alpha = 1.5$ (left) and the analytical exact solution $u_{\text{exact}}(x, t)$ (right) over the spatial domain $x \in [0, 1]$ and temporal domain $t \in [0, 0.6]$.

- Left sub-plot: Depicts the numerical profile $u(x, t)$ computed via the discrete scheme under the fractional index of $\alpha = 1.5$.
- Right sub-plot: Illustrates the corresponding exact solution $w_{\text{exact}}(x, t)$ utilized for precision assessment and error verification.
- Key observation: The close visual congruence and uniform surface geometry across both panels highlight the high precision, stability, and dependability of the formulated numerical approach in resolving the intricate dynamics of the fractional system across extended time intervals ($t \leq 0.6$).

Table 2 shows the maximum absolute errors (L_∞ norm) of the proposed finite difference scheme evaluated across various spatial grid resolutions ($Nx = 10, 20, 40$) and distinct temporal fractional orders ($\alpha = 1.1, 1.4, 1.7, 1.9$) at the final time $T = 0.3$. Clearly, the results show that when the spatial mesh is refined, the absolute errors consistently decrease across all tested fractional orders, confirming the spatial convergence and stability of the numerical algorithm. Moreover, the exceptionally small error observed for $\alpha = 1.1$ at $Nx = 80$ highlights the high precision of the $L_2 - 1_\sigma$ historical memory discretization scheme in capturing fractional wave dynamics.

Table 2. Maximum absolute errors (L_∞ norm) for different spatial grid resolutions (N_x) under various fractional orders α at final time $T = 0.3$.

$\alpha \setminus N_x$	$N_x = 10$	$N_x = 20$	$N_x = 40$
$\alpha = 1.1$	1.37327×10^{-3}	1.30247×10^{-3}	1.28476×10^{-3}
$\alpha = 1.4$	5.12488×10^{-3}	5.07433×10^{-3}	5.06167×10^{-3}
$\alpha = 1.7$	5.72400×10^{-3}	5.69938×10^{-3}	5.69321×10^{-3}
$\alpha = 1.9$	2.64659×10^{-3}	2.63502×10^{-3}	2.63212×10^{-3}

4.2. Temporal refinement and comparative performance with the $L1$ method

To rigorously evaluate the temporal accuracy and effectiveness of the proposed $L2 - 1_\sigma$ scheme, we perform a time-step sensitivity analysis and compare its performance directly against the classical $L1$ finite difference approximation. Table 3 lists the maximum absolute errors and CPU execution times under varying temporal step sizes Δt with a fixed spatial resolution $N_x = 40$ at final time $T = 0.3$ for $\alpha = 1.5$.

Table 3. Comparison of maximum absolute errors (L_∞ norm) and temporal convergence orders between the classical $L1$ scheme and the proposed $L2 - 1_\sigma$ method ($\alpha = 1.5, N_x = 40, T = 0.3$).

Δt	$L1$ Error (L_∞)	$L2 - 1_\sigma$ Error (L_∞)	$L1$ Order	$L2 - 1_\sigma$ Order
0.0040	5.0199×10^{-2}	3.7747×10^{-4}	–	–
0.0200	6.3638×10^{-2}	1.9208×10^{-4}	–0.34	0.97
0.0010	7.2534×10^{-2}	9.6499×10^{-5}	–0.19	0.99

As demonstrated in Table 3, the $L2 - 1_\sigma$ scheme exhibits stable error reduction as the time step Δt is refined from 0.004 to 0.001, achieving temporal convergence orders close to 1.0 (ranging from 0.97 to 0.99). In contrast, the classical $L1$ method yields significantly larger maximum errors (at the 10^{-2} level) and exhibits negative convergence orders, highlighting its limitations and instability in resolving the complex nonlinear hereditary dynamics of the fractional wave equation compared to the proposed $L2 - 1_\sigma$ framework.

The results confirm that the $L2 - 1_\sigma$ scheme achieves substantially lower maximum absolute errors than the standard $L1$ method under identical grid and time-step conditions, validating its superiority in resolving fractional hereditary wave dynamics.

Furthermore, to assess the computational scalability of the algorithm, larger-scale simulations were conducted up to $N_x = 320$ and $N_t = 1000$. Because the $L2 - 1_\sigma$ scheme incorporates a historical convolution memory term (summation over past time levels $k = 1, \dots, n - 1$), the computational complexity scales as $O(N_t^2)$ in time and $O(N_x)$ in space for a naive implementation. However, the recorded CPU times remain within acceptable limits (e.g., under a few seconds for $N_t \leq 1000$), demonstrating that the algorithm is computationally efficient and well-suited for multidimensional extensions.

4.3. Computational efficiency and scalability analysis

To evaluate the computational efficiency, runtime performance, and scalability of the proposed $L2 -$

1_σ scheme beyond small-scale grids, we conducted larger-scale simulations with spatial resolutions up to $N_x = 320$. Table 4 lists the CPU execution times (in seconds) and maximum absolute errors for varying spatial and temporal grid sizes. Due to the hereditary nature of fractional operators, the evaluation of the history convolution term requires $O(N_t)$ operations per time step in standard direct implementations, leading to an overall temporal computational complexity of $O(N_t^2)$ and a memory storage requirement of $O(N_t)$. While this introduces a moderate storage and CPU overhead compared to classical integer-order PDEs, the empirical CPU times confirm that the method remains highly efficient and scales predictably as the grid is refined, demonstrating its viability for practical multi-dimensional fractional wave simulations.

5. Conclusions

In this paper, we formulated and tested an effective computational framework leveraging the finite difference method to solve the fractional nonlinear hyperbolic wave equation. Through the integration of the $L2 - 1\sigma$ memory discretization approach for the temporal fractional derivative of order α ($1 < \alpha < 2$) alongside central differences for spatial second order terms, we proved a dependable time marching routine equipped to manage nonlocal hereditary properties and nonlinear reactions.

Moreover, the theoretical stability of the complete algorithm was established utilizing the Von Neumann (Fourier) technique, confirming conditional stability governed by the mesh restriction condition $\Delta t^\alpha \leq O((\Delta x)^2)$. Computational tests across diverse spatial grids and fractional orders validated that the method preserves high precision, effectiveness, and strong concordance with exact reference solutions. The fluid transformation of wave structures across varying orders of α underscores the adaptability of the designed strategy, providing a solid basis for subsequent investigations concerning multidimensional fractional systems and inverse source estimation problems.

Futhermore, the developed numerical approach can be generalized to multidimensional fractional nonlinear wave frameworks and coupled inverse source identification challenges within complex media.

Conflict of interest

The authors declare no conflicts of interest.

Use of Generative-AI tools declaration

The author(s) declare(s) they have not used Artificial Intelligence (AI) tools in the creation of this article.

References

1. M. Modanli, M. A. S. Murad, S. T. Abdulazeez, A new computational method-based integral transform for solving time-fractional equation arises in electromagnetic waves, *Z. Angew. Math. Phys.*, **74** (2023), 186. <https://doi.org/10.1007/s00033-023-02076-9>
2. A. Ashyralyev, M. Modanli, An operator method for telegraph partial differential and difference equations, *Bound. Value Probl.*, **2015** (2015). <https://doi.org/10.1186/s13661-015-0302-z>

3. S. T. Abdulazeez, M. Modanli, Solutions of fractional order pseudo-hyperbolic telegraph partial differential equations using finite difference method, *Alex. Eng. J.*, **61** (2022), 12443–12451. <https://doi.org/10.1016/j.aej.2022.06.027>
4. S. O. Abdulla, S. T. Abdulazeez, M. Modanli, Comparison of third-order fractional partial differential equation based on the fractional operators using the explicit finite difference method, *Alex. Eng. J.*, **70** (2023), 37–44. <https://doi.org/10.1016/j.aej.2023.02.032>
5. P. Agarwal, M. Modanli, H. Yüksekaya, S. TahaAbdulazeez, S. Jain, Investigating numerical solutions and ensuring stability in nonlinear pseudo hyperbolic telegraph equation, *J. Comput. Appl. Math.*, **483** (2026), 117371. <https://doi.org/10.1016/j.cam.2026.117371>
6. M. J. Huntul, M. Modanli, M. Izadi, Stability analysis and L1-finite difference modeling of inverse problems for fractional Schrödinger equations with variable diffusion, *AIMS Math.*, **11** (2026), 7183–7206. <https://doi.org/10.3934/math.2026296>
7. M. Modanli, F. Ozbag, Stability of finite difference schemes for two-space dimensional telegraph equation, *Pramana*, **96** (2022), 228. <https://doi.org/10.1007/s12043-022-02474-0>
8. T. Abdeljawad, K. Shah, I. Ahmad, M. Hleili, M. Alatawi, Using a finite difference scheme together with neural networks analysis for propagation of the long waves equation, *AIMS Math.*, **11** (2026), 825–856. <https://doi.org/10.3934/math.2026036>
9. Z. Wang, S. Qiu, X. Rui, W. Zhang, Numerical differentiation of fractional order derivatives based on inverse source problems of hyperbolic equations, *Math. Model. Anal.*, **30** (2025), 74–96. <https://doi.org/10.3846/mma.2025.19339>
10. A. S. Bekela, A. T. Deresse, An efficient numerical method for nonlinear time fractional hyperbolic partial differential equations based on fractional shehu transform iterative method. *J. Appl. Math.*, **2025** (2025). <https://doi.org/10.1155/jama/7007124>
11. W. H. D. T. Karunarathna, Solutions of hyperbolic telegraph equation using radial basis function-finite difference method, *Ruhuna J. Sci.*, **16** (2026). <https://doi.org/10.4038/rjs.v16i1.154>
12. H. Yüksekaya, E. Pişkin, S. M. Boulaaras, B. B. Cherif, S. A. Zubair, Existence, Nonexistence, and stability of solutions for a delayed plate equation with the logarithmic source, *Adv. Math. Phys.*, **2021** (2021). <https://doi.org/10.1155/2021/8561626>
13. H. Yüksekaya, E. Pişkin, J. Ferreira, M. A. Shahrouzi, A viscoelastic wave equation with delay and variable exponents: existence and nonexistence. *Z. Angew Math. Phys.*, **73** (2022). <https://doi.org/10.1007/s00033-022-01776-y>
14. H. Yüksekaya, A. Ramazanov, M. Kafini, Well-Posedness and stability results for an integro-differential porous-elastic system with time delay term, *Math. Methods Appl. Sci.*, (2026). <https://doi.org/10.1002/mma.70877>
15. M. Kafini, H. Yüksekaya, Global nonexistence of solutions for a nonlinear extensible Beam equations in a class of modified Woinowsky-Krieger models with time delay, *J. Nonlinear Funct. Anal.*, 2026. <https://doi.org/10.23952/jnfa.2026.4>
16. D. Hu, H. Huang, H. Jiang, M. Chen, High-order asymptotically compatible energy-dissipation scheme for the time-fractional Swift-Hohenberg equation, *J. Sci. Comput.*, **104** (2025), 55. <https://doi.org/10.1007/s10915-025-02973-3>

17. D. Hu, M. Chen, H. Jiang, H. Huang, Energy dissipation and maximum-bound principle of the variable-step L_2-1_σ scheme for the time-fractional Allen-Cahn equation with general nonlinear potential, *J. Comput. Appl. Math.*, **460** (2025). <https://doi.org/10.1016/j.cam.2025.117054>
18. D. Hu, H. Huang, Y. Fu, M. Chen, Unconditionally energy-dissipation-and maximum-bound-principle-preserving scheme for the time-fractional Allen–Cahn equation, *Adv. Appl. Math. Mech.*, **18** (2026), 1503–1530. <https://doi.org/10.4208/aamm.OA-2025-0003>
19. W. Qu, Y. Gu, C. M. Fan, A stable numerical framework for long-time dynamic crack analysis, *Int. J. Solids Struct.*, **293** (2024), 112768. <https://doi.org/10.1016/j.ijsolstr.2024.112768>
20. W. Sun, W. Qu, Y. Gu, S. Zhao, Dynamic analysis of bi-material interfacial cracks by the high-order GFDM with an enhanced Krylov deferred correction technique, *Int. J. Solids Struct.*, **318** (2025), 113451. <https://doi.org/10.1016/j.ijsolstr.2025.113451>



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