



Research article

A projection activation inertial BAM neural network for solving fixed points of extremal mappings with coupled constraints

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Abstract: This paper develops a projection activation inertial bidirectional associative memory (BAM) neural network allowing delayed feedback for computing fixed points of extremal mappings with coupled constraints. In this problem, the unknown point is both a parameter and an optimizer of an inner constrained optimization problem. By combining the Karush–Kuhn–Tucker (KKT) conditions with the projection form of normal cone inclusions, the fixed point problem is reformulated as a projection residual equation. Different from BAM neural networks with standard activation functions, the proposed model uses the projection mapping itself as the activation mapping, so that the constraint set, multiplier nonnegativity, and residual structure are embedded into the neural dynamics. Under a matching condition on the connection matrices, the equilibria of the proposed network are proved to coincide with the zeros of the projection residual, and therefore correspond to the fixed points of the original extremal mapping. A conditional exponential stability criterion on a prescribed operating domain is further established by using matrix measure estimates and Halanay’s inequality. Numerical examples and a multi-actuator output allocation application confirm the theoretical stability condition and demonstrate the convergence accuracy, boundary projection ability, scalability, and delay-dependent performance of the proposed method.

Keywords: fixed points of extremal mappings; coupled constraints; projection residual; projection activation; inertial BAM neural network; exponential stability

Mathematics Subject Classification: 34K20, 90C30, 68T07

1. Introduction

Nonlinear neural dynamics have become an important continuous time approach for solving optimization and equilibrium problems, since they can combine parallel implementation with dynamical stability analysis. In many engineering and machine learning systems, the underlying optimization process is not isolated or static. Instead, it often contains state dependent objectives, coupled constraints, and self consistent feedback structures. The current state may affect the objective function and the feasible set of the next optimization step, while the optimized decision is required to be consistent with the current state. Such feedback mechanisms naturally lead to nonlinear dynamical formulations of constrained optimization problems. Fixed points of extremal mappings with coupled constraints therefore provide a suitable mathematical framework for describing these self consistent optimization mechanisms.

Motivated by the above observations, this paper considers the problem of solving fixed points of extremal mappings with coupled constraints: find $v^* \in \Omega$ such that

$$v^* \in \operatorname{Argmin} \{ \Phi(v^*, w) \mid g(v^*, w) \leq 0, w \in \Omega \}. \quad (1.1)$$

Here, $\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, and $\Omega \subseteq \mathbb{R}^n$ is a nonempty closed convex set. For each fixed $v \in \Omega$, the function $w \mapsto \Phi(v, w)$ and each component $w \mapsto g_i(v, w)$, $i = 1, \dots, m$, are assumed to be convex and continuously differentiable on Ω . Moreover, the parameter dependent feasible set $K(v) := \{w \in \Omega \mid g(v, w) \leq 0\}$ is assumed to be nonempty for every $v \in \Omega$. The associated extremal mapping is given by $S(v) := \operatorname{Argmin} \{ \Phi(v, w) \mid g(v, w) \leq 0, w \in \Omega \}$, and the solution set is denoted by $\Omega^* := \{v^* \in \Omega \mid v^* \in S(v^*)\}$, which is assumed to be nonempty. In this formulation, v^* appears both as the parameter of the inner constrained optimization problem and as the optimizer to be determined.

The study of fixed points of extremal mappings is rooted in set valued analysis, since the solution set of a parametric optimization problem usually depends on the parameter in a set valued manner. The monograph of Aubin and Frankowska [1] laid an important foundation for the analysis of such solution mappings. In this direction, Antipin [2] studied proximal schemes for finding fixed points of extremal mappings and derived convergence rate estimates. Flam and Antipin [3] further connected this idea with equilibrium programming through proximal-like algorithms. Besides discrete iterative schemes, continuous time methods have also been explored. Antipin [4] introduced differential prediction type gradient methods for fixed points of extremal mappings, and later extended the dynamical system approach to equilibrium problems with coupled constraints [5]. For fixed point problems with coupled constraints, Wang, Shan and Zhang [6] proposed an implementable augmented Lagrange method, in which the inner fixed point subproblems are solved by a semismooth Newton procedure. These works provide useful theoretical and computational tools. Nevertheless, most existing approaches are developed from proximal, gradient, or augmented Lagrange viewpoints, and they do not directly yield a projection residual neural dynamics whose equilibria can be identified with the solutions of problem (1.1).

More recently, projection based neurodynamic methods have also been investigated for constrained optimization and nonlinear projection equations. Accelerated projection neurodynamic approaches have been developed for constrained convex optimization [7], while modified projection neural networks have been studied for nonlinear projection equations with prescribed convergence properties [8]. These developments further illustrate the usefulness of projection based neural dynamics for constrained

optimization problems.

Bidirectional associative memory (BAM) neural networks provide a natural dynamical architecture for describing bidirectional interactions between different groups of variables. This feature is particularly useful for constrained optimization problems, where primal variables and multipliers are intrinsically coupled through optimality conditions. Inertial BAM neural networks further introduce a second order dynamical structure, which can improve the flexibility of the transient behavior. Time delays may arise in neural and engineering systems due to memory effects, communication, or signal transmission. Cao and Wan [9] studied delayed inertial BAM neural networks and derived exponential stability criteria by transforming the second order model into a first order system and using matrix measures together with Halanay's inequality. Recent studies have continued to investigate the dynamical properties of delayed inertial neural networks and inertial BAM neural networks. Related stability and delay dependent behaviors have been studied in [10–15]. These studies show that inertial and delayed BAM dynamics remain an active topic of current research. However, the activation functions in existing BAM models are usually standard nonlinear functions, and such models are not directly tailored to the projection residual equations associated with problem (1.1).

Motivated by these observations, this paper develops a nonlinear projection activation inertial BAM neural dynamic model for solving problem (1.1). The main idea is to characterize the inner constrained optimization problem by its Karush–Kuhn–Tucker (KKT) conditions and then rewrite the resulting normal cone inclusion in a projection form. This leads to a projection mapping $T(z)$ and a residual equation $R(z) = z - T(z) = 0$, whose zeros characterize the solutions of problem (1.1). Different from neural networks with standard activation functions, the proposed model uses $T(z)$ itself as the activation mapping. In this way, the constraint set, multiplier nonnegativity, and projection structure are directly embedded into the neural dynamics. The resulting model combines the projection residual formulation with an inertial delayed BAM structure, which allows us to establish the equilibrium characterization and a conditional exponential stability criterion for the proposed solver.

The novelty and main contributions of this paper can be summarized as follows:

- A projection residual characterization is established for problem (1.1). By combining the KKT conditions of the inner convex problem with the projection form of normal cone inclusions, the fixed point condition is equivalently converted into a projection residual equation associated with $C = \Omega \times \mathbb{R}_+^m$.
- A nonlinear projection activation inertial BAM neural dynamic model allowing delayed feedback is proposed. Different from standard BAM neural networks with ordinary activation functions, the proposed model uses the projection mapping as the activation mapping, so that the primal constraint set, multiplier nonnegativity, and residual structure are directly embedded into the neural dynamics.
- The exact relationship between the proposed neural dynamics and problem (1.1) is proved. Under the matching condition $B_0 = C_0 + D_0$, the equilibria of the proposed model are shown to coincide with the residual zeros, and hence correspond to solutions of problem (1.1).
- A conditional exponential stability criterion on a prescribed operating domain is derived by using matrix measure estimates and Halanay's inequality. A verifiable stability margin is introduced, and the theoretical condition is explicitly checked in the numerical examples. Certified numerical tests further demonstrate boundary projection, scalability, delay dependent behavior, and finite time convergence performance.

The rest of this paper is organized as follows. Section 2 introduces the basic notation, definitions, lemmas, and assumptions. Section 3 derives the projection residual characterization of problem (1.1). Section 4 presents the nonlinear projection activation inertial BAM neural dynamic model. Section 5 analyzes the conditional exponential stability of the proposed neural dynamics. Section 6 reports comparative numerical experiments. Section 7 presents an engineering application to multi-actuator output allocation under power, thermal, and current constraints. Section 8 gives concluding comments and discusses future work.

2. Preliminaries

Notation 2.1. The following notation will be used throughout this paper.

- 1) The nonnegative orthant of $\mathbb{R}^m$ is denoted by

$$\mathbb{R}_+^m := \{\lambda \in \mathbb{R}^m \mid \lambda_i \geq 0, i = 1, \dots, m\}.$$

- 2) For a continuously differentiable function

$$g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m,$$

the symbol

$$\nabla_w g(v, w) \in \mathbb{R}^{m \times n}$$

denotes the Jacobian matrix of g with respect to its second argument w .

- 3) For a continuously differentiable function

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R},$$

the symbol

$$\nabla_w \Phi(v, w) \in \mathbb{R}^n$$

denotes the gradient of Φ with respect to its second argument w .

Definition 2.2. Let $C \subset \mathbb{R}^q$ be a nonempty closed convex set. For any $x \in \mathbb{R}^q$, the Euclidean projection of x onto C is defined by

$$\Pi_C(x) := \arg \min_{y \in C} \|x - y\|.$$

The normal cone of C at $x \in C$ is defined by

$$N_C(x) := \{d \in \mathbb{R}^q \mid \langle d, y - x \rangle \leq 0, \forall y \in C\}.$$

These notions are standard in convex analysis and variational inequality theory; see, for example, [16, 17].

Definition 2.3. Let $\|\cdot\|_p$ be an induced matrix norm on $\mathbb{R}^{q \times q}$. The matrix measure associated with $\|\cdot\|_p$ is defined by

$$\mu_p(A) := \lim_{h \rightarrow 0^+} \frac{\|I_q + hA\|_p - 1}{h}, \quad A \in \mathbb{R}^{q \times q}.$$

In particular,

$$\mu_1(A) = \max_{1 \leq j \leq q} \left\{ a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^q |a_{ij}| \right\},$$

$$\mu_\infty(A) = \max_{1 \leq i \leq q} \left\{ a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^q |a_{ij}| \right\},$$

and

$$\mu_2(A) = \lambda_{\max} \left(\frac{A + A^\top}{2} \right).$$

The matrix measure has been widely used in stability analysis of nonlinear dynamical systems and delayed neural networks [9, 18].

Definition 2.4. Consider a delayed second-order dynamical system of the form

$$\ddot{z}(t) = \mathcal{G}(z(t), \dot{z}(t), z(t - \tau(t))),$$

where $z(t) \in \mathbb{R}^q$ and $0 \leq \tau(t) \leq \bar{\tau}$. A point $z^* \in \mathbb{R}^q$ is called an equilibrium point of the system if the constant trajectory

$$z(t) \equiv z^*$$

satisfies the system. Equivalently, it holds that

$$\mathcal{G}(z^*, 0, z^*) = 0.$$

Lemma 2.1 (Projection-normal cone equivalence [16, 17]). Let $C \subset \mathbb{R}^q$ be nonempty, closed, and convex. Let $G : \mathbb{R}^q \rightarrow \mathbb{R}^q$ be a given mapping and let $\alpha > 0$. Then $x \in C$ satisfies

$$x = \Pi_C(x - \alpha G(x))$$

if and only if

$$0 \in G(x) + N_C(x).$$

Lemma 2.2. Let $A \in \mathbb{R}^{q \times q}$, and let $x : [t_0, \infty) \rightarrow \mathbb{R}^q$ be absolutely continuous. If

$$\dot{x}(t) = Ax(t) + r(t),$$

then, for any induced norm $\|\cdot\|_p$,

$$D^+ \|x(t)\|_p \leq \mu_p(A) \|x(t)\|_p + \|r(t)\|_p,$$

where D^+ denotes the upper-right Dini derivative.

Proof. The result follows from the standard differential inequality associated with matrix measures [18]. Indeed, by the definition of the upper-right Dini derivative and the induced matrix measure, the homogeneous part satisfies

$$D^+ \|x(t)\|_p \leq \mu_p(A) \|x(t)\|_p$$

when $r(t) = 0$. The additional term $r(t)$ is then estimated by the triangle inequality, which gives

$$D^+ \|x(t)\|_p \leq \mu_p(A) \|x(t)\|_p + \|r(t)\|_p.$$

□

Lemma 2.3 (Halanay inequality). *Let $q_1 > q_2 > 0$, and let $x(t)$ be a nonnegative continuous function on $[t_0 - \tau, \infty)$, where $\tau \geq 0$. If*

$$D^+ x(t) \leq -q_1 x(t) + q_2 \sup_{t-\tau \leq s \leq t} x(s), \quad t \geq t_0,$$

then

$$x(t) \leq \left(\sup_{t_0 - \tau \leq s \leq t_0} x(s) \right) e^{-r(t-t_0)}, \quad t \geq t_0,$$

where $r > 0$ is the unique positive solution of

$$r = q_1 - q_2 e^{r\tau}.$$

Proof. This is the classical Halanay inequality for delayed differential inequalities [19]. It has also been used in matrix-measure-based stability analysis of inertial BAM neural networks with time-varying delays [9]. □

For later use, write $g(v, w) = (g_1(v, w), \dots, g_m(v, w))^T$. The following additional assumptions are imposed throughout the paper.

Assumption 2.1 (Constraint qualification at solution points). *At every fixed point solution v^* considered in this paper, the corresponding inner convex optimization problem satisfies a constraint qualification. In particular, it is sufficient to assume the Slater condition: for every such v^* , there exists $\bar{w} \in \Omega$ such that*

$$g_i(v^*, \bar{w}) < 0, \quad i = 1, \dots, m.$$

Remark 2.1. *Assumption 2.1 is used only for the optimization-level characterization of problem (1.1). More precisely, it ensures that the fixed point condition can be characterized by the KKT conditions of the corresponding inner convex optimization problem, and then by the projection residual equation. This constraint qualification is imposed at the fixed point solutions under consideration. It is not imposed on arbitrary transient neural states generated by the neural dynamics and is not used in the matrix measure based stability estimate.*

Assumption 2.2. *Let*

$$F(z) := \begin{pmatrix} \nabla_w \Phi(v, v) + \nabla_w g(v, v)^T \lambda \\ -g(v, v) \end{pmatrix}, \quad z = \begin{pmatrix} v \\ \lambda \end{pmatrix}.$$

There exists a domain $\mathcal{D} \subset \mathbb{R}^{n+m}$ that contains the equilibrium pairs $z^* = (v^*, \lambda^*)^\top$ considered in this paper and the neural network trajectories used in the stability analysis. The mapping F is Lipschitz continuous on $\mathcal{D}$; that is, there exists a constant $L_F > 0$ such that, for all $z_1, z_2 \in \mathcal{D}$,

$$\|F(z_1) - F(z_2)\| \leq L_F \|z_1 - z_2\|.$$

Remark 2.2. Assumption 2.2 is imposed for the stability analysis of the projection-activation inertial BAM neural network. Together with the nonexpansiveness of Euclidean projections onto closed convex sets, it ensures the Lipschitz continuity of the projection-activation mapping introduced in Section 3.

3. Problem formulation and projection-residual characterization

In this section, we derive a projection-residual characterization for problem (1.1). The goal is to reformulate the original fixed point condition as an equivalent residual equation, which will serve as the basis for the inertial BAM-type neural network developed in the next section.

Recall from the Introduction that, for each $v \in \Omega$, the parameter-dependent feasible set is $K(v) := \{w \in \Omega \mid g(v, w) \leq 0\}$, and the associated extremal mapping is $S(v) := \text{Argmin}\{\Phi(v, w) \mid g(v, w) \leq 0, w \in \Omega\}$. Thus, problem (1.1) can be written compactly as $v^* \in S(v^*)$. Here, w denotes the decision variable of the inner optimization problem, while v^* acts both as the parameter and as the fixed point candidate. Hence, the first-order optimality conditions of the inner problem should be derived with respect to the second argument w .

For a fixed $v \in \Omega$, the inner optimization problem associated with $S(v)$ is

$$\begin{aligned} \min_{w \in \Omega} \quad & \Phi(v, w), \\ \text{s.t.} \quad & g(v, w) \leq 0. \end{aligned} \tag{3.1}$$

Under the standing assumptions stated in the Introduction, problem (3.1) is a convex optimization problem with respect to the decision variable w . When v is taken as a fixed point solution v^* under consideration, Assumption 2.1 guarantees the required constraint qualification for the corresponding inner problem. Therefore, the global minimizers of this inner convex problem can be characterized by the KKT conditions [16, 17]. This use of the constraint qualification is at the optimization characterization level and is unrelated to the transient neural states in the later stability analysis.

Theorem 3.1 (KKT characterization of fixed point solutions). *Suppose that the standing assumptions stated in the Introduction hold. Assume further that Assumption 2.1 holds at the fixed point solution under consideration. Then $v^* \in \Omega$ is a fixed point solution of problem (1.1), namely $v^* \in S(v^*)$, if and only if there exists a multiplier $\lambda^* \in \mathbb{R}_+^m$ such that*

$$0 \in \nabla_w \Phi(v^*, v^*) + \nabla_w g(v^*, v^*)^\top \lambda^* + N_\Omega(v^*), \tag{3.2}$$

$$g(v^*, v^*) \leq 0, \tag{3.3}$$

$$\lambda^* \geq 0, \tag{3.4}$$

and

$$\lambda_i^* g_i(v^*, v^*) = 0, \quad i = 1, \dots, m. \tag{3.5}$$

Proof. Assume first that $v^* \in S(v^*)$. Then v^* is a global solution of the convex inner problem

$$\min_{w \in \Omega} \Phi(v^*, w) \quad \text{s.t.} \quad g(v^*, w) \leq 0.$$

By the standing convexity assumptions stated in the Introduction, the objective function and the inequality constraint functions are convex with respect to w . By Assumption 2.1, the required constraint qualification holds for the inner problem associated with this fixed point solution v^* . Therefore, the KKT necessary conditions hold for this inner convex program. Hence, there exists $\lambda^* \in \mathbb{R}_+^m$ satisfying (3.2)–(3.5).

Conversely, suppose that there exists $\lambda^* \in \mathbb{R}_+^m$ satisfying (3.2)–(3.5). For any $w \in K(v^*)$, the normal-cone inclusion (3.2) implies

$$\langle \nabla_w \Phi(v^*, v^*) + \nabla_w g(v^*, v^*)^\top \lambda^*, w - v^* \rangle \geq 0. \quad (3.6)$$

By the convexity of $w \mapsto \Phi(v^*, w)$, one has

$$\Phi(v^*, w) - \Phi(v^*, v^*) \geq \langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle. \quad (3.7)$$

For each $i = 1, \dots, m$, the convexity of $w \mapsto g_i(v^*, w)$ gives

$$g_i(v^*, w) - g_i(v^*, v^*) \geq \langle \nabla_w g_i(v^*, v^*), w - v^* \rangle.$$

Multiplying the above inequality by $\lambda_i^* \geq 0$ and summing over i , we obtain

$$\langle \nabla_w g(v^*, v^*)^\top \lambda^*, w - v^* \rangle \leq \langle \lambda^*, g(v^*, w) - g(v^*, v^*) \rangle. \quad (3.8)$$

Since $w \in K(v^*)$, one has $g(v^*, w) \leq 0$. Together with $g(v^*, v^*) \leq 0$, $\lambda^* \geq 0$, and

$$\langle \lambda^*, g(v^*, v^*) \rangle = 0,$$

it follows that

$$\langle \lambda^*, g(v^*, w) - g(v^*, v^*) \rangle \leq 0. \quad (3.9)$$

Combining (3.6)–(3.9), we obtain

$$\Phi(v^*, w) - \Phi(v^*, v^*) \geq 0, \quad \forall w \in K(v^*).$$

Therefore,

$$v^* \in \operatorname{Argmin} \{ \Phi(v^*, w) \mid w \in K(v^*) \} = S(v^*).$$

This proves the assertion. $\square$

We now rewrite the KKT system in a compact generalized-equation form. Let

$$z := \begin{pmatrix} v \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+m}, \quad C := \Omega \times \mathbb{R}_+^m. \quad (3.10)$$

Consistent with Assumption 2.2, define the diagonal KKT-type mapping $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$ by

$$F(z) := \begin{pmatrix} F_v(v, \lambda) \\ F_\lambda(v, \lambda) \end{pmatrix} := \begin{pmatrix} \nabla_w \Phi(v, v) + \nabla_w g(v, v)^\top \lambda \\ -g(v, v) \end{pmatrix}. \quad (3.11)$$

Then the KKT system (3.2)–(3.5) can be written as

$$0 \in F(z^*) + N_C(z^*), \quad z^* = \begin{pmatrix} v^* \\ \lambda^* \end{pmatrix}. \quad (3.12)$$

Indeed, since

$$C = \Omega \times \mathbb{R}_+^m,$$

one has

$$N_C(z) = N_\Omega(v) \times N_{\mathbb{R}_+^m}(\lambda).$$

Thus, the first component of (3.12) gives the stationarity condition (3.2), while the second component is equivalent to

$$g(v^*, v^*) \leq 0, \quad \lambda^* \geq 0, \quad \lambda_i^* g_i(v^*, v^*) = 0, \quad i = 1, \dots, m.$$

For a given parameter $\alpha > 0$, define the projection-activation mapping

$$T(z) := \Pi_C[z - \alpha F(z)]. \quad (3.13)$$

Since $C = \Omega \times \mathbb{R}_+^m$, the mapping T can be decomposed as

$$T(z) = \begin{pmatrix} T_v(v, \lambda) \\ T_\lambda(v, \lambda) \end{pmatrix}, \quad (3.14)$$

where

$$T_v(v, \lambda) = \Pi_\Omega [v - \alpha (\nabla_w \Phi(v, v) + \nabla_w g(v, v)^\top \lambda)], \quad (3.15)$$

and

$$T_\lambda(v, \lambda) = \Pi_{\mathbb{R}_+^m} [\lambda + \alpha g(v, v)]. \quad (3.16)$$

The corresponding projection residual is defined by

$$R(z) := z - T(z). \quad (3.17)$$

Thus, the equation

$$R(z) = 0 \quad (3.18)$$

is equivalent to the fixed point equation

$$z = T(z). \quad (3.19)$$

Theorem 3.2 (Projection-residual characterization). *Suppose that the standing assumptions stated in the Introduction hold. Assume further that Assumption 2.1 holds at the fixed point solution under consideration. Let T and R be defined by (3.13) and (3.17), respectively. Then $v^* \in \Omega$ is a fixed point solution of problem (1.1), namely $v^* \in S(v^*)$, if and only if there exists $\lambda^* \in \mathbb{R}_+^m$ such that, with*

$$z^* = \begin{pmatrix} v^* \\ \lambda^* \end{pmatrix},$$

one has

$$R(z^*) = 0. \quad (3.20)$$

Proof. By Lemma 2.1, for any $\alpha > 0$,

$$z^* = \Pi_C[z^* - \alpha F(z^*)]$$

if and only if

$$0 \in F(z^*) + N_C(z^*).$$

By the definitions of T and R , the equation

$$z^* = \Pi_C[z^* - \alpha F(z^*)]$$

is equivalent to

$$z^* = T(z^*),$$

which is further equivalent to

$$R(z^*) = 0.$$

On the other hand, by Theorem 3.1, the generalized equation

$$0 \in F(z^*) + N_C(z^*)$$

is equivalent to the existence of a multiplier $\lambda^* \in \mathbb{R}_+^m$ such that $v^* \in S(v^*)$. Therefore,

$$v^* \in S(v^*)$$

if and only if there exists $\lambda^* \in \mathbb{R}_+^m$ such that, with

$$z^* = (v^*, \lambda^*)^\top,$$

one has

$$R(z^*) = 0.$$

This completes the proof. $\square$

Remark 3.1. The parameter $\alpha > 0$ in (3.13) is a projection stepsize. For the residual equation $R(z) = 0$, any positive α gives the same solution set, because the equivalence between the projection equation and the normal-cone inclusion holds for all $\alpha > 0$. However, in the neural-network model and its stability analysis, α affects the Lipschitz constant of T and hence influences the admissible ranges of the network parameters.

Remark 3.2. The residual equation (3.18) characterizes the fixed point condition of problem (1.1) under the convex inner-program setting and the constraint qualification at the solution point under consideration. The convexity assumptions with respect to w ensure that the KKT system is not merely a stationarity system, but an exact characterization of the global minimizers of the inner problem. Assumption 2.1 is used to justify the KKT characterization and is not part of the dynamical stability estimate.

Finally, we establish the Lipschitz property of the projection-activation mapping. This property will be used when T is embedded as the activation mapping in the inertial BAM-type neural network.

Lemma 3.1 (Lipschitz continuity of the projection-activation mapping). *Suppose that Assumption 2.2 holds. Then the projection-activation mapping T defined by (3.13) is Lipschitz continuous on $\mathcal{D}$. More precisely, for all $z_1, z_2 \in \mathcal{D}$, one has*

$$\|T(z_1) - T(z_2)\| \leq L_T \|z_1 - z_2\|,$$

where one may take

$$L_T := 1 + \alpha L_F.$$

Proof. Since C is nonempty, closed, and convex, the projection Π_C is nonexpansive. Thus, for any $z_1, z_2 \in \mathcal{D}$,

$$\begin{aligned} \|T(z_1) - T(z_2)\| &= \|\Pi_C[z_1 - \alpha F(z_1)] - \Pi_C[z_2 - \alpha F(z_2)]\| \\ &\leq \|z_1 - z_2 - \alpha(F(z_1) - F(z_2))\| \\ &\leq \|z_1 - z_2\| + \alpha\|F(z_1) - F(z_2)\|. \end{aligned}$$

By Assumption 2.2,

$$\|F(z_1) - F(z_2)\| \leq L_F \|z_1 - z_2\|.$$

Therefore,

$$\|T(z_1) - T(z_2)\| \leq (1 + \alpha L_F) \|z_1 - z_2\|.$$

This proves the claim. $\square$

Remark 3.3. *In the subsequent neural-network construction, T will be regarded as a projection-activation mapping. The residual $R(z) = z - T(z)$ measures the violation of the optimality, feasibility, and complementarity conditions associated with problem (1.1). Therefore, driving $R(z(t))$ to zero is equivalent to solving problem (1.1).*

4. Projection-activation inertial BAM neural network

In this section, we construct a projection-activation inertial BAM-type neural network for solving the projection-residual equation established in Section 3. To retain the inertial structure while allowing optional memory or transmission effects, we consider the following second-order neural network with delayed feedback:

$$\ddot{z}(t) = -A_0 \dot{z}(t) - B_0 z(t) + C_0 T(z(t)) + D_0 T(z(t - \tau(t))), \quad t \geq 0. \quad (4.1)$$

Here,

$$z(t) = \begin{pmatrix} v(t) \\ \lambda(t) \end{pmatrix} \in \mathbb{R}^q, \quad q := n + m,$$

where $v(t) \in \mathbb{R}^n$ is the primal state and $\lambda(t) \in \mathbb{R}^m$ is the multiplier state. The mapping T is given by

$$T(z) = \Pi_C[z - \alpha F(z)],$$

and is used as the activation mapping of the proposed neural network. The corresponding projection residual is

$$R(z) = z - T(z).$$

Thus, driving the neural trajectory toward the zero set of R amounts to solving problem (1.1).

The matrices

$$A_0, B_0, C_0, D_0 \in \mathbb{R}^{q \times q}$$

are constant connection matrices. Specifically, A_0 denotes the inertial damping matrix, B_0 denotes the resetting or leakage matrix, and C_0 and D_0 denote the instantaneous and delayed connection matrices, respectively. The delay function $\tau(t)$ satisfies

$$0 \leq \tau(t) \leq \bar{\tau}, \quad t \geq 0, \quad (4.2)$$

where $\bar{\tau} > 0$ is a given constant.

Remark 4.1 (Role of the delayed feedback). *The delayed feedback term is not required for the projection-residual formulation itself. It is included to describe possible memory effects or signal transmission lags in the neural implementation. When $D_0 = 0$, the matching condition reduces to $B_0 = C_0$, and system (4.1) becomes the corresponding non delayed projection-activation inertial BAM neural network. When $\tau(t) = 0$, the delayed activation becomes an instantaneous activation term. Therefore, the delayed formulation contains the non delayed model as a special case and does not change the equilibrium projection-residual characterization established below.*

The initial data are prescribed by

$$z(s) = \varphi(s), \quad \dot{z}(s) = \psi(s), \quad s \in [-\bar{\tau}, 0], \quad (4.3)$$

where φ and ψ are continuous initial functions.

To ensure that the equilibria of (4.1) are consistent with the projection-residual equation, we impose the following matching condition on the connection matrices:

$$B_0 = C_0 + D_0, \quad (4.4)$$

and assume that B_0 is nonsingular. Under the matching condition (4.4), system (4.1) can be written directly as

$$\ddot{z}(t) = -A_0 \dot{z}(t) - C_0[z(t) - T(z(t))] - D_0[z(t) - T(z(t - \tau(t)))]. \quad (4.5)$$

Hence, the instantaneous term is driven by the current projection residual, whereas the delayed term provides an additional memory feedback through $T(z(t - \tau(t)))$.

Since

$$T(z) = \begin{pmatrix} T_v(v, \lambda) \\ T_\lambda(v, \lambda) \end{pmatrix},$$

the primal state $v(t)$ and the multiplier state $\lambda(t)$ are coupled through the projection activation. More specifically, $T_v(v, \lambda)$ contains the multiplier variable through $\nabla_w g(v, v)^\top \lambda$, while $T_\lambda(v, \lambda)$ depends on the primal variable through $g(v, v)$. This gives the proposed model a bidirectional primal–multiplier coupling structure, which is consistent with the BAM-type architecture.

We next establish the relationship between the equilibria of the proposed network and the projection-residual equation.

According to Definition 2.4, a point $z^* \in \mathbb{R}^q$ is an equilibrium point of (4.1) if the constant trajectory $z(t) \equiv z^*$ satisfies (4.1).

Theorem 4.1 (Equilibrium-residual equivalence). *Suppose that the matching condition (4.4) holds and that B_0 is nonsingular. Then $z^* \in \mathbb{R}^q$ is an equilibrium point of the projection-activation inertial BAM neural network (4.1) if and only if*

$$R(z^*) = 0. \quad (4.6)$$

Proof. Assume first that z^* is an equilibrium point of (4.1). Then the constant trajectory $z(t) \equiv z^*$ satisfies

$$\dot{z}(t) = 0, \quad \ddot{z}(t) = 0, \quad z(t - \tau(t)) = z^*.$$

Substituting these identities into (4.1) gives

$$0 = -B_0 z^* + C_0 T(z^*) + D_0 T(z^*).$$

Using $B_0 = C_0 + D_0$, we have

$$0 = -B_0 z^* + B_0 T(z^*) = -B_0 [z^* - T(z^*)].$$

Since B_0 is nonsingular, it follows that

$$z^* = T(z^*),$$

which is equivalent to $R(z^*) = 0$.

Conversely, suppose that $R(z^*) = 0$. Then $z^* = T(z^*)$. For the constant trajectory $z(t) \equiv z^*$, one has

$$\dot{z}(t) = 0, \quad \ddot{z}(t) = 0, \quad z(t - \tau(t)) = z^*.$$

Therefore, by $B_0 = C_0 + D_0$,

$$-B_0 z^* + C_0 T(z^*) + D_0 T(z^*) = -(C_0 + D_0)z^* + (C_0 + D_0)z^* = 0.$$

Thus, $z(t) \equiv z^*$ satisfies (4.1), and hence z^* is an equilibrium point of the proposed neural network. $\square$

Combining Theorem 4.1 with the projection-residual characterization in Theorem 3.2, we obtain the following result.

Corollary 4.1. *Suppose that the standing assumptions stated in the Introduction and Assumption 2.1 hold. Suppose further that (4.4) holds and that B_0 is nonsingular. Then $z^* = (v^*, \lambda^*)^\top$ is an equilibrium point of (4.1) if and only if v^* solves problem (1.1).*

Proof. By Theorem 4.1, z^* is an equilibrium point of (4.1) if and only if $R(z^*) = 0$. By Theorem 3.2, $R(z^*) = 0$ holds if and only if v^* solves problem (1.1). Therefore, the assertion follows. $\square$

Remark 4.2. *Corollary 4.1 establishes the correspondence between equilibria of the proposed neural network and solutions of problem (1.1). This correspondence does not imply that the solution of problem (1.1) is unique. The standing convexity assumptions guarantee the KKT based projection residual characterization of the inner optimization problem for each fixed parameter, but they do not, by themselves, exclude the possibility that the associated extremal mapping has multiple fixed points.*

For the subsequent stability analysis, we use an equivalent first-order state-space representation of system (4.1). Introduce

$$\eta(t) := \dot{z}(t) + \Xi z(t), \quad (4.7)$$

where

$$\Xi = \text{diag}(\xi_1, \dots, \xi_q) > 0.$$

Define

$$\mathcal{A} := B_0 - A_0\Xi + \Xi^2, \quad \mathcal{B} := A_0 - \Xi. \quad (4.8)$$

Then system (4.1) is equivalently represented as

$$\begin{cases} \dot{z}(t) = -\Xi z(t) + \eta(t), \\ \dot{\eta}(t) = -\mathcal{A}z(t) - \mathcal{B}\eta(t) + C_0T(z(t)) + D_0T(z(t - \tau(t))). \end{cases} \quad (4.9)$$

System (4.1) is retained as the original inertial neural model because it explicitly displays the second-order inertial structure. Equation (4.9) is its equivalent state-space representation and will be used throughout the stability analysis in Section 5. This avoids repeated second-order manipulations while preserving the interpretation of the proposed inertial BAM dynamics.

The corresponding initial data are

$$z(s) = \varphi(s), \quad \eta(s) = \psi(s) + \Xi\varphi(s), \quad s \in [-\bar{\tau}, 0]. \quad (4.10)$$

Define the augmented state

$$x(t) := \begin{pmatrix} z(t) \\ \eta(t) \end{pmatrix} \in \mathbb{R}^{2q}.$$

Then the linear part of (4.9) is described by the block matrix

$$H := \begin{pmatrix} -\Xi & I_q \\ -\mathcal{A} & -\mathcal{B} \end{pmatrix}. \quad (4.11)$$

The nonlinear projection-activation terms enter only through the second block equation in (4.9). This block representation will be used in Section 5 to derive matrix-measure-based exponential stability criteria.

For the first-order representation (4.9), the equilibrium corresponding to z^* is

$$x^* = \begin{pmatrix} z^* \\ \eta^* \end{pmatrix}, \quad \eta^* = \Xi z^*.$$

Remark 4.3. *The results in this section show that the equilibria of the proposed projection-activation inertial BAM neural network exactly correspond to the solutions of problem (1.1). The convergence of trajectories toward such equilibria will be studied in the next section by using matrix measure estimates and Halanay-type delay inequalities.*

5. Exponential stability analysis

In this section, we analyze the exponential stability of the proposed projection-activation inertial BAM neural network through its equivalent first-order representation (4.9). The analysis is based on matrix measure estimates and Halanay's inequality. Since the projection-activation mapping T is constructed through the Euclidean projection, the stability analysis is carried out under the Euclidean norm and the corresponding matrix measure $\mu_2(\cdot)$.

The stability result below is formulated on the considered domain $\mathcal{D}$. Since the Lipschitz continuity of the projection activation mapping T is guaranteed only on $\mathcal{D}$, the analysis applies to trajectories whose z -components remain in this domain. Therefore, the result should be understood as a conditional exponential stability result on $\mathcal{D}$, rather than an unconditional global convergence result.

Throughout this section, the matching condition (4.4) is assumed to hold, and B_0 is assumed to be nonsingular. Let $z^* \in \mathcal{D}$ be an equilibrium point of (4.1). By Theorem 4.1, this is equivalent to

$$R(z^*) = 0,$$

or equivalently,

$$z^* = T(z^*).$$

For the first-order system (4.9), the corresponding equilibrium is

$$x^* = \begin{pmatrix} z^* \\ \eta^* \end{pmatrix}, \quad \eta^* = \Xi z^*.$$

In the following analysis, z^* is treated as a prescribed equilibrium point. The stability result is therefore formulated with respect to this equilibrium. It does not rely on the uniqueness of all residual zeros in the whole admissible domain.

Define the error variables

$$e_z(t) := z(t) - z^*, \quad e_\eta(t) := \eta(t) - \eta^*, \quad (5.1)$$

and the augmented error vector

$$e(t) := \begin{pmatrix} e_z(t) \\ e_\eta(t) \end{pmatrix}. \quad (5.2)$$

Moreover, denote

$$\Delta T(t) := T(z(t)) - T(z^*), \quad (5.3)$$

and

$$\Delta T_\tau(t) := T(z(t - \tau(t))) - T(z^*). \quad (5.4)$$

For the equilibrium $x^* = (z^*, \eta^*)^\top$, the first-order system (4.9) gives

$$0 = -\Xi z^* + \eta^*, \quad (5.5)$$

and

$$0 = -\mathcal{A}z^* - \mathcal{B}\eta^* + C_0T(z^*) + D_0T(z^*). \quad (5.6)$$

Subtracting (5.5) and (5.6) from (4.9), respectively, yields

$$\begin{cases} \dot{e}_z(t) = -\Xi e_z(t) + e_\eta(t), \\ \dot{e}_\eta(t) = -\mathcal{A}e_z(t) - \mathcal{B}e_\eta(t) + C_0\Delta T(t) + D_0\Delta T_\tau(t). \end{cases} \quad (5.7)$$

Define

$$\bar{C}_0 := \begin{pmatrix} 0 \\ C_0 \end{pmatrix}, \quad \bar{D}_0 := \begin{pmatrix} 0 \\ D_0 \end{pmatrix}, \quad (5.8)$$

where the zero block is of size $q \times q$. Then (5.7) can be written in the compact form

$$\dot{e}(t) = He(t) + \bar{C}_0\Delta T(t) + \bar{D}_0\Delta T_\tau(t), \quad (5.9)$$

where

$$H = \begin{pmatrix} -\Xi & I_q \\ -\mathcal{A} & -\mathcal{B} \end{pmatrix}$$

is defined in (4.11).

We first derive the key differential inequality for the error system.

Lemma 5.1 (Error estimate on $\mathcal{D}$). *Suppose that Assumption 2.2 holds. Let $z^* \in \mathcal{D}$ be an equilibrium point of (4.1). Assume that*

$$z(t) \in \mathcal{D}, \quad z(t - \tau(t)) \in \mathcal{D}, \quad t \geq 0.$$

Then the error vector $e(t)$ satisfies

$$D^+\|e(t)\|_2 \leq (\mu_2(H) + L_T\|\bar{C}_0\|_2)\|e(t)\|_2 + L_T\|\bar{D}_0\|_2 \sup_{t-\bar{\tau} \leq s \leq t} \|e(s)\|_2. \quad (5.10)$$

Proof. By Lemma 2.2 applied to the compact error system (5.9), we have

$$D^+\|e(t)\|_2 \leq \mu_2(H)\|e(t)\|_2 + \|\bar{C}_0\Delta T(t) + \bar{D}_0\Delta T_\tau(t)\|_2.$$

Using the triangle inequality and the induced matrix norm, we get

$$\begin{aligned} \|\bar{C}_0\Delta T(t) + \bar{D}_0\Delta T_\tau(t)\|_2 &\leq \|\bar{C}_0\Delta T(t)\|_2 + \|\bar{D}_0\Delta T_\tau(t)\|_2 \\ &\leq \|\bar{C}_0\|_2\|\Delta T(t)\|_2 + \|\bar{D}_0\|_2\|\Delta T_\tau(t)\|_2. \end{aligned}$$

Therefore,

$$D^+\|e(t)\|_2 \leq \mu_2(H)\|e(t)\|_2 + \|\bar{C}_0\|_2\|\Delta T(t)\|_2 + \|\bar{D}_0\|_2\|\Delta T_\tau(t)\|_2. \quad (5.11)$$

We next estimate the two activation-difference terms. Since T is Lipschitz continuous on $\mathcal{D}$ by Lemma 3.1, and since both $z(t)$ and z^* belong to $\mathcal{D}$, we have

$$\begin{aligned} \|\Delta T(t)\|_2 &= \|T(z(t)) - T(z^*)\|_2 \\ &\leq L_T\|z(t) - z^*\|_2 \\ &= L_T\|e_z(t)\|_2 \\ &\leq L_T\|e(t)\|_2. \end{aligned}$$

Thus,

$$\|\Delta T(t)\|_2 \leq L_T \|e(t)\|_2. \quad (5.12)$$

Similarly, since $z(t - \tau(t)) \in \mathcal{D}$, we obtain

$$\begin{aligned} \|\Delta T_\tau(t)\|_2 &= \|T(z(t - \tau(t))) - T(z^*)\|_2 \\ &\leq L_T \|z(t - \tau(t)) - z^*\|_2 \\ &= L_T \|e_z(t - \tau(t))\|_2. \end{aligned}$$

From $0 \leq \tau(t) \leq \bar{\tau}$, it follows that

$$t - \tau(t) \in [t - \bar{\tau}, t].$$

Hence,

$$\begin{aligned} \|\Delta T_\tau(t)\|_2 &\leq L_T \sup_{t-\bar{\tau} \leq s \leq t} \|e_z(s)\|_2 \\ &\leq L_T \sup_{t-\bar{\tau} \leq s \leq t} \|e(s)\|_2. \end{aligned}$$

Therefore,

$$\|\Delta T_\tau(t)\|_2 \leq L_T \sup_{t-\bar{\tau} \leq s \leq t} \|e(s)\|_2. \quad (5.13)$$

Substituting (5.12) and (5.13) into (5.11), we arrive at

$$D^+ \|e(t)\|_2 \leq (\mu_2(H) + L_T \|\bar{C}_0\|_2) \|e(t)\|_2 + L_T \|\bar{D}_0\|_2 \sup_{t-\bar{\tau} \leq s \leq t} \|e(s)\|_2.$$

This completes the proof. $\square$

We now present the exponential stability criterion.

Theorem 5.1 (Conditional exponential stability on $\mathcal{D}$ with respect to z^*). *Suppose that Assumption 2.2 holds. Let $z^* \in \mathcal{D}$ be an equilibrium point of (4.1). Assume that the delay function satisfies $0 \leq \tau(t) \leq \bar{\tau}$ for all $t \geq 0$, and that*

$$z(t) \in \mathcal{D}, \quad z(t - \tau(t)) \in \mathcal{D}, \quad t \geq 0.$$

If

$$-(\mu_2(H) + L_T \|\bar{C}_0\|_2) > L_T \|\bar{D}_0\|_2 \geq 0, \quad (5.14)$$

then the equilibrium

$$x^* = (z^*, \Xi z^*)^\top$$

of the first order system (4.9) is conditionally exponentially stable on $\mathcal{D}$. More precisely,

$$\|e(t)\|_2 \leq \left(\sup_{-\bar{\tau} \leq s \leq 0} \|e(s)\|_2 \right) e^{-rt}, \quad t \geq 0, \quad (5.15)$$

where the decay rate $r > 0$ is determined as follows. If

$$L_T \|\bar{D}_0\|_2 > 0,$$

then r is the unique positive solution of

$$r = k_1 - k_2 e^{r\bar{\tau}}, \quad (5.16)$$

where

$$k_1 := -(\mu_2(H) + L_T \|\bar{C}_0\|_2), \quad k_2 := L_T \|\bar{D}_0\|_2. \quad (5.17)$$

If $L_T \|\bar{D}_0\|_2 = 0$, one may take $r = k_1$.

Proof. By Lemma 5.1, we have

$$D^+ \|e(t)\|_2 \leq (\mu_2(H) + L_T \|\bar{C}_0\|_2) \|e(t)\|_2 + L_T \|\bar{D}_0\|_2 \sup_{t-\bar{\tau} \leq s \leq t} \|e(s)\|_2.$$

Using the constants k_1 and k_2 defined in (5.17), the above inequality becomes

$$D^+ \|e(t)\|_2 \leq -k_1 \|e(t)\|_2 + k_2 \sup_{t-\bar{\tau} \leq s \leq t} \|e(s)\|_2. \quad (5.18)$$

Condition (5.14) gives

$$k_1 > k_2 \geq 0.$$

If $k_2 > 0$, then Lemma 2.3 applies to the nonnegative function

$$y(t) := \|e(t)\|_2.$$

Consequently,

$$\|e(t)\|_2 \leq \left(\sup_{-\bar{\tau} \leq s \leq 0} \|e(s)\|_2 \right) e^{-rt}, \quad t \geq 0,$$

where $r > 0$ is the unique positive solution of

$$r = k_1 - k_2 e^{r\bar{\tau}}.$$

If $k_2 = 0$, then (5.18) reduces to

$$D^+ \|e(t)\|_2 \leq -k_1 \|e(t)\|_2.$$

By the standard comparison principle, we have

$$\|e(t)\|_2 \leq \|e(0)\|_2 e^{-k_1 t} \leq \left(\sup_{-\bar{\tau} \leq s \leq 0} \|e(s)\|_2 \right) e^{-k_1 t}.$$

Thus, in this case, one may take $r = k_1$. The proof is complete. $\square$

Remark 5.1 (Influence of the delay bound). *Let*

$$k_1 = -(\mu_2(H) + L_T \|\bar{C}_0\|_2), \quad k_2 = L_T \|\bar{D}_0\|_2.$$

The stability condition (5.14) is equivalent to $k_1 > k_2 \geq 0$. Under this condition, Halanay's inequality gives an exponential decay rate $r > 0$ determined by

$$r = k_1 - k_2 e^{r\bar{\tau}}.$$

This relation does not give a finite maximum admissible delay in the form of $\bar{\tau} < \tau_{\max}$. Instead, it shows how the delay bound affects the certified convergence rate. When $k_2 > 0$, for fixed k_1 and k_2 , a larger $\bar{\tau}$ leads to a smaller admissible rate r . When $k_2 = 0$, the delayed feedback term disappears and the decay rate reduces to $r = k_1$, independently of $\bar{\tau}$. Hence, the delay bound does not change the equilibrium-residual equivalence, but it weakens the theoretical convergence guarantee by reducing the certified exponential decay rate.

Remark 5.2 (Non delayed special case). When $D_0 = 0$, the delayed feedback term disappears and the matching condition reduces to

$$B_0 = C_0.$$

In this case,

$$k_2 = L_T \|\bar{D}_0\|_2 = 0,$$

and the stability condition reduces to

$$-(\mu_2(H) + L_T \|\bar{C}_0\|_2) > 0.$$

Accordingly, the Halanay relation becomes

$$r = k_1 = -(\mu_2(H) + L_T \|\bar{C}_0\|_2),$$

which is independent of the delay bound $\bar{\tau}$. Therefore, the corresponding non delayed PA-IBAM model is directly covered as a special case of the present stability result, and no separate stability analysis is required.

Remark 5.3. Condition (5.14) requires the contraction strength of the transformed linear part to dominate both the instantaneous and delayed projection-activation effects. Indeed, the quantity

$$-\mu_2(H)$$

reflects the contraction induced by the linear part, while

$$L_T \|\bar{C}_0\|_2 \quad \text{and} \quad L_T \|\bar{D}_0\|_2$$

measure the influence of the instantaneous and delayed activation couplings, respectively.

Remark 5.4 (Verification of the matrix-measure condition). The stability condition (5.14) can be checked directly from the selected network parameters. Define the stability margin

$$\mathcal{M} := -(\mu_2(H) + L_T \|\bar{C}_0\|_2) - L_T \|\bar{D}_0\|_2.$$

Then condition (5.14) is equivalent to

$$\mathcal{M} > 0.$$

Here,

$$\mu_2(H) = \lambda_{\max} \left(\frac{H + H^\top}{2} \right),$$

and

$$\|\bar{C}_0\|_2 = \|C_0\|_2, \quad \|\bar{D}_0\|_2 = \|D_0\|_2.$$

For affine numerical examples with

$$F(z) = M_F z + b_F,$$

the global nonexpansiveness estimate gives

$$\text{Lip}(T) \leq \|I_q - \alpha M_F\|_2.$$

For the local stability verification used in Sections 6 and 7, the operating domain $\mathcal{D}$ is chosen so that the projection pattern is fixed. Let $P_{\mathcal{D}}$ denote the corresponding diagonal selection matrix for the locally free projection components. Then T is affine on $\mathcal{D}$, and one may take

$$L_T = \|P_{\mathcal{D}}(I_q - \alpha M_F)\|_2.$$

The local value L_T is used in the stability margins reported in the numerical examples.

Remark 5.5. Theorem 5.1 is a conditional stability result on the considered domain $\mathcal{D}$. It does not claim that $\mathcal{D}$ is positively invariant for all admissible initial functions. If positive invariance of $\mathcal{D}$ can be verified for a specific application, then the theorem gives exponential convergence for all initial functions chosen in that domain. Otherwise, the result applies to trajectories that remain in $\mathcal{D}$.

Finally, we connect the stability result with problem (1.1).

Corollary 5.1 (Conditional convergence to a prescribed solution). *Suppose that the standing assumptions stated in the Introduction, Assumption 2.1 and 2.2 hold. Suppose further that (4.4) holds, B_0 is nonsingular, and the stability condition (5.14) is satisfied. Let $z^* = (v^*, \lambda^*)^\top \in \mathcal{D}$ be a prescribed equilibrium point of (4.1). Then every trajectory of (4.1) to which Theorem 5.1 applies satisfies*

$$z(t) \rightarrow z^*, \quad \dot{z}(t) \rightarrow 0, \quad t \rightarrow \infty, \quad (5.19)$$

with an exponential rate. Moreover, the primal component v^* solves problem (1.1).

Proof. By Theorem 5.1, we have

$$\|e(t)\|_2 \rightarrow 0 \quad \text{exponentially as } t \rightarrow \infty.$$

Since

$$e(t) = \begin{pmatrix} z(t) - z^* \\ \eta(t) - \eta^* \end{pmatrix},$$

it follows that

$$z(t) \rightarrow z^*, \quad \eta(t) \rightarrow \eta^* = \Xi z^*.$$

Using the transformation

$$\eta(t) = \dot{z}(t) + \Xi z(t),$$

we obtain

$$\dot{z}(t) = \eta(t) - \Xi z(t) \rightarrow \eta^* - \Xi z^* = 0.$$

This proves (5.19).

Since z^* is an equilibrium point of (4.1), Theorem 4.1 implies

$$R(z^*) = 0.$$

By Theorem 3.2, this is equivalent to v^* solving problem (1.1). The proof is complete. $\square$

Remark 5.6. Corollary 5.1 does not assert the uniqueness of the solution of problem (1.1). The standing convexity assumptions guarantee the convexity of the inner optimization problem for each fixed parameter and justify the KKT based projection residual characterization, but they do not guarantee that the extremal mapping has a unique fixed point. If several residual zeros exist in the admissible domain, the limiting equilibrium may depend on the initial function. When the residual zero is unique in the considered domain $\mathcal{D}$, the above convergence result reduces to convergence to this unique solution.

Remark 5.7. Theorem 5.1 proves the conditional exponential convergence of the proposed projection activation inertial BAM-type neural network under the matrix measure condition (5.14). Together with the projection residual characterization in Section 3 and the equilibrium residual equivalence in Section 4, this shows that the considered trajectory converges to the prescribed equilibrium solution z^* , provided that the trajectory remains in $\mathcal{D}$ and the conditions of Theorem 5.1 are satisfied.

6. Numerical experiments

This section evaluates the proposed projection activation inertial BAM neural network for solving problem (1.1). The examples test certified convergence, boundary projection, scalability, delay influence, and comparison with baseline methods. For a terminal state $z(T) = (v(T), \lambda(T))$, we report the relative state error

$$\text{relerr} = \frac{\|z(T) - z^*\|_2}{1 + \|z^*\|_2},$$

the projection residual $\|R(z(T))\|_2$, the feasibility violation $\|g_+(T)\|_2$, the complementarity residual $\|\lambda(T) \odot g(v(T), v(T))\|_2$, and the velocity norm $\|\dot{z}(T)\|_2$. The stability margin is

$$\mathcal{M} = -(\mu_2(H) + L_T \|\bar{C}_0\|_2) - L_T \|\bar{D}_0\|_2.$$

Thus, $\mathcal{M} > 0$ means that the sufficient stability condition in Theorem 5.1 is satisfied. In the certified examples below, L_T denotes the local Lipschitz constant of T on an operating domain around the prescribed equilibrium where the projection pattern is fixed. Therefore, the reported stability margins certify the local matrix measure condition of Theorem 5.1 on the corresponding operating domain. Unless otherwise stated, we set

$$C_0 = 3I_q, \quad D_0 = 0.02I_q, \quad B_0 = 3.02I_q, \quad A_0 = 2.85I_q, \quad \Xi = 1.4243I_q, \quad \tau(t) \equiv 0.02.$$

6.1. Example 1: Certified benchmark with objective and constraint coupling

This example verifies the matrix measure condition for a fixed point problem with both objective coupling and affine constraint coupling. Let $n = 8$, $m = 3$, and $\Omega = [0, 1]^8$. For $v, w \in \mathbb{R}^8$, consider

$$\Phi(v, w) = \frac{1}{2}(w - d)^\top Q(w - d) + \eta v^\top Pw + c^\top w, \quad g(v, w) = Aw + \Gamma v - b \leq 0. \quad (6.1)$$

Here $\eta = 0.35$,

$$Q = \text{diag}(8.0000, 8.0214, 8.0429, 8.0643, 8.0857, 8.1071, 8.1286, 8.1500),$$

and

$$c = (0.1279, 0.1361, 0.1435, 0.1499, 0.1548, 0.1582, 0.1599, 0.1597)^\top.$$

The objective coupling matrix is

$$P = \begin{pmatrix} 0.00 & 0.18 & 0.05 & 0.00 & 0.00 & 0.00 & 0.00 & 0.02 \\ 0.10 & 0.00 & 0.16 & 0.04 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.03 & 0.12 & 0.00 & 0.14 & 0.05 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.03 & 0.10 & 0.00 & 0.15 & 0.05 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.03 & 0.11 & 0.00 & 0.15 & 0.06 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.04 & 0.12 & 0.00 & 0.16 & 0.06 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.04 & 0.13 & 0.00 & 0.18 \\ 0.04 & 0.00 & 0.00 & 0.00 & 0.00 & 0.05 & 0.15 & 0.00 \end{pmatrix}.$$

The affine coupling matrices are

$$A = \begin{pmatrix} 1.00 & 0.90 & 0.70 & 0.40 & 0.20 & 0.10 & 0.00 & 0.00 \\ 0.00 & 0.10 & 0.30 & 0.60 & 0.80 & 1.00 & 0.70 & 0.40 \\ 0.40 & 0.20 & 0.50 & 0.10 & 0.30 & 0.20 & 0.90 & 1.00 \end{pmatrix},$$

and

$$\Gamma = 0.12 \begin{pmatrix} 0.20 & 0.10 & 0.10 & 0.20 & 0.10 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.10 & 0.10 & 0.20 & 0.20 & 0.10 & 0.10 \\ 0.10 & 0.00 & 0.10 & 0.00 & 0.10 & 0.00 & 0.20 & 0.20 \end{pmatrix}.$$

The prescribed solution and multiplier are

$$v^* = (0.25, 0.31, 0.38, 0.45, 0.52, 0.59, 0.66, 0.73)^\top, \quad \lambda^* = 0.$$

We construct

$$d = v^* + Q^{-1}(\eta P^\top v^* + c), \quad b = (A + \Gamma)v^* + \Delta_b, \quad \Delta_b = (0.06, 0.07, 0.075)^\top.$$

Then $\nabla_w \Phi(v^*, v^*) = 0$ and $g(v^*, v^*) = -\Delta_b < 0$. Thus, the KKT conditions hold with $\lambda^* = 0$. For this example,

$$F(z) = \begin{pmatrix} Q(v - d) + \eta P^\top v + c + A^\top \lambda \\ -((A + \Gamma)v - b) \end{pmatrix},$$

and $T(z)$ is obtained by projecting $v - \alpha F_v(z)$ onto $[0, 1]^8$ and $\lambda - \alpha F_\lambda(z)$ onto $\mathbb{R}_+^3$, with $\alpha = 0.1$. The simulation is performed on $[0, 80]$ with $h = 0.005$. Figure 1 shows that the state error and projection residual converge to machine precision, and the final state coincides with the prescribed fixed point.

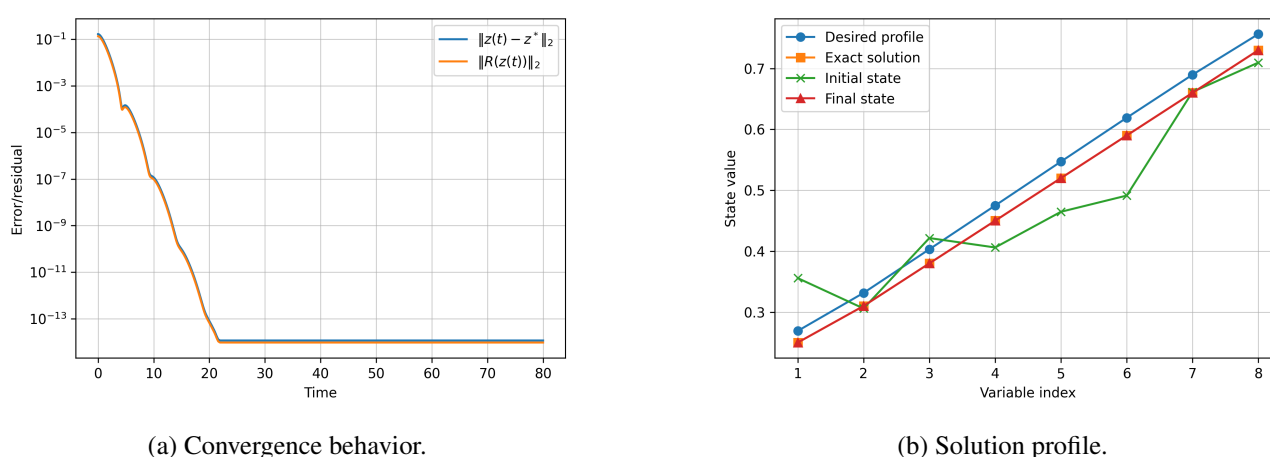


Figure 1. Convergence behavior and solution profile in Example 1.

6.2. Influence of the delay

To examine the influence of the delay term under certified parameters, we conduct a delay-sensitivity test based on Example 1. The constant delay is varied as

$$\tau = 0, 0.02, 0.05, 0.08, 0.10, 0.20, 0.50, 1.00,$$

while all other parameters and initial conditions are kept unchanged. For each delay value, the sufficient matrix-measure condition remains satisfied. Table 1 summarizes the certified decay rates and numerical errors for the tested delay values. The certified decay rate is computed from the Halanay-type relation in Remark 5.1.

Table 1. Delay sensitivity under the certified parameter setting in Example 1.

τ	Certified rate r	$\ z(T) - z^*\ _2$	$\ R(z(T))\ _2$	$t_{10^{-6}}$
0	0.552572	1.157×10^{-14}	9.411×10^{-15}	8.43
0.02	0.552508	1.163×10^{-14}	9.476×10^{-15}	8.43
0.05	0.552411	1.163×10^{-14}	9.476×10^{-15}	8.43
0.08	0.552313	1.171×10^{-14}	9.511×10^{-15}	8.43
0.10	0.552246	1.179×10^{-14}	9.590×10^{-15}	8.43
0.20	0.551902	1.199×10^{-14}	9.760×10^{-15}	8.43
0.50	0.550752	1.273×10^{-14}	1.039×10^{-14}	8.43
1.00	0.548378	1.505×10^{-14}	1.226×10^{-14}	8.46

Figure 2 shows the certified decay rate under different delay values. As τ increases from 0 to 1.0, the certified decay rate decreases from 0.552572 to 0.548378. This trend is consistent with the Halanay-type relation and shows that a larger delay weakens the theoretical convergence guarantee. The terminal projection residual remains at the order of 10^{-14} for all tested delay values, indicating stable numerical convergence under the tested delay range.

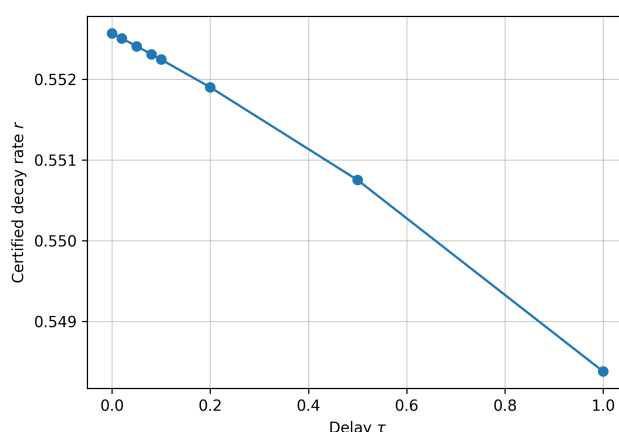


Figure 2. Certified decay rate under different constant delays in Example 1.

6.3. Example 2: Certified boundary projection with coupled objective

This example tests the projection behavior when the solution lies on the boundary of Ω . Let $n = 6$, $m = 3$, and $\Omega = [0, 1]^6$. For $v, w \in \mathbb{R}^6$, consider the same coupled form as in (6.1), with $Q = 8I_6$, $\eta = 0.08$, and

$$c = (0.020, 0.018, 0.016, 0.014, 0.012, 0.010)^\top.$$

The objective coupling matrix is

$$P = \begin{pmatrix} 0.00 & 0.15 & -0.05 & 0.00 & 0.04 & -0.02 \\ 0.10 & 0.00 & 0.08 & -0.04 & 0.00 & 0.03 \\ -0.03 & 0.06 & 0.00 & 0.09 & -0.02 & 0.00 \\ 0.00 & -0.05 & 0.07 & 0.00 & 0.10 & -0.04 \\ 0.02 & 0.00 & -0.03 & 0.08 & 0.00 & 0.12 \\ -0.04 & 0.05 & 0.00 & -0.02 & 0.09 & 0.00 \end{pmatrix}.$$

The affine coupling matrices are

$$A = \begin{pmatrix} 0.70 & 0.40 & 0.20 & 0.10 & 0.00 & 0.00 \\ 0.00 & 0.10 & 0.20 & 0.40 & 0.50 & 0.70 \\ 0.20 & 0.30 & 0.10 & 0.30 & 0.20 & 0.10 \end{pmatrix},$$

and

$$\Gamma = 0.10 \begin{pmatrix} 0.20 & 0.10 & 0.10 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.10 & 0.10 & 0.20 & 0.20 \\ 0.10 & 0.10 & 0.00 & 0.10 & 0.10 & 0.00 \end{pmatrix}.$$

The prescribed solution is

$$v^* = (0, 0.22, 0.38, 0.58, 0.78, 1.00)^\top, \quad \lambda^* = 0,$$

so that $v_1^* = 0$ and $v_6^* = 1$. We set

$$b = (A + \Gamma)v^* + (0.060, 0.070, 0.075)^\top.$$

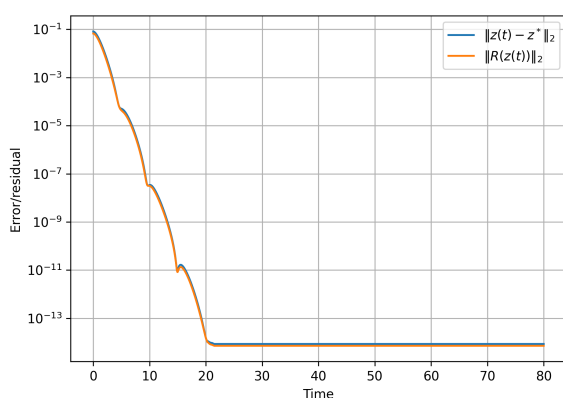
Thus, the affine coupled constraints are strictly inactive at the solution. To satisfy the boundary stationarity condition, we choose

$$s = (0.050, 0, 0, 0, 0, -0.050)^\top, \quad d = v^* + Q^{-1}(\eta P^\top v^* + c - s).$$

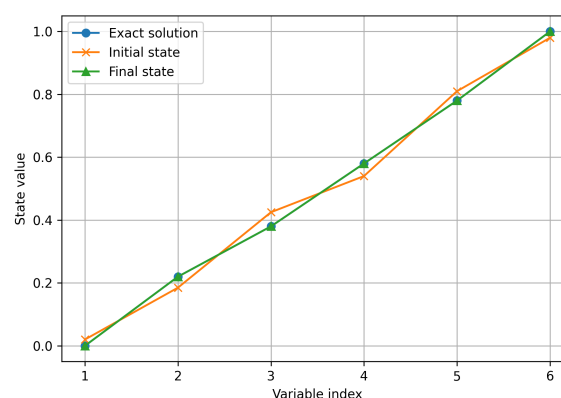
Then $\nabla_w \Phi(v^*, v^*) = s \in -N_\Omega(v^*)$, and the KKT conditions hold. With $\alpha = 0.1$, $[0, 80]$, and $h = 0.005$, the matrix measure condition and terminal accuracy are reported in Table 2. Figure 3 shows that the method accurately recovers the lower and upper boundary components. Since the solution is on the box boundary, the ordinary stationarity residual need not vanish; the projection residual is the proper optimality measure.

Table 2. Summary of certified results for Examples 1 and 2.

Example	L_T	$\mu_2(H)$	$\mathcal{M}$	$\ z(T) - z^*\ _2$	$\ R(z(T))\ _2$
1	0.287109	-1.419642	0.552572	1.163×10^{-14}	9.476×10^{-15}
2	0.216549	-1.419642	0.765664	8.601×10^{-15}	7.221×10^{-15}



(a) Convergence behavior.



(b) Boundary projection result.

Figure 3. Convergence behavior and boundary projection result in Example 2.

6.4. Example 3: High dimensional scalability under certified parameters

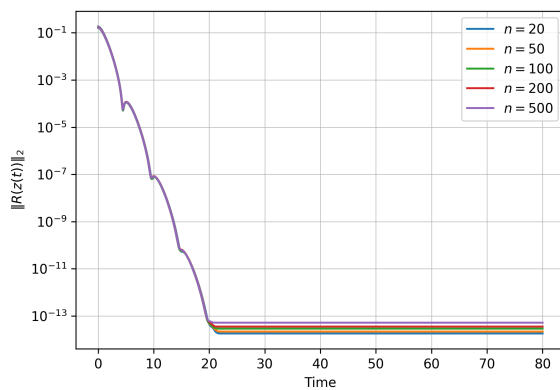
This example studies scalability. We use the same coupled structure as Example 1 and test $n = 20, 50, 100, 200, 500$ with $m = 3$ and $\Omega = [0, 1]^n$. For each dimension, $Q > 0$, P is a nonsymmetric tridiagonal matrix, and the rows of A and Γ are normalized to fixed Euclidean norms. The solution is chosen in the interior of Ω , and d and b are constructed so that

$$\nabla_w \Phi(v^*, v^*) = 0, \quad g(v^*, v^*) = (-0.060, -0.070, -0.075)^\top < 0.$$

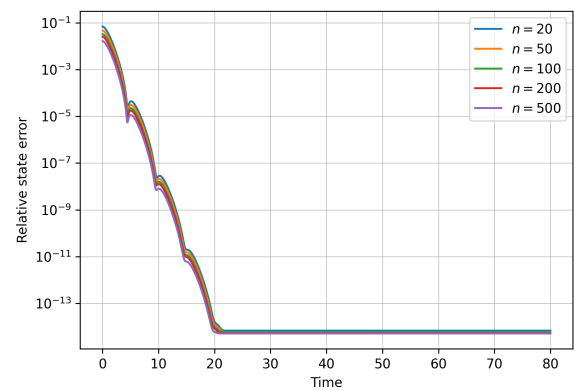
Thus, the KKT conditions hold with $\lambda^* = 0$ for all tested instances. Table 3 shows that the stability margin is positive in every case and that the projection residual remains at the order of 10^{-14} . Figure 4 illustrates the time evolution of the projection residual and relative state error for the tested problem dimensions.

Table 3. High dimensional results for Example 3.

n	q	$\mathcal{M}$	relerr	$\ R(z(T))\ _2$	CPU time (s)
20	23	0.651024	6.8431×10^{-15}	1.8178×10^{-14}	1.052
50	53	0.651123	5.8572×10^{-15}	2.1937×10^{-14}	1.029
100	103	0.651086	5.8424×10^{-15}	2.9029×10^{-14}	1.044
200	203	0.651067	5.3233×10^{-15}	3.5558×10^{-14}	1.142
500	503	0.651047	5.1607×10^{-15}	5.2059×10^{-14}	1.291



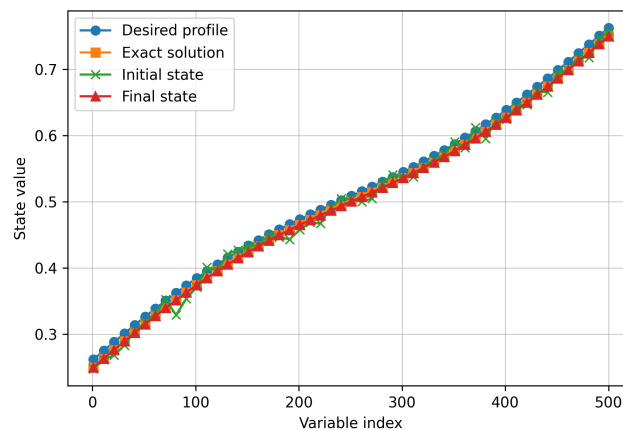
(a) Projection residual.



(b) Relative state error.

Figure 4. Projection residual and relative state error for Example 3.

For the largest case $n = 500$, the sampled final state almost coincides with the exact solution, as shown in Figure 5. This confirms the accuracy of the proposed method in the tested high dimensional setting.

**Figure 5.** Sampled solution profile for $n = 500$ in Example 3.

6.5. Example 4: Comparison with projection type baselines

This example compares the proposed PA-IBAM neural network with several projection type baseline methods on the certified coupled benchmark in Example 1. All methods are applied to the same

projection residual equation

$$R(z) = z - T(z),$$

and use the same initial state. The proposed PA-IBAM model is

$$\ddot{z}(t) = -A_0\dot{z}(t) - B_0z(t) + C_0T(z(t)) + D_0T(z(t - \tau(t))).$$

The no delay PA-IBAM model is obtained by setting $D_0 = 0$ and $B_0 = C_0$, so that the matching condition is preserved:

$$\ddot{z}(t) = -A_0\dot{z}(t) - B_0z(t) + C_0T(z(t)).$$

We also compare with a first order residual neural network

$$\dot{z}(t) = -\kappa R(z(t)),$$

a non inertial projection neural network

$$\dot{z}(t) = -B_0z(t) + C_0T(z(t)),$$

and a relaxed projection iteration

$$z^{k+1} = (1 - \theta)z^k + \theta T(z^k).$$

For the proposed PA-IBAM model, we use $D_0 = 0.20I_q$ and $\tau(t) \equiv 0.02$. The corresponding matrix measure quantities are

$$L_T = 0.287109, \quad \mu_2(H) = -1.340309,$$

$$L_T\|\bar{C}_0\|_2 = 0.861327, \quad L_T\|\bar{D}_0\|_2 = 0.057422,$$

and hence

$$\mathcal{M} = 0.421560 > 0.$$

Thus, the sufficient stability condition in Theorem 5.1 is satisfied.

Table 4. Terminal comparison results for Example 4.

Method	$\ R(z(T))\ _2$	relerr	$\ g_+(T)\ _2$	$t_{10^{-6}}$	$t_{10^{-8}}$	CPU time (s)
PA-IBAM	4.6076×10^{-9}	2.3331×10^{-9}	0	7.85	11.76	0.149
PA-IBAM without delay	2.2287×10^{-8}	1.1300×10^{-8}	0	8.83	—	0.132
First order residual NN	2.4070×10^{-2}	1.2208×10^{-2}	0	—	—	0.299
Non inertial projection NN	2.1377×10^{-3}	1.0843×10^{-3}	0	—	—	0.290
Relaxed projection iteration	1.3080×10^{-3}	6.6351×10^{-4}	0	—	—	0.011

Table 4 shows that PA-IBAM achieves the smallest terminal residual within the same simulation horizon. Its residual is about 4.84 times smaller than that of the non delayed PA-IBAM model, and it reaches the threshold 10^{-6} earlier. This indicates a finite time benefit of the delayed feedback term in this certified numerical setting.

6.6. Discussion

The numerical results show that the proposed projection activation inertial BAM neural network performs well under certified parameter settings. Example 1 verifies convergence for a coupled fixed point benchmark, Example 2 confirms the boundary projection ability, Example 3 shows accuracy up to $n = 500$, and Example 4 shows better finite time residual decay than the compared baselines. The delay test also agrees with the Halanay estimate, since a larger delay reduces the certified decay rate. Overall, the results verify the convergence accuracy, boundary projection ability, delay dependent behavior, and scalability of the proposed method.

7. Application to multi-actuator output allocation

Multi-actuator output allocation is a typical constrained engineering optimization problem in over-actuated systems, such as multi-motor drive systems, multi-thruster robotic platforms, and redundant actuation systems. In these systems, several actuators jointly provide the required control effort, and the actuator outputs should be allocated under physical limitations such as amplitude bounds, power budgets, current limits, and thermal safety constraints [20–23]. Fixed point based formulations have also been used in constrained control allocation problems with actuator saturation [24]. Recent studies have further considered constrained control allocation while accounting for actuator limitations, actuator dynamics, and heterogeneous actuation configurations [25, 26].

Motivated by this setting, this section considers a self-consistent multi-actuator output allocation problem. Let $v \in \mathbb{R}^n$ denote the actuator output state in the current control period, and let $w \in \mathbb{R}^n$ denote the optimized actuator output in the next control period. A fixed point $v^* \in S(v^*)$ represents a stable allocation state: when the current actuator output is v^* , the next allocation optimization still returns v^* . This interpretation naturally fits the fixed point problem of an extremal mapping with coupled constraints.

7.1. Multi-actuator output-allocation model

The normalized actuator output is constrained by

$$\Omega = [0, 1]^n, \quad (7.1)$$

where $w_i = 0$ means that the i -th actuator is inactive and $w_i = 1$ means that it reaches its normalized upper bound. The allocation cost is defined as

$$\Phi(v, w) = \frac{1}{2}(w - d)^\top Q(w - d) + \frac{\rho}{2}\|w - v\|^2 + c^\top w, \quad (7.2)$$

where $d \in \mathbb{R}^n$ is the desired actuator output profile, $Q > 0$ is a weighting matrix, $\rho > 0$ penalizes abrupt changes between two successive control periods, and $c^\top w$ denotes the actuator usage cost. Although the proximal smoothing term $\rho\|w - v\|^2/2$ vanishes in $\nabla_w \Phi(v, v)$, it is retained in the inner allocation model to penalize abrupt changes between two successive control periods.

The coupled engineering constraints are given by

$$g(v, w) = Aw + \Gamma v - b \leq 0. \quad (7.3)$$

Here, Aw represents the power, thermal, and current load generated by the next actuator command, Γv represents the residual load effect of the current actuator state, and b denotes the corresponding safety limits. Thus, the constraints couple the current actuator state and the next optimized actuator output.

The corresponding instance of problem (1.1) is

$$v^* \in \operatorname{Argmin} \{ \Phi(v^*, w) \mid g(v^*, w) \leq 0, w \in \Omega \}. \quad (7.4)$$

This fixed point condition means that, under the actuator output v^* , the next-period allocation optimization returns the same output. Hence, v^* represents a self-consistent stable actuator allocation.

For each fixed v , one has

$$\nabla_{ww}^2 \Phi(v, w) = Q + \rho I_n > 0,$$

and $w \mapsto g(v, w)$ is affine. Moreover, Ω is a nonempty closed convex set. Therefore, the application model satisfies the standing convexity assumptions used in the projection-residual characterization. In the certified operating case considered below, the prescribed equilibrium is strictly feasible, and hence the corresponding constraint qualification is satisfied.

For (7.4), the gradient with respect to w is

$$\nabla_w \Phi(v, w) = Q(w - d) + \rho(w - v) + c.$$

Thus, on the diagonal $w = v$, one obtains

$$\nabla_w \Phi(v, v) = Q(v - d) + c, \quad \nabla_w g(v, w)^\top = A^\top.$$

Let

$$z = \begin{pmatrix} v \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+m}.$$

Then the diagonal KKT-type mapping is

$$F(z) = \begin{pmatrix} Q(v - d) + c + A^\top \lambda \\ -((A + \Gamma)v - b) \end{pmatrix}, \quad (7.5)$$

and the projection-activation mapping is

$$T(z) = \begin{pmatrix} \Pi_{[0,1]^n} [v - \alpha(Q(v - d) + c + A^\top \lambda)] \\ \Pi_{\mathbb{R}_+^m} [\lambda + \alpha((A + \Gamma)v - b)] \end{pmatrix}. \quad (7.6)$$

The residual is $R(z) = z - T(z)$. By the projection-residual characterization in Section 3, solving $R(z^*) = 0$ is equivalent to solving the diagonal KKT system of (7.4). Since the inner problem is strongly convex, the KKT point gives the global solution of the inner actuator-allocation problem.

The proposed neural network is applied as

$$\ddot{z}(t) = -A_0 \dot{z}(t) - B_0 z(t) + C_0 T(z(t)) + D_0 T(z(t - \tau(t))). \quad (7.7)$$

7.2. Simulation results

In the simulation, we set

$$n = 20, \quad m = 3, \quad q = n + m = 23.$$

The three components of $g(v, w)$ are interpreted as the total power constraint, thermal load constraint, and current safety constraint, respectively. The prescribed equilibrium allocation is chosen as

$$v_i^* = 0.25 + 0.5 \frac{i-1}{n-1}, \quad i = 1, \dots, n,$$

and the associated multiplier is

$$\lambda^* = 0.$$

We choose

$$Q = 8I_n, \quad \alpha = 0.1,$$

and construct

$$d = v^* + Q^{-1}c.$$

Then

$$\nabla_w \Phi(v^*, v^*) = 0.$$

Moreover, the safety limit is selected as

$$b = (A + \Gamma)v^* + \Delta_b, \quad \Delta_b = (0.06, 0.07, 0.075)^\top.$$

Therefore,

$$g(v^*, v^*) = (-0.06, -0.07, -0.075)^\top < 0.$$

Hence, v^* is a strictly feasible equilibrium allocation, and the KKT conditions are satisfied with $\lambda^* = 0$.

The network parameters are set as

$$\begin{aligned} C_0 &= 3I_q, & D_0 &= 0.02I_q, & B_0 &= 3.02I_q, \\ A_0 &= 2.85I_q, & \Xi &= 1.4243I_q, & \tau(t) &\equiv 0.02. \end{aligned}$$

The simulation interval is $[0, 80]$, and the stepsize is $h = 0.005$.

The matrix-measure quantities are computed as

$$\begin{aligned} L_T &= 0.202464, & \mu_2(H) &= -1.419642, \\ L_T \|\bar{C}_0\|_2 &= 0.607392, & L_T \|\bar{D}_0\|_2 &= 0.004049. \end{aligned}$$

Thus, the stability margin is

$$\mathcal{M} = 0.808200 > 0.$$

Therefore, the sufficient stability condition in Theorem 5.1 is satisfied for this multi-actuator output allocation application. The computed matrix-measure quantities and the resulting positive stability margin are summarized in Table 5.

Table 5. Matrix-measure verification for the multi-actuator output allocation application.

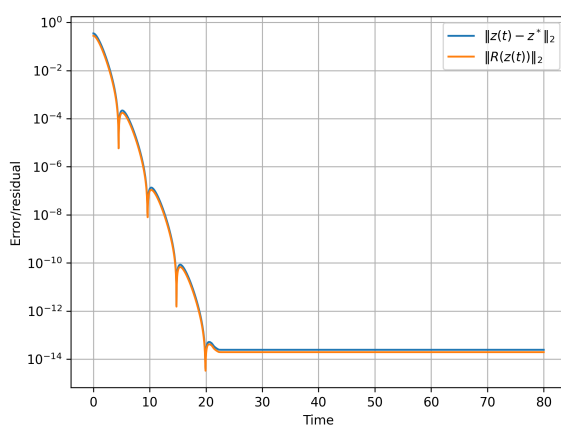
Case	L_T	$\mu_2(H)$	$L_T\ \bar{C}_0\ _2$	$L_T\ \bar{D}_0\ _2$	$\mathcal{M}$
Application	0.202464	-1.419642	0.607392	0.004049	0.808200

Figure 6(a) shows the convergence of the state error and projection residual. Both quantities decrease rapidly and reach the level of machine precision. At the terminal time, we obtain

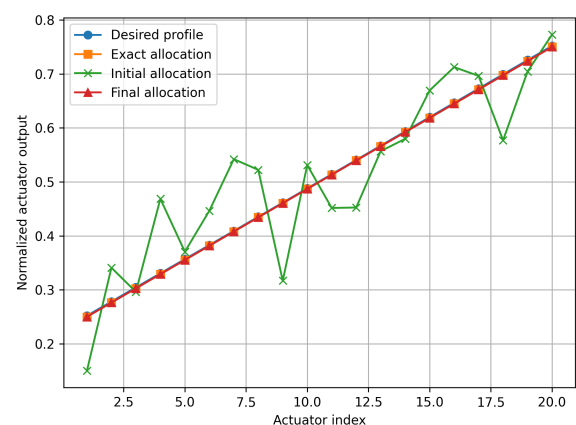
$$\|z(T) - z^*\|_2 = 2.448489 \times 10^{-14}, \quad \|R(z(T))\|_2 = 1.953866 \times 10^{-14}.$$

These results confirm that the proposed neural network converges to the prescribed equilibrium allocation with high numerical accuracy.

Figure 6(b) compares the desired, initial, exact, and final actuator output profiles. The initial allocation deviates from the desired profile, while the final allocation almost coincides with the exact equilibrium allocation. This indicates that the proposed neural dynamics can correct the initial actuator output and recover the self-consistent allocation state.



(a) Convergence behavior.



(b) Actuator output profiles.

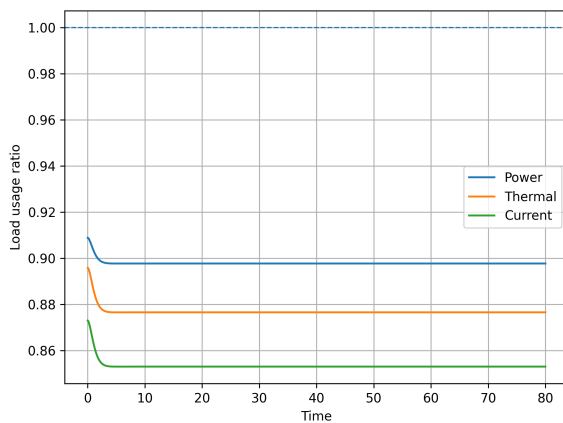
Figure 6. Convergence behavior and actuator output profiles for the multi-actuator output allocation problem.

Figure 7(a) reports the load usage ratios of the power, thermal, and current constraints. The final usage ratios are

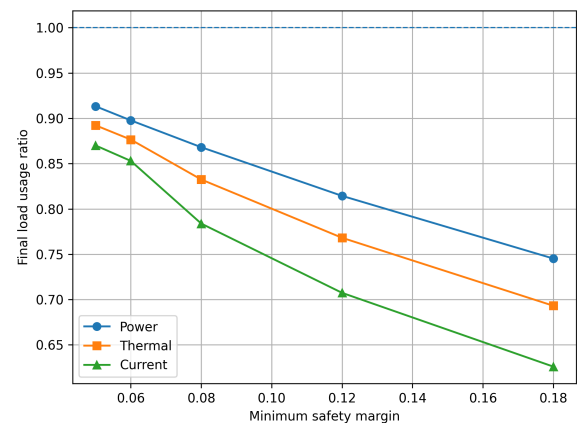
$$0.897712, \quad 0.876537, \quad 0.852987,$$

respectively. Hence, the obtained allocation makes relatively high use of the available engineering resources while remaining within the safe operating region. Since all three constraints are strictly inactive at the equilibrium, the associated multipliers converge to zero, which is consistent with the complementarity condition.

To further examine the effect of the safety margin, we vary Δ_b and record the final usage ratios of the three engineering constraints. The results are shown in Figure 7(b). As the safety margin increases, the final usage ratios decrease. This is consistent with the engineering interpretation that a larger safety margin means a more conservative operating region.



(a) Load usage during the neural dynamics



(b) Load usage under different safety margins.

Figure 7. Power, thermal, and current load usage ratios for the multi-actuator output allocation problem.

The application results show that the proposed projection-activation inertial BAM-type neural network can solve the multi-actuator output allocation problem under coupled power, thermal, and current constraints. The matrix-measure stability condition is explicitly verified, the projection residual converges to machine precision, and the final allocation remains in the certified safe operating region. These results support the applicability of the proposed neural-network method to engineering optimization problems with coupled constraints.

8. Conclusions

This paper proposed a nonlinear projection activation inertial BAM neural dynamic model for solving fixed points of extremal mappings with coupled constraints. By combining the KKT characterization with the projection form of normal cone inclusions, problem (1.1) was reformulated as the projection residual equation $R(z) = z - T(z) = 0$. Under the matching condition $B_0 = C_0 + D_0$, the equilibria of the proposed model were proved to coincide with the zeros of the projection residual, and therefore correspond to the solutions of problem (1.1). Moreover, by rewriting the second order dynamics as a first order error system, a conditional exponential stability criterion on the prescribed operating domain was established by using matrix measure estimates and Halanay's inequality. Numerical examples and a multi-actuator output allocation application showed that the theoretical matrix measure condition can be explicitly verified and that the proposed method maintains accurate convergence for boundary solutions and high dimensional problems. The numerical results also illustrated the delay dependent convergence behavior predicted by the Halanay estimate, while the comparison experiments demonstrated competitive finite time convergence performance under certified parameter settings. Future work will consider nonsmooth models, adaptive parameter selection, and sparse implementations for larger engineering applications.

Author contributions

Yunlai Zhong: Conceptualization, methodology, formal analysis, software, validation and writing—original draft. Li Wang: Supervision, methodology, writing—review and editing, and funding acquisition. Juhe Sun: Supervision, formal analysis and writing—review and editing. Xu Wu: Validation, numerical experiments and writing—review and editing. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have used Artificial Intelligence (AI) tools in the preparation and revision of this article.

AI tools used: ChatGPT (OpenAI).

How were the AI tools used? ChatGPT was used to improve grammar, sentence structure, and readability.

Where in the article is the information located? The AI-assisted English editing applies to the overall manuscript, particularly in refining the clarity and coherence of the text.

Data availability

The numerical data generated in this study can be made available by the corresponding author upon reasonable request.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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