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*Research article*

## On coefficient problems for bounded turning functions associated with a strip domain

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**Abstract:** In the paper, we investigate the upper bounds of the second and third Hankel determinants for bounded turning functions linked with a strip domain, whose results are sharp. We also obtain the sharp upper bounds of the second-order Hankel determinants of logarithmic coefficients. Finally, we provide the sharp upper bounds of the third-order Hankel determinant of inverse functions, as well as the sharp upper bounds of the second-order Hankel determinants of inverse logarithmic coefficients. All the other results are accurate, except for the estimation of the fourth logarithmic coefficient.

**Keywords:** univalent functions; Hankel determinants; bounded turning functions; logarithmic coefficients; inverse functions

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### 1. Introduction

The class of analytic functions in open unit disk  $\mathbb{D} = \{v \in \mathbb{C} : |v| < 1\}$  by  $\mathcal{A}$ , can be typically expressed as

$$F(v) = v + \sum_{m=2}^{\infty} \mu_m v^m \quad (v \in \mathbb{D}), \quad (1.1)$$

with the normalization conditions  $F(0) = F'(0) - 1 = 0$ . A univalent function  $F \in \mathcal{A}$  belongs to the distinguished class  $\mathcal{S}$ .

The Hankel determinant of  $F \in \mathcal{A}$ , the function for  $p, m \in \mathbb{N}$ , denoted by  $\mathcal{H}_{p,m}(F)$ , is defined as

follows:

$$\mathcal{H}_{p,m}(F) = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+p-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+p} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m+p-1} & \mu_{m+p} & \cdots & \mu_{m+2p-2} \end{vmatrix}.$$

For  $m = 1$  and  $p = 2$ , we obtain

$$\mathcal{H}_{2,1}(F) = \mu_3 - \mu_2^2. \quad (1.2)$$

The value of  $\mathcal{H}_{2,1}(F)$  is a special case of  $\mu_3 - \lambda\mu_2^2$  for  $\lambda = 1$ . This case is known as the Fekete-Szegő functional. For specific values of  $m$  and  $p$ , the second- and third-order Hankel determinants  $\mathcal{H}_{2,2}(F)$  and  $\mathcal{H}_{3,1}(F)$  are given by:

$$\mathcal{H}_{2,2}(F) = \mu_2\mu_4 - \mu_3^2 \quad (1.3)$$

and

$$\mathcal{H}_{3,1}(F) = -\mu_3^3 + 2\mu_4\mu_3\mu_2 - \mu_5\mu_2^2 + \mu_5\mu_3 - \mu_4^2. \quad (1.4)$$

Determining the upper bound of  $\mathcal{H}_{p,m}(F)$  for various subclasses of  $\mathcal{A}$  is a fascinating and well-studied problem in the field of geometric function theory. Several authors [1–3] have studied the determinant  $\mathcal{H}_{2,2}(F)$  for different subclasses of holomorphic functions. Babalola [4] studied the Hankel determinant  $\mathcal{H}_{3,1}(F)$  for some subclasses of analytic functions. Later, several researchers obtained the sharp bound of the third Hankel determinant for different subclasses of analytic functions. For example, Zaprawa et al. [5] obtained the third Hankel determinants for non-Bazilevič functions, Nandhini and Sruthakeerthi [6] investigated the third Hankel determinant for the bounded turning functions associated with a bean shaped domain, and Khan et al. [7] obtained the sharp bound of the third Hankel determinant bounded turning functions in the domain of cosine Hyperbolic function. Additionally, upper bounds for the third Hankel determinant for various subclasses of  $\mathcal{A}$  were obtained by Raza et al. [8], Shi and Arif [9], Srivastava et al. [10], and Rath et al. [11].

For  $F \in \mathcal{S}$ , the logarithmic coefficients  $\tau_m$  are defined by

$$\log \frac{F(v)}{v} = 2 \sum_{m=1}^{\infty} \tau_m v^m, \quad (v \in \mathbb{D}). \quad (1.5)$$

For a function  $F \in \mathcal{S}$ , as defined in Eq (1.1), differentiating Eq (1.5) allows the logarithmic coefficients to be determined:

$$\begin{cases} \tau_1 = \frac{1}{2}\mu_2, \\ \tau_2 = \frac{1}{2}(\mu_3 - \frac{1}{2}\mu_2^2), \\ \tau_3 = \frac{1}{2}(\mu_4 - \mu_2\mu_3 + \frac{1}{3}\mu_2^3), \\ \tau_4 = \frac{1}{2}(\mu_5 - \frac{1}{2}\mu_3^2 + \mu_3\mu_2^2 - \mu_4\mu_2 - \frac{1}{4}\mu_2^4). \end{cases} \quad (1.6)$$

Thus, we have

$$\tau_3\tau_1 - \tau_2^2 = \frac{1}{48}(-12\mu_3^2 + 12\mu_2\mu_4 + \mu_2^4) \quad (1.7)$$

and

$$\begin{aligned} \tau_2\tau_4 - \tau_3^2 = & \frac{1}{288}(72\mu_3\mu_5 - 6\mu_2^4\mu_3 + 72\mu_2\mu_3\mu_4 + 18\mu_2^2\mu_3^2 - 36\mu_3^3 \\ & - 36\mu_2^2\mu_5 - 72\mu_4^2 - 12\mu_2^3\mu_4 + \mu_2^6). \end{aligned} \quad (1.8)$$

Let  $G$  be the inverse function of  $F \in \mathcal{S}$  defined in a neighborhood of the origin with the Taylor series expansion

$$G(w) = F^{-1}(w) = w + \sum_{m=2}^{\infty} D_m w^m, \quad (1.9)$$

where we may choose  $|w| < 1/4$  from Koebe's  $1/4$ -theorem. Because

$$F(F^{-1}(w)) = w,$$

it follows from Eq (1.9) that

$$\begin{cases} D_2 = -\mu_2, \\ D_3 = -\mu_3 + 2\mu_2^2, \\ D_4 = -\mu_4 + 5\mu_2\mu_3 - 5\mu_2^3, \\ D_5 = -\mu_5 + 3\mu_3^2 - 21\mu_3\mu_2^2 + 6\mu_4\mu_2 + 14\mu_2^4. \end{cases} \quad (1.10)$$

From (1.4) and (1.10), we have

$$\begin{aligned} \mathcal{H}_{3,1}(F^{-1}) &= 2D_4D_3D_2 - D_3^3 + D_5D_3 - D_5D_2^2 - D_4^2 \\ &= -3\mu_2^4\mu_3 + \mu_2^6 + 3\mu_2^2\mu_3^2 + 2\mu_2\mu_3\mu_4 - 2\mu_3^3 - \mu_4^2 - \mu_2^2\mu_5 + \mu_3\mu_5. \end{aligned} \quad (1.11)$$

The notation of the logarithmic coefficient of inverse of  $F$  was introduced by Ponnusamy et al. [12]. As with  $F$ , the logarithmic inverse coefficients  $\Gamma_m := \Gamma_m(G)$ ,  $m \in \mathbb{N}$ , of  $G$  are defined by the equation

$$\log \frac{G(w)}{w} = 2 \sum_{m=1}^{\infty} \Gamma_m(G) w^m \quad (|w| < \frac{1}{4}). \quad (1.12)$$

Differentiating and using (1.10), we obtain

$$\begin{cases} \Gamma_1 = -\frac{1}{2}\mu_2, \\ \Gamma_2 = -\frac{1}{4}(2\mu_3 - 3\mu_2^2), \\ \Gamma_3 = -\frac{1}{6}(3\mu_4 - 12\mu_2\mu_3 + 10\mu_2^3), \\ \Gamma_4 = -\frac{1}{8}(4\mu_5 - 20\mu_4\mu_2 + 60\mu_3\mu_2^2 - 10\mu_3^2 - 35\mu_2^4). \end{cases}$$

Therefore,

$$\Gamma_1\Gamma_3 - \Gamma_2^2 = \frac{1}{48}(12\mu_2\mu_4 + 13\mu_2^4 - 12\mu_2^2\mu_3 - 12\mu_3^2) \quad (1.13)$$

and

$$\Gamma_2\Gamma_4 - \Gamma_3^2 = \frac{1}{288}(-72\mu_4^2 + 216\mu_2\mu_3\mu_4 + 72\mu_3\mu_5 - 180\mu_3^3 + 198\mu_2^2\mu_3^2 - 330\mu_2^4\mu_3 + 60\mu_2^3\mu_4 - 108\mu_2^2\mu_5 + 145\mu_2^6).$$
(1.14)

Kumar and Verma [13] introduced a set of starlike functions connected to a strip domain that is given by

$$\mathcal{S}_\tau^* = \left\{ F : F \in \mathcal{A} \quad \text{and} \quad \frac{\nu F'(\nu)}{F(\nu)} < \Phi(\nu) = 1 + \arctan \nu \quad (\nu \in \mathbb{D}) \right\},$$

where  $\Phi(\nu) = 1 + \arctan \nu$  is univalent in  $\mathbb{D}$ . We similarly introduce a class of bounded turning functions linked with a strip domain, which is given by

$$\mathcal{R}_\tau = \left\{ F : F \in \mathcal{A} \quad \text{and} \quad F'(\nu) < \Phi(\nu) = 1 + \arctan \nu \quad (\nu \in \mathbb{D}) \right\}.$$
(1.15)

Kumar and Verma [13] investigated the bounds of the initial coefficients  $\mu_m$ ,  $m = 1$  to 4 and established the following:

$$|\mu_m| \leq \frac{1}{m-1} (m = 2, 3, 4), \quad |\mathcal{H}_{2,2}(F)| \leq \frac{1}{4}, \quad |\mathcal{H}_{3,1}(F)| \leq \frac{1}{9}.$$

In this paper, for bounded turning functions linked with a strip domain, we obtain the upper bounds of the second- and third-order Hankel determinants in the second part. In the third to fifth parts, we investigate the second-order Hankel determinants of logarithmic coefficients, the Hankel determinants of inverse functions, and the second-order Hankel determinants of inverse logarithmic coefficients, the upper bounds of which are also sharp. The novelty lies in the systematic treatment of both direct and inverse coefficient problems for this class, especially the sharp inverse logarithmic Hankel determinant bound, which cannot be improved within the class.

Let  $\Omega$  be the class of Schwarz functions, that is analytic functions  $\varpi: \mathbb{D} \rightarrow \mathbb{D}$ ,  $\varpi(0) = 0$ . A function  $\varpi \in \Omega$  can be written as a power series,

$$\varpi(\nu) = \xi_1\nu + \xi_2\nu^2 + \xi_3\nu^3 + \xi_4\nu^4 + \cdots.$$
(1.16)

The following lemmas will be used in the sequel.

**Lemma 1.1.** ([14]) *If  $\varpi \in \Omega$ , then  $|\xi_n| \leq 1$  ( $n \geq 1$ ).*

**Lemma 1.2.** ([15]) *Let  $\varpi \in \Omega$  given in (1.16) and  $\kappa \in \mathbb{C}$ , then,*

$$|\xi_2 + \kappa\xi_1^2| \leq \max\{1, |\kappa|\}.$$
(1.17)

**Lemma 1.3.** ([16]) *Let  $\varpi \in \Omega$ , having the series representation given in (1.16). Then, for any real numbers  $\iota$  and  $\eta$  such that*

$$(\iota, \eta) \in \left\{ (\iota, \eta) : |\iota| \leq \frac{1}{2}, -1 \leq \eta \leq 1 \right\} \bigcup \left\{ (\iota, \eta) : \frac{1}{2} \leq |\iota| \leq 2, \frac{4}{27}(|\iota| + 1)^3 - (|\iota| + 1) \leq \eta \leq 1 \right\},$$

the following inequality holds:

$$|\xi_3 + \iota \xi_1 \xi_2 + \eta \xi_1^3| \leq 1.$$

**Lemma 1.4.** ([17]) Let  $\varpi \in \Omega$ ,  $\theta \in \mathbb{C}$ , and  $|\theta| \leq 1$ . Then,

$$|\xi_4 + 2\theta \xi_1 \xi_3 + 3\theta^2 \xi_1^2 \xi_2 + \theta \xi_2^2 + \theta^3 \xi_1^4| \leq 1.$$

**Lemma 1.5.** ([18]) If  $\varpi \in \Omega$ , then

$$|\xi_2| \leq 1 - |\xi_1|^2, \quad |\xi_4| \leq 1 - |\xi_1|^2 - |\xi_2|^2.$$

**Lemma 1.6.** ([19]) Let  $w \in \Omega$ , then, for  $\kappa_i \in \mathbb{C}$  with  $|\kappa_i| \leq 1$  ( $i = 1, 2, 3, 4$ ), we have

$$\begin{aligned} \xi_1 &= \kappa_1, \\ \xi_2 &= (1 - |\kappa_1|^2)\kappa_2, \\ \xi_3 &= -(1 - |\kappa_1|^2)\bar{\kappa}_1\kappa_2^2 + (1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\kappa_3, \\ \xi_4 &= (1 - |\kappa_1|^2)\bar{\kappa}_1^2\kappa_2^3 - (1 - |\kappa_1|^2)(1 - |\kappa_2|^2)(\bar{\kappa}_2\kappa_3^2 + 2\bar{\kappa}_1\kappa_2\kappa_3) \\ &\quad + (1 - |\kappa_1|^2)(1 - |\kappa_2|^2)(1 - |\kappa_3|^2)\kappa_4. \end{aligned}$$

## 2. The second- and third-order Hankel determinants

**Theorem 2.1.** Let  $F \in \mathcal{R}_T$  have the series representation given in (1.1). Then, we have

$$|\mu_2| \leq \frac{1}{2}, \quad |\mu_3| \leq \frac{1}{3}, \quad |\mu_4| \leq \frac{1}{4}, \quad |\mu_5| \leq \frac{1}{5}.$$

These inequalities are sharp, and sharpness for  $\mu_m$  is attained by  $F_m(v)$ , defined as

$$F'_m(v) = 1 + \arctan v^m \quad (m \in \mathbb{N}). \quad (2.1)$$

In particular, for  $m = 1, 2, 3, 4$ , we have

$$F_1(v) = \int_0^v (1 + \arctan t) dt = v + \frac{1}{2}v^2 + \cdots, \quad (2.2)$$

$$F_2(v) = \int_0^v (1 + \arctan t^2) dt = v + \frac{1}{3}v^3 + \cdots, \quad (2.3)$$

$$F_3(v) = \int_0^v (1 + \arctan t^3) dt = v + \frac{1}{4}v^4 + \cdots, \quad (2.4)$$

$$F_4(v) = \int_0^v (1 + \arctan t^4) dt = v + \frac{1}{5}v^5 + \cdots. \quad (2.5)$$

*Proof.* Let  $F \in \mathcal{R}_T$ . Then, by (1.15), we achieve

$$F'(v) = 1 + \arctan(w(v)), \quad (2.6)$$

where  $\varpi(v)$  is known as a Schwarz function.

From (1.1), we have

$$F'(v) = 1 + 2\mu_2 v + 3\mu_3 v^2 + 4\mu_4 v^3 + 5\mu_5 v^4 + \dots \quad (2.7)$$

Using series form of  $\varpi(v)$  in the right-hand side of (2.6), we yield

$$1 + \arctan(w(v)) = 1 + \xi_1 v + \xi_2 v^2 + \left(-\frac{1}{3}\xi_1^3 + \xi_3\right)v^3 + (-\xi_2\xi_1^2 + \xi_4)v^4 + \dots \quad (2.8)$$

It follows from (2.7) and (2.8), that

$$\mu_2 = \frac{1}{2}\xi_1, \quad (2.9)$$

$$\mu_3 = \frac{1}{3}\xi_2, \quad (2.10)$$

$$\mu_4 = \frac{1}{4}\xi_3 - \frac{1}{12}\xi_1^3, \quad (2.11)$$

$$\mu_5 = \frac{1}{5}\xi_4 - \frac{1}{5}\xi_1^2\xi_2. \quad (2.12)$$

Applying Lemma 1.1 on Eqs (2.9) and (2.10) leads to

$$|\mu_2| \leq \frac{1}{2}, \quad |\mu_3| \leq \frac{1}{3}.$$

From Lemma 1.3 and (2.11), we have

$$|\mu_4| = \frac{1}{4}|\xi_3 - \frac{1}{3}\xi_1^3| \leq \frac{1}{4}.$$

Similarly, by Lemma 1.5 on Eq (2.12), we have

$$|\mu_5| \leq \frac{1}{5}(|\xi_4| + |\xi_1|^2|\xi_2|) \leq \frac{1}{5}[1 - |\xi_1|^2 - |\xi_2|^2 + |\xi_1|^2(1 - |\xi_1|^2)] = \frac{1}{5}(1 - |\xi_2|^2 - |\xi_1|^4) \leq \frac{1}{5}.$$

This completes the proof.  $\square$

**Theorem 2.2.** If  $F \in \mathcal{R}_T$ , then

$$|\mathcal{H}_{2,1}(F)| \leq \frac{1}{3}.$$

The equality is sharp for the function  $F_2(v)$ .

*Proof.* Let  $F \in \mathcal{R}_T$ . From Lemma 1.2, Eqs (1.2), (2.9), and (2.10), we have

$$|\mu_3 - \mu_2^2| = \frac{1}{3}|\xi_2 - \frac{3}{4}\xi_1^2| \leq \frac{1}{3}\max\{1, |\frac{3}{4}|\} = \frac{1}{3}.$$

This completes the proof.  $\square$

**Theorem 2.3.** If  $F \in \mathcal{R}_T$ , then

$$|\mathcal{H}_{2,2}(F)| \leq \frac{1}{9}.$$

The result is sharp for the function given by (2.3).

*Proof.* Let  $F \in \mathcal{R}_T$ . From Eqs (1.3) and (2.9)–(2.11), we have

$$|\mu_2\mu_4 - \mu_3^2| = \frac{1}{72}|9\xi_1\xi_3 - 8\xi_2^2 - 3\xi_1^4|. \quad (2.13)$$

From Lemma 1.6 and (2.13), we obtain

$$|\mu_2\mu_4 - \mu_3^2| = \frac{1}{72} \left| -9|\kappa_1|^2(1 - |\kappa_1|^2)\kappa_2^2 + 9\kappa_1(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\kappa_3 - 3\kappa_1^4 - 8(1 - |\kappa_1|^2)^2\kappa_2^2 \right|.$$

Setting  $|\kappa_1| = \kappa$ ,  $|\kappa_2| = \sigma$ , and upon taking  $|\kappa_3| \leq 1$ , we yield

$$|\mu_2\mu_4 - \mu_3^2| \leq \frac{1}{72}[9\kappa^2(1 - \kappa^2)\sigma^2 + 9\kappa(1 - \kappa^2)(1 - \sigma^2) + 3\kappa^4 + 8(1 - \kappa^2)^2\sigma^2] = \frac{1}{72}\rho_1(\kappa, \sigma).$$

Differentiating  $\rho_1(\kappa, \sigma)$  with respect to  $\sigma$ , we obtain

$$\frac{\partial \rho_1}{\partial \sigma} = 2\sigma(1 - \kappa^2)(\kappa^2 - 9\kappa + 8) \geq 0.$$

Therefore,  $\rho_1(\kappa, \sigma)$  attains its maximum value at  $\sigma = 1$ . Thus, we achieve

$$|\mu_2\mu_4 - \mu_3^2| \leq \frac{1}{72}\rho_1(\kappa, 1) = \frac{1}{72}(8 - 7\kappa^2 + 2\kappa^4) \leq \frac{8}{72} = \frac{1}{9}.$$

This completes the proof.  $\square$

**Theorem 2.4.** If  $F \in \mathcal{R}_T$ , then

$$|\mathcal{H}_{3,1}(F)| \leq \frac{1}{16}.$$

The equality is sharp for the function given by (2.4).

*Proof.* Let  $F \in \mathcal{R}_T$ . From Eqs (1.4) and (2.9)–(2.12), we obtain

$$\begin{aligned} \mathcal{H}_{3,1}(F) = & \frac{1}{2160}(-135\xi_3^2 + 180\xi_1\xi_2\xi_3 + 48\xi_1^4\xi_2 - 80\xi_2^3 \\ & + 90\xi_1^3\xi_3 - 108\xi_1^2\xi_4 - 144\xi_1^2\xi_2^2 + 144\xi_2\xi_4 - 15\xi_1^6). \end{aligned} \quad (2.14)$$

From Lemma 1.6 and (2.14), it follows that

$$\begin{aligned} \mathcal{H}_{3,1}(F) = & \frac{1}{2160}[-180(1 - |\kappa_1|^2)^2|\kappa_1|^2\kappa_2^3 + 180(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2)\kappa_1\kappa_2\kappa_3 - 80(1 - |\kappa_1|^2)^3\kappa_2^3 \\ & - 5\kappa_1^6 - 144(1 - |\kappa_1|^2)^2\kappa_1^2\kappa_2^2 + 48(1 - |\kappa_1|^2)\kappa_1^4\kappa_2 + 9(1 - |\kappa_1|^2)^2\bar{\kappa}_1^2\kappa_2^4 - 144(1 - |\kappa_2|^2) \\ & \times (1 - |\kappa_1|^2)^2|\kappa_2|^2\kappa_3^2 - 18(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2)\bar{\kappa}_1\kappa_2^2\kappa_3 + 144(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2) \\ & \times (1 - |\kappa_3|^2)\kappa_2\kappa_4 - 135(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2)^2\kappa_3^2 - 90(1 - |\kappa_1|^2)\kappa_1^3\bar{\kappa}_1\kappa_2^2 - 108(1 - |\kappa_1|^2) \\ & \times \kappa_1^2\bar{\kappa}_1^2\kappa_2^3 + 90(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\kappa_1^3\kappa_3 + 108(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\kappa_1^2\bar{\kappa}_2\kappa_3^2 \\ & + 216(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\kappa_1^2\bar{\kappa}_1\kappa_2\kappa_3 - 108(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)(1 - |\kappa_3|^2)\kappa_1^2\kappa_4]. \end{aligned}$$

Thus, we yield

$$\mathcal{H}_{3,1}(F) = \frac{1}{2160}[\varrho_1(\kappa_1, \kappa_2) + \varrho_2(\kappa_1, \kappa_2)\kappa_3 + \varrho_3(\kappa_1, \kappa_2)\kappa_3^2 + \varrho_4(\kappa_1, \kappa_2, \kappa_3)\kappa_4],$$

where

$$\begin{aligned}\varrho_1(\kappa_1, \kappa_2) &= -15\kappa_1^6 + (1 - |\kappa_1|^2) \left[ (9\bar{\kappa}_1^2 \kappa_2^4 - 100|\kappa_1|^2 \kappa_2^3 - 80\kappa_2^3 - 144\kappa_1^2 \kappa_2^2)(1 - |\kappa_1|^2) + 48\kappa_1^4 \kappa_2 \right. \\ &\quad \left. - 90\kappa_1^3 \bar{\kappa}_1 \kappa_2^2 - 108\bar{\kappa}_1^2 \kappa_1^2 \kappa_2^3 \right], \\ \varrho_2(\kappa_1, \kappa_2) &= (1 - |\kappa_1|^2)(1 - |\kappa_2|^2) [(-18\bar{\kappa}_1 \kappa_2^2 + 180\kappa_1 \kappa_2)(1 - |\kappa_1|^2) + 216\bar{\kappa}_1 \kappa_1^2 \kappa_2 + 90\kappa_1^3], \\ \varrho_3(\kappa_1, \kappa_2) &= (1 - |\kappa_2|^2)(1 - |\kappa_1|^2) [(-9|\kappa_2|^2 - 135)(1 - |\kappa_1|^2) + 108\bar{\kappa}_2 \kappa_1^2], \\ \varrho_4(\kappa_1, \kappa_2, \kappa_3) &= (1 - |\kappa_3|^2)(1 - |\kappa_1|^2)(1 - |\kappa_2|^2) [144\kappa_2(1 - |\kappa_1|^2) - 108\kappa_1^2].\end{aligned}$$

Upon taking  $|\kappa_4| \leq 1$ , and setting  $|\kappa_1| = \kappa \in [0, 1]$ ,  $|\kappa_2| = \sigma \in [0, 1]$ ,  $|\kappa_3| = \aleph \in [0, 1]$ , we yield

$$\begin{aligned}|\mathcal{H}_{3,1}(F)| &\leq \frac{1}{2160} (|\varrho_1(\kappa_1, \kappa_2)| + |\varrho_2(\kappa_1, \kappa_2)| \aleph + |\varrho_3(\kappa_1, \kappa_2)| \aleph^2 + |\varrho_4(\kappa_1, \kappa_2, \kappa_3)|) \\ &\leq \frac{1}{2160} \Theta_1(\kappa, \sigma, \aleph),\end{aligned}$$

where

$$\Theta_1(\kappa, \sigma, \aleph) = \chi_1(\kappa, \sigma) + \chi_2(\kappa, \sigma) \aleph + \chi_3(\kappa, \sigma) \aleph^2 + \chi_4(\kappa, \sigma) (1 - \aleph^2),$$

with

$$\begin{aligned}\chi_1(\kappa, \sigma) &= 15\kappa^6 + (1 - \kappa^2) \left[ (9\kappa^2 \sigma^4 + 100\kappa^2 \sigma^3 + 80\sigma^3 + 144\kappa^2 \sigma^2)(1 - \kappa^2) + 48\kappa^4 \sigma \right. \\ &\quad \left. + 90\kappa^4 \sigma^2 + 108\kappa^4 \sigma^3 \right], \\ \chi_2(\kappa, \sigma) &= (1 - \kappa^2)(1 - \sigma^2) [(18\kappa \sigma^2 + 180\kappa \sigma)(1 - \kappa^2) + 216\kappa^3 \sigma + 90\kappa^3], \\ \chi_3(\kappa, \sigma) &= (1 - \sigma^2)(1 - \kappa^2) [(9\sigma^2 + 135)(1 - \kappa^2) + 108\sigma \kappa^2], \\ \chi_4(\kappa, \sigma) &= (1 - \kappa^2)(1 - \sigma^2) [144\sigma(1 - \kappa^2) + 108\kappa^2].\end{aligned}$$

Note that

$$\begin{aligned}\chi_3(\kappa, \sigma) &\leq (1 - \kappa^2)(1 - \sigma^2) [(1 - \kappa^2)(135 + 9\sigma^2) + 108\kappa^2] \\ &= \epsilon_3(\kappa, \sigma).\end{aligned}$$

Taking  $\epsilon_i = \chi_i$  for  $i = 1, 2, 4$  and

$$\delta_1(\kappa, \sigma, \aleph) = \epsilon_1(\kappa, \sigma) + \epsilon_2(\kappa, \sigma) \aleph + \epsilon_3(\kappa, \sigma) \aleph^2 + \epsilon_4(\kappa, \sigma) (1 - \aleph^2).$$

It is not difficult to see that

$$\Theta_1(\kappa, \sigma, \aleph) \leq \delta_1(\kappa, \sigma, \aleph).$$

Differentiating  $\delta_1(\kappa, \sigma, \aleph)$  partially with  $\aleph$ , we obtain

$$\begin{aligned}\frac{\partial \delta_1}{\partial \aleph} &= \epsilon_2(\kappa, \sigma) + 2\aleph [\epsilon_3(\kappa, \sigma) - \epsilon_4(\kappa, \sigma)] \\ &= \epsilon_2(\kappa, \sigma) + 18(1 - \sigma^2)(1 - \kappa^2)^2(\sigma^2 - 16\sigma + 15)\aleph \geq 0.\end{aligned}$$

This shows that  $\delta_1(\kappa, \sigma, \aleph)$  is an increasing function for  $\aleph \in [0, 1]$ . Therefore, we obtain

$$\begin{aligned}\Theta_1(\kappa, \sigma, \aleph) &\leq \delta_1(\kappa, \sigma, \aleph) \leq \delta_1(\kappa, \sigma, 1) \\ &= \epsilon_1(\kappa, \sigma) + \epsilon_2(\kappa, \sigma) + \epsilon_3(\kappa, \sigma) \\ &= (9\kappa^6 - 18\kappa^5 - 27\kappa^4 + 36\kappa^3 + 27\kappa^2 - 18\kappa - 9)\sigma^4 + (-8\kappa^6 + 36\kappa^5 - 12\kappa^4 + 144\kappa^3 - 60\kappa^2 \\ &\quad - 180\kappa + 80)\sigma^3 + (54\kappa^6 + 108\kappa^5 - 216\kappa^4 - 126\kappa^3 + 288\kappa^2 + 18\kappa - 126)\sigma^2 + (-48\kappa^6 \\ &\quad - 36\kappa^5 + 48\kappa^4 - 144\kappa^3 + 180\kappa^2)\sigma + 15\kappa^6 - 90\kappa^5 + 27\kappa^4 + 90\kappa^3 - 162\kappa^2 + 135 \\ &= \rho_2(\kappa, \sigma).\end{aligned}$$

Consider

$$\left\{ \begin{array}{l} \frac{\partial \rho_2}{\partial \kappa} = (54\kappa^5 - 90\kappa^4 - 108\kappa^3 + 108\kappa^2 + 54\kappa - 18)\sigma^4 + (-48\kappa^5 + 180\kappa^4 - 48\kappa^3 + 432\kappa^2 - 120\kappa \\ \quad - 180)\sigma^3 + (324\kappa^5 + 540\kappa^4 - 864\kappa^3 - 378\kappa^2 + 576\kappa + 18)\sigma^2 + (-288\kappa^5 - 180\kappa^4 + 192\kappa^3 \\ \quad - 432\kappa^2 + 180)\sigma + 90\kappa^5 - 450\kappa^4 + 108\kappa^3 + 270\kappa^2 - 324\kappa = 0, \\ \frac{\partial \rho_2}{\partial \sigma} = (36\kappa^6 - 72\kappa^5 - 108\kappa^4 + 144\kappa^3 + 108\kappa^2 - 72\kappa - 36)\sigma^3 + (-24\kappa^6 + 108\kappa^5 - 36\kappa^4 + 432\kappa^3 \\ \quad - 180\kappa^2 - 540\kappa + 240)\sigma^2 + (108\kappa^6 + 216\kappa^5 - 432\kappa^4 - 252\kappa^3 + 576\kappa^2 + 36\kappa - 252)\sigma \\ \quad + 180\kappa - 144\kappa^3 + 48\kappa^4 - 36\kappa^5 - 48\kappa^6 = 0. \end{array} \right.$$

By calculating, we obtain

$$\left\{ \begin{array}{l} \kappa_{1,1} \approx 1, \\ \sigma_{1,1} \approx 1.4213, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,2} \approx 1, \\ \sigma_{1,2} \approx -1.8953, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,3} \approx 1, \\ \sigma_{1,3} \approx -0.5259, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,4} \approx -1, \\ \sigma_{1,4} \approx 0.6955, \end{array} \right.$$

$$\left\{ \begin{array}{l} \kappa_{1,5} \approx -1, \\ \sigma_{1,5} \approx -0.7601, \end{array} \right. \quad \left\{ \begin{array}{l} \theta_{1,6} \approx -1, \\ \sigma_{1,6} \approx -0.1576, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,7} = 0, \\ \sigma_{1,7} = 0, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,8} \approx 1.3108, \\ \sigma_{1,8} \approx 11.6363, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,9} \approx 17.2105, \\ \sigma_{1,9} \approx 0.4263, \end{array} \right.$$

$$\left\{ \begin{array}{l} \kappa_{1,10} \approx -0.1343, \\ \sigma_{1,10} \approx 0.9520, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,11} \approx -0.7240, \\ \sigma_{1,11} \approx -42.7079, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,12} \approx -0.9075, \\ \sigma_{1,12} \approx 0.3851, \end{array} \right. \quad \left\{ \begin{array}{l} \kappa_{1,13} \approx -1.0505, \\ \sigma_{1,13} \approx -0.5089. \end{array} \right.$$

Thus, there are no optimal points in  $(0, 1) \times (0, 1)$ .

(1) For  $\kappa = 1$ ,

$$\rho_2(1, \sigma) = 15.$$

(2) For  $\kappa = 0$ ,

$$\rho_2(0, \sigma) = 135 - 126\sigma^2 + 80\sigma^3 - 9\sigma^4 \leq 135.$$

(3) For  $\sigma = 0$ ,

$$\rho_2(\kappa, 0) = 135 - 162\kappa^2 + 90\kappa^3 + 27\kappa^4 - 90\kappa^5 + 15\kappa^6 \leq 135.$$

(4) For  $\sigma = 1$ ,

$$\rho_2(\kappa, 1) = 80 + 93\kappa^2 - 180\kappa^4 + 22\kappa^6 = \varsigma_1(\kappa).$$

Let

$$\varsigma_1(\kappa) = \rho_2(\kappa, 1) = 80 + 93\kappa^2 - 180\kappa^4 + 22\kappa^6, \quad (2.15)$$

$$\varsigma_1'(\kappa) = 132\kappa^5 - 720\kappa^3 + 186\kappa, \quad (2.16)$$

$$\varsigma_1''(\kappa) = 186 - 2160\kappa^2 + 660\kappa^4. \quad (2.17)$$

For the optimum value of  $\varsigma_1(\kappa)$ , consider  $\varsigma_1'(\kappa) = 0$ . From Eq (2.16), we obtain  $\kappa = \delta_1 = 0$  and  $\kappa = \delta_2 \approx 0.5214$ . Substituting the value of  $\kappa = \delta_2$  in (2.17), we observe that  $\varsigma_1'(\delta_2) \approx -352.4347 < 0$ . Thus, by the second derivative test,  $\varsigma_1(\kappa)$  has maximum value at  $\kappa = \delta_2$ , and from (2.15), we obtain  $\max\{\varsigma_1(\kappa)\} \approx 92.4216$ .

Therefore, we yield

$$|\mathcal{H}_{3,1}(F)| \leq \frac{135}{2160} = \frac{1}{16}.$$

This completes the proof.  $\square$

### 3. The second-order Hankel determinants of logarithmic coefficients

**Theorem 3.1.** *If  $F \in \mathcal{R}_T$ , then*

$$|\tau_1| \leq \frac{1}{4}, \quad |\tau_2| \leq \frac{1}{6}, \quad |\tau_3| \leq \frac{1}{8}, \quad |\tau_4| \leq 0.100798.$$

*The first three bounds are sharp and obtained by (2.2)–(2.4), respectively.*

*Proof.* Let  $F \in \mathcal{R}_T$ . From Eqs (1.6) and (2.9)–(2.12), it follows that

$$\tau_1 = \frac{1}{4}\xi_1, \quad (3.1)$$

$$\tau_2 = \frac{1}{6}\left(-\frac{3}{8}\xi_1^2 + \xi_2\right), \quad (3.2)$$

$$\tau_3 = \frac{1}{8}\left(-\frac{1}{6}\xi_1^3 - \frac{2}{3}\xi_1\xi_2 + \xi_3\right), \quad (3.3)$$

$$\tau_4 = \frac{1}{5760}(75\xi_1^4 - 160\xi_2^2 - 336\xi_1^2\xi_2 - 360\xi_1\xi_3 + 576\xi_4). \quad (3.4)$$

From Eqs (3.1)–(3.3), Lemmas 1.1–1.3 ( $\iota = -2/3, \eta = -1/6$ ), we obtain

$$|\tau_1| \leq \frac{1}{4}, \quad |\tau_2| \leq \frac{1}{6}, \quad |\tau_3| \leq \frac{1}{8}.$$

From (3.4), Lemmas 1.2 and 1.4, we yield

$$\begin{aligned} |\tau_4| &= \frac{1}{5760} |180(-\xi_1^4 + 3\xi_2\xi_1^2 - \xi_2^2 - 2\xi_3\xi_1 + \xi_4) + 396\xi_4 - 255\xi_1^2(\xi_2 - \xi_1^2) + 20\xi_2^2 - 621\xi_1^2\xi_2| \\ &\leq \frac{1}{5760} [180 + 396(1 - |\xi_2|^2 - |\xi_1|^2) + 20|\xi_2|^2 + 255|\xi_1|^2 + 621|\xi_1|^2|\xi_2|] \\ &= \frac{1}{5760} [576 + (-141 + 621|\xi_2|)|\xi_1|^2 - 376|\xi_2|^2]. \end{aligned}$$

Setting  $|\xi_1| = \kappa \in [0, 1]$ ,  $|\xi_2| = x \in [0, 1]$ , we have

$$|\tau_4| \leq \frac{1}{5760} \varphi_1(\kappa, x),$$

where

$$\varphi_1(k, x) = 576 + (-141 + 621x)\kappa^2 - 376x^2.$$

Consider

$$\begin{cases} \frac{\partial \varphi_1}{\partial \kappa} = (-282 + 1242x)\kappa = 0, \\ \frac{\partial \varphi_1}{\partial x} = 621\kappa^2 - 752x = 0. \end{cases}$$

By calculating, we obtain

$$\begin{cases} \kappa_{1,1} = 0, \\ x_{1,1} = 0, \end{cases} \quad \begin{cases} \kappa_{1,2} = 0.5244, \\ x_{1,2} = 0.2271, \end{cases} \quad \begin{cases} \kappa_{1,3} = -0.5244, \\ x_{1,3} = 0.2271. \end{cases}$$

Thus, there is an optimal point of  $(\kappa_{1,2}, x_{1,2})$  in  $(0, 1) \times (0, 1 - \kappa^2)$ . Therefore, we get  $\max\{\varphi_1(\kappa, x)\} \leq \varphi_1(\kappa_{1,2}, x_{1,2}) \approx 556.616003$ .

(5) For  $\kappa = 0$ ,

$$\varphi_1(0, x) = 576 - 376x^2 \leq 576.$$

(6) For  $x = 0$ ,

$$\varphi_1(\kappa, 0) = 576 - 141\kappa^2 \leq 576.$$

(7) For  $x = 1 - \kappa^2$ ,

$$\varphi_1(\kappa, 1 - \kappa^2) = 200 + 1232\kappa^2 - 997\kappa^4 = \varsigma_2(\kappa).$$

It is easy to observe that the zero of  $\varsigma_2'(\kappa)$  in  $[0, 1]$  is  $\kappa = \delta_3 = 2\sqrt{153538}/997$  and satisfies  $\varsigma_2'(\delta_3) \approx -4927.3$ . Thus,

$$\varsigma_2(\kappa) \leq \varsigma_2\left(\frac{2\sqrt{153538}}{997}\right) \approx 580.5978.$$

Therefore, we obtain

$$|\tau_4| \leq \frac{580.5978}{5760} \approx 0.100798.$$

This completes the proof. □

**Theorem 3.2.** If  $F \in \mathcal{R}_T$ , then

$$|\tau_1\tau_3 - \tau_2^2| \leq \frac{1}{36}.$$

The inequality is sharp for (2.3).

*Proof.* Let  $F \in \mathcal{R}_T$ . From (1.7) and (2.9)–(2.11), we achieve

$$|\tau_1\tau_3 - \tau_2^2| = \frac{1}{2304}|72\xi_1\xi_3 - 21\xi_1^4 - 64\xi_2^2|. \quad (3.5)$$

From (3.5) and Lemma 1.6, we obtain

$$|\tau_1\tau_3 - \tau_2^2| = \frac{1}{2304} \left| -72(1 - |\kappa_1|^2)|\kappa_1|^2\kappa_2^2 + 72\kappa_1(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)\kappa_3 - 21\kappa_1^4 - 64\kappa_2^2(1 - |\kappa_1|^2)^2 \right|.$$

By putting  $\kappa = |\kappa_1| \in [0, 1]$ ,  $|\kappa_2| = \sigma \in [0, 1]$  and taking  $|\kappa_3| \leq 1$ , we achieve

$$\begin{aligned} |\tau_1\tau_3 - \tau_2^2| &\leq \frac{1}{2304} \left[ 72(1 - \kappa^2)\kappa^2\sigma^2 + 72\kappa(1 - \kappa^2)(1 - \sigma^2) + 21\kappa^4 + 64\sigma^2(1 - \kappa^2)^2 \right] \\ &= \frac{1}{2304}\rho_3(\kappa, \sigma). \end{aligned}$$

Differentiating  $\rho_3(\kappa, \sigma)$  with respect to  $\sigma$ , we have

$$\frac{\partial \rho_3}{\partial \sigma} = 16\sigma(1 - \kappa^2)(8 - 9\kappa + \kappa^2) \geq 0.$$

Thus,  $\rho_3(\kappa, \sigma)$  attains its maximum value at  $\sigma = 1$ . Therefore, we achieve

$$\begin{aligned} |\tau_1\tau_3 - \tau_2^2| &\leq \rho_3(\kappa, 1) = \frac{1}{2304}[64 - 56\kappa^2 + 13\kappa^4] \leq \frac{64}{2304} \\ &= \frac{1}{36}. \end{aligned}$$

This completes the proof. □

**Theorem 3.3.** Let  $F \in \mathcal{R}_T$ . Then,

$$|\tau_2\tau_4 - \tau_3^2| \leq \frac{1}{64}.$$

The equality is sharp for the function given by (2.4).

*Proof.* Let  $F \in \mathcal{R}_T$ . From (1.8) and (2.9)–(2.12), we obtain

$$\begin{aligned} \tau_2\tau_4 - \tau_3^2 &= \frac{1}{276480}(-4320\xi_3^2 + 2880\xi_1\xi_2\xi_3 - 4128\xi_1^2\xi_2^2 + 2520\xi_1^3\xi_3 + 648\xi_1^4\xi_2 - 345\xi_1^6 \\ &\quad + 4608\xi_2\xi_4 - 1280\xi_2^3 - 1728\xi_1^2\xi_4). \end{aligned} \quad (3.6)$$

From (3.6) and Lemma 1.6, it follows that

$$\begin{aligned} \tau_2\tau_4 - \tau_3^2 &= \frac{1}{276480} \left[ -345\kappa_1^6 + 648(1 - |\kappa_1|^2)\kappa_1^4\kappa_2 - 1280(1 - |\kappa_1|^2)^3\kappa_2^3 - 4128(1 - |\kappa_1|^2)^2\kappa_1^2\kappa_2^2 \right. \\ &\quad + 2880\kappa_1\kappa_2\kappa_3(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2) + 2520\kappa_1^3(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\kappa_3 - 2880|\kappa_1|^2\kappa_2^3 \\ &\quad \times (1 - |\kappa_1|^2)^2 - 2520(1 - |\kappa_1|^2)\kappa_1^3\bar{\kappa}_1\kappa_2^2 - 1728\kappa_1^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)(1 - |\kappa_3|^2)\kappa_4 \\ &\quad + 3456(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)\kappa_1^2\bar{\kappa}_1\kappa_2\kappa_3 + 1728\kappa_1^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)\bar{\kappa}_2\kappa_3^2 - 1728\bar{\kappa}_1^2\kappa_1^2 \\ &\quad \times \kappa_2^3(1 - |\kappa_1|^2) + 4608(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2(1 - |\kappa_3|^2)\kappa_2\kappa_4 - 4320\kappa_3^2(1 - |\kappa_2|^2)^2 \end{aligned}$$

$$\times (1 - |\kappa_1|^2)^2 - 4608|\kappa_2|^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2\kappa_3^2 + 288\bar{\kappa}_1^2\kappa_2^4(1 - |\kappa_1|^2)^2 \\ - 576\bar{\kappa}_1\kappa_2^2(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2)\kappa_3 \Big].$$

Hence, we obtain

$$\tau_2\tau_4 - \tau_3^2 = \frac{1}{276480} \left[ \varrho_5(\kappa_1, \kappa_2) + \varrho_6(\kappa_1, \kappa_2)\kappa_3 + \varrho_7(\kappa_1, \kappa_2)\kappa_3^2 + \varrho_8(\kappa_1, \kappa_2, \kappa_3)\kappa_4 \right],$$

where

$$\begin{aligned} \varrho_5(\kappa_1, \kappa_2) &= -345\kappa_1^6 + (1 - |\kappa_1|^2) \left[ (288\bar{\kappa}_1^2\kappa_2^4 - 1600\kappa_2^3|\kappa_1|^2 - 1280\kappa_2^3 - 4128\kappa_2^2\kappa_1^2)(1 - |\kappa_1|^2) \right. \\ &\quad \left. + 648\kappa_2\kappa_1^4 - 1728\kappa_2^3\kappa_1^2\bar{\kappa}_1^2 - 2520\bar{\kappa}_1\kappa_1^3\kappa_2^2 \right], \\ \varrho_6(\kappa_1, \kappa_2) &= (1 - |\kappa_2|^2)(1 - |\kappa_1|^2) \left[ (-576\bar{\kappa}_1\kappa_2^2 + 2880\kappa_1\kappa_2)(1 - |\kappa_1|^2) + 3456\kappa_2\kappa_1^2\bar{\kappa}_1 + 2520\kappa_1^3 \right], \\ \varrho_7(\kappa_1, \kappa_2) &= (1 - |\kappa_1|^2)(1 - |\kappa_2|^2) \left[ (-288|\kappa_2|^2 - 4320)(1 - |\kappa_1|^2) + 1728\kappa_1^2\bar{\kappa}_2 \right], \\ \varrho_8(\kappa_1, \kappa_2, \kappa_3) &= (1 - |\kappa_1|^2)(1 - |\kappa_3|^2)(1 - |\kappa_2|^2) [4608(1 - |\kappa_1|^2)\kappa_2 - 1728\kappa_1^2]. \end{aligned}$$

By setting  $\kappa = |\kappa_1|$ ,  $\sigma = |\kappa_2|$ ,  $\aleph = |\kappa_3|$  and using  $|\kappa_4| \leq 1$ , we yield

$$\begin{aligned} |\tau_2\tau_4 - \tau_3^2| &\leq \frac{1}{276480} \left( |\varrho_5(\kappa_1, \kappa_2)| + |\varrho_6(\kappa_1, \kappa_2)|\aleph + |\varrho_7(\kappa_1, \kappa_2)|\aleph^2 + |\varrho_8(\kappa_1, \kappa_2, \kappa_3)| \right) \\ &\leq \frac{1}{276480} \Theta_2(\kappa, \sigma, \aleph), \end{aligned}$$

where  $\kappa, \sigma, \aleph \in [0, 1]$ , and

$$\Theta_2(\kappa, \sigma, \aleph) = \chi_5(\kappa, \sigma) + \chi_6(\kappa, \sigma)\aleph + \chi_7(\kappa, \sigma)\aleph^2 + \chi_8(\kappa, \sigma)(1 - \aleph^2)$$

with

$$\begin{aligned} \chi_5(\kappa, \sigma) &= 345\kappa^6 + (1 - \kappa^2) \left[ (4128\kappa^2\sigma^2 + 1600\kappa^2\sigma^3 + 1280\sigma^3 + 288\kappa^2\sigma^4)(1 - \kappa^2) + 648\kappa^4\sigma \right. \\ &\quad \left. + 2520\kappa^4\sigma^2 + 1728\kappa^4\sigma^3 \right], \\ \chi_6(\kappa, \sigma) &= (1 - \kappa^2)(1 - \sigma^2) \left[ (2880\kappa\sigma + 576\kappa\sigma^2)(1 - \kappa^2) + 3456\kappa^3\sigma + 2520\kappa^3 \right], \\ \chi_7(\kappa, \sigma) &= (1 - \sigma^2)(1 - \kappa^2) \left[ (1 - \kappa^2)(288\sigma^2 + 4320) + 1728\kappa^2\sigma \right], \\ \chi_8(\kappa, \sigma) &= (1 - \kappa^2)(1 - \sigma^2) [4608(1 - \kappa^2)\sigma + 1728\kappa^2]. \end{aligned}$$

Note that

$$\chi_7(\kappa, \sigma) \leq (1 - \kappa^2)(1 - \sigma^2) [(4320 + 288\sigma^2)(1 - \kappa^2) + 1728\kappa^2] = \epsilon_7(\kappa, \sigma).$$

Take  $\epsilon_i = \chi_i$  for  $i = 5, 6, 8$  and

$$\delta_2(\kappa, \sigma, \aleph) = \epsilon_5(\kappa, \sigma) + \epsilon_6(\kappa, \sigma)\aleph + \epsilon_7(\kappa, \sigma)\aleph^2 + \epsilon_8(\kappa, \sigma)(1 - \aleph^2).$$

Differentiating  $\delta_2(\kappa, \sigma, \aleph)$  with respect to  $\aleph$ , we obtain

$$\frac{\partial \delta_2}{\partial \aleph} = \epsilon_6(\kappa, \sigma) + 2[\epsilon_7(\kappa, \sigma) - \epsilon_8(\kappa, \sigma)]\aleph = \epsilon_6(\kappa, \sigma) + 576\aleph(1 - \kappa^2)^2(1 - \sigma^2)(\sigma^2 - 16\sigma + 15) \geq 0.$$

Therefore,  $\delta_2(\kappa, \sigma, \aleph)$  is an increasing function for  $\aleph \in [0, 1]$ . Thus, we obtain

$$\begin{aligned}\Theta_2(\kappa, \sigma, \aleph) &\leq \delta_2(\kappa, \sigma, \aleph) \leq \delta_2(\kappa, \sigma, 1) \\ &= \epsilon_5(\kappa, \sigma) + \epsilon_6(\kappa, \sigma) + \epsilon_7(\kappa, \sigma) \\ &= 345\kappa^6 - 2520\kappa^5 + 2592\kappa^4 + 2520\kappa^3 - 6912\kappa^2 + 4320 + (288\kappa^6 - 576\kappa^5 - 864\kappa^4 + 1152\kappa^3 \\ &\quad + 864\kappa^2 - 576\kappa - 288)\sigma^4 + (-128\kappa^6 + 576\kappa^5 - 192\kappa^4 + 2304\kappa^3 - 960\kappa^2 - 2880\kappa + 1280)\sigma^3 \\ &\quad + (1608\kappa^6 + 3096\kappa^5 - 8040\kappa^4 - 3672\kappa^3 + 10464\kappa^2 + 576\kappa - 4032)\sigma^2 + (-648\kappa^6 - 576\kappa^5 \\ &\quad + 648\kappa^4 - 2304\kappa^3 + 2880\kappa)\sigma \\ &= \rho_4(\kappa, \sigma).\end{aligned}$$

Consider

$$\left\{ \begin{array}{l} \frac{\partial \rho_4}{\partial \kappa} = 2070\kappa^5 - 12600\kappa^4 + 10368\kappa^3 + 7560\kappa^2 - 13824\kappa + (1728\kappa^5 - 2880\kappa^4 - 3456\kappa^3 + 3456\kappa^2 \\ \quad + 1728\kappa - 576)\sigma^4 + (-768\kappa^5 + 2880\kappa^4 - 768\kappa^3 + 6912\kappa^2 - 1920\kappa - 2880)\sigma^3 + (9648\kappa^5 \\ \quad + 15480\kappa^4 - 32160\kappa^3 - 11016\kappa^2 + 20928\kappa + 576)\sigma^2 + (-3888\kappa^5 - 2880\kappa^4 + 2592\kappa^3 \\ \quad - 6912\kappa^2 + 2880)\sigma = 0, \\ \frac{\partial \rho_4}{\partial \sigma} = 2880\kappa - 2304\kappa^3 + 648\kappa^4 - 576\kappa^5 - 648\kappa^6 + (1152\kappa^6 - 2304\kappa^5 - 3456\kappa^4 + 4608\kappa^3 + 3456\kappa^2 \\ \quad - 2304\kappa - 1152)\sigma^3 + (-384\kappa^6 + 1728\kappa^5 - 576\kappa^4 + 6912\kappa^3 - 2880\kappa^2 - 8640\kappa + 3840)\sigma^2 \\ \quad + (3216\kappa^6 + 6192\kappa^5 - 16080\kappa^4 - 7344\kappa^3 + 20928\kappa^2 + 1152\kappa - 8064)\sigma = 0. \end{array} \right.$$

By calculating, we obtain

$$\left\{ \begin{array}{l} \kappa_{1,1} \approx 1, \\ \sigma_{1,1} \approx -1.7500, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,2} \approx 1, \\ \sigma_{1,2} \approx 1.4719, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,3} \approx 1, \\ \sigma_{1,3} \approx -0.7219, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,4} \approx -1, \\ \sigma_{1,4} \approx 0.7391, \end{array} \right.$$

$$\left\{ \begin{array}{l} \kappa_{1,5} = 0, \\ \sigma_{1,5} = 0, \end{array} \right. \left\{ \begin{array}{l} \theta_{1,6} \approx 1.2740, \\ \sigma_{1,6} \approx 6.5737, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,7} \approx 6.5641, \\ \sigma_{1,7} \approx 0.1929, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,8} \approx -0.4226, \\ \sigma_{1,8} \approx 0.9240, \end{array} \right.$$

$$\left\{ \begin{array}{l} \kappa_{1,9} \approx -0.7314, \\ \sigma_{1,9} \approx -21.4745, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,10} \approx -0.9123, \\ \sigma_{1,10} \approx 0.3093, \end{array} \right. \left\{ \begin{array}{l} \kappa_{1,11} \approx -0.9991, \\ \sigma_{1,11} \approx -0.6874, \end{array} \right.$$

and hence there is no solution in  $(0, 1) \times (0, 1)$ .

(8) For  $\kappa = 1$ ,

$$\rho_4(1, \sigma) = 345.$$

(9) For  $\kappa = 0$ ,

$$\rho_4(0, \sigma) = 4320 - 4032\sigma^2 + 1280\sigma^3 - 288\sigma^4 \leq 4320.$$

(10) For  $\sigma = 0$ ,

$$\rho_4(\kappa, 0) = 4320 - 6912\kappa^2 + 2520\kappa^3 + 2592\kappa^4 - 2520\kappa^5 + 345\kappa^6 \leq 4320.$$

(11) For  $\sigma = 1$ ,

$$\rho_4(\kappa, 1) = 1280 + 3456\kappa^2 - 5856\kappa^4 + 1465\kappa^6 = \varsigma_3(\kappa).$$

Now,  $\varsigma'_3(\kappa) = 0$  when  $\kappa = \delta_4 = 0$ , and  $\kappa = \delta_5 \approx 0.5813$  in the interval  $[0, 1]$  with  $\delta_4$  and  $\delta_5$  as points of minima and maxima, respectively. Therefore,

$$\rho_4(\kappa, 1) = \varsigma_3(\kappa) \leq \varsigma_3(\delta_5) \approx 1835.7.$$

Thus, we obtain

$$|\tau_2\tau_4 - \tau_3^2| \leq \frac{4320}{276480} = \frac{1}{64}.$$

For the Schwarz function  $\varpi(v) = v^3$ , Eqs (2.9)–(2.12) give  $\mu_4 = 1/4$  and  $\mu_2 = \mu_3 = \mu_5 = 0$ , and Eq (1.8) leads to  $\tau_2\tau_4 - \tau_3^2 = -1/64$ .  $\square$

#### 4. The third-order Hankel determinant of inverse function

**Theorem 4.1.** *If  $F \in \mathcal{R}_T$ , then*

$$|\mathcal{H}_{3,1}(F^{-1})| \leq \frac{2}{27}.$$

*The outcome is sharp for the function given by (2.3).*

*Proof.* Let  $F \in \mathcal{R}_T$ . From (1.11) and (2.9)–(2.12), it follows that

$$\begin{aligned} \mathcal{H}_{3,1}(F^{-1}) = & \frac{1}{8640} (576\xi_2\xi_4 + 144\xi_1^2\xi_2^2 + 720\xi_1\xi_2\xi_3 - 640\xi_2^3 - 348\xi_1^4\xi_2 - 432\xi_1^2\xi_4 + 360\xi_1^3\xi_3 \\ & - 540\xi_3^2 + 75\xi_1^6). \end{aligned} \quad (4.1)$$

From (4.1) and Lemma 1.6, we obtain

$$\begin{aligned} |\mathcal{H}_{3,1}(F^{-1})| = & \frac{1}{8640} \left[ 36\kappa_2^4\bar{\kappa}_1^2(1 - |\kappa_1|^2)^2 - 72\bar{\kappa}_1\kappa_2^2\kappa_3(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2 - 540\kappa_3^2(1 - |\kappa_2|^2)^2(1 - |\kappa_1|^2)^2 \right. \\ & - 576|\kappa_2|^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2\kappa_3^2 - 720|\kappa_1|^2\kappa_3^2(1 - |\kappa_1|^2)^2 + 576(1 - |\kappa_3|^2)(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2 \\ & \times \kappa_2\kappa_4 + 720\kappa_3\kappa_1\kappa_2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2 + 144\kappa_1^2(1 - |\kappa_1|^2)^2\kappa_2^2 + 432\kappa_1^2(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)\bar{\kappa}_2\kappa_3^2 \\ & - 640\kappa_2^3(1 - |\kappa_1|^2)^3 + 864\kappa_3\kappa_1^2\bar{\kappa}_1\kappa_2(1 - |\kappa_1|^2)(1 - |\kappa_2|^2) - 432\kappa_1^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)(1 - |\kappa_3|^2)\kappa_4 \\ & + 75\kappa_1^6 - 348\kappa_2(1 - |\kappa_1|^2)\kappa_1^4 - 360(1 - |\kappa_1|^2)\kappa_1^3\bar{\kappa}_1\kappa_2^2 + 360\kappa_1^3\kappa_3(1 - |\kappa_2|^2)(1 - |\kappa_1|^2) \\ & \left. - 432\kappa_1^2\bar{\kappa}_1^2(1 - |\kappa_1|^2)\kappa_2^3 \right]. \end{aligned}$$

Therefore, we achieve

$$|\mathcal{H}_{3,1}(F^{-1})| = \frac{1}{8640} \left[ \varrho_9(\kappa_1, \kappa_2) + \varrho_{10}(\kappa_1, \kappa_2)\kappa_3 + \varrho_{11}(\kappa_1, \kappa_2)\kappa_3^2 + \varrho_{12}(\kappa_1, \kappa_2, \kappa_3)\kappa_4 \right],$$

where

$$\varrho_9(\kappa_1, \kappa_2) = 75\kappa_1^6 + (1 - |\kappa_1|^2) \left[ (36\bar{\kappa}_1^2\kappa_2^4 - 80\kappa_2^3|\kappa_1|^2 - 640\kappa_2^3 + 144\kappa_2^2\kappa_1^2)(1 - |\kappa_1|^2) - 348\kappa_2\kappa_1^4 \right]$$

$$-432\kappa_2^2\kappa_1^2\bar{\kappa}_1^2 - 360\bar{\kappa}_1\kappa_1^3\kappa_2^2],$$

$$\varrho_{10}(\kappa_1, \kappa_2) = (1 - |\kappa_2|^2)(1 - |\kappa_1|^2)[(720\kappa_2\kappa_1 - 72\kappa_2^2\bar{\kappa}_1)(1 - |\kappa_1|^2) + 864\kappa_2\bar{\kappa}_1\kappa_1^2 + 360\kappa_1^3],$$

$$\varrho_{11}(\kappa_1, \kappa_2) = (1 - |\kappa_1|^2)(1 - |\kappa_2|^2)[(-540 - 36|\kappa_2|^2)(1 - |\kappa_1|^2) + 432\kappa_1^2\bar{\kappa}_2],$$

$$\varrho_{12}(\kappa_1, \kappa_2, \kappa_3) = (1 - |\kappa_1|^2)(1 - |\kappa_3|^2)(1 - |\kappa_2|^2)[-432\kappa_1^2 + 576(1 - |\kappa_1|^2)\kappa_2].$$

By writing  $|\kappa_3| = \aleph$ ,  $\sigma = |\kappa_2|$ ,  $|\kappa_1| = \kappa$  and using  $|\kappa_4| \leq 1$ , we obtain

$$\begin{aligned} |\mathcal{H}_{3,1}(F^{-1})| &\leq \frac{1}{8640} (|\varrho_9(\kappa_1, \kappa_2)| + |\varrho_{10}(\kappa_1, \kappa_2)|\aleph + |\varrho_{11}(\kappa_1, \kappa_2)|\aleph^2 + |\varrho_{12}(\kappa_1, \kappa_2, \kappa_3)|) \\ &\leq \frac{1}{8640} \Theta_3(\kappa, \sigma, \aleph), \end{aligned}$$

where  $(\kappa, \sigma, \aleph) \in [0, 1] \times [0, 1] \times [0, 1]$ , and

$$\Theta_3(\kappa, \sigma, \aleph) = \chi_9(\kappa, \sigma) + \chi_{10}(\kappa, \sigma)\aleph + \chi_{11}(\kappa, \sigma)\aleph^2 + \chi_{12}(\kappa, \sigma)(1 - \aleph^2),$$

with

$$\begin{aligned} \chi_9(\kappa, \sigma) &= 75\kappa^6 + (1 - \kappa^2)[(1 - \kappa^2)(144\kappa^2\sigma^2 + 80\kappa^2\sigma^3 + 640\sigma^3 + 36\sigma^4\kappa^2) + 348\kappa^4\sigma \\ &\quad + 360\sigma^2\kappa^4 + 432\sigma^3\kappa^4], \end{aligned}$$

$$\chi_{10}(\kappa, \sigma) = (1 - \sigma^2)(1 - \kappa^2)[(720\kappa\sigma + 72\sigma^2\kappa)(1 - \kappa^2) + 864\sigma\kappa^3 + 360\kappa^3],$$

$$\chi_{11}(\kappa, \sigma) = (1 - \kappa^2)(1 - \sigma^2)[(540 + 36\sigma^2)(1 - \kappa^2) + 432\kappa^2\sigma],$$

$$\chi_{12}(\kappa, \sigma) = (1 - \kappa^2)(1 - \sigma^2)[432\kappa^2 + 576(1 - \kappa^2)\sigma].$$

Note that

$$\chi_{11}(\kappa, \sigma) \leq (1 - \sigma^2)(1 - \kappa^2)[432\kappa^2 + (1 - \kappa^2)(36\sigma^2 + 540)] = \epsilon_{11}(\kappa, \sigma).$$

Taking  $\epsilon_i = \chi_i$  for  $i = 9, 10, 12$  and

$$\delta_3(\kappa, \sigma, \aleph) = \epsilon_9(\kappa, \sigma) + \epsilon_{10}(\kappa, \sigma)\aleph + \epsilon_{11}(\kappa, \sigma)\aleph^2 + \epsilon_{12}(\kappa, \sigma)(1 - \aleph^2).$$

It is easy to see that

$$\begin{aligned} \frac{\partial \delta_3}{\partial \aleph} &= \epsilon_{10}(\kappa, \sigma) + 2\aleph[\epsilon_{11}(\kappa, \sigma) - \epsilon_{12}(\kappa, \sigma)] \\ &= \epsilon_{10}(\kappa, \sigma) + 72(15 - 16\sigma + \sigma^2)(1 - \sigma^2)(1 - \kappa^2)^2\aleph \\ &\geq 0. \end{aligned}$$

Thus,  $\delta_3(\kappa, \sigma, \aleph)$  is an increasing function on  $\aleph \in [0, 1]$ . Thus, we obtain

$$\begin{aligned} \Theta_3(\kappa, \sigma, \aleph) &\leq \delta_3(\kappa, \sigma, \aleph) \leq \delta_3(\kappa, \sigma, 1) = \epsilon_9(\kappa, \sigma) + \epsilon_{10}(\kappa, \sigma) + \epsilon_{11}(\kappa, \sigma) \\ &= 540 - 648\kappa^2 + 360\kappa^3 + 108\kappa^4 - 360\kappa^5 + 75\kappa^6 + (-36 - 72\kappa + 108\kappa^2 + 144\kappa^3 - 108\kappa^4 \\ &\quad - 72\kappa^5 + 36\kappa^6)\sigma^4 + (640 - 720\kappa - 1200\kappa^2 + 576\kappa^3 + 912\kappa^4 + 144\kappa^5 - 352\kappa^6)\sigma^3 + (-504 \end{aligned}$$

$$+ 72\kappa + 720\kappa^2 - 504\kappa^3 + 432\kappa^5 - 216\kappa^6)\sigma^2 + (720\kappa - 576\kappa^3 + 348\kappa^4 - 144\kappa^5 - 348\kappa^6)\sigma \\ = \rho_5(\kappa, \sigma).$$

Consider

$$\begin{cases} \frac{\partial \rho_5}{\partial \kappa} = -1296\kappa + 1080\kappa^2 + 432\kappa^3 - 1800\kappa^4 + 450\kappa^5 + (-72 + 216\kappa + 432\kappa^2 - 432\kappa^3 - 360\kappa^4 \\ \quad + 216\kappa^5)\sigma^4 + (-720 - 2400\kappa + 1728\kappa^2 + 3648\kappa^3 + 720\kappa^4 - 2112\kappa^5)\sigma^3 + (72 + 1440\kappa \\ \quad - 1512\kappa^2 + 2160\kappa^4 - 1296\kappa^5)\sigma^2 + (720 - 1728\kappa^2 + 1392\kappa^3 - 720\kappa^4 - 2088\kappa^5)\sigma = 0, \\ \frac{\partial \rho_5}{\partial \sigma} = -348\kappa^6 - 144\kappa^5 + 348\kappa^4 - 576\kappa^3 + 720\kappa + (-144 - 288\kappa + 432\kappa^2 + 576\kappa^3 - 432\kappa^4 - 288\kappa^5 \\ \quad + 144\kappa^6)\sigma^3 + (1920 - 2160\kappa - 3600\kappa^2 + 1728\kappa^3 + 2736\kappa^4 + 432\kappa^5 - 1056\kappa^6)\sigma^2 + (-1008 \\ \quad + 144\kappa + 1440\kappa^2 - 1008\kappa^3 + 864\kappa^5 - 432\kappa^6)\sigma = 0. \end{cases}$$

By calculating, we obtain

$$\begin{cases} \kappa_{1,1} \approx 1, & \begin{cases} \kappa_{1,2} \approx 1, \\ \sigma_{1,2} \approx -2.0546, \end{cases} & \begin{cases} \kappa_{1,3} \approx 1, \\ \sigma_{1,3} \approx -0.4302, \end{cases} & \begin{cases} \kappa_{1,4} \approx -1, \\ \sigma_{1,4} \approx 0.6577, \end{cases} \\ \begin{cases} \kappa_{1,5} \approx -1, \\ \sigma_{1,5} \approx -0.5565, \end{cases} & \begin{cases} \kappa_{1,6} \approx -1 \\ \sigma_{1,6} \approx -0.3229, \end{cases} & \begin{cases} \kappa_{1,7} = 0, \\ \sigma_{1,7} = 0, \end{cases} & \begin{cases} \kappa_{1,8} \approx 0.2506 \\ \sigma_{1,8} \approx 0.5052, \end{cases} & \begin{cases} \kappa_{1,9} \approx 3.1709, \\ \sigma_{1,9} \approx 20.4224, \end{cases} \\ \begin{cases} \kappa_{1,10} \approx -0.7560, \\ \sigma_{1,10} \approx -46.8467, \end{cases} & \begin{cases} \kappa_{1,11} \approx -0.8987, \\ \sigma_{1,11} \approx 0.3692, \end{cases} & \begin{cases} \kappa_{1,12} \approx -1.0065, \\ \sigma_{1,12} \approx -0.4483, \end{cases} & \begin{cases} \kappa_{1,13} \approx -1.5517, \\ \sigma_{1,13} \approx 10.3879. \end{cases} \end{cases}$$

Therefore, there is an optimal point of  $(\kappa_{1,8}, \sigma_{1,8})$  in  $(0, 1) \times (0, 1)$ . Thus, we get  $\max\{\rho_5(\kappa, \sigma)\} \leq \rho_5(\kappa_{1,8}, \sigma_{1,8}) \approx 526.009022$ .

(12) For  $\kappa = 1$ ,

$$\rho_5(1, \sigma) = 75.$$

(13) For  $\kappa = 0$ ,

$$\rho_5(0, \sigma) = 540 - 504\sigma^2 + 640\sigma^3 - 36\sigma^4 \leq 640.$$

(14) For  $\sigma = 0$ ,

$$\rho_5(\kappa, 0) = 75\kappa^6 - 360\kappa^5 + 108\kappa^4 + 360\kappa^3 - 648\kappa^2 + 540 \leq 540.$$

(15) For  $\sigma = 1$ ,

$$\rho_5(\kappa, 1) = 640 - 1020\kappa^2 + 1260\kappa^4 - 805\kappa^6 \leq 640.$$

Thus, we get

$$|H_{3,1}(F^{-1})| \leq \frac{640}{8640} = \frac{2}{27}.$$

For the Schwarz function  $\varpi(v) = v^2$ , Eqs (2.9)–(2.12) give  $\mu_2 = \mu_4 = \mu_5 = 0$  and  $\mu_3 = 1/3$ , and Eq (1.11) leads to  $\mathcal{H}_{3,1}(\kappa^{-1}) = -2/27$ .  $\square$

## 5. The second-order Hankel determinants of inverse logarithmic coefficients

**Theorem 5.1.** *If  $F \in \mathcal{R}_T$ , then*

$$|\Gamma_1\Gamma_3 - \Gamma_2^2| \leq \frac{1}{36}.$$

*The bound is sharp.*

*Proof.* Let  $F \in \mathcal{R}_T$ . From (1.13) and (2.9)–(2.11), it follows that

$$|\Gamma_1\Gamma_3 - \Gamma_2^2| = \frac{1}{2304} |72\xi_3\xi_1 - 48\xi_1^2\xi_2 + 15\xi_1^4 - 64\xi_2^2|. \quad (5.1)$$

From (5.1) and Lemma 1.6, we obtain

$$\begin{aligned} |\Gamma_1\Gamma_3 - \Gamma_2^2| &= \frac{1}{2304} \left| -72(|\kappa_1|^2 - |\kappa_1|^4)\kappa_2^2 + 72\kappa_3\kappa_1(1 - |\kappa_1|^2)(1 - |\kappa_2|^2) + 15\kappa_1^4 \right. \\ &\quad \left. - 48\kappa_1^2(1 - |\kappa_1|^2)\kappa_2 - 64(1 - |\kappa_1|^2)^2\kappa_2^2 \right|. \end{aligned}$$

Putting  $|\kappa_2| = \sigma \in [0, 1]$ ,  $\kappa = |\kappa_1| \in [0, 1]$  and taking  $|\kappa_3| \leq 1$ , we obtain

$$\begin{aligned} |\Gamma_1\Gamma_3 - \Gamma_2^2| &\leq \frac{1}{2304} \left[ 72(\kappa^2 - \kappa^4)\sigma^2 + 72(\kappa - \kappa^3)(1 - \sigma^2) + 15\kappa^4 + 48(\kappa^2 - \kappa^4)\sigma + 64(1 - \kappa^2)^2\sigma^2 \right] \\ &= \frac{1}{2304} \rho_6(\kappa, \sigma). \end{aligned}$$

Differentiating  $\rho_6(\kappa, \sigma)$  with respect to  $\sigma$ , we yield

$$\frac{\partial \rho_6}{\partial \sigma} = 48(\kappa^2 - \kappa^4) + 16(8 - 9\kappa + \kappa^2)(1 - \kappa^2)\sigma \geq 0.$$

Therefore,  $\rho_6(\kappa, \sigma)$  attains its maximum value at  $\sigma = 1$ . Thus, we have

$$|\Gamma_1\Gamma_3 - \Gamma_2^2| \leq \rho_6(\kappa, 1) = \frac{1}{2304} [64 - 8\kappa^2 - 41\kappa^4] \leq \frac{64}{2304} = \frac{1}{36}.$$

The equality holds for  $\mu_2 = 0$  and  $\mu_3 = 1/3$ . □

**Theorem 5.2.** *If  $F \in \mathcal{R}_T$ , then*

$$|\Gamma_2\Gamma_4 - \Gamma_3^2| \leq \frac{5}{216}.$$

*The result is sharp.*

*Proof.* Let  $F \in \mathcal{R}_T$ . From (1.14) and (2.9)–(2.12), we obtain

$$\begin{aligned} \Gamma_2\Gamma_4 - \Gamma_3^2 &= \frac{1}{276480} (4608\xi_2\xi_4 + 672\xi_1^2\xi_2^2 + 8640\xi_1\xi_2\xi_3 - 6400\xi_2^3 - 4296\xi_1^4\xi_2 - 5184\xi_1^2\xi_4 \\ &\quad + 4680\xi_1^3\xi_3 - 4320\xi_3^2 + 1095\xi_1^6). \end{aligned} \quad (5.2)$$

From (5.2) and Lemma 1.6, it follows that

$$\Gamma_2\Gamma_4 - \Gamma_3^2 = \frac{1}{276480} \left[ 672(1 - |\kappa_1|^2)^2\kappa_2^2\kappa_1^2 - 8640\kappa_2^3(1 - |\kappa_1|^2)^2|\kappa_1|^2 + 8640\kappa_3\kappa_1\kappa_2(1 - |\kappa_1|^2)^2 \right]$$

$$\begin{aligned}
& \times (1 - |\kappa_2|^2) + 288\kappa_2^4(1 - |\kappa_1|^2)^2\bar{\kappa}_1^2 - 4608\kappa_3^2|\kappa_2|^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2 - 576\kappa_3 \\
& \times \kappa_2^2\bar{\kappa}_1(1 - |\kappa_1|^2)^2(1 - |\kappa_2|^2) + 4608\kappa_2(1 - |\kappa_3|^2)(1 - |\kappa_2|^2)(1 - |\kappa_1|^2)^2\kappa_4 \\
& - 4320(1 - |\kappa_2|^2)^2(1 - |\kappa_1|^2)^2\kappa_3^2 - 4680\kappa_2^2(1 - |\kappa_1|^2)\kappa_1^3\bar{\kappa}_1 + 4680\kappa_1^3\kappa_3(1 - |\kappa_2|^2) \\
& \times (1 - |\kappa_1|^2) - 5184\kappa_2^3(1 - |\kappa_1|^2)\kappa_1^2\bar{\kappa}_1^2 + 1095\kappa_1^6 + 5184\kappa_3^2\bar{\kappa}_2\kappa_1^2(1 - |\kappa_2|^2)(1 - |\kappa_1|^2) \\
& + 10368\kappa_3\bar{\kappa}_1\kappa_1^2\kappa_2(1 - |\kappa_1|^2)(1 - |\kappa_2|^2) - 5184\kappa_4\kappa_1^2(1 - |\kappa_3|^2)(1 - |\kappa_1|^2)(1 - |\kappa_2|^2) \\
& - 4296\kappa_2(1 - |\kappa_1|^2)\kappa_1^4 - 6400(1 - |\kappa_1|^2)^3\kappa_2^3 \Big].
\end{aligned}$$

Therefore, we obtain

$$\Gamma_2\Gamma_4 - \Gamma_3^2 = \frac{1}{276480} \left[ \varrho_{13}(\kappa_1, \kappa_2) + \varrho_{14}(\kappa_1, \kappa_2)\kappa_3 + \varrho_{15}(\kappa_1, \kappa_2)\kappa_3^2 + \varrho_{16}(\kappa_1, \kappa_2, \kappa_3)\kappa_4 \right],$$

where

$$\begin{aligned}
\varrho_{13}(\kappa_1, \kappa_2) &= 1095\kappa_1^6 + (1 - |\kappa_1|^2) \left[ (288\bar{\kappa}_1^2\kappa_2^4 - 2240\kappa_2^3|\kappa_1|^2 - 6400\kappa_2^3 + 672\kappa_2^2\kappa_1^2)(1 - |\kappa_1|^2) \right. \\
&\quad \left. - 4296\kappa_2\kappa_1^4 - 5184\kappa_2^3\kappa_1^2\bar{\kappa}_1^2 - 4680\bar{\kappa}_1\kappa_1^3\kappa_2^2 \right], \\
\varrho_{14}(\kappa_1, \kappa_2) &= (1 - |\kappa_2|^2)(1 - |\kappa_1|^2) \left[ (8640\kappa_2\kappa_1 - 576\kappa_2^2\bar{\kappa}_1)(1 - |\kappa_1|^2) + 10368\kappa_2\kappa_1^2\bar{\kappa}_1 + 4608\kappa_1^3 \right], \\
\varrho_{15}(\kappa_1, \kappa_2) &= (1 - |\kappa_1|^2)(1 - |\kappa_2|^2) \left[ (-4320 - 288|\kappa_2|^2)(1 - |\kappa_1|^2) + 5184\bar{\kappa}_2\kappa_1^2 \right], \\
\varrho_{16}(\kappa_1, \kappa_2, \kappa_3) &= (1 - |\kappa_2|^2)(1 - |\kappa_1|^2)(1 - |\kappa_3|^2) \left[ -5184\kappa_1^2 + 4608(1 - |\kappa_1|^2)\kappa_2 \right].
\end{aligned}$$

Setting  $\sigma = |\kappa_2| \in [0, 1]$ ,  $|\kappa_3| = \aleph \in [0, 1]$ ,  $|\kappa_1| = \kappa \in [0, 1]$  and using  $|\kappa_4| \leq 1$ , we achieve

$$\begin{aligned}
|\Gamma_2\Gamma_4 - \Gamma_3^2| &\leq \frac{1}{276480} \left( |\varrho_{13}(\kappa_1, \kappa_2)| + |\varrho_{14}(\kappa_1, \kappa_2)|\aleph + |\varrho_{15}(\kappa_1, \kappa_2)|\aleph^2 + |\varrho_{16}(\kappa_1, \kappa_2, \kappa_3)| \right) \\
&\leq \frac{1}{276480} \Theta_4(\kappa, \sigma, \aleph),
\end{aligned}$$

where

$$\Theta_4(\kappa, \sigma, \aleph) = \chi_{13}(\kappa, \sigma) + \chi_{14}(\kappa, \sigma)\aleph + \chi_{15}(\kappa, \sigma)\aleph^2 + \chi_{16}(\kappa, \sigma)(1 - \aleph^2),$$

with

$$\begin{aligned}
\chi_{13}(\kappa, \sigma) &= 1095\kappa^6 + (1 - \kappa^2) \left[ (672\kappa^2\sigma^2 + 2240\kappa^2\sigma^3 + 6400\sigma^3 + 288\sigma^4\kappa^2)(1 - \kappa^2) + 4296\kappa^4\sigma \right. \\
&\quad \left. + 4680\sigma^2\kappa^4 + 5184\sigma^3\kappa^4 \right], \\
\chi_{14}(\kappa, \sigma) &= (1 - \sigma^2)(1 - \kappa^2) \left[ (8640\kappa\sigma + 576\sigma^2\kappa)(1 - \kappa^2) + 10368\sigma\kappa^3 + 4680\kappa^3 \right], \\
\chi_{15}(\kappa, \sigma) &= (1 - \kappa^2)(1 - \sigma^2) \left[ (4320 + 288\sigma^2)(1 - \kappa^2) + 5184\kappa^2\sigma \right], \\
\chi_{16}(\kappa, \sigma) &= (1 - \kappa^2)(1 - \sigma^2) \left[ 5184\kappa^2 + 4608\sigma(1 - \kappa^2) \right].
\end{aligned}$$

Note that

$$\chi_{15}(\kappa, \sigma) \leq (1 - \sigma^2)(1 - \kappa^2) \left[ 5184\kappa^2 + (1 - \kappa^2)(288\sigma^2 + 4320) \right] = \epsilon_{15}(\kappa, \sigma).$$

Take  $\chi_i = \epsilon_i$  for  $i = 13, 14, 16$  and

$$\delta_4(\kappa, \sigma, \aleph) = \epsilon_{13}(\kappa, \sigma) + \epsilon_{14}(\kappa, \sigma)\aleph + \epsilon_{15}(\kappa, \sigma)\aleph^2 + \epsilon_{16}(\kappa, \sigma)(1 - \aleph^2).$$

Differentiating  $\delta_4(\kappa, \sigma, \aleph)$  with respect to  $\aleph$ , we obtain

$$\begin{aligned}\frac{\partial \delta_4}{\partial \aleph} &= \epsilon_{14}(\kappa, \sigma) + 2[\epsilon_{15}(\kappa, \sigma) - \epsilon_{16}(\kappa, \sigma)]\aleph \\ &= \epsilon_{14}(\kappa, \sigma) + 576\aleph(\sigma^2 - 16\sigma + 15)(1 - \sigma^2)(1 - \kappa^2)^2 \geq 0.\end{aligned}$$

Therefore,  $\delta_4(\kappa, \sigma, \aleph)$  is an increasing function for  $\aleph \in [0, 1]$ . Thus, we obtain

$$\begin{aligned}\Theta_4(\kappa, \sigma, \aleph) &\leq \delta_4(\kappa, \sigma, \aleph) \leq \delta_4(\kappa, \sigma, 1) = \epsilon_{13}(\kappa, \sigma) + \epsilon_{14}(\kappa, \sigma) + \epsilon_{15}(\kappa, \sigma) \\ &= 4320 - 3456\kappa^2 + 4680\kappa^3 - 864\kappa^4 - 4680\kappa^5 + 1095\kappa^6 + (-288 - 576\kappa + 864\kappa^2 \\ &\quad + 1152\kappa^3 - 864\kappa^4 - 576\kappa^5 + 288\kappa^6)\sigma^4 + (6400 - 8640\kappa - 10560\kappa^2 + 6912\kappa^3 + 7104\kappa^4 \\ &\quad + 1728\kappa^5 - 2944\kappa^6)\sigma^3 + (-4032 + 576\kappa + 3552\kappa^2 - 5832\kappa^3 + 4488\kappa^4 + 5256\kappa^5 \\ &\quad - 4008\kappa^6)\sigma^2 + (8640\kappa - 6912\kappa^3 + 4296\kappa^4 - 1728\kappa^5 - 4296\kappa^6)\sigma = \rho_7(\kappa, \sigma).\end{aligned}$$

Consider

$$\left\{ \begin{array}{l} \frac{\partial \rho_7}{\partial \kappa} = -6912\kappa + 14040\kappa^2 - 3456\kappa^3 - 23400\kappa^4 + 6570\kappa^5 + (-576 + 1728\kappa + 3456\kappa^2 - 3456\kappa^3 \\ \quad - 2880\kappa^4 + 1728\kappa^5)\sigma^4 + (-8640 - 21120\kappa + 20736\kappa^2 + 28416\kappa^3 + 8640\kappa^4 - 17664\kappa^5)\sigma^3 \\ \quad + (576 + 7104\kappa - 17496\kappa^2 + 17952\kappa^3 + 26280\kappa^4 - 24048\kappa^5)\sigma^2 + (8640 - 20736\kappa^2 \\ \quad + 17184\kappa^3 - 8640\kappa^4 - 25776\kappa^5)\sigma = 0, \\ \frac{\partial \rho_7}{\partial \sigma} = -4296\kappa^6 - 1728\kappa^5 + 4296\kappa^4 - 6912\kappa^3 + 8640\kappa + (-1152 - 2304\kappa + 3456\kappa^2 + 4608\kappa^3 \\ \quad - 3456\kappa^4 - 2304\kappa^5 + 1152\kappa^6)\sigma^3 + (19200 - 25920\kappa - 31680\kappa^2 + 20736\kappa^3 + 21312\kappa^4 \\ \quad + 5184\kappa^5 - 8832\kappa^6)\sigma^2 + (-8064 + 1152\kappa + 7104\kappa^2 - 11664\kappa^3 + 8976\kappa^4 + 10512\kappa^5 \\ \quad - 8016\kappa^6)\sigma = 0. \end{array} \right.$$

By calculating, we yield

$$\begin{array}{llll} \left\{ \begin{array}{l} \kappa_{1,1} \approx 1, \\ \sigma_{1,1} \approx 1.4835, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,2} \approx 1, \\ \sigma_{1,2} \approx -2.0703, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,3} \approx 1, \\ \sigma_{1,3} \approx -0.4132, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,4} \approx -1, \\ \sigma_{1,4} \approx 0.6842, \end{array} \right. \\ \left\{ \begin{array}{l} \kappa_{1,5} = 0, \\ \sigma_{1,5} = 0, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,6} \approx 3.0534 \\ \sigma_{1,6} \approx 21.7586, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,7} \approx -0.0654, \\ \sigma_{1,7} \approx -0.0601, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,8} \approx -0.7457, \\ \sigma_{1,8} \approx -68.1509, \end{array} \right. \\ \left\{ \begin{array}{l} \kappa_{1,9} \approx -0.8372, \\ \sigma_{1,9} \approx 0.3880, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,10} \approx -0.9796, \\ \sigma_{1,10} \approx -0.4535, \end{array} \right. & \left\{ \begin{array}{l} \kappa_{1,11} \approx -1.6359, \\ \sigma_{1,11} \approx 12.7029. \end{array} \right. & \end{array}$$

Hence, there is no solution in  $(0, 1) \times (0, 1)$ .

(16) For  $\kappa = 1$ ,

$$\rho_7(1, \sigma) = 1095.$$

(17) For  $\kappa = 0$ ,

$$\rho_7(0, \sigma) = -288\sigma^4 + 6400\sigma^3 - 4032\sigma^2 + 4320 \leq 6400.$$

(18) For  $\sigma = 0$ ,

$$\rho_7(\kappa, 0) = 1095\kappa^6 - 4680\kappa^5 - 864\kappa^4 + 4680\kappa^3 - 3456\kappa^2 + 4320 \leq 4320.$$

(19) For  $\sigma = 1$ ,

$$\rho_7(\kappa, 1) = 6400 - 9600\kappa^2 + 14160\kappa^4 - 9865\kappa^6 \leq 6400.$$

Therefore, we obtain

$$|\Gamma_2\Gamma_4 - \Gamma_3^2| \leq \frac{6400}{276480} = \frac{5}{216}.$$

For the Schwarz function  $\varpi(v) = v^2$ , Eqs (2.9)–(2.12) give  $\mu_3 = 1/3$  and  $\mu_5 = \mu_2 = \mu_4 = 0$ , and Eq (1.14) leads to  $\Gamma_2\Gamma_4 - \Gamma_3^2 = -5/216$ .  $\square$

## 6. Conclusions

In this paper, we studied the class of analytic functions  $F$  in the unit disk satisfying  $F'(v) < 1 + \arctan v$ . We obtained coefficient estimates for

$$F(v) = v + \sum_{m=2}^{\infty} \mu_m v^m,$$

upper bounds for the second- and third-order Hankel determinants  $\mathcal{H}_{2,1}(F)$  and  $\mathcal{H}_{3,1}(F)$ , a bound for the second-order Hankel determinant of the logarithmic coefficients, a bound for the third-order Hankel determinant of the inverse function, and a sharp upper bound for the second-order Hankel determinant of the inverse logarithmic coefficients. These results describe the initial coefficient and Hankel determinant behavior of the class. Possible directions for further work include sharpness of the remaining estimates, higher-order Hankel determinants, generalized subordination, and other functionals such as the Zalcman functionals and the initial higher-order Schippers-Schwarzian derivatives.

## Author contributions

Dong Guo: writing–original draft preparation, editing and writing–review; Zongtao Li: conceptualization, software, formal analysis, writing–review; Xi Luo: software, investigation, writing–original draft preparation. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there is no competing interest.

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