



*Research article***Blending-type modifications of α -Bernstein polynomials and their approximation behavior****Harun ÇİÇEK***

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Abstract: This paper introduced a modified version of the classical Bernstein operators that involved a nonnegative mixing parameter denoted by α . We also demonstrated that these operators satisfy important mathematical properties, such as linearity and positivity. By applying the Bohman-Korovkin theorem, we established conditions under which the operators uniformly approximated a given continuous function. Based on this, we derived formulas that described the rate of convergence. Additionally, we proved a Voronovskaya type theorem, which provided an asymptotic description of the approximation error. We also provided estimates related to the smoothness of the function. By using the classical modulus of continuity, we explained how the error behaved when the function had local variations, and expressed this in terms of first-order properties like monotonicity and convexity. Finally, we showed that our operators preserved these shape-related features, which gave theoretical strength to the results and highlighted the practical value of the operator.

Keywords: approximation operators; blending-type operators; Voronovskaja theorem; modulus of continuity; shape-preserving properties

Mathematics Subject Classification: 41A10, 41A25, 41A36

1. Introduction

Linear positive operators play a central and essential role in approximation theory. Their common properties, such as stability and positivity, along with their ability to reproduce basic test functions, make them very useful. A careful study of these operators has greatly influenced the development of approximation processes and provided an important path to the classical Weierstrass approximation theorem [1]. The simplest proof of this classical result was first introduced by Bernstein [2] using

binomial distribution as follows:

$$B_r(h, u) = \sum_{j=0}^r w_r^j(u) h\left(\frac{j}{r}\right), \quad (1.1)$$

where $w_r^j(u) = \binom{r}{j} u^j (1-u)^{r-j}$ and $h \in C[0, 1]$. In the last twenty years, this field has grown rapidly. This growth is due to the development of computational mathematics and the increasing need for unified and high-performance approximation methods. These advances have allowed for better control over convergence behavior, approximation rates, and the geometric shapes of the resulting operators, largely through the use of parameter-dependent generalizations. This changing approach has led to many important generalizations of classical operators, e.g., Nasir et al. investigated the approximation properties of generalized q -Szász [3], α -Bernstein-Schurer [4], and Szász-Mirakjan [5] operators through wavelet-based, Kantorovich, and Stancu-type modifications. His research focused on convergence behavior, rates of approximation, and Voronovskaja-type asymptotic results using moduli of continuity and Ditzian-Totik functionals, supported by numerical and graphical analyses. Rao et al. contributions to approximation theory include the construction and analysis of modified Szász-Beta [6] and Szász-Integral operators [7] based on general Appell polynomials, alongside Stancu-type generalized Baskakov operators [8]. These investigations provide quantitative approximation estimates, weighted and statistical convergence results, and Voronovskaja-type asymptotic formulas through various smoothness measures and functional techniques. Berrera et al. [9] developed a normalized B-spline-like basis for C^2 cubic splines and super-convergent quasi-interpolation operators. Bouhiri et al. [10] developed polynomial and spline quasi-interpolants based on the Bernstein basis, utilizing polar forms and symmetric polynomials to achieve effective approximation. Cai et al. [11] introduced generalized (α, q) -Bernstein operators and established their convergence rates, monotonicity, and convexity-preserving properties. Walczak et al. [12] investigated the convergence rates of modified Baskakov operators for polynomial-type functions, emphasizing their computational applicability. Aktuğlu et al. [13] introduced ψ -Baskakov-Kantorovich operators, establishing improved approximation, weighted convergence rates, and shape-preserving properties. Ayman-Mursaleen et al. [14] investigated shifted-knot α -Bernstein operators, deriving approximation error bounds, direct estimates, and shape-preserving properties, and Baytunç et al. [15] developed Riemann-Liouville fractional Bernstein-Kantorovich operators, establishing convergence rates, affine function preservation, and bivariate approximation results.

In 2012, Izgi [16] defined a class of new Bernstein-type operators on a moving interval as

$$F_{r,a,b}(h, u) = \sum_{j=0}^r v_{r,a,b}(u) h\left(\frac{j(r+a)}{r(r+b)}\right), \quad 0 \leq u \leq \frac{(r+a)}{(r+b)}, \quad (1.2)$$

where $v_{r,a,b}(u) = \left(\frac{r+a}{r+b}\right)^r \binom{r}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j}$, $0 \leq a \leq b$, and he studied some approximation properties of this new type of operators for one and two variable functions.

Remark 1.1. For $a = b$, the sequence of modified Bernstein operators given in (1.2) reduces to the classical Bernstein operators introduced in (1.1).

In 2020, Usta [17] defined a new modification of Bernstein operators as

$$B_r^*(h, u) = \frac{1}{r} \sum_{j=0}^r \binom{r}{j} h\left(\frac{j}{r}\right) (j - ru)^2 u^{j-1} (1-u)^{r-j-1}, \quad 0 \leq u \leq 1, \quad (1.3)$$

and he showed some approximation properties of these operators.

In [18], a new family of generalized Bernstein operators has been studied as

$$T_r^\alpha(h, u) = \sum_{j=0}^r w_{r,j}^\alpha(u) h\left(\frac{j}{r}\right), \quad 0 \leq u \leq 1, \quad (1.4)$$

where

$$w_{r,j}^\alpha(u) = \left[(1-\alpha) \binom{r-2}{j} u + (1-\alpha) \binom{r-2}{j-2} (1-u) + \alpha u (1-u) \binom{r}{j} \right] \times \binom{r}{j} u^{j-1} (1-u)^{r-j-1},$$

$r \geq 2$, $w_{1,0}^\alpha(u) = 1-u$, and $w_{1,1}^\alpha(u) = u$. For $\alpha = 1$, (1.4) reduces to (1.1).

2. Construction of new generalized Bernstein operators

Motivated by the above literature, we construct a new modification of Bernstein operators as

$$F_{r,a,b}^\alpha(h, u) = \sum_{j=0}^r v_{r,j,a,b}^\alpha(u) h_j, \quad 0 \leq u \leq \frac{r+a}{r+b}, \quad (2.1)$$

where $h_j = h\left(\frac{j(r+a)}{r(r+b)}\right)$, $v_{1,0,a,b}^\alpha(u) = \frac{1+a}{1+b} - u$, $v_{1,1,a,b}^\alpha(u) = u$, and

$$v_{r,j,a,b}^\alpha(u) = \left(\frac{r+b}{r+a}\right)^r \left[(1-\alpha) \frac{r+a}{r+b} \binom{r-2}{j} u + (1-\alpha) \frac{r+a}{r+b} \binom{r-2}{j-2} \left(\frac{r+a}{r+b} - u\right) + \alpha u \left(\frac{r+a}{r+b} - u\right) \binom{r}{j} \right] \times u^{j-1} \left(\frac{r+a}{r+b} - u\right)^{r-j-1},$$

where $r \geq 2$ and the binomial coefficients $\binom{r}{j}$ are given by

$$\binom{r}{j} = \begin{cases} \frac{r!}{j!(r-j)!}, & \text{if } 0 \leq j \leq r, \\ 0, & \text{else.} \end{cases}$$

The calculations lead directly to the following low-degree formulas:

$$v_{r,0,a,b}^\alpha(u) = \left(1 - \alpha \frac{r+b}{r+a} u\right) \left(1 - \frac{r+b}{r+a} u\right)^{r-1},$$

$$v_{r,r,a,b}^\alpha(u) = \left(1 - \alpha + \alpha \frac{r+b}{r+a} u\right) \left(\frac{r+b}{r+a} u\right)^{r-1}.$$

For example,

$$v_{2,0,a,b}^\alpha(u) = \left(1 - \alpha \frac{2+b}{2+a} u\right) \left(1 - \frac{2+b}{2+a} u\right),$$

$$v_{2,1,a,b}^\alpha(u) = 2\alpha \frac{2+b}{2+a} u \left(1 - \frac{2+b}{2+a} u\right),$$

$$v_{2,2,a,b}^\alpha(u) = \left(1 - \alpha + \alpha \frac{2+b}{2+a} u\right) \frac{2+b}{2+a} u,$$

and

$$\begin{aligned}v_{3,0,a,b}^\alpha(u) &= \left(1 - \alpha \frac{3+b}{3+a}u\right) \left(1 - \frac{3+b}{3+a}u\right)^2, \\v_{3,1,a,b}^\alpha(u) &= \left(1 + 2\alpha - 3\alpha \frac{3+b}{3+a}u\right) \frac{3+b}{3+a}u \left(1 - \frac{3+b}{3+a}u\right), \\v_{3,2,a,b}^\alpha(u) &= \left(1 - \alpha + 3\alpha \frac{3+b}{3+a}u\right) \frac{3+b}{3+a}u \left(1 - \frac{3+b}{3+a}u\right), \\v_{3,3,a,b}^\alpha(u) &= \left(1 - \alpha + \alpha \frac{3+b}{3+a}u\right) \left(\frac{3+b}{3+a}u\right)^2.\end{aligned}$$

These formulas satisfy $\sum_{j=0}^r v_{r,j,a,b}^\alpha(u) = 1$ on $0 \leq u \leq \frac{r+a}{r+b}$.

Remark 2.1. The operator $F_{r,a,b}^\alpha(\cdot; u)$ defined in (2.1) is positive under the parameter restrictions.

3. Novelty and contributions

The primary novelty of this work lies in the introduction of a new family of generalized Bernstein operators, denoted by $F_{r,a,b}^\alpha(h, u)$ and defined in Eq (2.1). This construction unifies and extends several existing approximation operators through the incorporation of three independent parameters: a non-negative mixing parameter α , and two domain parameters a and b that shift the approximation interval.

Generalization of known operators

The proposed operators satisfy the following reduction properties:

- For $\alpha = 1$, the operators reduce to the modified Bernstein operators on a moving interval introduced by Izgi [16] (Eq (1.2)).
- For $a = b = 0$, they coincide with the family of generalized Bernstein operators studied by Chen et al. [18] (Eq (1.4)).
- When $\alpha = 1$ and $a = b$, the classical Bernstein operators (Eq (1.1)) are recovered.

Thus, the new operators serve as a unifying framework that encompasses these important special cases.

Shape parameter α

The parameter α acts as a shape parameter influencing the approximation behavior. For $0 \leq \alpha \leq \frac{r+a}{r+b}$, the operators are positive linear operators that reproduce constants and linear functions. Numerical experiments (see Figures 1–3) show that varying α produces moderate changes in the approximation curves, particularly in regions of higher curvature, whereas the parameter r plays the principal role in the convergence process.

Key theoretical properties

Beyond the unifying structure, the paper establishes several new theoretical results for this family:

- A Voronovskaya-type asymptotic formula (Theorem 4.9) that quantifies the rate of convergence for twice differentiable functions.

- A uniform error estimate in terms of the modulus of continuity (Theorem 4.12), which provides a constructive bound on the approximation error.
- Shape-preserving properties (Theorem 5.1) showing that the operators preserve both monotonicity and convexity, which describe the shape-preserving behavior of the proposed operators.

These results are not direct corollaries of the earlier special cases; they are derived specifically for the α -Bernstein structure blended and demonstrate the robustness of the new operators.

Practical relevance

The flexible interval $[0, \frac{r+a}{r+b}]$ and the parameter α provide a parameter-dependent extension of the corresponding Bernstein-type construction. In the present paper, we restrict our claims to the approximation-theoretic and shape-preserving properties that are established analytically and numerically. We now investigate several properties of the family of operators constructed in (2.1).

Lemma 3.1. *For all $r \geq 2$, independent of α :*

- (i) (End point interpolation) *The $F_{r,a,b}^\alpha(\cdot, u)$ α -Bernstein operator for $h(u)$ interpolates $h(u)$ at both endpoints of the interval $[0, \frac{r+a}{r+b}]$, namely,*

$$F_{r,a,b}^\alpha(h; 0) = h(0) \quad \text{and} \quad F_{r,a,b}^\alpha\left(h; \frac{r+a}{r+b}\right) = h\left(\frac{r+a}{r+b}\right).$$

- (ii) (Linearity) *The $F_{r,a,b}^\alpha(\cdot, u)$ α -Bernstein operator is linear, that is,*

$$F_{r,a,b}^\alpha(mh + ng, u) = mF_{r,a,b}^\alpha(h, u) + nF_{r,a,b}^\alpha(g, u),$$

for all functions h and g defined on $[0, \frac{r+a}{r+b}]$, and all real m and n .

We can express (2.1) as follows to discuss some other properties of this operator:

$$F_{r,a,b}^\alpha(h, u) = (1 - \alpha) \left(\frac{r+b}{r+a}\right)^{r-1} (f_1 + f_2) + \alpha \left(\frac{r+b}{r+a}\right)^r \sum_{j=0}^r h_j \binom{r}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j},$$

where

$$f_1 = \sum_{j=0}^r h_j \binom{r-2}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-1}, \quad f_2 = \sum_{j=0}^r h_j \binom{r-2}{j-2} u^{j-1} \left(\frac{r+a}{r+b} - u\right)^{r-j}.$$

It is imperative to recall that both terms corresponding to $j = r$ in f_1 and $j = 0$ in f_2 are equivalent to zero. Therefore, the following expressions can be used to represent f_1 and f_2 :

$$f_1 = \sum_{j=0}^{r-1} h_j \binom{r-2}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-1}, \quad f_2 = \sum_{j=1}^r h_j \binom{r-2}{j-2} u^{j-1} \left(\frac{r+a}{r+b} - u\right)^{r-j},$$

where $\binom{r-2}{r-1} = 0$ in f_1 and $\binom{r-2}{-1} = 0$ in f_2 . Replacing j with $j+1$ in the summation indicated by f_2 yields the ensuing result

$$f_2 = \sum_{j=0}^{r-1} h_{j+1} \binom{r-2}{j-1} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-1}.$$

Consequently, the following result can be obtained:

$$f_1 + f_2 = \sum_{j=0}^{r-1} \left[\binom{r-2}{j} h_j + \binom{r-2}{j-1} h_{j+1} \right] u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-1}.$$

Since

$$\binom{r-2}{j} = \left(1 - \frac{j}{r-1}\right) \binom{r-1}{j} \quad \text{and} \quad \binom{r-2}{j-1} = \frac{j}{r-1} \binom{r-1}{j},$$

as a result

$$f_1 + f_2 = \sum_{j=0}^{r-1} F_j \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-1},$$

where

$$F_j = \left(1 - \frac{j}{r-1}\right) h_j + \frac{j}{r-1} h_{j+1}. \quad (3.1)$$

It has been demonstrated that F_j is a linear combination of h_j and h_{j+1} .

Therefore, there exist alternative representations of the operator $F_{r,a,b}^\alpha$.

Theorem 3.2. *The newly defined modified α -Bernstein operator for $h(u)$ can be expressed as follows:*

$$\begin{aligned} F_{r,a,b}^\alpha(h, u) &= (1 - \alpha) \left(\frac{r+b}{r+a} \right)^{r-1} \sum_{j=0}^{r-1} F_j \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-1} \\ &\quad + \alpha \left(\frac{r+b}{r+a} \right)^r \sum_{j=0}^r h_j \binom{r}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j}, \end{aligned} \quad (3.2)$$

where

$$F_j = \left(1 - \frac{j}{r-1}\right) h_j + \frac{j}{r-1} h_{j+1}.$$

Lemma 3.3. *The following identities hold:*

$$(i) \quad F_{r,a,b}^\alpha(1, u) = 1, \quad (3.3)$$

$$(ii) \quad F_{r,a,b}^\alpha(u, u) = u. \quad (3.4)$$

Proof. (i) It follows from (3.1) and the representation (3.2) that, if $h_j \equiv 1$, then $F_j = 1$ and

$$F_{r,a,b}^\alpha(1, u) = (1 - \alpha) \left(\frac{r+b}{r+a} \right)^{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-1} + \alpha \left(\frac{r+b}{r+a} \right)^r \sum_{j=0}^r \binom{r}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j} = 1,$$

by the binomial theorem.

(ii) For the linear function $h(t) = t$, we have

$$h_j = \frac{j(r+a)}{r(r+b)}, \quad F_j = \frac{j(r+a)}{(r-1)(r+b)}.$$

Substitution into (3.2), followed by the standard first-moment identity for the binomial basis on $[0, \frac{r+a}{r+b}]$, gives

$$F_{r,a,b}^\alpha(t, u) = (1 - \alpha)u + \alpha u = u.$$

Thus, the operator reproduces both constants and linear functions.

4. Approximation by $F_{r,a,b}^\alpha(h, u)$ operators

Definition 4.1. Let I be a compact subset of $\mathbb{R}$, let $h : I \rightarrow \mathbb{R}$ be a real-valued function, and let $m > 0$. The forward difference operator is defined as

$$\Delta_m h(u) = h(u + m) - h(u).$$

The higher-order forward difference is further defined as

$$\Delta_m^n h(u) = \Delta_m^{n-1} \Delta_m h(u).$$

The closed form is

$$\Delta_m^n h(u) = \sum_{j=0}^n (-1)^{n+j} \binom{n}{j} h(u + jm).$$

These higher-order forward differences will be employed to reformulate the operator and to streamline the computation of its moments. In particular, only the fourth-order forward difference is required for the evaluation of the fourth moment of the operator. So, let us take $m = \frac{(r+a)}{r(r+b)}$ and the polynomial function of degree $kh(u) = u^k$, where $r \geq m$. Then, we have

$$\Delta^t h(0) = 0 \quad \text{for } m > t$$

and

$$\Delta^k h\left(\frac{(r+a)}{r(r+b)}j\right) = \frac{(r+a)^k}{r^k(r+b)^k} h^k(\xi_j) = \frac{(r+a)^k k!}{(r+b)^k r^k},$$

where $\xi_j \in \left(\frac{(r+a)}{r(r+b)}j, \frac{(r+a)}{r(r+b)}(j+k)\right)$.

We consider now the following result.

Lemma 4.2. In the context of the present framework, the higher-order forward difference of F_j admits the representation

$$\Delta^k F_j = \left(1 - \frac{j}{r-1}\right) \Delta^k h_j + \frac{j+k}{r-1} \Delta^k h_{j+1}, \quad (4.1)$$

where

$$F_j = \left(1 - \frac{j}{r-1}\right) h_j + \frac{j}{r-1} h_{j+1}.$$

Now, from Theorem 3.1 in the work of Chen et al. [18], and consistently with the representation (3.2), we obtain the following theorem:

Theorem 4.3. The new blending-type modification of the α -Bernstein polynomials can be represented in terms of difference operators as follows:

$$F_{r,a,b}^\alpha(h, u) = \sum_{k=0}^r \left[(1-\alpha) \binom{r-1}{k} \Delta^k F_0 + \alpha \binom{r}{k} \Delta^k h_0 \right] \left(\frac{r+b}{r+a} u \right)^k.$$

Furthermore, Theorem 4.3 and Lemma 4.2 prove that the new modification of the α -Bernstein operator is degree-preserving. In the particular case, for $h(u) = u^k$ and $j \geq k + 1$, it follows that

$$F_{r,a,b}^\alpha(u^k, u) = c_k \left(\frac{r+b}{r+a} u \right)^k + c_{k-1} \left(\frac{r+b}{r+a} u \right)^{k-1} + \dots + c_1 \left(\frac{r+b}{r+a} u \right) + c_0, \quad (4.2)$$

where

$$c_k = (1 - \alpha) \binom{j-1}{k} \Delta^j F_0 + \alpha \binom{r}{j} \Delta^j h_0.$$

Lemma 4.4. *The following identities hold:*

(i)

$$F_{r,a,b}^\alpha(u^2, u) = u^2 + \frac{r+2(1-\alpha)}{r^2} u \left(\frac{r+a}{r+b} - u \right),$$

(ii)

$$F_{r,a,b}^\alpha(u^3, u) = u^3 + \frac{3[r+2(1-\alpha)]}{r^2} u^2 \left(\frac{r+a}{r+b} - u \right) + \frac{r+6(1-\alpha)}{r^3} u \left(\frac{r+a}{r+b} - u \right) \left(\frac{r+a}{r+b} - 2u \right),$$

(iii)

$$\begin{aligned} F_{r,a,b}^\alpha(u^4, u) = & u^4 + \frac{6[r+2(1-\alpha)]}{r^2} u^3 \left(\frac{r+a}{r+b} - u \right) + \frac{4[r+6(1-\alpha)]}{r^3} u^2 \left(\frac{r+a}{r+b} - u \right) \left(\frac{r+a}{r+b} - 2u \right) \\ & + \frac{[3r(r-2) + 12(r-6)(1-\alpha)] u \left(\frac{r+a}{r+b} - u \right) [r+14(1-\alpha)]}{r^4} u \left(\frac{r+a}{r+b} - u \right). \end{aligned}$$

Proof. With $h(u) = u^2$, we have

$$\begin{aligned} h(0) &= 0, & \Delta h(0) &= \frac{(r+a)^2}{r^2(r+b)^2}, & \Delta^2 h(0) &= 2 \cdot \frac{(r+a)^2}{r^2(r+b)^2}, \\ h\left(\frac{(r+a)}{r(r+b)}\right) &= \frac{(r+a)^2}{r^2(r+b)^2}, & \Delta h\left(\frac{(r+a)}{r(r+b)}\right) &= 3 \cdot \frac{(r+a)^2}{r^2(r+b)^2}, & \Delta^2 h\left(\frac{(r+a)}{r(r+b)}\right) &= 2 \cdot \frac{(r+a)^2}{r^2(r+b)^2}, \\ h\left(\frac{2(r+a)}{r(r+b)}\right) &= 4 \cdot \frac{(r+a)^2}{r^2(r+b)^2}, & \Delta h\left(\frac{2(r+a)}{r(r+b)}\right) &= 5 \cdot \frac{(r+a)^2}{r^2(r+b)^2}, \\ h\left(\frac{3(r+a)}{r(r+b)}\right) &= 9 \cdot \frac{(r+a)^2}{r^2(r+b)^2}, \end{aligned}$$

and

$$\begin{aligned} F_0 &= 0, \quad \Delta F_0 = \Delta h(0) + \frac{1}{r-1} h\left(\frac{(r+a)}{r(r+b)}\right) = \frac{r+2}{r-1} \frac{(r+a)^2}{r^2(r+b)^2}, \\ \Delta^2 F_0 &= \Delta^2 h(0) + \frac{2}{r-1} h\left(\frac{(r+a)}{r(r+b)}\right) = 2 \cdot \frac{r+1}{r-1} \frac{(r+a)^2}{r^2(r+b)^2}, \end{aligned}$$

for $n \geq 3$. Thus, it follows from (4.2) that

$$\begin{aligned} F_{r,a,b}^\alpha(u^2, u) &= \left[(1-\alpha) \binom{r-1}{1} \frac{(r+a)^2}{r^2(r+b)^2} \frac{r+2}{r-1} + \alpha \binom{r}{1} \frac{(r+a)^2}{r^2(r+b)^2} \right] \frac{(r+b)}{(r+a)} u \\ &\quad + \left[(1-\alpha) \binom{r-1}{2} 2 \cdot \frac{(r+a)^2}{r^2(r+b)^2} \frac{r+1}{r-1} + \alpha \binom{r}{2} 2 \cdot \frac{(r+a)^2}{r^2(r+b)^2} \right] \frac{(r+b)^2}{(r+a)^2} u^2 \\ &= \frac{(r+a)^2}{r^2(r+b)^2} \cdot \frac{(r+b)}{(r+a)} u \cdot (r+2(1-\alpha)) + \frac{(r+a)^2}{r^2(r+b)^2} \cdot \frac{(r+b)^2}{(r+a)^2} u^2 \cdot (r^2 - r - 2(1-\alpha)). \end{aligned}$$

Thus, when $h(u) = u^2$, $F_{r,a,b}^\alpha(u^2, u)$ is the quadratic polynomial

$$F_{r,a,b}^\alpha(u^2, u) = u^2 + \frac{r+2(1-\alpha)}{r^2} u \left(\frac{r+a}{r+b} - u \right). \quad (4.3)$$

Similarly, we can prove (ii) and (iii).

The following lemma introduces a concise yet highly effective criterion for characterizing the convergence of a linear positive operator to the identity operator.

Lemma 4.5. [19, 20] For any nonnegative integer r , let $\{L_r h; r = 0, 1, \dots\}$ be a sequence of linear positive operators from $C[c, d]$ to $C[c, d]$. If the sequence $\{L_r h; r = 0, 1, \dots\}$ converges uniformly to $h(u)$ for the functions $h(u) = 1, u, u^2$ on $[c, d]$ and $u \in [c, d]$,

$$\lim_{r \rightarrow \infty} L_r(h; u) = h(u)$$

uniformly on $[c, d]$.

This cornerstone result in the theory of uniform approximation by linear positive operators is known as the Bohman-Korovkin theorem. Relation (4.3) implies the uniform convergence of the α -Bernstein operators to u^2 for all real α . In view of the Bohman-Korovkin theorem, the convergence statements below follow directly from (3.3), (3.4), and (4.3).

Theorem 4.6. (Bohman-Korovkin type convergence) Let $h(u)$ be a continuous function on the interval $\left[0, \frac{r+a}{r+b}\right]$. If the $F_{r,a,b}^\alpha(h, u)$, new modification α -Bernstein operators satisfy the Bohman-Korovkin test conditions on this interval, then for every $\alpha \in \left[0, \frac{r+a}{r+b}\right]$, the sequence of α -Bernstein operators converges uniformly to $h(u)$ on $\left[0, \frac{r+a}{r+b}\right]$.

We next examine the error term $F_{r,a,b}^\alpha(h(u), u) - h(u)$ and, for this purpose, introduce the two lemmas below.

Lemma 4.7. Let

$$\mu_{r,m}^\alpha(u) = F_{r,a,b}^\alpha((t-u)^m, u), \quad m = 1, 2, 3, 4,$$

where $c_r = (r+a)/(r+b)$. Then,

$$\mu_{r,1}^\alpha(u) = 0, \quad (4.4)$$

$$\mu_{r,2}^\alpha(u) = \frac{r+2(1-\alpha)}{r^2} u(c_r - u), \quad (4.5)$$

$$\mu_{r,3}^\alpha(u) = \frac{r+6(1-\alpha)}{r^3} u(c_r - u)(c_r - 2u), \quad (4.6)$$

$$\mu_{r,4}^\alpha(u) = \frac{u(c_r - u)}{r^4} \{ [3r(r-2) + 12(r-6)(1-\alpha)]u(c_r - u) + [r + 14(1-\alpha)]c_r^2 \}. \quad (4.7)$$

Proof. The identities follow by expanding $(t - u)^m$ and using Lemma 3.3 together with Lemma 4.4. The use of the actual sampling nodes $t_{r,j} = jc_r/r$ removes the scale mismatch present in the previous formulation.

Lemma 4.8. *There exists a constant $C > 0$, independent of r , such that for $u \in [0, c_r]$ and $\lambda \in (0, 1/4)$,*

$$\sum_{|t_{r,j}-u| \geq r^{-\lambda}} v_{r,j,a,b}^\alpha(u) \leq Cr^{4\lambda-2}, \quad t_{r,j} = \frac{jc_r}{r}.$$

Indeed, Markov's inequality together with (4.7) gives

$$\sum_{|t_{r,j}-u| \geq r^{-\lambda}} v_{r,j,a,b}^\alpha(u) \leq r^{4\lambda} \mu_{r,4}^\alpha(u) = O(r^{4\lambda-2}).$$

We establish two theorems devoted to the estimation of the approximation error $F_{r,a,b}^\alpha(h(u), u) - h(u)$. For functions possessing continuous second derivatives, the first theorem establishes an asymptotic representation of the error associated with the new α -Bernstein polynomials. A new variant of the Voronovskaja theorem [21], applicable to a wide class of linear positive operators, is established in the following theorem.

Theorem 4.9. (*Voronovskaja theorem*) *Let $h(u)$ be bounded on $[0, \frac{r+a}{r+b}]$. Then, for any $u \in [0, \frac{r+a}{r+b}]$ at which $h''(u)$ exists,*

$$\lim_{r \rightarrow \infty} r \left[F_{r,a,b}^\alpha(h(u), u) - h(u) \right] = \frac{1}{2} u(1-u) h''(u),$$

where $0 \leq \alpha \leq \frac{r+a}{r+b}$.

Proof. For $j \leq r$, we have by Taylor's formula

$$h(t) = h(u) + (t - u)h'(u) + \frac{1}{2}(t - u)^2 h''(u) + s(t)(t - u)^2,$$

where $\lim_{t \rightarrow u} s(t) = 0$. In this setting, if one chooses $t = \frac{j(r+a)}{r(r+b)}$,

$$h\left(\frac{j(r+a)}{r(r+b)}\right) = h(u) + \left(\frac{j(r+a)}{r(r+b)} - u\right)h'(u) + \frac{1}{2}\left(\frac{j(r+a)}{r(r+b)} - u\right)^2 h''(u) + s\left(\frac{j(r+a)}{r(r+b)}\right)\left(\frac{j(r+a)}{r(r+b)} - u\right)^2.$$

Consequently, we obtain

$$\begin{aligned} r \left[F_{r,a,b}^\alpha(h, u) - h(u) \right] &= r \sum_{j=0}^r v_{r,j,a,b}^\alpha(u) \left[h\left(\frac{j(r+a)}{r(r+b)}\right) - h(u) \right] \\ &= r \sum_{j=0}^r v_{r,j,a,b}^\alpha(u) \left[\left(\frac{j(r+a)}{r(r+b)} - u\right)h'(u) + \frac{1}{2}\left(\frac{j(r+a)}{r(r+b)} - u\right)^2 h''(u) \right. \\ &\quad \left. + s\left(\frac{j(r+a)}{r(r+b)}\right)\left(\frac{j(r+a)}{r(r+b)} - u\right)^2 \right] \\ &= H_1(u)h'(u) + \frac{1}{2r}H_2(u)h''(u) + r \sum_{j=0}^r s\left(\frac{j(r+a)}{r(r+b)}\right)\left(\frac{j(r+a)}{r(r+b)} - u\right)^2 v_{r,j,a,b}^\alpha(u). \end{aligned}$$

Then, use of (4.4) and (4.5) gives

$$r[F_{r,a,b}^\alpha(h, u) - h(u)] = \left(\frac{1}{2} + \frac{1-\alpha}{r}\right) u \left(\frac{r+a}{r+b} - u\right) h''(u) + rK_r(u),$$

where

$$K_r(u) = \sum_{j=0}^r s\left(\frac{j(r+a)}{r(r+b)}\right) \left(\frac{j(r+a)}{r(r+b)} - u\right)^2 v_{r,j,a,b}^\alpha(u).$$

When $0 \leq \alpha \leq \frac{r+a}{r+b}$, one can derive the following inequality:

$$\begin{aligned} |K_r(u)| &\leq \sum_{\left|\frac{j}{r-u}\right| < r^{-\frac{1}{8}}} \left| s\left(\frac{j(r+a)}{r(r+b)}\right) \right| \left(\frac{j(r+a)}{r(r+b)} - u\right)^2 v_{r,j,a,b}^\alpha(u) \\ &\quad + \sum_{\left|\frac{j}{r-u}\right| \geq r^{-\frac{1}{8}}} \left| s\left(\frac{j(r+a)}{r(r+b)}\right) \right| \left(\frac{j(r+a)}{r(r+b)} - u\right)^2 v_{r,j,a,b}^\alpha(u). \end{aligned}$$

Let any $\epsilon > 0$ be given. We can find r sufficiently large such that $\left|\frac{j}{r-u}\right| < r^{-\frac{1}{8}}$ implies $\left|s\left(\frac{j(r+a)}{r(r+b)}\right)\right| < \epsilon$. Hence,

$$|K_r(u)| \leq \frac{\epsilon}{r^2} H_2(u) + N \sum_{\left|\frac{j}{r-u}\right| \geq r^{-\frac{1}{8}}} v_{r,j,a,b}^\alpha(u),$$

where $N = \sup_{0 \leq t \leq \frac{r+a}{r+b}} s(t)(t-u)^2$. By (4.5) and Lemma 4.8 (when $\lambda = \frac{1}{8}$),

$$r|K_r(u)| \leq \epsilon \left[1 + \frac{2(1-\alpha)}{r}\right] u \left(\frac{r+a}{r+b} - u\right) + \frac{NC}{r^{\frac{1}{2}}}.$$

As the choice of ϵ is arbitrary, the argument concludes the proof.

In the ensuing analysis, we will examine the function defined by

$$h(u) = \frac{1}{20} \left(\frac{1}{1 + 120\left(u - \frac{1}{4}\right)^2} - \frac{1}{1 + 120\left(u - \frac{3}{4}\right)^2} \right), \quad (4.8)$$

approximating it with the α -Bernstein operator for various values of the parameters a, b, r , and α .

Figure 1 illustrates the approximation behavior of the operators $F_{r,a,b}^\alpha(h, u)$, defined for the special case $a = b$, with respect to the test function $h(u)$. In Figure 1(a), the parameter α is fixed at $\alpha = 0.5$, while the parameter r varies. It is clearly observed that, as r increases, the corresponding operator curves converge uniformly to the target function $h(u)$, indicating an improvement in the approximation quality. In Figure 1(b), the parameter r is fixed at $r = 20$, and the influence of different values of α is examined. The results show that increasing α significantly enhances the approximation accuracy, particularly in regions where the test function exhibits higher curvature. These observations confirm that the proposed operator family possesses both asymptotic convergence properties and a flexible, parameter-dependent structure that allows effective control over the approximation behavior.

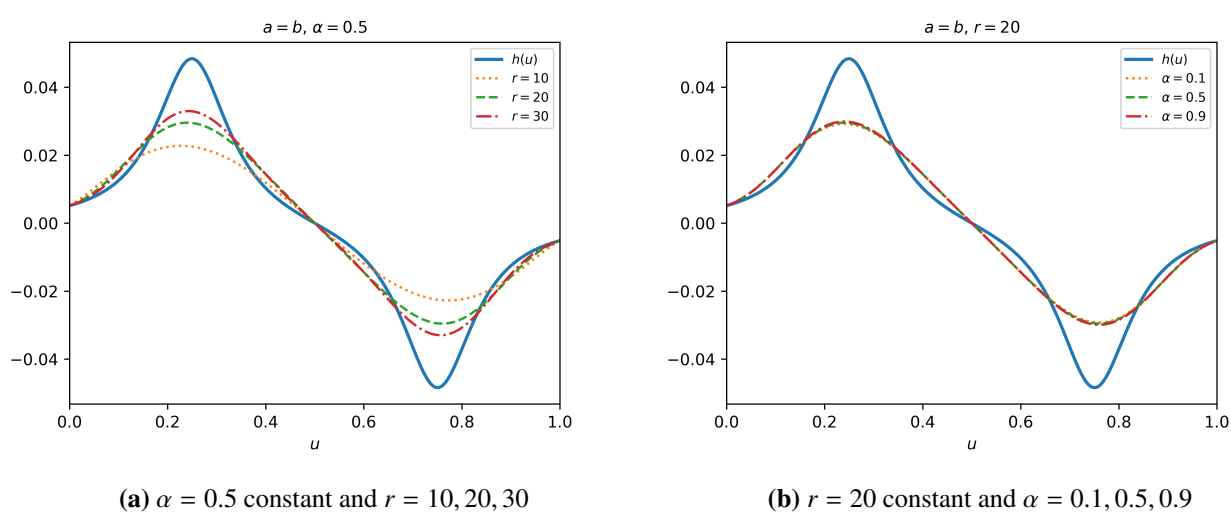


Figure 1. Approximation of $h(u)$ for $a = b$ using $F_{r,a,b}^\alpha(h, u)$ operators.

Figure 2 illustrates the approximation of the test function $h(u)$ by the operators $F_{r,a,b}^\alpha(h, u)$ for $a = 2$, $b = 5$, and $\alpha = 0.4$, while r varies as $r = 10, 20, 30$.

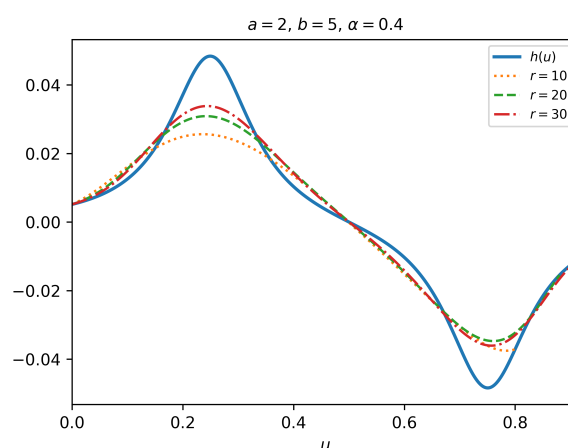


Figure 2. Approximation to $h(u)$ for $a = 2$, $b = 5$, $\alpha = 0.4$, and $r = 10, 20, 30$ using $F_{r,a,b}^\alpha(h, u)$ operators.

Figure 3 illustrates the approximation behavior of the operators $F_{r,a,b}^\alpha(h, u)$ for different pairs a, b with the parameter r fixed at $r = 20$. In Figure 3(a), the values $a = 2$, $b = 5$, and $\alpha = 0.088, 0.440, 0.792$ are considered, whereas in Figure 3(b), $a = 5$, $b = 12$ with $\alpha = 0.078, 0.391, 0.703$ are used. In both cases, it is observed that the approximation improves as α increases, with the operator curves approaching $h(u)$ more closely, particularly in regions of higher curvature. These results indicate that the parameters a, b indirectly influence the approximation process by determining the admissible range of the parameter α .

The α -Bernstein operators yield explicit approximations for a given continuous function. It has been established that the sequence $F_{r,a,b}^\alpha(h, u)$ converges uniformly to the target function $h(u)$ on the interval $[0, \frac{r+a}{r+b}]$. A natural subsequent problem is the quantitative assessment of the approximation error, for which the modulus of continuity serves as an effective analytical tool.

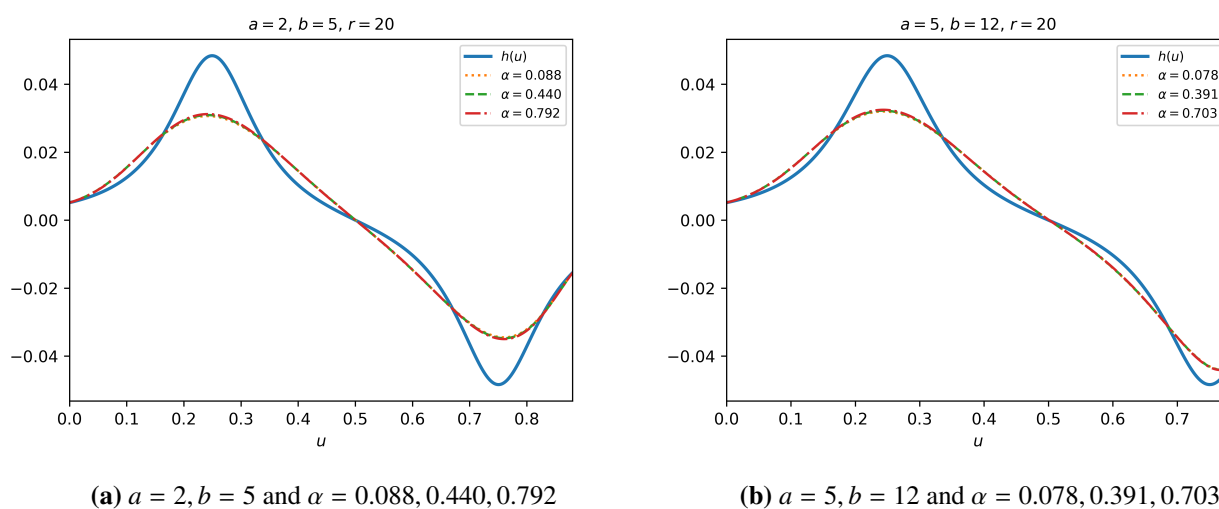


Figure 3. Approximation to $h(u)$ for $r = 20$ using $F_{r,a,b}^\alpha(h, u)$ operators.

Definition 4.10. [22] Let h be a function defined on the interval $[c, d]$. For $\delta > 0$, the modulus of continuity of h on $[c, d]$, denoted by $\omega(\delta)$, is defined as follows:

$$\omega(\delta) = \sup_{\substack{u_1, u_2 \in [c, d] \\ |u_1 - u_2| \leq \delta}} |h(u_1) - h(u_2)|.$$

We need some properties of the modulus of continuity.

Lemma 4.11. [22] The modulus of continuity of $h(u)$ on the interval $[c, d]$ satisfies the following properties:

- (i) If $0 < \delta_1 \leq \delta_2$, then $\omega(\delta_1) \leq \omega(\delta_2)$.
- (ii) The function $h(u)$ is uniformly continuous on $[c, d]$ if, and only if, $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$.
- (iii) For $\lambda > 0$, the inequality $\omega(\lambda\delta) \leq (1 + \lambda)\omega(\delta)$ holds.

We consider an estimate for the approximation error measured in the uniform norm, namely,

$$\|h(u) - F_{r,a,b}^\alpha(h; u)\| = \max_{0 \leq u \leq \frac{r+a}{r+b}} |h(u) - F_{r,a,b}^\alpha(h, u)|,$$

where $\|\cdot\|$ denotes the uniform norm on the interval $[0, \frac{r+a}{r+b}]$. The following theorem gives an upper bound.

Theorem 4.12. Assume that $h \in C[0, \frac{r+a}{r+b}]$. Then, for each α satisfying $0 \leq \alpha \leq \frac{r+a}{r+b}$, the following assertion holds:

$$\|h(u) - F_{r,a,b}^\alpha(h, u)\| \leq \frac{3}{2} \omega(\delta_{r,\alpha}),$$

where $\delta_{r,\alpha} = \frac{\sqrt{r(r+a)^2 + 2(1-\alpha)(r+a)^2}}{r(r+b)}.$

Proof. The linearity of the F operator and (3.3), for $0 \leq \alpha \leq \frac{r+a}{r+b}$,

$$\begin{aligned} |h(u) - F_{r,a,b}^\alpha(h, u)| &= \left| \sum_{j=0}^r v_{r,j,a,b}^\alpha(u) \left(h(u) - h\left(\frac{j(r+a)}{r(r+b)}\right) \right) \right| \\ &\leq \sum_{j=0}^r \left| h(u) - h\left(\frac{j(r+a)}{r(r+b)}\right) \right| v_{r,j,a,b}^\alpha(u) \\ &\leq \sum_{j=0}^r \omega\left(u - \frac{j(r+a)}{r(r+b)}\right) v_{r,j,a,b}^\alpha(u). \end{aligned}$$

Now, by invoking Lemma 4.11, we obtain

$$\omega\left(u - \frac{j(r+a)}{r(r+b)}\right) = \omega\left(\frac{1}{\delta_{r,\alpha}} \left| u - \frac{j(r+a)}{r(r+b)} \right| \delta_{r,\alpha}\right) \leq \left(1 + \frac{1}{\delta_{r,\alpha}} \left| u - \frac{j(r+a)}{r(r+b)} \right| \right) \omega(\delta_{r,\alpha})$$

so that

$$\begin{aligned} |h(u) - F_{r,a,b}^\alpha(h, u)| &\leq \sum_{j=0}^r \left(1 + \frac{1}{\delta_{r,\alpha}} \left| u - \frac{j(r+a)}{r(r+b)} \right| \right) \omega(\delta_{r,\alpha}) v_{r,j,a,b}^\alpha(u) \\ &\leq \omega(\delta_{r,\alpha}) \left(1 + \frac{1}{\delta_{r,\alpha}} \sum_{j=0}^r \left| u - \frac{j(r+a)}{r(r+b)} \right| v_{r,j,a,b}^\alpha(u) \right). \end{aligned}$$

By an application of the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \sum_{j=0}^r \left| u - \frac{j(r+a)}{r(r+b)} \right| v_{r,j,a,b}^\alpha(u) &= \sum_{j=0}^r \left| u - \frac{j(r+a)}{r(r+b)} \right| \sqrt{v_{r,j,a,b}^\alpha(u)} \sqrt{v_{r,j,a,b}^\alpha(u)} \\ &\leq \left[\sum_{j=0}^r \left(u - \frac{j(r+a)}{r(r+b)} \right)^2 v_{r,j,a,b}^\alpha(u) \right]^{\frac{1}{2}} \left[\sum_{j=0}^r v_{r,j,a,b}^\alpha(u) \right]^{\frac{1}{2}} \\ &= \left[\sum_{j=0}^r \left(u - \frac{j(r+a)}{r(r+b)} \right)^2 v_{r,j,a,b}^\alpha(u) \right]^{\frac{1}{2}}. \end{aligned}$$

In view of (4.5), we obtain

$$\sum_{j=0}^r \left(u - \frac{j(r+a)}{r(r+b)} \right)^2 v_{r,j,a,b}^\alpha(u) = \frac{1}{r^2} H_2(u) = \frac{r+2(1-\alpha)}{r^2} u \left(\frac{r+a}{r+b} - u \right) \leq \frac{r+2(1-\alpha)}{4r^2} \left(\frac{r+a}{r+b} \right)^2.$$

Thus,

$$\begin{aligned} |h(u) - F_{r,a,b}^\alpha(h, u)| &\leq \omega(\delta_{r,\alpha}) \left(1 + \frac{1}{\delta_{r,\alpha}} \sqrt{\frac{r+2(1-\alpha)}{4r^2} \left(\frac{r+a}{r+b} \right)^2} \right) \\ &= \omega(\delta_{r,\alpha}) \left(1 + \frac{1}{\delta_{r,\alpha}} \frac{\sqrt{r(r+a)^2 + 2(1-\alpha)(r+a)^2}}{2r(r+b)} \right) \\ &= \frac{3}{2} \omega(\delta_{r,\alpha}), \end{aligned}$$

with the choice $\delta_{r,\alpha} = \frac{\sqrt{r(r+a)^2 + 2(1-\alpha)(r+a)^2}}{r(r+b)}$, and the proof of the theorem is complete.

The preceding theorem provides an estimate for the approximation error

$$h(u) - F_{r,a,b}^\alpha(h, u)$$

expressed in terms of the modulus of continuity. On the other hand, assuming that the function $h(u)$ is continuous on the interval $[0, 1]$, it follows from Lemma 4.11 that

$$\lim_{r \rightarrow \infty} \omega \left(\frac{\sqrt{r(r+a)^2 + 2(1-\alpha)(r+a)^2}}{r(r+b)} \right) = 0.$$

Consequently, this observation yields an alternative proof of Theorem 4.6.

5. Shape preserving properties of $F_{r,a,b}^\alpha(h, u)$

This section is devoted to the study of shape-preserving properties, where it is shown that the α -Bernstein operator preserves monotonicity as well as convexity. The following theorem establishes that the α -Bernstein operator $F_{r,a,b}^\alpha(h, u)$ preserves monotonicity.

Theorem 5.1. *Suppose that $h \in C[0, \frac{r+a}{r+b}]$. For any α with $0 \leq \alpha \leq \frac{r+a}{r+b}$, if $h(u)$ is monotone increasing (respectively, monotone decreasing) on $[0, \frac{r+a}{r+b}]$, then each α -Bernstein operator associated with h retains the same monotonic behavior on $[0, \frac{r+a}{r+b}]$.*

Proof. We write the α -Bernstein operator in the form

$$\begin{aligned} F_{r,a,b}^\alpha(h, u) &= (1-\alpha) \left(\frac{r+b}{r+a} \right)^{r-1} \sum_{j=0}^{r-1} F_j \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-1} \\ &\quad + \alpha \left(\frac{r+b}{r+a} \right)^r \sum_{j=0}^r h_j \binom{r}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j}, \end{aligned}$$

where

$$h_j = h \left(\frac{j(r+a)}{r(r+b)} \right), \quad F_j = \left(1 - \frac{j}{r-1} \right) h_j + \frac{j}{r-1} h_{j+1}.$$

By computing the derivative of the α -Bernstein operator $F_{r,a,b}^\alpha(h, u)$, we obtain

$$\left(F_{r,a,b}^\alpha \right)'(h, u) = (1-\alpha)T_1 + \alpha T_2,$$

where

$$T_1 = \left(\frac{r+b}{r+a} \right)^{r-1} \sum_{j=0}^{r-1} F_j \binom{r-1}{j} \left[j u^{j-1} \left(\frac{r+a}{r+b} - u \right)^{r-j-1} - (r-j-1) u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-2} \right]$$

and

$$T_2 = \left(\frac{r+b}{r+a} \right)^r \sum_{j=0}^r h_j \frac{d}{du} \left[u^j \left(\frac{r+a}{r+b} - u \right)^{r-j} \right] \binom{r}{j}.$$

The term T_1 can be written in the form

$$T_1 = \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-1} j \binom{r-1}{j} F_j u^{j-1} \left(\frac{r+a}{r+b} - u\right)^{r-j-1} \\ - \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-1} (r-j-1) \binom{r-1}{j} F_j u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-2}.$$

Observe that the term associated with $j = 0$ in the first summation and the term corresponding to $j = r - 1$ in the second summation both vanish. Hence, after shifting the index in the first sum,

$$T_1 = \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-2} (j+1) \binom{r-1}{j+1} F_{j+1} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-2} \\ - \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-2} (r-j-1) \binom{r-1}{j} F_j u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-2}.$$

Since

$$(j+1) \binom{r-1}{j+1} = (r-1) \binom{r-2}{j} = (r-j-1) \binom{r-1}{j},$$

we have

$$T_1 = (r-1) \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-2} \Delta F_j \binom{r-2}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-2},$$

where Δ denotes the forward difference operator. Consequently, by Lemma 4.2,

$$\Delta F_j = \left(1 - \frac{j}{r-1}\right) \Delta h_j + \frac{j+1}{r-1} \Delta h_{j+1},$$

and, therefore,

$$T_1 = \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-2} [(r-j-1) \Delta h_j + (j+1) \Delta h_{j+1}] \binom{r-2}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-2}.$$

By an analogous argument, one can show that

$$T_2 = r \left(\frac{r+b}{r+a}\right)^r \sum_{j=0}^{r-1} \Delta h_j \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-1}.$$

Consequently, the derivative of the α -Bernstein operator $F_{r,a,b}^\alpha(h, u)$ takes the form

$$\left(F_{r,a,b}^\alpha\right)'(h, u) = (1-\alpha) \left(\frac{r+b}{r+a}\right)^{r-1} \sum_{j=0}^{r-2} [(r-j-1) \Delta h_j + (j+1) \Delta h_{j+1}] \binom{r-2}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-2} \\ + \alpha r \left(\frac{r+b}{r+a}\right)^r \sum_{j=0}^{r-1} \Delta h_j \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u\right)^{r-j-1}.$$

The monotonic character of the α -Bernstein operator is dictated by the sign of the forward differences Δh_j . Indeed, when $h(u)$ is monotone increasing, all forward differences are nonnegative, which ensures that the derivative of $F_{r,a,b}^\alpha(h, u)$ is nonnegative for every $u \in [0, \frac{r+a}{r+b}]$. Hence, $F_{r,a,b}^\alpha$ inherits the increasing nature of the function h . In a similar manner, if $h(u)$ is monotone decreasing on $[0, \frac{r+a}{r+b}]$, then the α -Bernstein operator likewise preserves this decreasing behavior.

The following theorem establishes that the α -Bernstein operator $F_{r,a,b}^\alpha(h; u)$ possesses the convexity-preserving property.

Theorem 5.2. *Let $h \in C[0, \frac{r+a}{r+b}]$. If $h(u)$ is convex on the interval $[0, \frac{r+a}{r+b}]$, then for every α satisfying $0 \leq \alpha \leq \frac{r+a}{r+b}$, all corresponding α -Bernstein operators preserve the convexity of h .*

Proof. It follows from (3.2) that

$$F_{r,a,b}^\alpha(h, u) = (1 - \alpha) \left(\frac{r+b}{r+a} \right)^{r-1} \sum_{j=0}^{r-1} F_j \binom{r-1}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j-1} \\ + \alpha \left(\frac{r+b}{r+a} \right)^r \sum_{j=0}^r h_j \binom{r}{j} u^j \left(\frac{r+a}{r+b} - u \right)^{r-j},$$

where

$$h_j = h\left(\frac{j(r+a)}{r(r+b)}\right), \quad F_j = \left(1 - \frac{j}{r-1}\right) h_j + \frac{j}{r-1} h_{j+1}.$$

Following the same line of reasoning adopted in the derivation of the first derivative in the proof of Theorem 5.1, we arrive at the equation

$$(F_{r,a,b}^\alpha)''(h, u) = (1 - \alpha)(r-1)(r-2) \left(\frac{r+b}{r+a} \right)^{r-1} \sum_{k=0}^{r-3} \Delta^2 F_k \binom{r-3}{k} u^k \left(\frac{r+a}{r+b} - u \right)^{r-k-3} \\ + \alpha r(r-1) \left(\frac{r+b}{r+a} \right)^r \sum_{k=0}^{r-2} \Delta^2 h_k \binom{r-2}{k} u^k \left(\frac{r+a}{r+b} - u \right)^{r-k-2}.$$

In accordance with (4.1), we have

$$\Delta^2 F_k = \left(1 - \frac{k}{r-1}\right) \Delta^2 h_k + \frac{k+2}{r-1} \Delta^2 h_{k+1}.$$

Since the function $h(u)$ is convex on the interval $[0, \frac{r+a}{r+b}]$, all of its second-order divided differences are nonnegative. By exploiting the well-known relationship between finite differences and divided differences at equidistant nodes, it follows that

$$\Delta^2 h_k = 2 \left(\frac{r+a}{r(r+b)} \right)^2 h \left[\frac{k(r+a)}{r(r+b)}, \frac{(k+1)(r+a)}{r(r+b)}, \frac{(k+2)(r+a)}{r(r+b)} \right] \geq 0,$$

and $\Delta^2 F_k \geq 0$. Hence,

$$(F_{r,a,b}^\alpha)''(h, u) \geq 0,$$

which establishes the convexity of $F_{r,a,b}^\alpha(h, u)$ and thereby completes the proof.

The results established in Theorems 5.1 and 5.2 demonstrate the shape-preserving nature of the operator. More precisely, if the function $h(u)$ is monotone increasing (respectively, monotone decreasing) on the interval $[0, \frac{r+a}{r+b}]$, then the associated operator preserves the same monotonic behavior on $[0, \frac{r+a}{r+b}]$. Likewise, whenever $h(u)$ is convex (or concave) on $[0, \frac{r+a}{r+b}]$, the operator retains this convexity (or concavity). These results show that the corrected α -Bernstein polynomials $v_{r,j,a,b}^\alpha(u)$ retain the monotonicity and convexity properties of the approximated function on the prescribed interval.

6. Conclusions

This study introduced a new family of generalized α -Bernstein operators that preserve fundamental properties analogous to those of the classical Bernstein operators (1.1) as well as the Izgi operators (1.2). As discussed earlier, the α -Bernstein operators under consideration admit the following representation:

$$\begin{aligned} F_{r,j,a,b}^\alpha(h,u) &= \left(\frac{r+b}{r+a}\right)^r \sum_{j=0}^r h_j \left[(1-\alpha) \frac{r+a}{r+b} \binom{r-2}{j} u + (1-\alpha) \frac{r+a}{r+b} \binom{r-2}{j-2} \left(\frac{r+a}{r+b} - u\right) \right. \\ &\quad \left. + \alpha u \left(\frac{r+a}{r+b} - u\right) \binom{r}{j} \right] u^{j-1} \left(\frac{r+a}{r+b} - u\right)^{r-j-1} \\ &= \sum_{k=0}^r \left[(1-\alpha) \binom{r-1}{k} \Delta^k F_0 + \alpha \binom{r}{k} \Delta^k h_0 \right] \left(\frac{r+b}{r+a} u\right)^k. \end{aligned}$$

The generalized blending-type Bernstein operators possess the following properties:

- (i) For $\alpha = 1$, Eq (2.1) reduces to İzgi operator (1.2).
- (ii) If $a = b = 0$, then Eq (2.1) coincides with (1.4) by Chen et al. [18].
- (iii) Moreover, when $\alpha = 1$ and $a = b$, Eq (2.1) reduces to the classical Bernstein operator given in (1.1).
- (iv) The new α -Bernstein operators constitute a class of positive linear operators that exactly reproduce linear polynomials and exhibit shape-preserving characteristics.
- (v) For every $\alpha \in [0, \frac{r+a}{r+b}]$, the newly introduced operators converge uniformly to any continuous function on the interval $[0, \frac{r+a}{r+b}]$, and this convergence is independent of the parameter α .
- (vi) For any α with $0 \leq \alpha \leq \frac{r+a}{r+b}$, the operator $F_{r,j,a,b}^\alpha(h,u)$ preserves both monotonicity and convexity.

Accordingly, the parameter α may be interpreted as a shape parameter of the newly introduced operators, since the form of the approximation curve produced by the operator can be effectively controlled by varying α within the interval $[0, \frac{r+a}{r+b}]$. This flexibility renders the α -Bernstein operator significant from both theoretical and practical perspectives.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. K. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen, *Sitzungsberichte Königlich Preussischen Akad. Wiss. Berl.*, **2** (1885), 633–639.
2. S. N. Bernstein, Démonstration du théoreme de Weierstrass fondée sur le calcul des probabilités, *Commun. Soc. Math. Kharkov*, **13** (1912), 1–2.
3. M. Nasiruzzaman, Wavelets associated approximation enhanced by Szász-type operators initiated by q -properties, *J. Inequal. Appl.*, **2026** (2026), 1. <https://doi.org/10.1186/s13660-025-03417-0>
4. M. Nasiruzzaman, A. Alotaibi, Kantorovich extension of parametric generalized q -Schurer operators and their approximation properties, *Mathematics*, **13** (2025), 1–22. <https://doi.org/10.3390/math13233770>
5. M. Nasiruzzaman, M. Farid, N. Rao, Convergence by class of Kantorovich-type q -Szász operators and comprehensive results, *Mathematics*, **13** (2025), 1–22. <https://doi.org/10.3390/math13223586>
6. N. Rao, N. K. Jha, Approximations: modified Szász-Beta operators via general-Appell polynomials, *Filomat*, **39** (2025), 9385–9401. <https://doi.org/10.2298/FIL2527385R>
7. N. Rao, M. Farid, N. K. Jha, Szász-integral operators linking general-Appell polynomials and approximation, *AIMS Math.*, **10** (2025), 13836–13854. <https://doi.org/10.3934/math.2025623>
8. N. Rao, A. Wafi, Stancu-variant of generalized Baskakov operators, *Filomat*, **31** (2017), 2625–2632. <https://doi.org/10.2298/FIL1709625R>
9. D. Barrera, S. Eddargani, M. J. Ibáñez, A. Lamnii, A new approach to deal with C^2 cubic splines and its application to super-convergent quasi-interpolation, *Math. Comput. Simul.*, **194** (2022), 401–415. <https://doi.org/10.1016/j.matcom.2021.12.003>
10. S. Bouhiri, A. Lamnii, M. Lamnii, A. Zidna, Quasi-interpolant operators in Bernstein basis, *Math. Comput. Simul.*, **186** (2021), 80–90. <https://doi.org/10.1016/j.matcom.2020.07.001>
11. Q. B. Cai, X. W. Xu, Shape-preserving properties of a new family of generalized Bernstein operators, *J. Inequal. Appl.*, **2018** (2018), 241. <https://doi.org/10.1186/s13660-018-1821-9>
12. Z. Walczak, V. Gupta, A note on the convergence of Baskakov type operators, *Appl. Math. Comput.*, **202** (2008), 370–375. <https://doi.org/10.1016/j.amc.2008.02.028>
13. H. Aktuğlu, M. Kara, E. Baytunç, Approximation by ψ -Baskakov-Kantorovich operators, *Math. Methods Appl. Sci.*, **48** (2025), 13552–13562. <https://doi.org/10.1002/mma.11123>
14. M. Ayman-Mursaleen, E. Alshaban, M. Nasiruzzaman, Shape preserving approximation properties to family of α -Bernstein shifted knot operators, *J. Inequal. Appl.*, **2025** (2025), 107. <https://doi.org/10.1186/s13660-025-03360-0>
15. E. Baytunç, H. Aktuğlu, N. I. Mahmudov, Approximation properties of Riemann-Liouville type fractional Bernstein-Kantorovich operators of order α , *Math. Found. Comput.*, **7** (2024), 544–567. <https://doi.org/10.3934/mfc.2023030>
16. A. İzgi, Approximation by a class of new type Bernstein polynomials of one and two variables, *Global J. Pure Appl. Math.*, **8** (2012), 55–71.

17. F. Usta, On new modification of Bernstein operators: theory and applications, *Iran. J. Sci. Techno. Trans. A Sci.*, **44** (2020), 1119–1124. <https://doi.org/10.1007/s40995-020-00919-y>
18. X. Y. Chen, J. Q. Tan, Z. Liu, J. Xie, Approximation of functions by a new family of generalized Bernstein operators, *J. Math. Anal. Appl.*, **450** (2017), 244–261. <https://doi.org/10.1016/j.jmaa.2016.12.075>
19. H. Bohman, On approximation of continuous and of analytic functions, *Ark. Mat.*, **2** (1952), 43–56. <https://doi.org/10.1007/BF02591381>
20. P. P. Korovkin, On convergence of linear positive operators in the space of continuous functions (Russian), *Dokl. Akad. Nauk SSSR (NS)*, **90** (1953), 961–964.
21. E. Voronovskaja, Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Bernstein, *CR Acad. Sci. URSS*, **79** (1932), 79–85.
22. T. J. Rivlin, *An introduction to the approximation of functions*, Courier Corporation, 1981.



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