



Research article**Linear codes over a new commutative non-unital ring of characteristic 2 and order 16: structure, duality, and Gray map****Fawziah Alharthi and Tamador Alihia***

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* **Correspondence:** Email: T.alyahya@qu.edu.sa.**Abstract:** We introduce and investigate linear codes over a new finite commutative non-unital ring

$$\mathcal{R} = \mathcal{I} + u\mathcal{I}, \quad u^2 = 0,$$

where

$$\mathcal{I} = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

The ring $\mathcal{R}$ has characteristic 2, order 16, and a rich nilpotent structure, providing a new algebraic setting for coding theory beyond finite fields and Frobenius rings. We establish a comprehensive framework for linear codes over $\mathcal{R}$ by describing the module structure of $\mathcal{R}$ over $\mathbb{F}_2 + u\mathbb{F}_2$ and introducing residue and torsion codes. Using this residue-torsion framework, we derive characterizations of self-orthogonal, quasi-self-dual, self-dual, and free codes together with explicit descriptions of their duals. Furthermore, we construct an $\mathbb{F}_2 + u\mathbb{F}_2$ -linear Gray map and prove that it is a Lee-weight-preserving isometry from $\mathcal{R}^n$ to $(\mathbb{F}_2 + u\mathbb{F}_2)^{2n}$. We also develop the theory of Hamming and Lee weights, establish the corresponding weight enumerators, and investigate their structural properties. Finally, several explicit examples are constructed and verified using the computational algebra system MAGMA, illustrating the existence of quasi-self-dual, self-dual, and Type IV codes together with their weight distributions. These results demonstrate that the ring $\mathcal{R}$ provides a natural and effective framework for constructing and studying new families of linear codes over finite non-unital rings, thereby enriching the algebraic foundations of coding theory.

Keywords: additive codes; non-unital rings; self-dual codes; quasi self-dual codes; numerical results**Mathematics Subject Classification:** 94B05, 16D10

1. Introduction

The theory of linear codes has traditionally been rooted in finite fields, forming the backbone of modern digital communication and error correction. Over the past few decades, this foundation has been significantly broadened by the study of codes over finite rings. Notably, the realization that many exceptional nonlinear binary codes, such as the Kerdock and Preparata codes, can be described as linear codes over $\mathbb{Z}_4$ [1–3] sparked a major shift in the field. This insight opened new directions in algebraic coding theory, leading to extensive research into codes over more general ring structures.

In particular, Frobenius rings, Galois rings, and chain rings have attracted substantial attention due to their rich algebraic structure, well-behaved duality theory, and close connections to classical results over fields. More recently, research has expanded into less conventional algebraic systems, including non-Frobenius and non-unital rings—rings that lack a multiplicative identity and exhibit nontrivial annihilator structures. These rings challenge many of the standard assumptions in coding theory but offer fresh opportunities for discovering new classes of codes with unusual symmetry, duality behavior, and weight distributions.

The classification and algebraic properties of non-unital rings have been studied in [4], while their applications in coding theory have gained traction through several recent works [5, 6]; see also [7, 8]. These studies have demonstrated that even without a multiplicative identity, non-unital rings can support robust module structures and meaningful duality theories—especially when enhanced with Gray maps, residue torsion frameworks, and specialized trace functions.

In this work, we introduce and investigate a new finite commutative ring of characteristic 2 and order 16, constructed as $\mathcal{R} = \mathcal{I} + u\mathcal{I}$, where $u^2 = 0$. The extension by the nilpotent element u results in a layered ring structure that is non-unital and algebraically rich. To our knowledge, this specific ring structure has not been previously studied in the context of coding theory.

We develop a complete framework for studying linear codes over this ring $\mathcal{R}$. Our approach includes the following:

- A residue-torsion framework for codes.
- A trace function compatible with the coefficient map $\alpha_{\mathcal{R}}$.
- A detailed analysis of duality, including self-orthogonal, self-dual, and quasi-self-dual codes.
- The construction of Gray map preserving Lee weight and supporting binary image analysis.

In support of the theoretical development, we provide explicit examples and classifications of linear codes over $\mathcal{R}$ for small lengths. These constructions demonstrate the viability of the ring for producing quasi-self-dual and self-dual codes and reveal structural phenomena not observable in more conventional settings.

This paper is organized as follows. Section 2 presents necessary background on modules over rings, duality, and weight enumerators. Section 3 develops the algebraic structure of the ring $\mathcal{R}$, the theory of linear codes over it, and the associated trace and weight frameworks. Section 4 concludes with a summary of our findings and outlines future directions, including the study of cyclic and quasi-cyclic codes over non-unital rings.

2. Preliminaries

This section provides the foundational concepts necessary for our development of linear codes over the ring $\mathcal{R}$. We begin by recalling basic definitions from module theory, which serve as the algebraic framework for coding theory over rings.

Definition 2.1. Let R be a ring. A left R -module is a set K equipped with the following:

- (i) An abelian group structure under addition.
- (ii) A scalar multiplication $R \times K \rightarrow K$, denoted by ax , satisfying the following axioms for all $a, b \in R$ and $x, y \in K$:

- (1) $(a + b)x = ax + bx$
- (2) $a(x + y) = ax + ay$
- (3) $(ab)x = a(bx)$

If R is unital (i.e., contains a multiplicative identity 1), we also require the following:

- (4) $1 \cdot x = x$, for all $x \in K$

A right R -module is defined similarly, with scalar multiplication on the right.

Remark 2.1. If R is a commutative ring, then any left R -module can naturally be regarded as a right R -module by defining $xa = ax$ for all $a \in R, x \in K$. In this case, we simply refer to K as an R -module.

Definition 2.2. Let K be an R -module. A subset $S \subseteq K$ is called an R -submodule if:

- S is a subgroup of the additive group K .
- For all $a \in R$ and $s \in S$, we have $as \in S$.

Definition 2.3. We define an inner product on R^n by

$$(x, y) = \sum_{i=1}^n x_i y_i, \quad \text{for } x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in R^n.$$

Definition 2.4 ([9]). Let $C \subseteq R^n$ be a linear code. The dual code of C is the submodule

$$C^\perp = \{y \in R^n \mid (x, y) = 0 \text{ for all } x \in C\}.$$

We say that C is

- self-orthogonal if $C \subseteq C^\perp$
- self-dual if $C = C^\perp$

Definition 2.5. A linear code $C \subseteq R^n$ is called quasi-self-dual (QSD) if it is self-orthogonal and has size $|C| = 4^n$.

Definition 2.6. A Type IV code over R is a quasi-self-dual code in which all complete weights (in a suitable symbolic sense) are even.

Definition 2.7. A quasi-Type IV code over R is a quasi-self-dual code whose torsion code

$$\text{Tor}(C) \subseteq (\mathbb{F}_2 + u\mathbb{F}_2)^n$$

is even, that is, every codeword in $\text{Tor}(C)$ has even Hamming weight.

Definition 2.8. Let $x = (x_1, \dots, x_n) \in R^n$ be a codeword. The Hamming weight of x , denoted by $\text{wt}_H(x)$, is defined as the number of components x_i that are nonzero, that is,

$$\text{wt}_H(x) = |\{i \mid x_i \neq 0, 1 \leq i \leq n\}|.$$

Definition 2.9. Let C be a code of length n over R . The minimum Hamming distance of C , denoted by $d_H(C)$, is the smallest Hamming distance between any two distinct codewords of C . More precisely,

$$d_H(C) = \min_{\substack{x, y \in C \\ x \neq y}} d_H(x, y),$$

where

$$d_H(x, y) = \text{wt}_H(x - y).$$

Rings of order 4

For a prime p , the classification of rings of order p^2 was first established by Raghavendran [10]. An explicit list and a self-contained proof can be found in [4]. Specializing to the case $p = 2$, the eleven rings of order 4 can be presented by generators and relations as follows:

$$\begin{aligned} \mathcal{A} &= \mathbb{Z}_4, \\ \mathcal{B} &= \langle a \mid 4a = 0, a^2 = 2a \rangle, \\ \mathcal{C} &= \langle a \mid 4a = 0, a^2 = 0 \rangle, \\ \mathcal{D} &= \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = ba = 0 \rangle, \\ \mathcal{E} &= \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle, \\ \mathcal{F} &= \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = b, ba = a \rangle, \\ \mathcal{G} &= \mathbb{F}_2[x]/(x^2), \\ \mathcal{H} &= \langle a, b \mid 2a = 2b = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle, \\ \mathcal{I} &= \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle, \\ \mathcal{J} &= \langle a, b \mid 2a = 2b = 0, a^2 = b^2 = 0 \rangle, \\ \mathcal{K} &= \mathbb{F}_4. \end{aligned}$$

The rings $\mathcal{A}, \mathcal{D}, \mathcal{G}$, and $\mathcal{K}$ are the only unital rings. Rings $\mathcal{E}$ and $\mathcal{F}$ are non-commutative, while the remaining rings are commutative.

We now focus on the ring $\mathcal{I}$, which plays a central role in our construction. It is defined by

$$\mathcal{I} = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

This ring is commutative, of characteristic 2, and consists of exactly four elements:

$$\mathcal{I} = \{0, a, b, a + b\}.$$

The following tables describe the addition and multiplication operations in $\mathcal{I}$.

Addition table:

+	0	a	b	$a + b$
0	0	a	b	$a + b$
a	a	0	$a + b$	b
b	b	$a + b$	0	a
$a + b$	$a + b$	b	a	0

Multiplication table:

$\times$	0	a	b	$a + b$
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
$a + b$	0	b	0	b

We observe the following key properties:

- $a^2 = b$, $b^2 = 0$, $ab = 0$.
- $(a + b)^2 = a^2 + ab + ba + b^2 = b + 0 + 0 + 0 = b$.
- The ring has no multiplicative identity.

The ring $\mathcal{I}$ provides a foundational algebraic structure for the extension $\mathcal{R} = \mathcal{I} + u\mathcal{I}$, where $u^2 = 0$. Understanding the properties of $\mathcal{I}$ is essential before analyzing the more complex structure and code-theoretic behavior of $\mathcal{R}$, which we explore in the next section.

3. Main results

In the current section, we build up the ring $\mathcal{R}$ and provide structure and results related to linear codes, including residue-torsion structure and duality, as well as special properties of codes and weight enumerators over $\mathcal{R}$. Finally, we provide numerical examples using $\mathcal{R}$.

3.1. Construction of the ring $\mathcal{R}$

Consider the ring $\mathcal{I}$ as presented above. Now we define the natural structure of a ring $\mathcal{R} = \mathcal{I} + u\mathcal{I}$ constructed by extending the ring $\mathcal{I}$ with an indeterminate u that commutes with elements of $\mathcal{I}$ and satisfies $u^2 = 0$. Notice that $\mathcal{R}$ is a finite ring of characteristic 2 and of order 16. Explicitly, the elements of $\mathcal{R}$ are

$$\begin{aligned} &0, \quad a, \quad b, \quad c = a + b, \quad ua, \quad ub, \quad uc, \\ &a + ua, \quad a + ub, \quad a + uc, \\ &b + ua, \quad b + ub, \quad b + uc, \\ &c + ua, \quad c + ub, \quad c + uc. \end{aligned}$$

(Note: $u0 = 0$, so it is not listed separately.)

Therefore, we can concisely write the elements of $\mathcal{R}$ as

$$\mathcal{R} = \{x + uy \mid x, y \in \mathcal{I}\}.$$

For two elements $x_1 + uy_1, x_2 + uy_2 \in \mathcal{R}$, the addition is defined component-wise as follows:

$$(x_1 + uy_1) + (x_2 + uy_2) = (x_1 + x_2) + u(y_1 + y_2),$$

while the multiplication uses the distributive property along with the rules for $\mathcal{I}$ and $u^2 = 0$:

$$(x_1 + uy_1)(x_2 + uy_2) = x_1x_2 + u(x_1y_2 + y_1x_2).$$

The multiplication table is as follows:

$\times$	0	a	b	c	ua	ub	uc	$a + ua$	$a + ub$	$a + uc$	$b + ua$	$b + ub$	$b + uc$	$c + ua$	$c + ub$	$c + uc$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	b	0	b	ub	0	ub	$b + ub$	b	$b + ub$	ub	0	ub	$b + ub$	b	$b + ub$
b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c	0	b	0	b	ub	0	ub	$b + ub$	b	$b + ub$	ub	0	ub	$b + ub$	b	$b + ub$
ua	0	ub	0	ub	0	0	0	ub	ub	ub	0	0	0	ub	ub	ub
ub	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
uc	0	ub	0	ub	0	0	0	ub	ub	ub	0	0	0	ub	ub	ub
$a + ua$	0	$b + ub$	0	$b + ub$	ub	0	ub	b	$b + ub$	b	ub	0	ub	b	$b + ub$	b
$a + ub$	0	b	0	b	ub	0	ub	$b + ub$	b	$b + ub$	ub	0	ub	$b + ub$	b	$b + ub$
$a + uc$	0	$b + ub$	0	$b + ub$	ub	0	ub	b	$b + ub$	b	ub	0	ub	b	$b + ub$	b
$b + ua$	0	ub	0	ub	0	0	0	ub	ub	ub	0	0	0	ub	ub	ub
$b + ub$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$b + uc$	0	ub	0	ub	0	0	0	ub	ub	ub	0	0	0	ub	ub	ub
$c + ua$	0	$b + ub$	0	$b + ub$	ub	0	ub	b	$b + ub$	b	ub	0	ub	b	$b + ub$	b
$c + ub$	0	b	0	b	ub	0	ub	$b + ub$	b	$b + ub$	ub	0	ub	$b + ub$	b	$b + ub$
$c + uc$	0	$b + ub$	0	$b + ub$	ub	0	ub	b	$b + ub$	b	ub	0	ub	b	$b + ub$	b

From the multiplication structure, we observe that $\mathcal{R}$ is a finite commutative non-unital ring of characteristic 2 and order 16. Let

$$\mathcal{S} = \mathbb{F}_2 + u\mathbb{F}_2 = \{0, 1, u, 1 + u\}, \quad u^2 = 0.$$

Every element $r \in \mathcal{R}$ can be uniquely expressed in the form

$$r = (as_1 + bt_1) + u(as_2 + bt_2) = a(s_1 + us_2) + b(t_1 + ut_2),$$

where

$$s_1, t_1, s_2, t_2 \in \mathbb{F}_2.$$

Consequently, every element of $\mathcal{R}$ has a unique representation

$$r = a\lambda + b\mu, \quad \lambda, \mu \in \mathcal{S}.$$

Hence, as an $\mathcal{S}$ -module,

$$\mathcal{R} = a\mathcal{S} \oplus b\mathcal{S}.$$

We define the coefficient map

$$\alpha_{\mathcal{R}}: \mathcal{R} \longrightarrow \mathcal{S}$$

by

$$\alpha_{\mathcal{R}}(a\lambda + b\mu) = \lambda, \quad \lambda, \mu \in \mathcal{S}.$$

The map $\alpha_{\mathcal{R}}$ extends componentwise to

$$\alpha_{\mathcal{R}}: \mathcal{R}^n \longrightarrow \mathcal{S}^n.$$

This map will be used to associate an $\mathcal{S}$ -linear residue code with every linear code over $\mathcal{R}$.

3.2. Linear codes over $\mathcal{R}$

A linear $\mathcal{R}$ -code C of length n is an $\mathcal{S}$ -submodule of $\mathcal{R}^n$ satisfying

$$rC \subseteq C \quad \text{for all } r \in \mathcal{R}.$$

To each linear $\mathcal{R}$ -code C , we associate an additive code

$$\psi(C) \subseteq (\mathbb{F}_4 + u\mathbb{F}_4)^n$$

via the following alphabet substitution:

$$\begin{array}{lll} 0 \mapsto 0, & a \mapsto \omega, & b \mapsto 1, & c \mapsto \omega^2, \\ ub \mapsto u, & ua \mapsto u\omega, & uc \mapsto u\omega^2, & \\ b + ub \mapsto 1 + u, & b + ua \mapsto 1 + u\omega, & b + uc \mapsto 1 + u\omega^2, & \\ a + ua \mapsto \omega + u\omega, & a + ub \mapsto \omega + u, & c + ub \mapsto \omega^2 + u, & \\ a + uc \mapsto \omega + u\omega^2, & c + ua \mapsto \omega^2 + u\omega, & c + uc \mapsto \omega^2 + u\omega^2. & \end{array}$$

Here, $\omega \in \mathbb{F}_4$ satisfies $\omega^2 + \omega + 1 = 0$ and ψ extends coordinatewise to $\mathcal{R}^n$. Codes over the ring $\mathbb{F}_4 + u\mathbb{F}_4$, $u^2 = 0$, were studied in [11].

For a linear $\mathcal{R}$ -code C , define the residue and torsion codes, respectively, by

$$\text{Res}(C) = \{\alpha_{\mathcal{R}}(c) \mid c \in C\}, \quad \text{Tor}(C) = \{x \in \mathcal{S}^n \mid bx \in C\},$$

where

$$\alpha_{\mathcal{R}}(a\lambda + b\mu) = \lambda, \quad \lambda, \mu \in \mathcal{S}.$$

Both $\text{Res}(C)$ and $\text{Tor}(C)$ are $\mathcal{S}$ -submodules of $\mathcal{S}^n$. Moreover,

$$\text{Res}(C) \subseteq \text{Tor}(C),$$

since $av + bw \in C$ implies

$$a(av + bw) = bv \in C.$$

Assume that $\text{Res}(C)$ and $\text{Tor}(C)$ are free $\mathcal{S}$ -modules. If

$$\text{rank}_{\mathcal{S}}(\text{Res}(C)) = k_1, \quad \text{rank}_{\mathcal{S}}(\text{Tor}(C)) = k_1 + k_2,$$

then C is said to be of type (k_1, k_2) , with

$$|\text{Res}(C)| = 4^{k_1}, \quad |\text{Tor}(C)| = 4^{k_1+k_2}.$$

Let

$$\text{Tr}(x + uy) = \text{Tr}(x) + u \text{Tr}(y), \quad x, y \in \mathbb{F}_4,$$

where $\text{Tr}(z) = z + z^2$ is the usual trace from $\mathbb{F}_4$ to $\mathbb{F}_2$. Since

$$\text{Tr}(\psi(r)) = \alpha_{\mathcal{R}}(r),$$

it follows that

$$\text{Res}(C) = \text{Tr}(\psi(C)).$$

Finally, for

$$\alpha_C = \alpha_{\mathcal{R}}|_C,$$

we have

$$\text{Res}(C) = \text{Im}(\alpha_C), \quad \ker(\alpha_C) = b \text{Tor}(C).$$

Since $x \mapsto bx$ is a bijection from $\mathcal{S}^n$ onto $b\mathcal{S}^n$, the first isomorphism theorem gives

$$|C| = |\text{Res}(C)| |\text{Tor}(C)|.$$

Therefore, for a code of type (k_1, k_2) ,

$$|C| = 4^{2k_1+k_2}.$$

3.3. Decomposition, duality, and special code properties over $\mathcal{R}$

Theorem 3.1. *Let $C \subseteq \mathcal{R}^n$ be a linear code. Then,*

- (i) *Every codeword $c \in C$ can be expressed as $c = av + bw$, if and only if $v \in \text{Res}(C)$, and $w \in \mathcal{S}^n$.*
- (ii) *C is self-orthogonal if and only if $\text{Res}(C)$ is self-orthogonal.*

Proof.

- (i) It is well-known that every codeword $c \in C$ can be written as $c = av + bw$, where $v, w \in \mathcal{S}^n$. Since

$$\alpha_C(c) = v \quad \text{and} \quad \text{Res}(C) = \text{Im}(\alpha_C),$$

it follows that

$$v \in \text{Res}(C).$$

Conversely, for any $v \in \text{Res}(C)$, there exists $c \in C$ such that $\alpha_C(c) = v$. Writing $c = ax + bw$ for some $x, w \in \mathcal{S}^n$, we have $\alpha_C(c) = x$. Hence, $x = v$, and so $av + bw \in C$.

- (ii) Let $v_1, v_2 \in \text{Res}(C)$. Then, there exist codewords

$$c_1 = av_1 + bw_1, \quad c_2 = av_2 + bw_2,$$

where $w_1, w_2 \in \mathcal{S}^n$. If C is self-orthogonal, then $0 = (c_1, c_2)$. Using the bilinearity of the inner product together with the multiplication rules $a^2 = b$ and $ab = b^2 = 0$, we obtain

$$(c_1, c_2) = (av_1 + bw_1, av_2 + bw_2) = b(v_1, v_2).$$

Hence, $b(v_1, v_2) = 0$, which implies that $(v_1, v_2) = 0$. Therefore, $\text{Res}(C)$ is self-orthogonal. □

Theorem 3.2. *A linear code $C \subseteq \mathcal{R}^n$ is QSD if and only if the following conditions hold:*

- (i) *$\text{Res}(C)$ is self-orthogonal with $\text{rank}_{\mathcal{S}}(\text{Res}(C)) = k_1$.*
- (ii) *$\text{rank}_{\mathcal{S}}(\text{Tor}(C)) = n - k_1$.*

Proof. According to Theorem 3.1, the inclusion $C \subseteq C^\perp$ holds if and only if $\text{Res}(C) \subseteq \text{Res}(C)^\perp$. Furthermore, since

$$|C| = |\text{Res}(C)| \cdot |\text{Tor}(C)| = 4^{k_1} \cdot 4^{n-k_1} = 4^n,$$

it follows that

$$\text{rank}_{\mathcal{S}}(\text{Tor}(C)) = n - k_1,$$

completing the proof. □

Remark 3.1. The residue code of every self-orthogonal code over $\mathcal{R}$ is an even code. Therefore, no such code can have a residue containing vectors of odd weight.

Theorem 3.3. A code $C \subseteq \mathcal{R}^n$ is free precisely when $\text{Tor}(C) = \text{Res}(C)$; equivalently, $k_2 = 0$.

Proof. Since

$$\text{rank}_S(\text{Res}(C)) = k_1 \quad \text{and} \quad \text{rank}_S(\text{Tor}(C)) = k_1 + k_2,$$

we have $\text{Tor}(C) = \text{Res}(C)$ precisely when $k_2 = 0$. $\square$

Corollary 3.1. Let C be a free QSD code over $\mathcal{R}$. Then, C is quasi Type IV.

Proof. Since C is free, we have $\text{Res}(C) = \text{Tor}(C)$. Moreover, as C is QSD, it is self-orthogonal. Therefore, by Theorem 3.2, the residue code $\text{Res}(C)$ is even. Since $\text{Res}(C) = \text{Tor}(C)$, it follows that $\text{Tor}(C)$ is also even. Hence, C is quasi Type IV. $\square$

We now examine how duality interacts with the residue and torsion codes of linear codes over the ring $\mathcal{R}$.

Theorem 3.4. For a linear code $C \subseteq \mathcal{R}^n$, the following properties are satisfied:

(1) The residue of the dual code coincides with the dual of the residue code; namely,

$$\text{Res}(C^\perp) = \text{Res}(C)^\perp.$$

(2) The torsion of the dual code is the entire space, that is, $\text{Tor}(C^\perp) = \mathcal{S}^n$.

Proof.

(1) Let $v \in \text{Res}(C^\perp)$. By Theorem 3.1, there exists a codeword such that $(av + bw) \in C^\perp$. For any $x \in \text{Res}$, there exists a codeword $(ax + by) \in C$. Since

$$(ax + by, av + bw) = b(x, v) = 0,$$

we conclude $v \in \text{Res}(C)^\perp$, so $\text{Res}(C^\perp) \subseteq \text{Res}(C)^\perp$.

Conversely, if $v \in \text{Res}(C)^\perp$, then for every codeword $ax + by \in C$, we have

$$(av, ax + by) = b(v, x) = 0.$$

Hence, $av \in C^\perp$, and therefore $v \in \text{Res}(C^\perp)$. Consequently, $\text{Res}(C^\perp) = \text{Res}(C)^\perp$.

(2) Let $v \in \mathcal{S}^n$. For any codeword $c = ax + by \in C$, we compute

$$(bv, c) = 0,$$

because $ab = b^2 = 0$ in $\mathcal{R}$. Therefore, $bv \in C^\perp$, implying $v \in \text{Tor}(C^\perp)$. Since v was arbitrary, we conclude $\text{Tor}(C^\perp) = \mathcal{S}^n$. $\square$

Theorem 3.5. For a linear code $C \subseteq \mathcal{R}^n$, one has the following.

$$C^\perp = a \cdot \text{Res}(C)^\perp \oplus b \cdot \mathcal{S}^n.$$

Proof. We prove

$$C^\perp = a \cdot \text{Res}(C)^\perp + b \cdot \mathcal{S}^n.$$

Each $c \in C^\perp$ can be written as $c = ax + by$, where $x \in \text{Res}(C)^\perp$ and $y \in \mathcal{S}^n$. Since

$$\text{Res}(C^\perp) = \text{Res}(C)^\perp,$$

from Theorem 3.4, it follows that

$$C^\perp \subseteq a \cdot \text{Res}(C)^\perp + b \cdot \mathcal{S}^n.$$

To prove the reverse inclusion, let

$$c = ax + by \in a \cdot \text{Res}(C)^\perp + b \cdot \mathcal{S}^n.$$

For any $c' \in C$, $c' = ax' + by'$ where $x' \in \text{Res}(C)$ and $y' \in \mathcal{S}^n$. Notice that since $ab = b^2 = 0$, we have

$$(c, c') = b(x, x') = 0.$$

Hence, c is orthogonal to every codeword of C , which implies that $c \in C^\perp$. Therefore,

$$a \cdot \text{Res}(C)^\perp + b \cdot \mathcal{S}^n \subseteq C^\perp.$$

Combining the two inclusions, we conclude that

$$C^\perp = a \cdot \text{Res}(C)^\perp + b \cdot \mathcal{S}^n.$$

Moreover, by Theorem 3.4,

$$|C^\perp| = |\text{Res}(C)^\perp| |\mathcal{S}^n|.$$

This completes the proof. □

Theorem 3.6. *A linear code $C \subseteq \mathcal{R}^n$ is self-dual precisely when:*

- (1) *The residue code $\text{Res}(C)$ is self-dual.*
- (2) *The torsion code $\text{Tor}(C) = \mathcal{S}^n$.*

Proof. Assume $C = C^\perp$. Then, by Theorem 3.4, we have $\text{Res}(C) = \text{Res}(C)^\perp$, so $\text{Res}(C)$ is self-dual. Also, $\text{Tor}(C) = \text{Tor}(C^\perp) = \mathcal{S}^n$.

Conversely, suppose both conditions hold. Applying Theorem 3.5, we obtain

$$C^\perp = a \cdot \text{Res}(C)^\perp \oplus b \cdot \mathcal{S}^n = a \cdot \text{Res}(C) \oplus b \cdot \mathcal{S}^n = C,$$

where the second equality follows from the self-duality of $\text{Res}(C)$. Therefore, C is self-dual. □

Corollary 3.2. *A linear code $C \subseteq \mathcal{R}^n$ cannot be both self-dual and quasi-self-dual.*

Proof. Assume that C is self-dual. By Theorem 3.6,

$$\text{Res}(C) = \text{Res}(C)^\perp \quad \text{and} \quad \text{Tor}(C) = S^n.$$

Hence,

$$\text{rank}_S(\text{Res}(C)) = \frac{n}{2}, \quad \text{rank}_S(\text{Tor}(C)) = n.$$

Using

$$|C| = |\text{Res}| |\text{Tor}(C)|,$$

we obtain

$$|C| = 4^{n/2} \cdot 4^n = 4^{3n/2}.$$

On the other hand, if C were quasi-self-dual, then by definition $|C| = 4^n$. Since $4^{3n/2} > 4^n$ for $n > 0$, this is impossible. Therefore, C cannot be both self-dual and quasi-self-dual. $\square$

Proposition 3.1. *Let $C \subseteq \mathcal{R}$ be a nonzero code. Then, $C \cap C^\perp \neq \{0\}$.*

Proof. Since C is nonzero, there exists a nonzero codeword

$$x = a(s + uv) + b(t + uw) \in C.$$

If $s + uv = 0$, then

$$x = b(t + uw),$$

which belongs to $C^\perp$. Hence, $x \in C \cap C^\perp$, and $x \neq 0$.

Now suppose $s + uv \neq 0$. Multiplying x by a , we obtain

$$ax = a^2(s + uv) + ab(t + uw) = b(s + uv),$$

where we use the identities $a^2 = b$ and $ab = 0$. Since C is an $\mathcal{R}$ -submodule, $ax \in C$. Moreover, $b(s + uv) \in C^\perp$. Because $s + uv \neq 0$, we have $ax \neq 0$. Therefore,

$$ax \in C \cap C^\perp.$$

Thus, in either case, the intersection $C \cap C^\perp$ contains a nonzero element, and consequently

$$C \cap C^\perp \neq \{0\}.$$

$\square$

3.4. Lee weight and Gray map

For any $r = a\alpha + b\beta \in \mathcal{R}$, where $\alpha, \beta \in \mathcal{S}$, define the Gray map

$$\Phi: \mathcal{R} \rightarrow \mathcal{S}^2, \quad \Phi(a\alpha + b\beta) = (\beta, \alpha + \beta).$$

The Lee weight of $(a\alpha + b\beta) \in \mathcal{R}$ is defined by

$$\begin{aligned} \text{wt}_{\text{Lee}}(a\alpha + b\beta) &= \text{wt}_{\text{H}}(\Phi(a\alpha + b\beta)) \\ &= \text{wt}_{\text{H}}(\beta, \alpha + \beta) \\ &= \text{wt}_{\text{H}}(\beta) + \text{wt}_{\text{H}}(\alpha + \beta), \end{aligned}$$

where wt_H denotes the ordinary Hamming weight on S^2 . That is for an element $x \in S$,

$$\text{wt}_H(x) = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0. \end{cases}$$

The Gray map Φ extends coordinatewise to

$$\Phi: R^n \longrightarrow S^{2n},$$

for every $n \in \mathbb{N}$.

The following Table 1 illustrates the decomposition of elements $r = a\alpha + b\beta$ in $\mathcal{R}$, their images under the Gray map Φ , and the corresponding Lee weights. This provides a clear overview of the weight structure fundamental to analyzing codes over $\mathcal{R}$.

Table 1. Gray images and Lee weights of all elements in $\mathcal{R}$, using $\Phi(a\alpha + b\beta) = (\beta, \alpha + \beta)$.

r	$r = a\alpha + b\beta$	α	β	$\Phi(r)$	$\text{wt}_{\text{Lee}}(r)$
0	0	0	0	(0, 0)	0
a	$a(1 + u \cdot 0) + b(0 + u \cdot 0)$	1	0	(0, 1)	1
b	$a(0 + u \cdot 0) + b(1 + u \cdot 0)$	0	1	(1, 1)	2
c	$a(1 + u \cdot 0) + b(1 + u \cdot 0)$	1	1	(1, 0)	1
ua	$a(0 + u \cdot 1) + b(0 + u \cdot 0)$	u	0	(0, u)	1
ub	$a(0 + u \cdot 0) + b(0 + u \cdot 1)$	0	u	(u , u)	2
uc	$a(0 + u \cdot 1) + b(0 + u \cdot 1)$	u	u	(u , 0)	1
$a + ua$	$a(1 + u \cdot 1) + b(0 + u \cdot 0)$	$1 + u$	0	(0, $1 + u$)	1
$a + ub$	$a(1 + u \cdot 0) + b(0 + u \cdot 1)$	1	u	(u , $1 + u$)	2
$a + uc$	$a(1 + u \cdot 1) + b(0 + u \cdot 1)$	$1 + u$	u	(u , 1)	2
$b + ua$	$a(0 + u \cdot 1) + b(1 + u \cdot 0)$	u	1	(1, $1 + u$)	2
$b + ub$	$a(0 + u \cdot 0) + b(1 + u \cdot 1)$	0	$1 + u$	($1 + u$, $1 + u$)	2
$b + uc$	$a(0 + u \cdot 1) + b(1 + u \cdot 1)$	u	$1 + u$	($1 + u$, 1)	2
$c + ua$	$a(1 + u \cdot 1) + b(1 + u \cdot 0)$	$1 + u$	1	(1, u)	2
$c + ub$	$a(1 + u \cdot 0) + b(1 + u \cdot 1)$	1	$1 + u$	($1 + u$, u)	2
$c + uc$	$a(1 + u \cdot 1) + b(1 + u \cdot 1)$	$1 + u$	$1 + u$	($1 + u$, 0)	1

For any linear code $C \subseteq \mathcal{R}^n$ and a codeword $c = (c_1, c_2, \dots, c_n) \in C$ we have:

- The Lee weight

$$\text{wt}_{\text{Lee}}(c) = \sum_{i=1}^n \text{wt}_{\text{Lee}}(c_i).$$

- The Lee distance between two different codewords, say $c, c' \in C$ is defined as

$$d_{\text{Lee}}(c, c') = \text{wt}_{\text{Lee}}(c - c').$$

- The Lee distance of the code C is defined by

$$d_{\text{Lee}}(C) = \min d_{\text{Lee}}(c, c').$$

Theorem 3.7. *The Gray map Φ is $\mathcal{S}$ -linear, weight-preserving map from $(\mathcal{R}^n, \text{wt}_{\text{Lee}})$ to $(\mathcal{S}^{2n}, \text{wt}_H)$. Moreover, Φ is an isometry from $(\mathcal{R}^n, d_{\text{Lee}})$ to $(\mathcal{S}^{2n}, d_H)$.*

Proof. It is straightforward to verify that Φ is $\mathcal{S}$ -linear. Notice that each $q \in \mathcal{R}^n$ can be written as $q = (q_1, q_2, \dots, q_n)$, where $q_i = a\alpha_i + b\beta_i$, and $1 \leq i \leq n$. By the definition of the Lee weight,

$$\begin{aligned} \text{wt}_{\text{Lee}}(q) &= \sum_{i=1}^n \text{wt}_{\text{Lee}}(q_i) \\ &= \sum_{i=1}^n \text{wt}_H(\Phi(q_i)) \\ &= \text{wt}_H(\Phi(q_1), \dots, \Phi(q_n)) \\ &= \text{wt}_H(\Phi(q)). \end{aligned}$$

Thus, $\text{wt}_{\text{Lee}}(q) = \text{wt}_H(\Phi(q))$, for every $q \in \mathcal{R}^n$. Furthermore, for any $q, q' \in \mathcal{R}^n$,

$$\begin{aligned} d_{\text{Lee}}(q, q') &= \text{wt}_{\text{Lee}}(q - q') \\ &= \text{wt}_H(\Phi(q - q')) \\ &= \text{wt}_H(\Phi(q) - \Phi(q')) \\ &= d_H(\Phi(q), \Phi(q')). \end{aligned}$$

Hence, Φ is an isometry. □

Definition 3.1. Let C be a code of length n over $\mathcal{R}$. The Lee weight enumerator associated with C is the polynomial

$$\text{LWE}_C(X, Y) = \sum_{c \in C} X^{2n - \text{wt}_{\text{Lee}}(c)} Y^{\text{wt}_{\text{Lee}}(c)},$$

where $\text{wt}_{\text{Lee}}(c)$ denotes the Lee weight of the codeword c .

Example 3.1. Let $C \subseteq \mathcal{R}^2$ be the code generated by

$$G = \begin{pmatrix} a & a \end{pmatrix}.$$

There are 16 codewords in C specifically,

$$C = \left\{ \begin{array}{l} (0, 0), (a, a), (b, b), (c, c), \\ (ua, ua), (ub, ub), (uc, uc), (a + ua, a + ua), \\ (b + ub, b + ub), (c + uc, c + uc), (b + ua, b + ua), (c + ua, c + ua), \\ (a + ub, a + ub), (c + ub, c + ub), (a + uc, a + uc), (b + uc, b + uc) \end{array} \right\}.$$

We now compute the Lee weights of the codewords in C , which are listed in Table 2, and the Lee weight enumerator of C .

Table 2. Lee weights of codewords in the code $C \subseteq \mathcal{R}^2$.

c	$\text{wt}_{\text{Lee}}(c) = \sum_{i=1}^n \text{wt}_{\text{Lee}}(c_i)$
$(0, 0)$	0
(a, a)	2
(ua, ua)	2
$(a + ua, a + ua)$	2
(b, b)	4
(c, c)	2
(ub, ub)	4
(uc, uc)	2
$(b + ub, b + ub)$	4
$(c + ua, c + ua)$	4
$(c + uc, c + uc)$	2
$(a + ub, a + ub)$	4
$(c + ub, c + ub)$	4
$(a + uc, a + uc)$	4
$(b + uc, b + uc)$	4
$(b + ua, b + ua)$	4

Thus, by Table 2, the Lee weight enumerator is

$$\text{LWE}_C(X, Y) = X^4 + 6X^2Y^2 + 9Y^4.$$

3.5. Examples

In this subsection, we present a collection of representative linear codes over the ring $\mathcal{R}$. All examples were generated and their parameters were verified using the computational algebra system MAGMA [12]. For each code, we list a generator matrix, identify its type, and present its Hamming weight distribution.

Example 3.2. The code generated by

$$G = (b)$$

is a quasi-self-dual code with parameters $(1, 4^1, 1)$. Its Hamming weight distribution is given in Table 3.

Table 3. Weight distribution of Example 3.2.

Weight	Frequency
0	1
1	3

Example 3.3. Let C_1 and C_2 be the S -submodules of R^2 generated by the rows of

$$G_1 = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \quad G_2 = \begin{pmatrix} c & a \\ b & b \end{pmatrix},$$

respectively, where $c = a + b$.

Both C_1 and C_2 are Type IV codes with parameters $(2, 4^2, 2)$. Their Hamming weight distribution is shown in Table 4.

Table 4. Weight distribution of Example 3.3.

Weight	Frequency
0	1
2	15

Example 3.4. Consider the code generated by

$$G = \begin{pmatrix} a & a \\ 0 & b \end{pmatrix}.$$

This is a self-dual code with parameters $(2, 4^3, 1)$. Its Hamming weight distribution is listed in Table 5.

Table 5. Weight distribution of Example 3.4.

Weight	Frequency
0	1
1	6
2	57

Example 3.5. Consider the code generated by

$$G = \begin{pmatrix} a & 0 & c \\ b & 0 & b \\ 0 & ub & b \end{pmatrix}.$$

This is a quasi-self-dual code with parameters $(3, 4^3, 1)$. Its Hamming weight distribution is listed in Table 6.

Table 6. Weight distribution of Example 3.5.

Weight	Frequency
0	1
1	2
2	33
3	28

4. Conclusions

This paper has introduced a new family of linear codes over the finite commutative non-unital ring

$$\mathcal{R} = \mathcal{I} + u\mathcal{I} \quad u^2 = 0,$$

and established the first systematic study of their algebraic and combinatorial properties. Despite the absence of a multiplicative identity, the ring admits a rich module structure that allows many fundamental concepts of coding theory to be developed in a natural way.

The residue-torsion framework provides a simple and effective approach for describing the structure of linear codes over $\mathcal{R}$, leading to characterizations of self-orthogonal, quasi-self-dual, self-dual, and free codes. Moreover, the Gray map constructed in this work maps codes over $\mathcal{R}$ into $(\mathbb{F}_2 + u\mathbb{F}_2)^{2n}$ while preserving Lee distance, making it possible to study their metric properties through their Gray images. The examples computed with the aid of MAGMA demonstrate that the ring $\mathcal{R}$ supports several interesting families of codes, including quasi-self-dual, self-dual, and Type IV codes, illustrating both the applicability of the theoretical results and the richness of the underlying algebraic structure.

The results presented here show that finite commutative non-unital rings of higher order provide a promising setting for algebraic coding theory. They also open several directions for future investigation, including the study of cyclic and constacyclic codes, classification of optimal self-dual codes, automorphism groups, MacWilliams-type identities, decoding algorithms, and applications to quantum error-correcting codes.

Author contributions

Fawziah Alharthi and Tamador Alihia: Conceptualization, methodology, formal analysis, investigation, software, validation, writing—original draft, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. S. Ling, C. Xing, *Coding theory: a first course*, Cambridge: Cambridge University Press, 2010. <https://doi.org/10.1017/CBO9780511755279> .
2. M. Shi, A. Alahmadi, P. Solé, *Codes and rings: theory and practice*, Amsterdam: Elsevier/Academic Press, 2017. <https://doi.org/10.1016/C2016-0-04429-7> .
3. V. Pless, On the uniqueness of the Golay codes, *J. Combin. Theory*, **5** (1968), 215–228. [https://doi.org/10.1016/S0021-9800\(68\)80067-5](https://doi.org/10.1016/S0021-9800(68)80067-5) .

4. B. Fine, Classification of finite rings of order p^2 , *Math. Mag.*, **66** (1993), 248–252. <https://doi.org/10.2307/2690742>.
5. A. Alahmadi, A. Altassan, W. Basaffar, A. Bonnetcaze, H. Shoaib, P. Solé, Quasi type IV codes over a non-unital ring, *Appl. Algebra Eng. Commun. Comput.*, **32** (2021), 217–228. <https://doi.org/10.1007/s00200-021-00488-6>.
6. A. Alahmadi, A. Melaibari, P. Solé, Duality of codes over non-unital rings of order four, *IEEE Access*, **11** (2023), 53120–53133. <https://doi.org/10.1109/ACCESS.2023.3261131>.
7. A. Alahmadi, T. Alihia, R. A. Betty, L. Galvez, P. Solé, The build-up construction for codes over a commutative non-unitary ring of order 9, *Mathematics*, **12** (2024), 860. <https://doi.org/10.3390/math12060860>.
8. T. Alihia, On the duality of codes over non-unital commutative ring of order p^2 , *Symmetry*, **17** (2025), 690. <https://doi.org/10.3390/sym17050690>
9. E. M. Rains, N. J. A. Sloane, Self-dual codes, In: *Handbook of coding theory*, Amsterdam: Elsevier, 1998.
10. R. Raghavendran, A class of finite rings, *Compos. Math.*, **22** (1970), 49–57.
11. S. Ling, P. Solé, Type II codes over $\mathbb{F}_4 + u\mathbb{F}_4$, *European J. Combin.*, **22** (2001), 983–997.
12. W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I: the user language, *J. Symb. Comput.*, **24** (1997), 235–265. <https://doi.org/10.1006/jSCO.1996.0125>



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