
Research article

Power-mean homogenization of dependent PHR systems: Exact parallel-system thresholds under Archimedean copulas

Thamer Manshi* and Maha Al-Mutairi

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: tmunshi@ksu.edu.sa; Tel: +966507466147.

Abstract: Replacing heterogeneous dependent component models with tractable homogeneous surrogates while preserving system-level stochastic comparisons is a recurring problem in reliability analysis. Existing mean-based results typically fix the surrogate parameter and do not identify the generator-dependent power-mean boundary for universal validity. In this paper, we developed a power-mean homogenization framework for parallel and series systems with proportional hazard rate (PHR) margins under Archimedean dependence. For parallel systems whose generator was valid in every dimension, we established an exact necessary-and-sufficient criterion: For every system size, every positive PHR parameter vector, and every non-degenerate baseline, the heterogeneous system dominated its homogeneous power-mean surrogate in the usual stochastic order if and only if a generator-induced one-variable transform was convex. The criterion yielded an exact critical power index and a constructive two-component counterexample below the admissible region. A relative-curvature condition characterized a dimension-free sufficient Jensen mechanism. For the Gumbel, Clayton, Ali-Mikhail-Haq, and Joe families, the exact and curvature-based criteria produced the same admissible sets over the studied nonzero power-index range with absolute value at most one. Strictness and equality conditions and a second-order local expansion quantified the heterogeneity gain over the arithmetic-mean benchmark. Complementary stochastic comparisons were established for series systems under appropriate convexity assumptions. Under decreasing-failure-rate baselines and additional generator-shape conditions, both system architectures admitted dispersive-order extensions; these results were sufficient rather than exact. The framework provided a rigorous basis for selecting homogeneous PHR benchmarks and comparing dependent heterogeneous systems.

Keywords: reliability modeling; system performance; Archimedean copula; proportional hazard rate

model; power-mean homogenization; stochastic order; dispersive order

Mathematics Subject Classification: 60E15, 90B25, 62H05, 62N05

Abbreviations and symbols

The following abbreviations and symbols are used in this manuscript:

Abbreviation	Meaning
PHR	Proportional hazard rate
DFR	Decreasing failure rate
AMH	Ali-Mikhail-Haq
PRHR	Proportional reversed hazard rate
Symbol	Meaning
$S(t)$	Baseline survival function
$h_S(t) = -\frac{d}{dt} \log S(t)$	Baseline hazard rate
$\lambda_i > 0$	PHR parameter of component i
$\lambda = (\lambda_1, \dots, \lambda_n)$	Vector of heterogeneous PHR parameters
r	Power-mean index
$\check{\lambda}_r = M_r(\lambda_1, \dots, \lambda_n), r \neq 0$	Homogeneous power-mean PHR benchmark
ϕ	Archimedean copula generator
$\psi = \phi^{-1}$	Inverse Archimedean generator
$X_{(n)}, Y_{(n)}$	Heterogeneous and homogeneous parallel-system lifetimes
$X_{(1)}, Y_{(1)}$	Heterogeneous and homogeneous series-system lifetimes
$\geq_{st}$	Usual stochastic order
$\geq_{disp}$	Dispersive order
$H_\phi(a) = \psi(1 - e^{-a})$	Basic transform in the exact parallel criterion
$G_{\phi,r}(z) = \psi(1 - e^{-z^{1/r}})$	Power-indexed transform governing universal homogenization
$\mathcal{K}_\phi(x) = \frac{\phi(x)\phi''(x)}{\{\phi'(x)\}^2}$	Relative generator-curvature functional
$B_\phi(a)$	Pointwise critical-index function in the exact threshold criterion
$\rho_\phi = \sup_{a>0} B_\phi(a)$	Exact generator-dependent critical power index
$\kappa_\phi = \inf_{x>0} \mathcal{K}_\phi(x), \quad r_c(\phi) = 1 - \kappa_\phi$	Curvature-based threshold quantities

1. Introduction

Reliability analysis of series and parallel systems must often account simultaneously for component heterogeneity and dependence, because both features can materially alter system-level lifetime behavior. In these canonical architectures, system lifetimes are represented by the minimum and maximum of component lifetimes, respectively. Components may differ because of manufacturing variation, age, stress, or operating environment, while common loads and shared environments induce dependence among their lifetimes. Replacing a heterogeneous system by a tractable homogeneous surrogate can simplify analytical comparison, but the replacement is meaningful only when it preserves a rigorous probabilistic relation between system lifetimes. Stochastic orders provide a natural framework for such guarantees because they compare lifetime distributions rather than isolated summary measures.

Among semiparametric reliability models, the proportional hazard rate (PHR) family represents component-specific risk through positive parameters while leaving the baseline distribution unspecified. Archimedean copulas provide a tractable description of dependence between such margins. Foundational comparisons of order statistics and spacings from heterogeneous distributions were developed by Pledger and Proschan [1]. Dykstra et al. [2] investigated stochastic comparisons for parallel systems with heterogeneous exponential components. Khaledi and Kochar extended comparisons of parallel systems [3] and examined order statistics and sample spacings [4]. Li and Fang [5] studied ordering properties of Archimedean-dependent order statistics, including applications to PHR models. Fang et al. [6] compared sample extremes of heterogeneous dependent observations, whereas Zhang et al. [7] considered series and parallel systems with resilience-scaled components. Li and Li [8] investigated hazard-rate and reversed-hazard-rate comparisons for heterogeneous dependent extremes.

Subsequent studies developed complementary, but distinct, comparison problems. Lando et al. [9] investigated second-order stochastic comparisons of order statistics. Mesfioui and Trufin [10] studied dispersive comparisons for extremes of homogeneous dependent vectors. Panja et al. [11] derived stochastic comparisons of series and parallel lifetimes with dependent heterogeneous components, while Sahoo and Hazra [12] investigated ordering and aging under Archimedean dependence. Amini-Seresht et al. [13] focused on heterogeneous dependent exponential parallel systems. Datta and Gupta [14] examined extreme order statistics under PHR, proportional reversed-hazard-rate, and location models. Shrahili [15] studied stochastic ordering of extremes from Archimedean-dependent samples, whereas Das et al. [16] considered extremes under Archimedean copulas and survival copulas. Zhong et al. [17] investigated extremes of generalized scaled PHR samples under Archimedean dependence. Arab et al. [18] developed stochastic comparisons for dependent transformation models, including PHR models. Alomani et al. [19] treated dispersive and hazard-rate comparisons of dependent exponential parallel systems, and Manshi et al. [20] studied preservation of hazard-rate and relative average hazard-rate orders in dependent series and parallel systems. These contributions address particular models, system architectures, or stochastic orders; they do not provide the exact generator-dependent power-mean boundary for universally valid homogeneous PHR parallel-system comparisons established here.

The closest mean-based results typically fix a particular homogeneous parameter, most often an arithmetic or geometric mean, and derive sufficient conditions for a stochastic comparison. In this paper, we address a different structural question: How far can the homogeneous benchmark be varied within the power-mean family while preserving the desired stochastic ordering? For positive PHR parameters $\lambda_1, \dots, \lambda_n$, define the r th power mean by

$$\tilde{\lambda}_r = \left(\frac{1}{n} \sum_{i=1}^n \lambda_i^r \right)^{1/r}, \quad r \neq 0.$$

The distinction from earlier mean-based comparisons is therefore structural rather than merely notational. A fixed-mean result asks whether one preselected surrogate (for example, the arithmetic or geometric mean) produces a valid comparison under stated conditions. This problem instead asks for the set of power indices for which validity is universal over system size, positive PHR parameter vectors, and baseline survival functions. Theorem 2.3 identifies the boundary of that set exactly and supplies a constructive two-component reversal below the boundary. Thus, the critical index is a sufficiency threshold and a sharp impossibility boundary, a feature not supplied by benchmark-specific sufficient comparisons.

The cases $r = 1$ and $r = -1$ correspond to the arithmetic and harmonic means, respectively, while the limit as $r \rightarrow 0$ yields the geometric mean. Introducing the power-mean benchmark is not merely a formal reformulation. The admissibility of a given power index depends jointly on the Archimedean generator and the sign of r . In particular, a negative power index that is valid for one copula family may fail for another. Classical properties of power means and majorization may be found in [21]. We use $M_r(\lambda_1, \dots, \lambda_n)$ as shorthand for the same power mean when a functional notation is convenient.

Main contributions. First, we establish usual stochastic comparisons between heterogeneous and homogeneous parallel and series systems with PHR margins under common Archimedean dependence, together with stronger dispersive-order comparisons under a decreasing-failure-rate (DFR) baseline and additional generator-shape conditions. Second, for parallel systems with an Archimedean generator valid in every dimension, we reduce the universal homogenization problem exactly to the ordinary Jensen convexity of the one-variable transform

$$G_{\phi,r}(z) = \psi\left(1 - e^{-z^{1/r}}\right), \quad z > 0.$$

This yields a necessary-and-sufficient generator-dependent critical index ρ_ϕ : When $r \geq \rho_\phi$, the stochastic comparison holds for every system dimension, every positive PHR parameter vector, and every non-degenerate baseline survival function, whereas when $r < \rho_\phi$, a two-component counterexample can be constructed directly from a Jensen violation. Third, we introduce the relative generator-curvature functional

$$\mathcal{K}_\phi(x) = \frac{\phi(x)\phi''(x)}{\{\phi'(x)\}^2},$$

which yields the simpler dimension-free threshold $r_c(\phi) = 1 - \inf_{x>0} \mathcal{K}_\phi(x)$ for the sufficient Jensen mechanism. For the Gumbel, Clayton, Ali-Mikhail-Haq, and Joe families, the exact and curvature-based criteria generate identical admissible power-index sets over $[-1, 0) \cup (0, 1]$.

Fourth, we establish strictness and equality conditions for the parallel-system comparison. Fifth, we derive a second-order local heterogeneity expansion that quantifies the gain over the arithmetic-mean benchmark in terms of PHR-parameter dispersion and the distance of r from one. Finally, numerical examples distinguish valid power-mean improvements from inadmissible extrapolations and illustrate the sharpness of the theoretical conditions. Table 1 positions this contribution relative to several closely related stochastic-comparison studies.

Table 1. Positioning of this work within related stochastic-comparison literature.

Study	Setting and principal orders	Relation to the present work
Li and Fang [5]	Extremes and order statistics under Archimedean dependence; likelihood-ratio and (reversed) hazard-rate orders with PHR applications	Provides a foundational dependent-ordering framework but does not consider a power-indexed homogeneous PHR benchmark.
Zhang et al. [7]	Heterogeneous resilience-scaled series and parallel systems; usual stochastic comparisons	Establishes heterogeneity effects for a different semiparametric model.
Panja et al. [11]	Dependent heterogeneous proportional-odds systems under Archimedean survival copulas	Treats a different marginal model and several system-level stochastic orders.
Arab et al. [18]	General transformation models, including PHR, PRHR, and proportional-odds models, under Archimedean copulas	Provides a broader transformation-model framework. The present paper focuses specifically on homogeneous power-mean PHR surrogates and derives an exact universal parallel-system criterion, a generator-dependent critical power index, and a constructive two-component converse.
Amini-Seresht et al. [13]; Alomani et al. [19]	Dependent heterogeneous exponential parallel systems; dispersive and hazard-rate comparisons	Provide the closest recent mean-based lower-bound results for parallel systems. The present work replaces the fixed benchmark by an r -indexed PHR family and studies admissibility as a function of the copula generator.
Present paper	Dependent heterogeneous PHR parallel and series systems; usual stochastic and dispersive orders	Uses the power mean as the homogeneous benchmark and derives an exact generator-dependent stochastic threshold together with a dimension-free curvature criterion, strictness conditions, and local-sharpness results.

1.1. Preliminaries and model setup

Let S denote the survival function of a non-negative continuous lifetime. A random variable X follows a proportional hazard rate (PHR) model with PHR parameter $\lambda > 0$ if

$$S_X(t) = S^\lambda(t), \quad t \geq 0.$$

Let $X = (X_1, \dots, X_n)$ have marginal distribution functions $F_1, \dots, F_n$. By Sklar's theorem, its joint distribution can be represented through a copula C as

$$F_X(x_1, \dots, x_n) = C\{F_1(x_1), \dots, F_n(x_n)\}.$$

Likewise, the joint survival function may be represented by a survival copula. Throughout this paper, dependence is modeled by Archimedean copulas. Let ϕ be a continuous, strictly decreasing function

from $[0, \infty)$ into $[0, 1]$, satisfying $\phi(0) = 1$ and $\phi(x) \rightarrow 0$ as $x \rightarrow \infty$, and let $\psi = \phi^{-1}$ denote its inverse. Then

$$C(u_1, \dots, u_n) = \phi\left(\sum_{i=1}^n \psi(u_i)\right), u_i \in [0, 1], i = 1, \dots, n. \quad (1.1)$$

For results at a fixed system dimension, the usual n -monotonicity conditions on ϕ are assumed whenever ensuring that (1.1) defines an n -dimensional Archimedean copula. For Theorem 2.3, whose statement ranges over all system dimensions, we additionally assume that ϕ is monotone and hence generates an Archimedean copula in every dimension. In particular, ϕ is decreasing and convex, and its inverse ψ is decreasing and convex on $(0, 1)$. Standard references for stochastic-order definitions and properties include [22–24].

Definition 1.1. Let X and Y be non-negative random variables with distribution functions F_X and F_Y and, when needed, densities f_X and f_Y .

1. X is larger than Y in the usual stochastic order, written $X \geq_{\text{st}} Y$, if $F_X(t) \leq F_Y(t)$, $t \geq 0$.
2. X is larger than Y in the dispersive order, written $X \geq_{\text{disp}} Y$, if $F_X^{-1}(u) - F_Y^{-1}(u)$ is increasing in $u \in (0, 1)$. Under the regularity conditions used below, this is equivalent to

$$f_X(t) \leq f_Y(F_Y^{-1}\{F_X(t)\}), \quad t \geq 0. \quad (1.2)$$

Throughout the paper, $X_1, \dots, X_n$ denote heterogeneous component lifetimes with survival functions $S^{\lambda_1}, \dots, S^{\lambda_n}$, whereas $Y_1, \dots, Y_n$ denote homogeneous component lifetimes with common survival function S^{λ_r} . Unless stated otherwise, $r \in [-1, 0) \cup (0, 1]$.

For parallel systems, both component vectors are coupled by the same Archimedean copula with generator ϕ , and their system lifetimes are denoted by $X_{(n)}$ and $Y_{(n)}$. For series systems, the corresponding vectors are coupled by the same Archimedean survival copula, and the system lifetimes are $X_{(1)}$ and $Y_{(1)}$. This common-dependence construction isolates the effect of replacing the heterogeneous PHR parameter vector $(\lambda_1, \dots, \lambda_n)$ by the homogeneous power-mean benchmark λ_r .

2. Power-mean stochastic ordering and exact parallel homogenization

In this section, we develop stochastic comparisons between heterogeneous PHR systems and homogeneous power-mean surrogates. The parallel result is first stated in a convenient sufficient form. We then solve the universal parallel homogenization problem exactly by reducing it to Jensen convexity of a one-variable generator transform, and relate the exact stochastic threshold to a simpler dimension-free generator-curvature criterion. The standard copula families follow as corollaries.

We begin with the technical monotonicity fact used in the parallel-system proof. For later reference, for each nonzero power index r , define the generator-composition map $J_{\phi, r}(u) = \psi(u^{1/r})$ on J_r . This is the core Jensen map used in Theorems 2.2 and 2.4.

Lemma 2.1. If $a \leq 1/2$, then $k(x; a) = \frac{xe^{ax}}{e^x - 1}$ is strictly decreasing in $x > 0$.

Proof. Differentiating directly gives

$$\frac{\partial}{\partial x} \log k(x; a) = \frac{1}{x} - \frac{e^x}{e^x - 1} + a \leq \frac{1}{x} - \frac{1}{1 - e^{-x}} + \frac{1}{2} < 0, \quad x > 0,$$

where the last inequality follows from $1 - e^{-x} < 2x/(2 + x)$ for $x > 0$.

Theorem 2.2. Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be non-negative random vectors having the same Archimedean copula with generator ϕ and inverse generator ψ . Suppose that X_i has survival

function $S^{\lambda_i}(t)$ and Y_i has survival function $S^{\check{\lambda}_r}(t)$, $i = 1, \dots, n$, where

$$\check{\lambda}_r = \left(\frac{1}{n} \sum_{i=1}^n \lambda_i^r \right)^{1/r}, \quad r \in [-1, 0) \cup (0, 1].$$

Let $J_r = (0, 1)$ for $r > 0$ and $J_r = (1, \infty)$ for $r < 0$. If $u \mapsto \psi(u^{1/r})$ is convex on J_r , then

$$X_{(n)} \geq_{st} Y_{(n)}.$$

Proof. The relation $X_{(n)} \geq_{st} Y_{(n)}$ is equivalent to

$$\phi \left(\sum_{i=1}^n \psi \{1 - S^{\lambda_i}(t)\} \right) \leq \phi(n\psi\{1 - S^{\check{\lambda}_r}(t)\}), \quad t \geq 0, \quad (2.1)$$

and, because ϕ is decreasing, to

$$\frac{1}{n} \sum_{i=1}^n \psi \{1 - S^{\lambda_i}(t)\} \geq \psi\{1 - S^{\check{\lambda}_r}(t)\}. \quad (2.2)$$

Jensen's inequality applied to the convex map $u \mapsto \psi(u^{1/r})$ gives

$$\frac{1}{n} \sum_{i=1}^n \psi \{1 - S^{\lambda_i}(t)\} \geq \psi \left[\left(\frac{1}{n} \sum_{i=1}^n (1 - S^{\lambda_i}(t))^r \right)^{\frac{1}{r}} \right]. \quad (2.3)$$

Since ψ is decreasing, it remains to establish

$$\left\{ \frac{1}{n} \sum_{i=1}^n (1 - S^{\lambda_i}(t))^r \right\}^{1/r} \leq 1 - S^{\check{\lambda}_r}(t). \quad (2.4)$$

For $0 < r \leq 1$, (2.4) is equivalent to

$$\frac{1}{n} \sum_{i=1}^n (1 - S^{\lambda_i}(t))^r \leq (1 - S^{\check{\lambda}_r}(t))^r. \quad (2.5)$$

For $0 < r \leq 1$, put $w(\lambda) = (1 - S^\lambda(t))^r$ and $u = \lambda^r$. With $x = -u^{1/r} \log S(t)$,

$$\frac{\partial}{\partial u} w(u^{1/r}) = \{-\log S(t)\}^r S^{ru^{1/r}}(t) k^{1-r}(x; 0).$$

By Lemma 2.1 (with $a = 0$), the derivative is decreasing in u , so $u \mapsto w(u^{1/r})$ is concave and Jensen's inequality yields (2.4). For $-1 \leq r < 0$, (2.4) is equivalent to

$$\frac{1}{n} \sum_{i=1}^n (1 - S^{\lambda_i}(t))^r \geq (1 - S^{\check{\lambda}_r}(t))^r. \quad (2.6)$$

For $-1 \leq r < 0$,

$$\frac{\partial}{\partial u} w(u^{1/r}) = \{-\log S(t)\}^r k^{1-r} \left(x; \frac{-r}{1-r} \right)$$

and $-r/(1-r) \leq 1/2$. Since x decreases with u and k decreases with x (by Lemma 2.1), this derivative increases with u ; hence, $u \mapsto w(u^{1/r})$ is convex and Jensen's inequality again gives (2.4). Combining (2.3) and (2.4) proves (2.2), and therefore the result.

Reliability interpretation. Theorem 2.2 provides an easily checked sufficient route: Once $J_{\phi,r}$ is convex on J_r , replacing heterogeneous PHR parameters by their power mean of order r yields a homogeneous parallel system whose survival function is a guaranteed lower benchmark.

Theorem 2.3. (exact universal parallel criterion) Assume that the Archimedean generator is completely monotone (hence valid in every dimension) and that $\phi \in C^2(0, \infty)$ satisfies $\phi'(x) < 0$, and let $\psi = \phi^{-1}$. Fix $r \neq 0$ and define

$$H_\phi(a) = \psi(1 - e^{-a}), \quad a > 0$$

and

$$G_{\phi,r}(z) = H_\phi(z^{1/r}) = \psi(1 - e^{-z^{1/r}}), \quad z > 0.$$

Here, H_ϕ is an auxiliary baseline-scale transform used to make the proof transparent, while the subscript ϕ in $G_{\phi,r}$ emphasizes that ψ is the inverse associated uniquely with the Archimedean generator ϕ .

The following statements are equivalent:

1. For every $n \geq 2$, every positive PHR parameter vector $(\lambda_1, \dots, \lambda_n)$, and every non-degenerate baseline survival function S , the parallel-system comparison $X_{(n)} \geq_{st} Y_{(n)}$ holds when the homogeneous PHR parameter is $\check{\lambda}_r = M_r(\lambda_1, \dots, \lambda_n)$.
2. The function $G_{\phi,r}$ is convex on $(0, \infty)$.
3. For every $a > 0$, $r \geq B_\phi(a) := 1 - a - \frac{a}{e^a - 1} \mathcal{K}_\phi(\psi(1 - e^{-a}))$, where

$$\mathcal{K}_\phi(x) = \frac{\phi(x)\phi''(x)}{\{\phi'(x)\}^2}.$$

Define ρ_ϕ as the supremum of $B_\phi(a)$ over all $a > 0$. Within the power-index range studied in this paper,

$$X_{(n)} \geq_{st} Y_{(n)} \text{ universally} \Leftrightarrow r \geq \rho_\phi.$$

Under the assumptions of Theorem 2.3, complete monotonicity gives $\phi''(x) \geq 0$ and hence $\mathcal{K}_\phi(x) \geq 0$. Consequently, $B_\phi(a) \leq 1 - a < 1$ for every $a > 0$, so the supremum defining ρ_ϕ is bounded above (in particular, $\rho_\phi \leq 1$). This observation is also useful for numerical evaluation.

Proof. Fix an interior time t and write $c = -\log S(t) > 0$ and $a_i = c\lambda_i$. Positive homogeneity of power means gives

$$M_r(a_1, \dots, a_n) = cM_r(\lambda_1, \dots, \lambda_n).$$

The Archimedean copula formulas yield

$$F_{X_{(n)}}(t) = \phi\left(\sum_{i=1}^n H_\phi(a_i)\right), \quad F_{Y_{(n)}}(t) = \phi(nH_\phi\{M_r(a_1, \dots, a_n)\}).$$

Because ϕ is strictly decreasing, $F_{X_{(n)}}(t) \leq F_{Y_{(n)}}(t)$ is equivalent to

$$\frac{1}{n} \sum_{i=1}^n H_{\phi}(a_i) \geq H_{\phi}\{M_r(a_1, \dots, a_n)\}.$$

Set $z_i = a_i^r$. Since $a \mapsto a^r$ is a bijection of $(0, \infty)$ for every $r \neq 0$ and

$$M_r(a_1, \dots, a_n)^r = \frac{1}{n} \sum_{i=1}^n z_i,$$

the preceding inequality becomes

$$\frac{1}{n} \sum_{i=1}^n G_{\phi,r}(z_i) \geq G_{\phi,r}\left(\frac{1}{n} \sum_{i=1}^n z_i\right).$$

Thus, universal stochastic ordering is equivalent to Jensen convexity of the continuous function $G_{\phi,r}$. In particular, midpoint convexity for $n = 2$ is already sufficient by the standard continuity theorem for midpoint-convex functions [25].

For the differential characterization, put $a = z^{1/r}$ and $q = 1 - e^{-a}$. Direct differentiation gives

$$G_{\phi,r}''(z) = \frac{ae^{-a}}{r^2 z^2} [ae^{-a}\psi''(q) + (1 - r - a)\psi'(q)].$$

Since the prefactor is positive and $\psi'(q) < 0$, the condition $G_{\phi,r}''(z) \geq 0$ is equivalent to

$$r \geq 1 - a - \frac{a}{e^a - 1} \left\{ -\frac{q\psi''(q)}{\psi'(q)} \right\}.$$

With $x = \psi(q)$, the quantity in braces equals $\mathcal{K}_{\phi}(x)$, which proves the equivalence with $r \geq B_{\phi}(a)$ for every $a > 0$ and hence with $r \geq \rho_{\phi}$.

Converse and sharpness.

Finally, if $r < \rho_{\phi}$, then $G_{\phi,r}$ is not convex. Since the function is continuous, there exist $z_1, z_2 > 0$ with

$$\frac{G_{\phi,r}(z_1) + G_{\phi,r}(z_2)}{2} < G_{\phi,r}\left(\frac{z_1 + z_2}{2}\right).$$

Take $a_i = z_i^{1/r}$, choose the exponential baseline $S(t) = e^{-t}$, and set $\lambda_i = a_i$. At $t = 1$, this produces a two-component system for which $F_{X_{(2)}}(1) > F_{Y_{(2)}}(1)$. Hence, the universal stochastic comparison fails.

Theorem 2.4. (relative-curvature sufficient criterion) Assume $\phi \in C^2(0, \infty)$, $\phi' < 0$, and let $\psi = \phi^{-1}$. Define

$$\mathcal{K}_{\phi}(x) = \frac{\phi(x)\phi''(x)}{\{\phi'(x)\}^2}, \quad x > 0.$$

For $r \neq 0$, the map $u \mapsto \psi(u^{1/r})$ is convex on J_r if and only if

$$\mathcal{K}_{\phi}(x) \geq 1 - r \quad \text{for every } x > 0.$$

Equivalently, define

$$\kappa_{\phi} = \inf_{x>0} \mathcal{K}_{\phi}(x), \quad r_c(\phi) = 1 - \kappa_{\phi},$$

Then the convexity mechanism in Theorem 2.2 applies exactly for $r \geq r_c(\phi)$ within the manuscript's range $-1 \leq r < 0$ or $0 < r \leq 1$.

The subscript c in $r_c(\phi)$ stands for curvature: $r_c(\phi)$ is the threshold obtained from the global relative-curvature bound, not a second exact threshold.

Proof. Let $f_r(u) = \psi(u^{1/r})$ and put $v = u^{1/r}$. Direct differentiation gives

$$f_r''(u) = \frac{u^{1/r-2}}{r^2} \{v\psi''(v) + (1-r)\psi'(v)\}.$$

Because $\psi'(v) < 0$, convexity is equivalent to

$$-\frac{v\psi''(v)}{\psi'(v)} \geq 1 - r.$$

Writing $x = \psi(v)$ and using

$$\psi'(v) = \frac{1}{\phi'(x)}, \quad \psi''(v) = -\frac{\phi''(x)}{\{\phi'(x)\}^3},$$

shows that the left side equals $\mathcal{K}_\phi(x)$. Since v ranges over $(0,1)$ and x over $(0,\infty)$, the condition is necessary and sufficient.

Remark 2.5. Theorem 2.4 characterizes the sufficient Jensen mechanism used in Theorem 2.2, whereas Theorem 2.3 characterizes universal stochastic validity. Let $\mathcal{D}$ denote the manuscript's range $-1 \leq r < 0$ or $0 < r \leq 1$. Within $\mathcal{D}$, the admissible sets generated by the two criteria coincide for the families studied below. No ordering relationship between the unrestricted numerical values $r_c(\phi)$ and ρ_ϕ is asserted outside $\mathcal{D}$. For the four standard families below, the two criteria generate exactly the same admissible subsets of $\mathcal{D}$.

2.1. Practical interpretation and numerical evaluation of the critical index

Reliability interpretation. The critical index ρ_ϕ is the generator-dependent boundary for universal validity of the parallel-system lower bound. Since power means increase with r , choosing a smaller admissible r lowers the homogeneous PHR parameter and raises the corresponding system-survival lower bound. Thus, ρ_ϕ identifies the sharp boundary for benchmark selection: Moving r downward toward ρ_ϕ tightens the guaranteed bound, whereas crossing below ρ_ϕ sacrifices universal validity. When ρ_ϕ is computed numerically, a small tolerance buffer above the estimated boundary is prudent.

Exact versus curvature criterion. Theorem 2.4 imposes convexity on the simpler map $J_{\phi,r}(u) = \psi(u^{1/r})$ before the PHR baseline transformation is introduced. It is therefore a convenient generator-only sufficient mechanism. Theorem 2.3 tests the full composite transform $G_{\phi,r}$ generated after the baseline-scale reduction and is necessary and sufficient for universal parallel-system validity. The curvature criterion is preferable for rapid analytic screening; the exact criterion is preferable when the boundary matters or when the two mechanisms need not coincide. For the four families studied here, they yield the same admissible subsets of the manuscript range, even though their unrestricted numerical thresholds need not be identical.

Algorithm 1. (numerical evaluation of ρ_ϕ) Given ϕ , its first two derivatives, and a numerically stable inverse ψ , evaluate $B_\phi(a)$ from Theorem 2.3 on a logarithmic grid $a = \exp(y)$. Expand the y -window until both boundary values lie safely below the current maximum; bracket every apparent local maximizer; refine each bracket with a one-dimensional bounded optimizer (for example, Brent's

method); and take the largest refined value, including any available analytic boundary limits as candidates. The inequality $B_\varphi(a) < 1$ supplies a finite global upper bound, while repeated interval expansion and grid refinement provide a practical convergence diagnostic. A candidate r is treated as numerically admissible when it exceeds the computed maximum by more than the chosen numerical tolerance; values within tolerance of the maximum should be treated as boundary cases.

The same procedure applies to any monotone Archimedean generator for which φ , its first two derivatives, and its inverse can be evaluated reliably. Hence, the method is not restricted to the four closed-form families used for illustration. Extending the exact reduction to non-Archimedean dependence would require a different structural representation and is not automatic. The numerical evaluation and corresponding analytical checks are reported in Table 2.

Table 2. Numerical check of the exact critical-index computation. The Clayton value at $\theta = 2$ is obtained by one-dimensional maximization of $B_\varphi(a)$; the remaining entries agree with the analytic limits in Corollary 2.6.

Generator setting	Numerically/analytically recovered ρ_φ	Theoretical check
Clayton, $\theta = 0.5$	-0.500000	-0.5
Clayton, $\theta = 1$	-1.000000	-1
Clayton, $\theta = 2$	-1.745644	in $(-2, -1)$
Gumbel, $\theta \geq 1$	0	0
AMH, $0 \leq \theta < 1$	0	0
Joe, $\theta \geq 1$	0	0

Corollary 2.6. For the four families used in this paper, the exact universal parallel-system admissible regions in the manuscript's range are summarized below. Each listed generator is valid in every dimension under the stated parameter range. The exact family-specific regions are summarized in Table 3.

Table 3. Exact admissible power-index regions for the four Archimedean copula families in Corollary 2.6.

Family	$r_c(\varphi)$	ρ_φ	Exact admissible r
Clayton, $0 < \theta \leq 1$	$-\theta$	$-\theta$	$[-\theta, 0) \cup (0, 1]$
Clayton, $\theta > 1$	$-\theta$	$\rho_\theta \in (-\theta, -1)$	$[-1, 0) \cup (0, 1]$
Gumbel, $\theta \geq 1$	0	0	$(0, 1]$
AMH, $0 \leq \theta < 1$	0	0	$(0, 1]$
Joe, $\theta \geq 1$	0	0	$(0, 1]$

For Clayton with $\theta > 1$, let ρ_θ denote the supremum, over $a > 0$, of the function

$$B_\theta(a) = 1 - a - (1 + \theta) \frac{a}{e^a - 1}.$$

Then $-\theta < \rho_\theta < -1$. Hence, the difference between the raw thresholds $r_c(\varphi) = -\theta$ and ρ_θ occurs below the lower endpoint $r = -1$ used in this paper.

Proof. Clayton has $\mathcal{K}_\phi \equiv 1 + \theta$, so

$$B_\phi(a) = 1 - a - (1 + \theta) \frac{a}{e^a - 1}.$$

If $0 < \theta \leq 1$, elementary differentiation shows that the supremum is the boundary limit $-\theta$ as $a \downarrow 0$. If $\theta > 1$, the derivative at the origin is positive, so $\rho_\theta > -\theta$; while $B_\theta(a) < B_1(a) \leq -1$ for every $a > 0$, hence $\rho_\theta < -1$. For Gumbel, AMH, and Joe, $\mathcal{K}_\phi(x) \geq 1$ and tends to one as $x \rightarrow \infty$. Therefore,

$$B_\phi(a) \leq 1 - a - \frac{a}{e^a - 1} < 0,$$

whereas $B_\phi(a) \rightarrow 0$ as $a \downarrow 0$. Thus, $\rho_\phi = 0$. The stated exact regions follow from Theorem 2.3.

Remark 2.7. Theorem 2.2 yields the lower bound

$$S_{X_{(n)}}(t) \geq 1 - \phi[n\psi\{1 - S^{\check{\lambda}_r}(t)\}], \quad t \geq 0.$$

For $0 < r \leq 1$, $\check{\lambda}_r \leq \bar{\lambda}$, where $\bar{\lambda}$ is the arithmetic mean. For $-1 \leq r < 0$, $\check{\lambda}_r \leq \tilde{\lambda}$, where $\tilde{\lambda}$ is the geometric mean. More generally, if $r_1 < r_2$ are admissible, then $\check{\lambda}_{r_1} \leq \check{\lambda}_{r_2}$ and the corresponding homogeneous lower bounds satisfy $L_{r_1}(t) \geq L_{r_2}(t)$. Thus, moving toward the lower edge of the exact admissible set sharpens the valid bound.

Remark 2.8. For $r = 1$, $\check{\lambda}_1 = \bar{\lambda}$ and Theorem 2.2 recovers the arithmetic-mean comparison in [5]. The exact criterion in Theorem 2.3 also confirms universal validity at this endpoint.

Proposition 2.9. Assume the conditions of Theorem 2.2 and suppose that the baseline is non-degenerate in the sense that $0 < S(t) < 1$ at every interior time point. If the PHR parameters are not all equal, then $F_{X_{(n)}}(t) < F_{Y_{(n)}}(t)$ throughout the interior of the support. Conversely, if all PHR parameters are equal, the heterogeneous and homogeneous systems coincide in distribution.

Proof. In the proof of Theorem 2.2, the map $u \mapsto w(u^{1/r})$ is strictly concave for $r > 0$ and strictly convex for $r < 0$ whenever $0 < S(t) < 1$. Therefore, the auxiliary power-mean inequality is strict for a nonconstant PHR parameter vector. Since ψ and ϕ are strictly decreasing, strictness propagates to the system distribution functions.

The next result gives a local, quantitative interpretation of bound tightness near a homogeneous component vector. The reference value λ is the common PHR level, ε_i are relative perturbations with zero mean, V measures quadratic heterogeneity, and δ controls the perturbation size.

Proposition 2.10. Let $\lambda_i = \lambda(1 + \varepsilon_i)$, $\sum_{i=1}^n \varepsilon_i = 0$, and write $V = \sum_{i=1}^n \varepsilon_i^2$ and $\delta = \max_i |\varepsilon_i|$. For fixed admissible r and $\delta \rightarrow 0$,

$$\check{\lambda}_r = \lambda \left\{ 1 + \frac{r-1}{2n} V + O(\delta^3) \right\}.$$

Let $q(t) = 1 - S^\lambda(t)$ and let $L_r(t)$ denote the homogeneous lower bound. For fixed t with $0 < S(t) < 1$,

$$L_r(t) - L_1(t) = \frac{\lambda(1-r)V}{2} \{-\log S(t)\} S^\lambda(t) \phi'[n\psi\{q(t)\}] \psi'\{q(t)\} + O(\delta^3).$$

Because $\phi' < 0$ and $\psi' < 0$, the leading coefficient is positive for $r < 1$. Thus, near homogeneity, the gain over the arithmetic-mean benchmark is second order in PHR-parameter dispersion and increases linearly with $1 - r$.

Proof. Expanding $(1 + \varepsilon_i)^r$ and using $\sum_i \varepsilon_i = 0$ gives

$$\frac{1}{n} \sum_{i=1}^n (1 + \varepsilon_i)^r = 1 + \frac{r(r-1)}{2n} V + O(\delta^3),$$

which yields the power-mean expansion. For

$$L(m; t) = 1 - \phi[n\psi\{1 - S^m(t)\}],$$

direct differentiation gives

$$\frac{\partial L}{\partial m} = n\phi'[n\psi\{1 - S^m(t)\}]\psi'\{1 - S^m(t)\}S^m(t)\log S(t).$$

Substitution of $m = \check{\lambda}_r$ and expansion about $m = \lambda = \check{\lambda}_1$ complete the proof.

Interpretation. For fixed time and admissible $r < 1$, the improvement over the arithmetic-mean benchmark is of order V , and its leading coefficient is proportional to $1 - r$. This local result anticipates the global sensitivity patterns reported below.

We next illustrate Theorem 2.2 and Corollary 2.6 with one positive-index Gumbel example and one valid negative-index Clayton example.

Example 2.11. Let X_1, X_2, X_3 be dependent non-negative lifetimes coupled by a Gumbel copula with generator $\phi(x) = \exp\left(-x^{\frac{1}{\theta}}\right)$, $\theta \geq 1$, such that X_i has survival function $S^{\lambda_i}(t) = e^{-\lambda_i t^2}$ with $t \geq 0$ for $i = 1, 2, 3$. Set $\theta = 3$ and observe that $\psi\left(u^{\frac{1}{r}}\right) = \frac{1}{r^3}(-\log(u))^3$ is convex in $u \in (0, 1)$ for $0 < r \leq 1$. Let Y_1, Y_2, Y_3 have the same Gumbel dependence structure and common survival function $e^{-\check{\lambda}_1 t^2}$, $t \geq 0$. Therefore, using Theorem 2.2, it follows that $X_{(3)} \geq_{st} Y_{(3)}$. For $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9$, and $r = \frac{1}{2}$, we obtain $\check{\lambda}_{\frac{1}{2}} = 4$ and $\bar{\lambda} = \frac{14}{3}$. Figure 1 compares $S_{X_{(3)}}(t)$ versus the lower bound $1 - \phi\left(3\psi\left(1 - S^{\lambda}(t)\right)\right)$ for $\lambda = \check{\lambda}_{\frac{1}{2}}$ and $\lambda = \bar{\lambda}$ over the interval $0 < t < 3$.

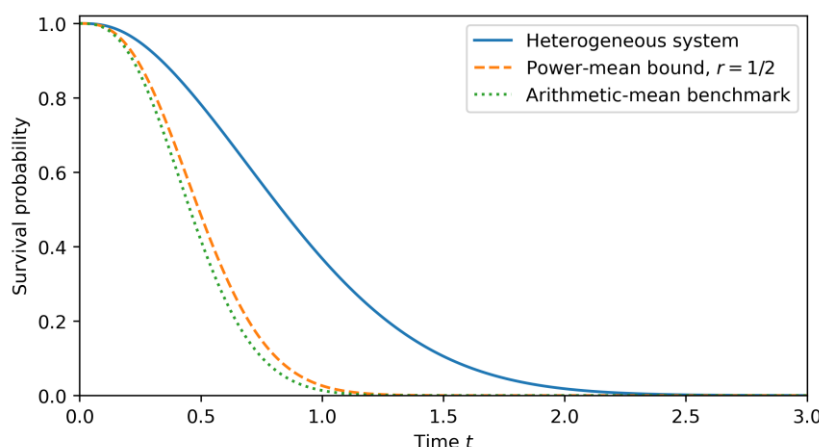


Figure 1. Survival function of the heterogeneous Gumbel parallel system in Example 2.11 (solid line), the power-mean lower bound based on $\check{\lambda}_{1/2} = 4$ (dashed line), and the arithmetic-mean benchmark based on $\bar{\lambda} = 14/3$ (dotted line).

Example 2.12. Consider dependent non-negative random variables X_1, X_2, X_3 coupled by a Clayton copula with generator

$$\phi(x) = (1 + \theta x)^{-1/\theta}, \quad \theta > 0,$$

and inverse generator

$$\psi(u) = \frac{1}{\theta}(u^{-\theta} - 1), \quad 0 < u < 1.$$

Assume that X_i has survival function $S^{\lambda_i}(t) = e^{-\lambda_i t^2}$, $t \geq 0$, for $i = 1, 2, 3$. Take $r = -1$ and $\theta = 2$. Since

$$\psi(u^{1/r}) = \frac{1}{2}(u^2 - 1), \quad u > 1,$$

is convex, Theorem 2.2 applies. Let $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 4)$. Then

$$\check{\lambda}_{-1} = \left\{ \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{4} \right) \right\}^{-1} = \frac{12}{7}, \quad \tilde{\lambda} = 2.$$

The survival function of the heterogeneous parallel-system lifetime is

$$S_{X_{(3)}}(t) = 1 - \left[\sum_{i=1}^3 (1 - e^{-\lambda_i t^2})^{-2} - 2 \right]^{-1/2},$$

whereas the homogeneous lower bound based on $\check{\lambda}_{-1} = 12/7$ is

$$L_{-1}(t) = 1 - \left[3(1 - e^{-(12/7)t^2})^{-2} - 2 \right]^{-1/2}.$$

For comparison, the geometric-mean benchmark is

$$L_G(t) = 1 - \left[3(1 - e^{-2t^2})^{-2} - 2 \right]^{-1/2}.$$

Theorem 2.2 gives $S_{X_{(3)}}(t) \geq L_{-1}(t)$ for all $t \geq 0$, and the power-mean inequality $\check{\lambda}_{-1} \leq \tilde{\lambda}$ yields $L_{-1}(t) \geq L_G(t)$. Figure 2 illustrates both inequalities.

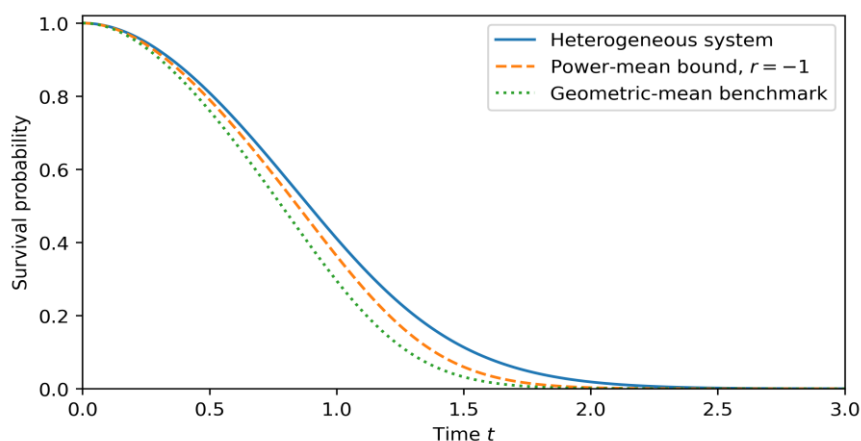


Figure 2. Survival function of the heterogeneous Clayton parallel system in Example 2.12 (solid line), the valid negative- r lower bound based on $\check{\lambda}_{-1} = 12/7$ (dashed line), and the geometric-mean benchmark based on $\tilde{\lambda} = 2$ (dotted line).

Cautionary negative- r Gumbel case. The Gumbel family cannot be used to justify Theorem 2.2 for $r < 0$. For $\theta = 4$, $r = -1$, $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 4)$, and $S(t) = e^{-t^2}$,

$$\frac{d^2}{du^2}(\log u)^4 = \frac{4(\log u)^2(3 - \log u)}{u^2},$$

which becomes negative for $u > e^3$. Moreover, the stochastic-order conclusion fails for these values: at $t \approx 0.11959248$,

$$S_{X_{(3)}}(t) \approx 0.9922765278 < 0.9925276309 \approx S_{Y_{(3)}}(t).$$

This calculation is now explained by Theorem 2.3. For Gumbel dependence, the exact universal threshold is $\rho_\phi = 0$, so every negative power index lies below the universal admissible region. The displayed numerical reversal is therefore a concrete instance of the exact Jensen converse rather than merely a failure of the sufficient proof technique.

2.2. Sensitivity analysis and quantitative bound tightness

To quantify the practical benefit of power-mean selection, consider the Clayton parallel system of Example 2.12 with $\theta = 2$, $\lambda = (1, 2, 4)$, and baseline $S(t) = \exp(-t^2)$. For a homogeneous benchmark with power index r , let $L_r(t)$ denote its survival lower bound and define the integrated bound deficit $D_r = \int_0^3 [S_{X_{(n)}}(t) - L_r(t)] dt$. Smaller D_r means a tighter lower bound over the displayed mission-time interval. The value $r = 0$ below denotes the geometric-mean limit and is included only as a benchmark. The power-index sensitivity results are summarized in Table 4.

Table 4. Power-index sensitivity for the Clayton parallel system in Example 2.12. The admissible lower edge $r = -1$ reduces the integrated deficit by about 64% relative to the arithmetic-mean benchmark.

r	Power mean M_r	Integrated deficit D_r	Maximum pointwise deficit	Reduction in D_r vs $r = 1$ (%)
-1	1.714286	0.069064	0.061176	64.0
-0.750000	1.778265	0.084636	0.074693	55.8
-0.500000	1.847548	0.100577	0.089384	47.5
-0.250000	1.921698	0.116674	0.104902	39.1
0	2.000000	0.132691	0.120894	30.7
0.250000	2.081492	0.148391	0.137018	22.6
0.500000	2.165031	0.163556	0.152965	14.6
0.750000	2.249384	0.178004	0.168466	7.1
1	2.333333	0.191598	0.183307	0.0

Dependence and heterogeneity sensitivity. Holding $r = -0.5$ and $\lambda = (1, 2, 4)$ fixed, the integrated deficit increases from 0.078761 at $\theta = 0.5$ to 0.125207 at $\theta = 4$. Holding $\theta = 2$ and $r = -1$ fixed while using $\lambda = (2/h, 2, 2h)$, the deficit rises with h , as expected when component heterogeneity increases; nevertheless, the power-mean benchmark retains a substantial advantage over $r = 1$. The associated dependence and heterogeneity sensitivity results are reported in Table 5.

These calculations show that the practical gain is not a cosmetic reformulation. In the representative Clayton setting, the harmonic-mean endpoint $r = -1$ reduces the integrated gap to the

heterogeneous system by roughly two thirds relative to the arithmetic benchmark, while the advantage persists and becomes more pronounced as heterogeneity grows. Dependence changes the absolute size of the gap but does not alter the threshold rule: Only values in the generator-specific admissible region carry the universal guarantee. Figure 3 displays the integrated lower-bound deficit as a function of the power index and illustrates its increase as the benchmark moves away from the lower admissible edge.

We next derive the corresponding usual stochastic comparison for series systems whose dependent component lifetimes are coupled by a common Archimedean survival copula.

Table 5. Sensitivity to the Clayton dependence parameter and to a controlled heterogeneity factor h . For the heterogeneity block, $\lambda = (2/h, 2, 2h)$, $\theta = 2$, and $r = -1$.

Sensitivity factor	Setting	Integrated deficit	Maximum deficit	Arithmetic deficit (if applicable)	Reduction vs arithmetic (%)
Dependence θ	$\theta = 0.5$; $r = -0.5$	0.078761	0.087761	—	—
Dependence θ	$\theta = 1$; $r = -0.5$	0.086094	0.088061	—	—
Dependence θ	$\theta = 2$; $r = -0.5$	0.100577	0.089384	—	—
Dependence θ	$\theta = 4$; $r = -0.5$	0.125207	0.097005	—	—
Heterogeneity h	1.25; $\lambda = (1.6, 2, 2.5)$	0.008208	0.008126	0.021334	61.5
Heterogeneity h	1.5; $\lambda = (1.333, 2, 3)$	0.026017	0.024987	0.068958	62.3
Heterogeneity h	2; $\lambda = (1, 2, 4)$	0.069064	0.061176	0.191598	64.0
Heterogeneity h	3; $\lambda = (0.667, 2, 6)$	0.147478	0.111760	0.441138	66.6

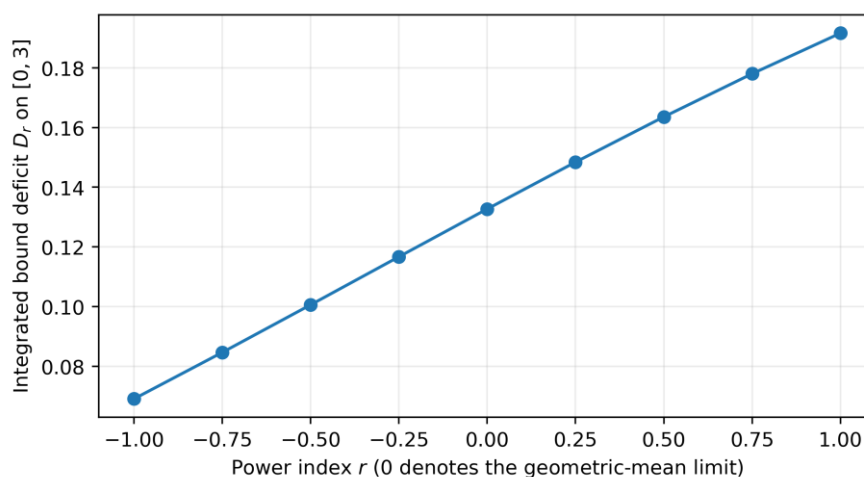


Figure 3. Integrated lower-bound deficit as a function of the power index for the Clayton system in Example 2.12. The monotone increase confirms quantitatively that moving toward the lower admissible edge sharpens the valid bound; $r = 0$ denotes the geometric-mean limit.

Theorem 2.13. Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two non-negative random vectors

having the same Archimedean survival copula with generator ϕ and inverse generator ψ . Suppose that X_i has survival function $S^{\lambda_i}(t)$ and Y_i has survival function $S^{\check{\lambda}_r}(t)$ for $i = 1, \dots, n$, where $r \in [-1, 0) \cup (0, 1]$. If, for every $t \geq 0$, the function $u \mapsto \psi(S^{u^{1/r}}(t))$ is convex on $(0, \infty)$, then $X_{(1)} \leq_{st} Y_{(1)}$.

Proof. Because ϕ is decreasing, it suffices to show that

$$\frac{1}{n} \sum_{i=1}^n \psi(S^{\lambda_i}(t)) \geq \psi(S^{\check{\lambda}_r}(t)), \quad t \geq 0. \quad (2.7)$$

Set $u_i = \lambda_i^r$ and note that $\bar{u}^{1/r} = \check{\lambda}_r$. By the convexity of $\psi(S^{u^{1/r}}(t))$ with respect to $u > 0$, let $B(t) = n^{-1} \sum_{i=1}^n \psi(S^{\lambda_i}(t))$. Jensen's inequality gives

$$B(t) = \frac{1}{n} \sum_{i=1}^n \psi(S^{u_i^{1/r}}(t)) \geq \psi(S^{\bar{u}^{1/r}}(t)) = \psi(S^{\check{\lambda}_r}(t)), \quad t \geq 0.$$

This completes the proof.

Reliability interpretation. In view of Theorem 2.13, for a series architecture, the direction reverses: Under the stated convexity condition, the homogeneous power-mean surrogate stochastically dominates the heterogeneous series lifetime, so it acts as an upper benchmark for system survival.

The following example illustrates Theorem 2.13 for a Clayton survival copula.

Example 2.14. Let X_1, X_2, X_3 be dependent non-negative lifetimes coupled by a Clayton survival copula with generator $\phi(x) = (1 + \theta x)^{-\frac{1}{\theta}}$, $\theta > 0$ and inverse generator $\psi(y) = \frac{1}{\theta}(y^{-\theta} - 1)$, $y \in (0, 1)$, and suppose that X_i has survival function $S^{\lambda_i}(t)$ for $t \geq 0$ and $i = 1, 2, 3$. For all $t \geq 0$ and $r \in [-1, 0) \cup (0, 1]$,

$$\psi(S^{u^{1/r}}(t)) = \frac{1}{\theta} [\exp\{-\theta \log S(t) u^{1/r}\} - 1].$$

Because $-\theta \log S(t) \geq 0$, the map $u \mapsto u^{1/r}$ is convex for the power-index range considered here, and the exponential function is convex and non-decreasing. Hence, $u \mapsto \psi(S^{u^{1/r}}(t))$ is convex on $(0, \infty)$ for every $t \geq 0$. Moreover, assume that Y_1, Y_2, Y_3 are dependent non-negative random variables with the same Clayton copula such that $Y_i, i = 1, 2, 3$ have a common survival function $S^{\check{\lambda}_r}(t), t \geq 0$ in which $r \neq 0$. Hence, Theorem 2.13 yields $X_{(1)} \leq_{st} Y_{(1)}$. Considering $\theta = 4, S(t) = e^{-t(t+1)}, t \geq 0$, $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 8$, and $r = 1$ for which $\check{\lambda}_r = \bar{\lambda} = 4$, we have

$$S_{X_{(1)}}(t) = \left(\sum_{i=1}^3 e^{4\lambda_i t(t+1)} - 2 \right)^{-\frac{1}{4}}, \quad t \geq 0$$

and

$$S_{Y_{(1)}}(t) = (3e^{16t(t+1)} - 2)^{-\frac{1}{4}}, \quad t \geq 0.$$

Figure 4 displays the two survival functions and illustrates the ordering $S_{X_{(1)}}(t) \leq S_{Y_{(1)}}(t)$ over $0 < t < 3$.

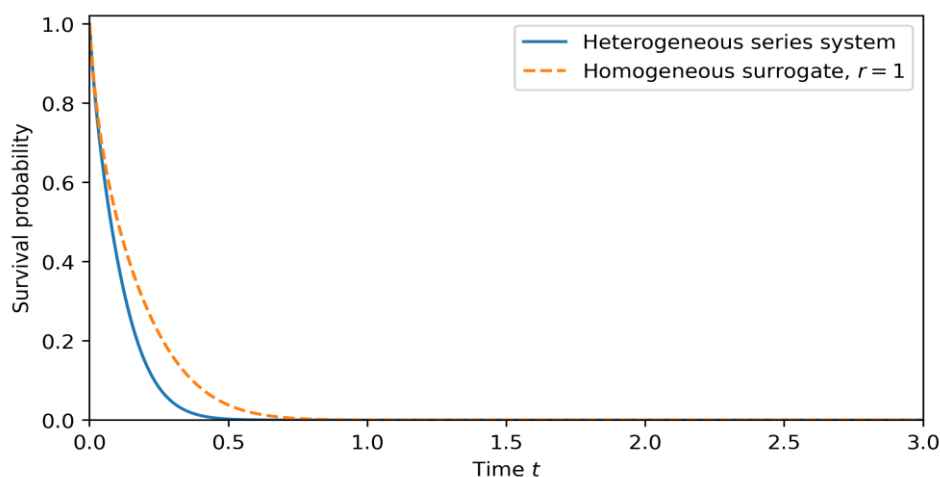


Figure 4. Survival functions in Example 2.14: $S_{X(1)}(t)$ (solid line) and $S_{Y(1)}(t)$ (dashed line).

For the four standard families considered above, the series-system convexity condition in Theorem 2.13 is satisfied throughout the stated nonzero power-index range. Specifically, define $G(z) = \psi(e^{-z})$ for $z \geq 0$. Then

$$G(z) = \begin{cases} z^\theta, & \text{Gumbel,} \\ \theta^{-1}(e^{\theta z} - 1), & \text{Clayton,} \\ \log\{\theta + (1 - \theta)e^z\}, & \text{AMH,} \\ -\log\{1 - (1 - e^{-z})^\theta\}, & \text{Joe.} \end{cases}$$

Each of these functions is non-decreasing and convex for $z \geq 0$ under its standard parameter range. For the Joe family, convexity follows from

$$G''(z) = \frac{\theta x(1-x)^{\theta-2}\{\theta x + (1-x)^\theta - 1\}}{\{(1-x)^\theta - 1\}^2} \geq 0, \quad x = e^{-z},$$

where $(1-x)^\theta \geq 1 - \theta x$ by Bernoulli's inequality. Since $u \mapsto u^{1/r}$ is convex on $(0, \infty)$ whenever $0 < r \leq 1$ or $-1 \leq r < 0$, the composition $u \mapsto \psi(S(t)^{u^{1/r}}) = G(-\log S(t) u^{1/r})$ is convex for every nonzero r , satisfying $r \leq 1$. Consequently, Theorem 2.13 applies to Gumbel, Clayton, AMH, and Joe survival copulas throughout the full power-index range considered in this paper. Here, G is a local series-system auxiliary and is distinct from the subscripted transform $G_{\varphi,r}$ in Theorem 2.3.

Why no generator-only exact series threshold is claimed. In the parallel case, the substitution $a_i = -\lambda_i \log S(t)$ followed by $z_i = a_i^r$ removes the baseline level and converts universal validity into Jensen convexity of the fixed function $G_{\varphi,r}$; midpoint failure then immediately yields a two-component converse. For a series system under a survival copula, the corresponding Jensen map is $u \mapsto \psi(S(t)^{u^{1/r}})$, whose curvature depends jointly on the generator and on the baseline level $S(t)$. Universal validity would therefore require a family of convexity conditions indexed by t (or, equivalently, by the baseline level), rather than a single generator-only condition. Theorem 2.13 deliberately states the resulting tractable sufficient condition; deriving a sharp baseline-free threshold for series systems is a separate problem.

3. Power-mean dispersive ordering under DFR baselines

The dispersive order compares the spread of two lifetime distributions and is stronger than the usual stochastic order for non-negative lifetimes with a common left endpoint. Establishing dispersive comparisons under dependence is more demanding because the argument involves the system density and the corresponding quantile transformation. In this section, we strengthen the stochastic comparisons of Section 2 under a DFR baseline and additional shape conditions on the Archimedean generator. Let $h_S(t) = -\frac{d}{dt} \log S(t)$ denote the baseline hazard rate. The baseline is DFR when h_S is non-increasing. The auxiliary functions H and K introduced below are local to the dispersive-order arguments and are distinct from H_ϕ and the curvature functional $\mathcal{K}_\phi$ used in Section 2.

Scope of the DFR assumption. DFR is a substantive aging restriction rather than a generic regularity condition. It includes the exponential model as the constant-hazard boundary, Weibull baselines with shape parameter at most one, and Pareto/Lomax-type baselines with decreasing hazard. It excludes, for example, Weibull models with shape greater than one and does not cover non-monotone hazards such as the lognormal hazard over its full support. The assumption is therefore natural for early-failure or decreasing-risk regimes but should not be interpreted as universal. In the proofs below, it is used only at the step converting the stochastic comparison into the hazard comparison required by the dispersive-order characterization; no necessity claim is made.

Lemma 3.1. Let $X_1, \dots, X_n$ be heterogeneous non-negative lifetimes with survival functions S^{λ_i} and a common Archimedean copula generated by ϕ . Let $Y_1, \dots, Y_n$ be homogeneous lifetimes with the same copula and common survival function $S^{\tilde{\lambda}_r}$, where $r \in [-1, 0) \cup (0, 1]$. Define

$$W_i(t) = \psi(1 - S^{\lambda_i}(t)), \quad \bar{W}(t) = \frac{1}{n} \sum_{i=1}^n W_i(t),$$

and introduce the non-negative functions

$$H(x) = \frac{\{1 - \phi(x)\} \log\{1 - \phi(x)\}}{\phi'(x)}, \quad R(x) = \frac{1 - \phi(x)}{-\phi'(x)}.$$

Then $X_{(n)} \geq_{disp} Y_{(n)}$ if, for every $t \geq 0$,

- (i) $h_S(t) \leq h_S[S^{-1}(\{1 - \phi(\bar{W}(t))\}^{1/\tilde{\lambda}_r})]$;
- (ii) $\frac{1}{-n \log S(t)} \sum_{i=1}^n H\{W_i(t)\} \leq \tilde{\lambda}_r R\{\bar{W}(t)\}$.

Proof. By the density-quantile characterization in (1.2), it suffices to prove

$$f_{X_{(n)}}(t) \leq f_{Y_{(n)}}(F_{Y_{(n)}}^{-1}\{F_{X_{(n)}}(t)\}), \quad t \geq 0. \quad (3.1)$$

The heterogeneous parallel-system distribution is

$$F_{X_{(n)}}(t) = \phi(n\bar{W}(t)). \quad (3.2)$$

Differentiation, together with $S^{\lambda_i}(t) = 1 - \phi\{W_i(t)\}$ and $\lambda_i = -\log[1 - \phi\{W_i(t)\}]/[-\log S(t)]$, gives

$$f_{X_{(n)}}(t) = \frac{h_S(t)\{-\phi'(n\bar{W}(t))\}}{-\log S(t)} \sum_{i=1}^n H\{W_i(t)\}. \quad (3.3)$$

For the homogeneous system,

$$F_{Y_{(n)}}(t) = \phi[n\psi\{1 - S^{\check{\lambda}_r}(t)\}].$$

Solving $F_{Y_{(n)}}(t) = u$ gives

$$F_{Y_{(n)}}^{-1}(u) = S^{-1} \left[\left\{ 1 - \phi \left(\frac{\psi(u)}{n} \right) \right\}^{1/\check{\lambda}_r} \right], \quad 0 \leq u \leq 1.$$

Therefore,

$$f_{Y_{(n)}} \left(F_{Y_{(n)}}^{-1} \{ F_{X_{(n)}}(t) \} \right) = n\check{\lambda}_r h_S(t^*) \{ -\phi'(n\bar{W}(t)) \} R\{\bar{W}(t)\}, \quad (3.4)$$

where

$$t^* = S^{-1} \left[\{ 1 - \phi(\bar{W}(t)) \}^{1/\check{\lambda}_r} \right].$$

After cancelling the common positive factor $n\{-\phi'(n\bar{W}(t))\}$, condition (3.1) reduces to

$$\frac{h_S(t)}{-n \log S(t)} \sum_{i=1}^n H\{W_i(t)\} \leq \check{\lambda}_r h_S(t^*) R\{\bar{W}(t)\}. \quad (3.5)$$

Conditions (i) and (ii) imply (3.5), which completes the proof.

Lemma 3.2. Let $\check{\lambda}_r = (n^{-1} \sum_{i=1}^n \lambda_i^r)^{1/r}$. Suppose that S has the DFR property and that $u \mapsto \psi(u^{1/r})$ is convex on J_r , where $J_r = (0,1)$ for $0 < r \leq 1$ and $J_r = (1, \infty)$ for $-1 \leq r < 0$. Then

$$h_S(t) \leq h_S \left[S^{-1} \left(\{ 1 - \phi(\bar{W}(t)) \}^{\frac{1}{\check{\lambda}_r}} \right) \right], \quad t \geq 0. \quad (3.6)$$

Proof. By the convexity assumption on J_r , Theorem 2.2 gives $X_{(n)} \geq_{st} Y_{(n)}$. Because the baseline is DFR, (3.6) follows once

$$t \geq S^{-1} \left(\{ 1 - \phi(\bar{W}(t)) \}^{\frac{1}{\check{\lambda}_r}} \right).$$

Since S is decreasing, this is equivalent to

$$1 - S^{\check{\lambda}_r}(t) \geq \phi\{\bar{W}(t)\}.$$

Because ψ is decreasing, the last inequality is equivalent to

$$\psi \left(1 - S^{\check{\lambda}_r}(t) \right) \leq \bar{W}(t) = \frac{1}{n} \sum_{i=1}^n \psi \left(1 - S^{\lambda_i}(t) \right),$$

which is exactly the comparison established in (2.2). Hence, (3.6) holds.

The next theorem combines Lemmas 3.1 and 3.2 with the generator concavity condition to obtain the parallel-system dispersive comparison.

Theorem 3.3. Let S be a DFR baseline survival function. Let X_i have survival function S^{λ_i} and Y_i have common survival function $S^{\check{\lambda}_r}$, and suppose that the two component vectors have the same Archimedean copula with generator ϕ and inverse generator ψ . Let $r \in [-1,0) \cup (0,1]$ and define J_r as in Theorem 2.2. If $u \mapsto \psi(u^{1/r})$ is convex on J_r and

$$H(x) = \frac{\{1 - \phi(x)\} \log\{1 - \phi(x)\}}{\phi'(x)}$$

is concave on $(0, \infty)$, then

$$X_{(n)} \geq_{disp} Y_{(n)}.$$

Proof. Condition (i) of Lemma 3.1 follows from Lemma 3.2. By concavity of H ,

$$\frac{1}{n} \sum_{i=1}^n H\{W_i(t)\} \leq H\{\bar{W}(t)\}.$$

Since $H(x) = [-\log\{1 - \phi(x)\}]R(x)$, condition (ii) of Lemma 3.1 is therefore satisfied whenever

$$\frac{-\log\{1 - \phi(\bar{W}(t))\}}{-\log S(t)} \leq \check{\lambda}_r. \quad (3.7)$$

Because ψ is decreasing, (3.7) is equivalent to

$$\psi\left(1 - S^{\check{\lambda}_r}(t)\right) \leq \bar{W}(t) = \frac{1}{n} \sum_{i=1}^n \psi\left(1 - S^{\lambda_i}(t)\right),$$

which is (2.2). Hence, Theorem 2.2 establishes (3.7), and the result follows. $\square$

Reliability interpretation. In view of Theorem 3.3, under DFR and the additional generator-shape condition, the parallel benchmark controls not only survival probabilities but also lifetime spread in the stronger dispersive sense.

When $r = 1$, Theorem 3.3 reduces to in [5, Theorem 4.5]; the convexity of ψ follows from the standard Archimedean-generator assumptions, as noted in Section 1.1. For the Clayton generator, direct differentiation shows that the required concavity of H holds exactly for $0 < \theta \leq 1$; for $\theta > 1$, $H''(x) > 0$ for all sufficiently large x . The corresponding Gumbel and Joe concavity conditions are verified in [13], and the AMH condition in [19]. Combining these facts with the copula-specific admissibility analysis in Section 2 yields a sharper applicability statement: For $0 < r \leq 1$, Theorem 3.3 is available for Gumbel, Joe, and AMH subject to the cited parameter ranges, and for Clayton only when $0 < \theta \leq 1$; whereas for $-1 \leq r < 0$, the Jensen convexity requirement singles out Clayton with $\theta \geq -r$, and the additional H -concavity condition restricts this further to $-r \leq \theta \leq 1$. In particular, the Gumbel family cannot be invoked in Theorem 3.3 for negative r on the basis of this condition.

The immediate consequence of Theorem 3.3 is the exact-threshold refinement. The convexity assumption used in Theorem 3.3 enters its proof only through the inner comparison (2.2), which is equivalent to the parallel usual stochastic order. Therefore, under the hypotheses of Theorem 2.3, Theorem 3.3 remains valid if that convexity assumption is replaced by $r \geq \rho_\phi$ while retaining the DFR baseline and the concavity condition on H . Thus, ρ_ϕ supplies the exact stochastic component of the parallel dispersive argument. For Gumbel, Clayton, AMH, and Joe, this refinement does not change the admissible set within $|r| \leq 1$, but it removes the unnecessary dependence of the dispersive proof on the stronger sufficient mechanism of Theorem 2.2.

In Theorem 3.3, the DFR condition enters through Lemma 3.2, where it converts the usual stochastic comparison into the hazard comparison needed for the quantile-density characterization of dispersive order. It is used as a sufficient condition and is not claimed to be necessary. The exponential baseline is included as the constant-hazard boundary case. Determining weaker baseline conditions

under which the same dispersive comparison remains valid is a separate question.

We next derive the analogous dispersive comparison for series systems under an Archimedean survival copula.

Theorem 3.4. Let S be a DFR baseline survival function. Suppose that X_i has survival function S^{λ_i} and Y_i has common survival function $S^{\check{\lambda}_r}$ for $i = 1, \dots, n$, with $r \in [-1, 0) \cup (0, 1]$. Let $\mathbf{X}$ and $\mathbf{Y}$ have the same Archimedean survival copula with generator ϕ and inverse generator ψ . Assume that $u \mapsto \psi(S^{u^{1/r}}(t))$ is convex on $(0, \infty)$ for every $t \geq 0$. If $K(x) = \frac{\phi(x)\log\phi(x)}{\phi'(x)}$ is convex on $(0, \infty)$, then $X_{(1)} \leq_{disp} Y_{(1)}$.

Proof. Put

$$V_i(t) = \psi(S^{\lambda_i}(t)), \quad \bar{V}(t) = \frac{1}{n} \sum_{i=1}^n V_i(t), \quad R_S(x) = \frac{\phi(x)}{-\phi'(x)}.$$

By (1.2), it suffices to show

$$f_{X_{(1)}}(t) \geq f_{Y_{(1)}}(F_{Y_{(1)}}^{-1}\{F_{X_{(1)}}(t)\}), \quad t \geq 0. \quad (3.8)$$

Differentiating $S_{X_{(1)}}(t) = \phi\{n\bar{V}(t)\}$ and using $S^{\lambda_i}(t) = \phi\{V_i(t)\}$ gives

$$f_{X_{(1)}}(t) = \frac{h_S(t)\{-\phi'(n\bar{V}(t))\}}{-\log S(t)} \sum_{i=1}^n K\{V_i(t)\}. \quad (3.9)$$

By convexity of K ,

$$f_{X_{(1)}}(t) \geq \frac{nh_S(t)\{-\phi'(n\bar{V}(t))\}}{-\log S(t)} K\{\bar{V}(t)\}. \quad (3.10)$$

For the homogeneous system, solving the quantile equation yields

$$F_{Y_{(1)}}^{-1}(u) = S^{-1} \left[\left\{ \phi \left(\frac{\psi(1-u)}{n} \right) \right\}^{1/\check{\lambda}_r} \right], \quad 0 \leq u \leq 1.$$

Consequently,

$$f_{Y_{(1)}}(F_{Y_{(1)}}^{-1}\{F_{X_{(1)}}(t)\}) = n\check{\lambda}_r h_S(s^*) \{-\phi'(n\bar{V}(t))\} R_S\{\bar{V}(t)\}, \quad (3.11)$$

where

$$s^* = S^{-1}[\{\phi(\bar{V}(t))\}^{1/\check{\lambda}_r}].$$

Since $K(x) = [-\log\phi(x)]R_S(x)$, comparison of (3.10) and (3.11) reduces to

$$\frac{-\log\phi\{\bar{V}(t)\}}{-\log S(t)} h_S(t) \geq \check{\lambda}_r h_S(s^*). \quad (3.12)$$

This follows from the two inequalities

$$h_S(t) \geq h_S(s^*) \quad (3.13)$$

and

$$\frac{-\log\phi\{\bar{V}(t)\}}{-\log S(t)} \geq \check{\lambda}_r, \quad (3.14)$$

Because S is DFR, (3.13) follows once $t \leq s^*$. The latter is equivalent to $\phi\{\bar{V}(t)\} \leq S^{\check{\lambda}_r}(t)$, which is the usual stochastic comparison in Theorem 2.13 after applying the decreasing inverse generator ψ . The same inequality also gives (3.14). Hence, (3.12) holds, and therefore $X_{(1)} \leq_{disp} Y_{(1)}$.

Reliability interpretation. Theorem 3.4 yields the corresponding series-system result: Under the stated survival-copula conditions, the homogeneous surrogate has the more dispersed lifetime in the sense of the dispersive order.

Example 3.5. Let $\mathbf{X}$ and $\mathbf{Y}$ have the same Archimedean survival copula, with X_i having survival function $S^{\lambda_i}(t)$ and Y_i having survival function $S^{\check{\lambda}_r}(t)$ for $i = 1, \dots, n$ where $r \neq 0$. Suppose S is DFR. First consider the Clayton generator $\phi(x) = (1 + \theta x)^{-\frac{1}{\theta}}$, $\theta > 0$. As verified in Example 2.14, $\psi\left(S^{u^{\frac{1}{r}}}(t)\right)$ is convex in $u > 0$, for all $t \geq 0$. Direct calculation gives $\frac{d^2}{dx^2} \left\{ \frac{\phi(x)}{\phi'(x)} \log(\phi(x)) \right\} = \frac{\theta}{1+\theta x}$, which is positive for $x \geq 0$ and $\theta > 0$; hence, the required generator function is convex and Theorem 3.4 gives $X_{(1)} \leq_{disp} Y_{(1)}$. For comparison, if the underlying Archimedean survival copula is Gumbel, which has generator $\phi(x) = e^{-x^{\frac{1}{\theta}}}$ and inverse generator $\psi(u) = (-\log(u))^{\theta}$, where $\theta \geq 1$, then $\frac{\phi(x)\log(\phi(x))}{\phi'(x)} = \theta x$, which is linear, and hence convex. Direct differentiation gives

$$\frac{\partial^2}{\partial u^2} \psi\left(S^{u^{\frac{1}{r}}}(t)\right) = \frac{\theta}{r} \left(\frac{\theta}{r} - 1 \right) \{-\log(S(t))\}^{\theta} u^{\frac{\theta}{r}-2}.$$

For the range $r \in [-1, 0) \cup (0, 1]$ and $\theta \geq 1$, this second derivative is non-negative. Thus, the convexity condition of Theorem 3.4 is satisfied, and $X_{(1)} \leq_{disp} Y_{(1)}$.

4. Conclusions and future directions

From a reliability-modeling perspective, the central question in homogenizing a dependent system is not merely which familiar mean should summarize the component parameters, but which homogeneous PHR benchmark preserves a provable comparison of system lifetimes. For parallel systems under Archimedean dependence, we resolve the universal stochastic-order problem by reducing it to the convexity of a generator-induced one-variable transform. The result yields a sharp generator-dependent critical power index and shows that every power index below the admissible boundary admits a constructive two-component counterexample. Power-mean homogenization is therefore converted from a benchmark-specific sufficient construction into an exact decision rule for the parallel-system comparison.

For the Gumbel, Clayton, Ali-Mikhail-Haq, and Joe families, the exact criterion and the relative-curvature criterion generate the same admissible power-index sets over the studied nonzero range with absolute value at most one. The curvature condition remains a convenient dimension-free verification mechanism, whereas the exact criterion identifies the true universal boundary. The strictness and equality results determine when the bound is informative, and the local expansion quantifies the leading gain over the arithmetic-mean benchmark in terms of PHR-parameter dispersion and the departure of the selected power mean from the arithmetic-mean endpoint.

The parallel and series analyses also reveal an important structural asymmetry. For series systems, stochastic comparisons are obtained under the stated convexity assumptions for the standard copula

families, but no exact universal threshold analogous to the parallel criterion is claimed. Under a decreasing-failure-rate baseline and additional generator-shape conditions, the usual stochastic comparisons extend to the dispersive order for both system architectures. These dispersive conditions are sufficient rather than exact. This separation makes clear which parts of the theory are sharp and which remain open.

Natural directions for further work include exact thresholds for hazard-rate and likelihood-ratio orders, weaker baseline conditions and possible exact criteria for dispersive comparisons, computational evaluation of the critical index for broader generator classes, and extensions beyond Archimedean dependence. A complementary statistical direction is to develop estimation procedures for PHR and dependence parameters from censored lifetime data and to propagate estimation uncertainty into the admissible power-index region. A hazard-rate-order extension is not asserted here: Unlike the usual stochastic comparison, differentiating the system-survival ratio introduces additional baseline- and generator-derivative terms, so these assumptions alone do not produce an exact generator-only threshold. Establishing sharp extra conditions, or systematic counterexample classes when they fail, is therefore a distinct problem.

Author contributions

Thamer Manshi: Conceptualization, methodology, formal analysis, validation, writing—original draft, writing—review & editing; Maha Al-Mutairi: Methodology, formal analysis, validation, visualization, writing—original draft, writing—review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors thank the editors and the two anonymous reviewers for their careful reading and constructive comments, which helped improve the clarity and rigor of the manuscript.

Funding

This research was supported by the Ongoing Research Funding program (ORF-2026-1454), King Saud University, Riyadh, Saudi Arabia.

Conflicts of interest

The authors declare no conflicts of interest.

References

1. G. Pledger, F. Proschan, *Comparisons of order statistics and of spacings from heterogeneous distributions*, In: Optimizing Methods in Statistics, J. S. Rustagi, Eds., New York: Academic Press, 1971, 89–113. <https://doi.org/10.1016/B978-0-12-604550-5.50011-0>
2. R. Dykstra, S. C. Kochar, J. Rojo, Stochastic comparisons of parallel systems of heterogeneous exponential components, *J. Stat. Plan. Infer.*, **65** (1997), 203–211. [https://doi.org/10.1016/S0378-3758\(97\)00058-X](https://doi.org/10.1016/S0378-3758(97)00058-X)
3. B. E. Khaledi, S. C. Kochar, Some new results on stochastic comparisons of parallel systems, *J. Appl. Probab.*, **37** (2000), 1123–1128. <https://doi.org/10.1239/jap/1014843091>
4. B. E. Khaledi, S. C. Kochar, *Stochastic orderings among order statistics and sample spacings*, In: Uncertainty and Optimality: Probability, Statistics and Operations Research, J. C. Misra, Eds., World Scientific, Singapore, 2002, 167–203. https://doi.org/10.1142/9789812777010_0004
5. X. Li, R. Fang, Ordering properties of order statistics from random variables of Archimedean copulas with applications, *J. Multivariate Anal.*, **133** (2015), 304–320. <https://doi.org/10.1016/j.jmva.2014.09.016>
6. R. Fang, C. Li, X. Li, Stochastic comparisons on sample extremes of dependent and heterogeneous observations, *Statistics*, **50** (2016), 930–955. <https://doi.org/10.1080/02331888.2015.1119151>
7. Y. Zhang, X. Cai, P. Zhao, H. Wang, Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components, *Statistics*, **53** (2019), 126–147. <https://doi.org/10.1080/02331888.2018.1546705>
8. C. Li, X. Li, Hazard rate and reversed hazard rate orders on extremes of heterogeneous and dependent random variables, *Stat. Probabil. Lett.*, **146** (2019), 104–111. <https://doi.org/10.1016/j.spl.2018.11.005>
9. T. Lando, I. Arab, P. E. Oliveira, Second-order stochastic comparisons of order statistics, *Statistics*, **55** (2021), 561–579. <https://doi.org/10.1080/02331888.2021.1960527>
10. M. Mesfioui, J. Trufin, Dispersive order comparisons on extreme order statistics from homogeneous dependent random vectors, *Depend. Model.*, **9** (2021), 385–393. <https://doi.org/10.1515/demo-2021-0118>
11. A. Panja, P. Kundu, B. Pradhan, Stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components, *Oper. Res. Lett.*, **49** (2021), 176–183. <https://doi.org/10.1016/j.orl.2020.12.009>
12. T. Sahoo, N. K. Hazra, Ordering and aging properties of systems with dependent components governed by the Archimedean copula, *Probabab. Eng. Inform. Sc.*, **37** (2023), 1–28. <https://doi.org/10.1017/S0269964821000425>
13. E. Amini-Seresht, B. E. Khaledi, S. Izadkhah, Stochastic comparison of parallel systems with heterogeneous dependent exponential components, *Stat. Probabil. Lett.*, **215** (2024), 110242. <https://doi.org/10.1016/j.spl.2024.110242>
14. M. Datta, N. Gupta, Stochastic comparison for extreme order statistics arising from PHR, PRHR or location model, *REVSTAT*, **22** (2024), 387–410. <https://doi.org/10.57805/revstat.v22i3.528>
15. M. Shrahili, Stochastic ordering results on extreme order statistics from dependent samples with Archimedean copula, *J. Inequal. Appl.*, **2024** (2024), 120. <https://doi.org/10.1186/s13660-024-03201-6>
16. S. Das, S. Kayal, N. Torrado, Ordering results between extreme order statistics in models with dependence defined by Archimedean [survival] copulas, *Ric. Mat.*, **73** (2024), 1997–2033. <https://doi.org/10.1007/s11587-022-00715-3>

17. J. Zhong, R. Fang, Y. Chen, Stochastic comparisons on extremes of generalized scaled proportional hazard rate samples associated with Archimedean copula, *Statistics*, **60** (2026), 1208–1239. <https://doi.org/10.1080/02331888.2025.2598385>
18. I. Arab, M. Hadjikyriakou, P. E. Oliveira, Stochastic ordering of dependent systems under transformation models and Archimedean copulas, *arXiv Preprint*, 2026. <https://doi.org/10.48550/arXiv.2604.26026>
19. G. Alomani, A. A. Alqefari, M. Kayid, Dispersive and hazard rate orderings of parallel systems with exponential dependent components, *Statist. Probabil. Lett.*, **227** (2026), 110514. <https://doi.org/10.1016/j.spl.2025.110514>
20. T. Manshi, A. A. Alqefari, M. Kayid, Preservation of hazard rate and relative average hazard rate orders in dependent series and parallel systems via Archimedean copulas, *J. Inequal. Appl.*, **2026** (2026), 52. <https://doi.org/10.1186/s13660-026-03453-4>
21. A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of majorization and its applications*, 2Eds., New York: Springer, 2011. <https://doi.org/10.1007/978-0-387-68276-1>
22. A. Müller, D. Stoyan, *Comparison methods for stochastic models and risks*, Chichester: Wiley, 2002.
23. M. Shaked, J. G. Shanthikumar, *Stochastic orders*, New York: Springer, 2007. <https://doi.org/10.1007/978-0-387-34675-5>
24. F. Belzunce, C. Martínez-Riquelme, J. Mulero, *An introduction to stochastic orders*, London: Academic Press, 2015. <https://doi.org/10.1016/C2014-0-04775-2>
25. C. P. Niculescu, L. E. Persson, *Convex functions and their applications: A contemporary approach*, 2Eds., Springer, Cham, 2018. <https://doi.org/10.1007/978-3-319-78337-6>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)