



Research article

A 4-coloring problem for triangle-free induced- $K_{3,3}^-$ -free graphs

Caibing Chang, Jinsen Xiao*, Mingshan Li, Haowen Zeng and Han Lin

Faculty of Science, Guangdong University of Petrochemical Technology, Guangdong 525000, China

* **Correspondence:** Email: jinsenxiao@gdupt.edu.cn.

Abstract: Let C_6 be the class of triangle-free graphs with no induced $K_{3,3}^-$ and no hole longer than six. We study whether every graph in C_6 is 4-colorable. We prove that every 4-hole is contained in a unique maximal induced complete bipartite subgraph, that two such subgraphs intersect in at most one edge, and that each vertex outside one of them has at most one neighbor in it. For vertex-critical graphs, we also bound the distances between attachment fibers. A Kempe-chain argument shows that every vertex of a triangle-free vertex- k -critical graph lies in at least $\binom{k-1}{2}$ induced odd holes. A complete computation identifies exactly three 4-critical graphs in C_6 with at most 15 vertices. Using the catalogs of triangle-free 5-critical graphs of orders 22, 23, and 24, we further prove that every triangle-free, induced- $K_{3,3}^-$ -free graph with at most 24 vertices is 4-colorable. Hence any counterexample in C_6 has at least 25 vertices. The conjecture remains open in general; the results provide structural restrictions and verification within a finite range of orders.

Keywords: graph coloring; triangle-free graph; induced subgraph; critical graph; complete bipartite graph

Mathematics Subject Classification: 05C15, 05C17, 05C35, 05C75

1. Introduction

Triangle-free graphs can have arbitrarily large chromatic number, as shown by Mycielski's construction [12]. Thus, a coloring bound requires conditions beyond the exclusion of triangles. Here we combine a forbidden induced subgraph with a bound on the length of induced cycles.

A hole is an induced cycle of length at least four. The induced circumference of a graph is the maximum length of a hole, with value zero if the graph has no hole. Write $K_{3,3}^-$ for the graph obtained from $K_{3,3}$ by deleting one edge. We investigate the hereditary class

$$C_6 = \{G : G \text{ is triangle-free, induced-}K_{3,3}^- \text{-free, and every hole of } G \text{ has length at most 6}\}.$$

The cutoff at six is motivated by the preceding three-colorable case: Randerath and Schiermeyer proved that every triangle-free graph whose holes have lengths only four and five is 3-colorable [13]. Allowing 6-holes is the next hole-length threshold. It retains 5-holes as the only possible odd holes, while 4-holes generate the complete bipartite subgraphs studied below. The induced- $K_{3,3}^-$ restriction controls these subgraphs and their external neighborhoods. This leads to a specific structural problem: How do the blocks generated by 4-holes interact with the 5-holes forced by criticality?

We consider the following conjecture.

Conjecture 1.1. *Every graph in C_6 is 4-colorable.*

A 4-hole in a triangle-free graph is an induced $K_{2,2}$. If the graph is also induced- $K_{3,3}^-$ -free, each 4-hole lies in a unique maximal induced complete bipartite subgraph, which we call a block. Distinct blocks share at most one edge, and a vertex outside a block has at most one neighbor in it. For a vertex-critical graph in C_6 , the bound on induced cycle length also limits the distance between vertices that attach to the same block.

Criticality gives a different restriction. After one vertex is deleted from a vertex- k -critical graph, Kempe chains in a $(k-1)$ -coloring produce induced odd holes through the deleted vertex. We prove that there are at least $\binom{k-1}{2}$ such holes through each vertex. In C_6 , each of these holes has length five.

We then use complete graph catalogs to obtain two finite results. Up to isomorphism, exactly three 4-critical graphs in C_6 have at most 15 vertices. For every triangle-free 5-critical graph in the complete catalogs of orders 22, 23, and 24, a sequence of forced edge completions ends in a triangle obstruction. These certificates exclude every triangle-free, induced- $K_{3,3}^-$ -free supergraph of a catalog entry. It follows that every triangle-free, induced- $K_{3,3}^-$ -free graph with at most 24 vertices is 4-colorable. This result does not require a bound on induced circumference. It does not prove Conjecture 1.1, but it shows that a counterexample has at least 25 vertices. The electronic supplement contains machine-readable graph data and two independent verification programs.

Related results impose different restrictions on induced cycles. Chudnovsky, Scott, and Seymour obtain chromatic bounds from bounds on clique number and hole length [2]. Lagoutte treats triangle-free graphs with no even hole of length at least six [9]; our class allows 6-holes but excludes all holes of length at least seven. Xu, Yu, and Zha prove 4-colorability when triangles, 4-cycles, and odd holes of length at least seven are excluded [14]. Here 4-cycles are allowed, and the induced- $K_{3,3}^-$ condition controls the blocks they generate. Charoenpanitseri studies $(2, t)$ -choosability under triangle and $K_{3,3}^-$ exclusions as ordinary subgraphs [1]; both the coloring parameter and the forbidden-subgraph condition differ from ours.

Local graph structure is also used in computational settings. TianheGraph uses topology-aware partitioning and storage for parallel graph processing [5], while Guo et al. study the detection of faulty substructures in hypercube networks [6]. The computation here uses explicit induced-subgraph certificates to make exhaustive checks of small graphs directly verifiable.

The computation uses Goedgebeur's complete generation of triangle-free 5-chromatic graphs [4] and the House of Graphs collection [3].

The contribution of this paper is the block and attachment analysis for the combined induced-subgraph restrictions, together with the finite classifications and verification described below. The odd-hole incidence bound is an induced-hole form of the classical color-pair Kempe-chain argument [10, 11], included with a self-contained proof for its application to C_6 . These results give restrictions on a potential counterexample; they do not resolve Conjecture 1.1 in general.

2. Definitions and preliminary results

All graphs are finite, simple, and undirected. For a graph G , write $V(G)$ and $E(G)$ for its vertex and edge sets, $N_G(v)$ for the open neighborhood of v , and $G[X]$ for the subgraph induced by $X \subseteq V(G)$. We omit the subscript on N_G when the ambient graph is clear.

A graph is k -critical if $\chi(G) = k$ and every proper subgraph is $(k-1)$ -colorable. It is *vertex- k -critical* if $\chi(G) = k$ and $G-v$ is $(k-1)$ -colorable for every vertex v . These notions are different: Vertex-criticality alone does not assert that deleting an edge lowers the chromatic number. Every graph with chromatic number at least k has an induced vertex- k -critical subgraph. Indeed, take an inclusion-minimal vertex set whose induced subgraph has chromatic number at least k ; deleting a vertex lowers the chromatic number to at most $k-1$, and adding that vertex uses at most one additional color. We use this induced reduction whenever we need to preserve forbidden induced subgraphs.

We use the elementary facts that a vertex-critical graph is connected, has minimum degree at least $k-1$, and has no clique cutset. The last fact follows by coloring the pieces separated by the clique and permuting colors so that their colorings agree on the cutset.

An induced biclique is an induced complete bipartite subgraph. We call an induced biclique $G[A \cup B]$ a block if $|A|, |B| \geq 2$ and it is maximal under vertex inclusion among induced complete bipartite subgraphs with both parts of size at least two. Triangle-freeness makes the bipartition of a block unique up to interchanging A and B .

The following six-vertex observation will be used throughout.

Lemma 2.1 (completion of a missing cross-edge). *Let G be triangle-free and induced- $K_{3,3}^-$ -free. Suppose that a_1, a_2, x and b_1, b_2, y are independent triples and every cross-edge between the triples except possibly xy is present. Then $xy \in E(G)$.*

Proof. If xy is absent, the subgraph on the six displayed vertices is induced and isomorphic to $K_{3,3}^-$, contrary to the hypothesis. $\square$

3. Maximal induced complete bipartite subgraphs

We begin with the block containing a fixed 4-hole.

Theorem 3.1 (unique maximal biclique containing a 4-hole). *Let G be triangle-free and induced- $K_{3,3}^-$ -free. Every 4-hole of G is contained in a unique block.*

Proof. Let $a_1 b_1 a_2 b_2 a_1$ be a 4-hole and put

$$A = N(b_1) \cap N(b_2), \quad B = N(a_1) \cap N(a_2).$$

Triangle-freeness implies that A and B are disjoint independent sets.

Take $x \in A$ and $y \in B$. If $x \in \{a_1, a_2\}$ or $y \in \{b_1, b_2\}$, then xy is present by the definitions of A and B . Otherwise the six vertices are distinct, and all eight cross-edges between $\{a_1, a_2, x\}$ and $\{b_1, b_2, y\}$ other than possibly xy are present. Lemma 2.1 gives $xy \in E(G)$.

Hence, $G[A \cup B]$ is an induced complete bipartite graph containing the original 4-hole.

Any induced complete bipartite graph containing that 4-hole has every vertex in one of the two common-neighbor sets A and B . Consequently $G[A \cup B]$ is maximal and is the unique block containing the hole. $\square$

A vertex outside a block has a particularly simple neighborhood in that block.

Lemma 3.2 (external neighborhood of a maximal biclique). *Let $Q = G[A \cup B]$ be a block in a triangle-free, induced- $K_{3,3}$ -free graph. Every vertex in $V(G) \setminus V(Q)$ has at most one neighbor in Q .*

Proof. Let $x \notin V(Q)$. It cannot have neighbors in both A and B , since every cross-pair in $A \times B$ is adjacent and would form a triangle with x . Suppose that x has two neighbors a_1, a_2 in A . If $A = \{a_1, a_2\}$, then x is complete to A and anticomplete to B , so it can be added to the B -side of Q , contradicting maximality.

If A has a third vertex a_3 , choose distinct $b_1, b_2 \in B$. The two triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, x\}$ induce $K_{3,3}$ with only the edge a_3x possibly missing. If that edge is missing, an induced $K_{3,3}^-$ occurs; if it is present, repeat the argument for every $a_3 \in A$ to see that x is complete to A , again contradicting maximality. The case of two neighbors in B is symmetric. $\square$

The uniqueness of the block containing a 4-hole also restricts the intersection of two blocks.

Theorem 3.3 (intersection of two distinct blocks). *Let Q_1 and Q_2 be distinct blocks in a triangle-free, induced- $K_{3,3}^-$ -free graph. Then $Q_1 \cap Q_2$ is empty, a single vertex, or a single edge. If the intersection is an edge, then $V(Q_1) \setminus V(Q_2)$ is anticomplete to $V(Q_2) \setminus V(Q_1)$.*

Proof. Suppose that Q_1 and Q_2 share distinct nonadjacent vertices u, v . They lie in the same part of each block. Every vertex in the opposite part of Q_2 is adjacent to both u and v , so Lemma 3.2 forces it to belong to Q_1 . Choose two such vertices. Every vertex in the remaining part of Q_2 is adjacent to both of them, and the same lemma again forces it into Q_1 . Hence, $V(Q_2) \subseteq V(Q_1)$, and maximality gives $Q_1 = Q_2$, a contradiction.

Any three vertices of a bipartite graph contain a nonadjacent pair. Thus, distinct blocks share at most two vertices, and a two-vertex intersection must be an edge.

Now, let the shared edge be ab , and let $x \in V(Q_1) \setminus V(Q_2)$. In the complete bipartite graph Q_1 , x is adjacent to exactly one of a, b . An edge from x to any $y \in V(Q_2) \setminus V(Q_1)$ would give x two neighbors in Q_2 , contrary to Lemma 3.2. The exclusive vertex sets are therefore anticomplete. $\square$

Remark 3.4. Theorem 3.3 restricts the interface between two blocks to a vertex or an edge. A coloring argument can therefore record the colors on this small interface when combining two pieces. These pairwise restrictions do not by themselves give a decomposition: Several blocks may form a cyclic arrangement, and colors must agree at all shared vertices. Figure 1 illustrates the edge-intersection case.

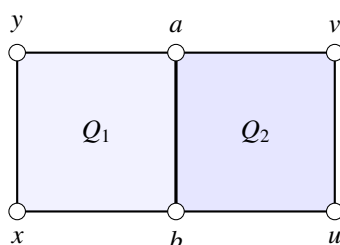


Figure 1. Two $K_{2,2}$ blocks meeting in the edge ab . The exclusive sets $\{x, y\}$ and $\{u, v\}$ are anticomplete. This graph illustrates the block-intersection theorem; it is not asserted to be vertex-critical.

4. Attachments to blocks in vertex-critical graphs

Let $Q = G[A \cup B]$ be a block and let C be a component of $G - V(Q)$. Its attachment set is

$$S_Q(C) = N_G(C) \cap V(Q).$$

For $q \in S_Q(C)$, the corresponding attachment fiber is

$$F_C(q) = N_G(q) \cap V(C).$$

Lemma 3.2 implies that the fibers are pairwise disjoint.

Proposition 4.1 (attachment sets in vertex-critical graphs). *If G is vertex-critical, triangle-free, and induced- $K_{3,3}$ -free, then every vertex of a block has a neighbor outside that block. Moreover, for every component C of $G - V(Q)$, the set $S_Q(C)$ is not a clique.*

Proof. Suppose that $q \in A$ has no neighbor outside Q . For any other $q' \in A$ we have $N(q) = B \subseteq N(q')$. In a vertex-critical graph there are no distinct nonadjacent vertices u, v with $N(u) \subseteq N(v)$: A coloring of $G - u$ extends to u by assigning it the color of v . This contradiction proves the first statement.

If $S_Q(C)$ were a clique, then it would separate C from the rest of G . Thus, it would be a clique cutset (or, if all remaining vertices lie in C , the same coloring-gluing argument applies directly). A vertex-critical graph has no clique cutset, so $S_Q(C)$ is not a clique. $\square$

The bound on hole length now gives distances between attachment fibers.

Proposition 4.2 (distance bounds between attachment fibers). *Let $G \in \mathcal{C}_6$ be vertex-critical, let $Q = G[A \cup B]$ be a block, and let C be a component of $G - V(Q)$.*

- (i) *If $a \in A \cap S_Q(C)$ and $b \in B \cap S_Q(C)$, then $\text{dist}_C(F_C(a), F_C(b)) \leq 3$.*
- (ii) *There exist two distinct attachment vertices in the same part of Q whose fibers have distance at most two in C .*

Proof. For (i), choose a shortest path $P = x_0 \cdots x_d$ in C from $F_C(a)$ to $F_C(b)$. It is induced, and no internal vertex lies in either endpoint fiber, since that would shorten the path. Lemma 3.2 excludes attachment of either endpoint to the other endpoint's block vertex. Thus, $ax_0 \cdots x_dba$ is an induced cycle of length $d + 3$. The hole-length bound gives $d \leq 3$.

For (ii), Proposition 4.1 ensures that $S_Q(C)$ is not a clique, so it contains a same-part pair. Among all same-part pairs in $S_Q(C)$, considering both parts of Q , choose a pair whose fibers have minimum distance d . Interchange the parts if necessary so that the chosen vertices are $a_1, a_2 \in A$, and take a shortest path $P = x_0 \cdots x_d$ between their fibers.

Suppose that $d \geq 3$. If a vertex $b \in B$ has no neighbor in the interior of P , then $a_1x_0 \cdots x_d a_2 b a_1$ is an induced cycle of length $d + 4$. Shortestness excludes chords from a_1, a_2 to the interior, and Lemma 3.2 excludes edges from b to x_0, x_d . Such a hole is too long. Consequently every vertex of B has a neighbor in the interior of P .

Choose distinct $b_1, b_2 \in B$ and internal neighbors $x_i \in F_C(b_1)$, $x_j \in F_C(b_2)$. The neighbors are distinct by Lemma 3.2. The subpath of P gives

$$\text{dist}_C(F_C(b_1), F_C(b_2)) \leq |i - j| \leq d - 2 < d.$$

This contradicts the choice of a minimum over all same-part pairs. Hence $d \leq 2$. $\square$

A prescribed same-part pair: The assertion in (ii) is existential. For specified distinct $a_1, a_2 \in A \cap S_Q(C)$, the same cycle argument proves a distance-two bound if some $b \in B$ has no neighbor in the interior of a shortest path between their fibers. Without that extra condition, an edge from b to an internal vertex is a chord, and the argument does not bound the distance for that specified pair. We therefore use only the existential conclusion in (ii).

Corollary 4.3. *Under the hypotheses of Proposition 4.2, every component of $G - V(Q)$ attaches to two nonadjacent vertices of Q whose fibers have distance at most two in that component.*

Proof. The pair supplied by Proposition 4.2(ii) lies in one independent part of Q and is therefore nonadjacent. $\square$

More than two attachments: The corollary places no upper bound of two on $|S_Q(C)|$. A single outside vertex has at most one neighbor in Q , but different vertices of a component can attach to different block vertices. For example, in the verified graph H_{11} of Section 6, deleting any of its blocks leaves one component attached to all four vertices of that block.

Where the hole-length bound enters: The proofs use the exact lengths $d + 3$ and $d + 4$ of the induced cycles formed from an outside path and a return path in the block. Without a bound on hole length these comparisons give no numerical distance bound. The block-closure and intersection results of Section 3, and the attachment coverage in Proposition 4.1, do not require any restriction on hole length. The restriction is needed for the cross-part bound: Appendix A gives a triangle-free, induced- $K_{3,3}^-$ -free, vertex-4-critical graph with a cross-part fiber distance of four.

5. Induced odd cycles in vertex-critical triangle-free graphs

The next result uses only triangle-freeness and vertex-criticality. It follows from the color-pair Kempe-chain argument underlying Stong's odd-cycle criterion [10]; see also Moore and West [11]. We make explicit that the cycles can be chosen induced through a prescribed vertex. Its role here is that every resulting odd hole in C_6 has length five.

Theorem 5.1 (induced odd holes through a prescribed vertex). *Let G be a triangle-free vertex- k -critical graph. Every vertex of G lies in at least*

$$\binom{k-1}{2}$$

distinct induced odd holes.

Proof. Fix $v \in V(G)$ and a proper $(k-1)$ -coloring φ of $G - v$. Every color occurs in $N(v)$, since otherwise the coloring extends to v .

Fix two colors i and j . Consider the components of the subgraph induced by these two color classes. Some such component contains both an i -colored neighbor and a j -colored neighbor of v . Indeed, if no component did, interchanging i and j on every component containing an i -colored neighbor of v would remove color i from $N(v)$, and then v could receive color i .

Choose an i -colored neighbor x and a j -colored neighbor y of v at minimum distance in such a component, and let P be a shortest x - y path. The path P is induced and alternates the two colors. Minimality ensures that v has no neighbor in the interior of P . Therefore, the cycle obtained from P by

adding the edges vx and yv is induced. Its length is odd, because the differently colored endpoints of P are at odd distance, and it has length at least five because G is triangle-free.

We obtain one induced odd hole for each unordered color pair $\{i, j\}$. A produced hole determines its pair of colors on $G - v$, so holes arising from distinct pairs are distinct. $\square$

Sharpness and scope: For $k = 3$, equality holds at every vertex of each odd cycle C_{2r+1} with $r \geq 2$: There is exactly one induced odd hole. For $k \geq 4$, we do not establish sharpness here. Neither the exclusion of $K_{3,3}^-$ nor an upper bound on hole length is required for Theorem 5.1; those assumptions enter only in its applications below.

Corollary 5.2 (incidence bound for induced 5-cycles). *If $G \in C_6$ is vertex- k -critical, then every vertex lies in at least $\binom{k-1}{2}$ induced 5-cycles. Consequently, if $h_5(G)$ denotes the number of induced 5-cycles, then*

$$h_5(G) \geq \frac{|V(G)|}{5} \binom{k-1}{2}.$$

In particular, every vertex of a vertex-5-critical member of C_6 lies in at least six induced 5-cycles.

Proof. Every induced odd hole in G has length five: Triangles are excluded and holes of length at least seven are excluded. Sum the local incidences from Theorem 5.1; each 5-hole is counted at exactly five vertices. $\square$

6. Classification of 4-critical graphs through order 15

We now classify the 4-critical members of C_6 of order at most 15. Their machine-readable encodings and verification data are included in the electronic supplementary material.

Theorem 6.1 (classification through order 15). *Up to isomorphism, exactly three 4-critical graphs in C_6 have at most 15 vertices. They have orders 11, 13, and 14; denote them by H_{11} , H_{13} , and H_{14} , respectively.*

Proof. There is no triangle-free 4-chromatic graph of order below 11; the unique example of order 11 is the Mycielski graph of C_5 [4]. We screened the complete House of Graphs vertex-4-critical catalogs at orders 11–15 [7]. The order-11 input is supplied there as the unique maximal triangle-free 4-chromatic graph of that order. For each input, the script checks triangle-freeness, detects induced $K_{3,3}^-$, and, if none is present, enumerates induced cycles to reject holes of length at least seven. This gives exactly the three survivors in Table 1.

For each survivor, exact backtracking verifies non-3-colorability and 3-colorability after deleting each vertex and each edge. Thus, each survivor is 4-critical under the definition of Section 2. Conversely, every 4-critical graph is vertex-4-critical and so appears in the screened input. This proves the stated classification. Isomorphism rejection is inherited from the complete catalogs. The input files, filtering script, and three labeled graph6 records are included in the accompanying verification package. $\square$

Table 1. Sequential screening of the small triangle-free input catalogs. Columns three, four, and five are disjoint outcomes and sum to the input count in each row.

Order	Input	Contains induced $K_{3,3}^-$	Induced- $K_{3,3}^-$ -free, has a hole of length ≥ 7	In C_6
11	1	0	0	1
12	4	2	2	0
13	31	16	14	1
14	1,080	762	317	1
15	49,015	38,491	10,524	0

Table 2 gives further invariants of the three graphs.

Here $n = |V(G)|$, $m = |E(G)|$, and h_i counts induced cycles of length i . In the degree multiset, d^r means that exactly r vertices have degree d ; it is not a power of a degree value. The quantity ℓ_{ip} is the maximum number of vertices in an induced path.

Table 2. Verified invariants of the three small critical graphs.

Graph	n	m	h_4	h_5	h_6	Degree multiset	ℓ_{ip}
H_{11}	11	20	10	31	5	$\{3^5, 4^5, 5^1\}$	5
H_{13}	13	26	13	52	26	$\{4^{13}\}$	6
H_{14}	14	27	15	48	22	$\{3^8, 4^2, 5^2, 6^2\}$	7

One 4-coloring of each graph is

$$\begin{aligned}
 H_{11} : & \{0, 2, 4\} \mid \{1, 3, 6, 8\} \mid \{5, 7, 9\} \mid \{10\}, \\
 H_{13} : & \{0, 2, 5, 7\} \mid \{1, 6, 8, 9\} \mid \{3, 4, 10, 11\} \mid \{12\}, \\
 H_{14} : & \{0, 3, 4, 11\} \mid \{1, 5, 8, 9\} \mid \{2, 6, 7, 10, 12\} \mid \{13\}.
 \end{aligned}$$

These color classes use the vertex labels of the accompanying graph6 records. The exact deletion checks include both vertices and edges: Each graph is not 3-colorable, whereas deletion of any vertex or any edge gives a 3-colorable graph. A vertex-deletion test alone would establish only vertex-4-criticality.

The block data also show how the structural bounds behave on these examples. H_{11} and H_{13} have, respectively, ten and thirteen $K_{2,2}$ blocks. The graph H_{14} has nine $K_{2,2}$ blocks and two $K_{2,3}$ blocks. Across all 34 blocks, deletion of the block leaves one outside component whose attachment set is the whole block. For cross-part attachment fibers, all 40 measured pairs in H_{11} and all 52 in H_{13} have distance one; in H_{14} , 39 pairs have distance one and nine have distance two. Thus, every measured cross-part distance is below the upper bound of three in Proposition 4.2(i).

7. 4-colorability through order 24

The hole-length bound is not needed for this finite result. Care is required in using critical catalogs: Induced- $K_{3,3}^-$ -freeness is preserved by vertex deletion, but not by arbitrary edge deletion. For example, $K_{3,3}$ is induced- $K_{3,3}^-$ -free, whereas deleting one edge creates $K_{3,3}^-$. We use the subgraph-critical catalogs together with the following completion argument.

7.1. Forced completion and obstruction certificates

Forced-completion lemma. Let H be triangle-free and let $H[S]$ be an induced $K_{3,3}^-$. Let uv be its missing cross-edge. Every triangle-free, induced- $K_{3,3}^-$ -free supergraph J of H must contain uv . If u and v already have a common neighbor in H , no such supergraph exists.

Proof. Every pair of vertices in the same part of $H[S]$ has a common neighbor in the opposite part. Adding an edge within a part therefore creates a triangle. All eight cross-edges of $H[S]$ remain in J . The only way for $J[S]$ to cease being an induced $K_{3,3}^-$ is to add uv . If u, v already have a common neighbor in H , this forced edge creates a triangle in J . $\square$

Starting from a catalog graph H , search for an induced $K_{3,3}^-$. If its missing-edge endpoints have a common neighbor, the lemma certifies that H has no permitted supergraph. Otherwise add the missing cross-edge and repeat. By induction, every added edge must occur in every permitted supergraph of the original H . There are at most $\binom{|V(H)|}{2} - |E(H)|$ edge additions. If the search reaches a graph with no induced $K_{3,3}^-$ and no triangle, that record has not been excluded by this procedure; it must be reported as a survivor. No such survivor occurs in the verified inputs below.

Each obstruction certificate records the input index, the six vertices of each successive induced $K_{3,3}^-$, and a common neighbor of the final missing-edge endpoints. A checker can replay the forced edges and verify the final contradiction without repeating the search. The maximum number of added edges in our computation is eight.

7.2. Complete catalogs and independent verification

The House of Graphs collection distinguishes vertex-critical graphs from graphs critical under all proper subgraphs [7]. The latter catalogs contain 21, 4,192, and 625,812 triangle-free 5-critical graphs of orders 22, 23, and 24, respectively, and are complete up to isomorphism [4]. These are the inputs used here. They are not the larger vertex-critical catalogs. Jensen and Royle's minimum-order result excludes triangle-free 5-chromatic graphs of order less than 22 [8].

The primary program uses a graph6 decoder and bit-mask adjacency. It finds a candidate by choosing two vertices with three common neighbors and a third vertex adjacent to exactly two of these neighbors. It then checks all 15 pairs on the six selected vertices, verifying that exactly the eight required cross-edges are present.

Theorem 7.1 (4-colorability through order 24). *Every triangle-free, induced- $K_{3,3}^-$ -free graph on at most 24 vertices is 4-colorable.*

Proof. Suppose that G satisfies the hypotheses and $\chi(G) \geq 5$. Choose a 5-critical subgraph H of G . It is triangle-free; we do not assume that it is induced- $K_{3,3}^-$ -free. The minimum-order theorem of Jensen and Royle [8] gives $|V(H)| \geq 22$, while $|V(H)| \leq |V(G)| \leq 24$.

Thus, H has order 22, 23, or 24, and completeness of the relevant catalog identifies it with one of the screened records. Table 3 and its checked certificates show that H has no triangle-free, induced- $K_{3,3}^-$ -free supergraph. This contradicts $H \subseteq G$. $\square$

Table 3. Forced-completion screens of the complete triangle-free 5-critical catalogs. Every input has a certificate excluding all triangle-free, induced- $K_{3,3}^-$ -free supergraphs. The independent checker verifies the same counts.

Order	Input graphs	Certified obstructions	Survivors	Maximum added edges
22	21	21	0	0
23	4,192	4,192	0	3
24	625,812	625,812	0	8

Corollary 7.2. *Conjecture 1.1 holds for graphs on at most 24 vertices. Every counterexample to the conjecture therefore has at least 25 vertices.*

Theorem 7.1 is stronger than this corollary because it allows induced cycles of arbitrary length. Its proof depends on both the completeness of the input catalogs and the certificate checks; testing an arbitrary sample would not establish the theorem. The minimum-order result is precisely why checking orders 22, 23, and 24 covers every possible critical obstruction through order 24.

8. Constraints on a minimal counterexample

Suppose that G is a counterexample to Conjecture 1.1 with the fewest vertices. Since C_6 is hereditary, every graph $G - v$ is 4-colorable. It follows that $\chi(G) = 5$ and G is vertex-5-critical. This argument does not justify assuming edge criticality, because deleting edges need not preserve the forbidden induced subgraph conditions. By Corollary 7.2, $|V(G)| \geq 25$. The preliminary criticality facts give minimum degree at least four and no clique cutset, while Corollary 5.2 gives at least six induced 5-holes through each vertex.

The proved block constraints are as follows. Every 4-hole belongs to a unique block; distinct blocks intersect in at most one vertex or one edge; and an outside vertex has at most one neighbor in a given block. Every vertex of a block has a neighbor outside it. For each outside component, Corollary 4.3 supplies two same-part attachment vertices whose fibers have distance at most two. Fibers attached to opposite parts have distance at most three. For a specified same-part pair, the discussion following Proposition 4.2 gives a distance-two conclusion when an opposite-part vertex misses the interior of a shortest path; the conclusion is not asserted for every specified pair.

Three obstacles remain before these facts can yield a proof of the conjecture. First, the pairwise intersection theorem does not control the global arrangement of the blocks: Their incidence graph may contain cycles, and the present results do not bound the sizes of attachment interfaces. Second, attachments from the opposite part can create chords in a cycle used to analyze a same-part path. The existential short-pair statement above does not control every pair of fibers. Third, the lower bound on 5-hole incidences gives no corresponding upper bound on their overlaps. An induced 5-hole through a block vertex need not use an edge of that block, so a direct count cannot assign the required incidences to disjoint local configurations.

A possible next step is a reduction lemma for block attachments. Such a lemma would have to specify which local configurations can be reduced, prove that the reduced graph remains in C_6 , and show that every 4-coloring of the reduced graph extends to the original one. None of these reduction claims is proved here. The verified small graphs offer test cases for proposed reductions, while the

finite verification through order 24 excludes smaller counterexamples.

9. Conclusions

The induced- $K_{3,3}^-$ restriction gives a precise local description of the maximal bicliques generated by 4-holes: They are uniquely determined, their pairwise intersections are small, and each outside vertex has at most one neighbor in a given block. Within C_6 , the hole-length bound also limits cross-part attachment distances and guarantees a close same-part attachment pair in each outside component of a vertex-critical graph. The Kempe-chain incidence bound requires at least six induced 5-holes through each vertex of a smallest counterexample.

The computation classifies the three small critical members through order 15 and, by checked forced-completion certificates, establishes 4-colorability through order 24 without a hole-length restriction. Conjecture 1.1 remains open for larger graphs. A proof still requires control of the interactions between blocks, their attachment fibers, and overlapping 5-holes. A coloring-extension reduction that preserves membership in C_6 would address this gap.

Author contributions

Caibing Chang: Conceptualization, development of the theoretical framework, mathematical analysis, and preparation of the initial manuscript; Jinsen Xiao: Methodology, supervision, critical assessment of the proofs, and manuscript revision; Mingshan Li: Investigation of structural properties, development of mathematical proofs, and validation of theoretical results; Haowen Zeng: Computational verification, and organization of graph data; Han Lin: Verification of computational results, preparation of tables and figures, and manuscript editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors used GPT-6.1 to assist with checking proofs and improving the exposition. All mathematical arguments, results, and conclusions were reviewed and verified by the authors, who take full responsibility for the content of this paper.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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A. An example without the hole-length restriction

Take two copies of the 4-critical Mycielski graph $M(C_5)$, with vertices v_i, u_i, z and v'_i, u'_i, z' for $i \in \mathbb{Z}/5\mathbb{Z}$. In each copy the edges are $v_i v_{i+1}$, $u_i v_{i-1}$, $u_i v_{i+1}$, and $z u_i$. Form their Hajós join by identifying v_0 with v'_0 , deleting $v_0 v_1$ and $v'_0 u'_4$, and adding $v_1 u'_4$. The resulting graph is vertex-4-critical and triangle-free. Every pair of vertices has at most two common neighbors, so it is induced- $K_{3,3}^-$ -free.

The vertices $\{z', v'_3\} \cup \{u'_2, u'_4\}$ induce a block Q . In the connected graph outside Q , the fibers of the adjacent block vertices z' and u'_4 are $\{u'_0, u'_1, u'_3\}$ and $\{v_1\}$, respectively, and their distance is four. Indeed,

outside Q the two copies meet only at $x = v_0 = v'_0$; the distance from v_1 to x is three, and x is adjacent to u'_1 . The resulting cycle $z'u'_1xv_4u_0v_1u'_4z'$ is an induced 7-cycle. This shows that the cross-part bound fails when the restriction on hole length is removed.

Verification. Each copy of $M(C_5)$ is 4-critical. Any 3-coloring of one copy with its selected edge removed assigns the same color to the endpoints of that edge, since otherwise it would color the original graph. Consequently, a 3-coloring of the join would give x , v_1 , and u'_4 the same color, contradicting the edge $v_1u'_4$. For a vertex deletion in one copy other than x , combine a 3-coloring of that vertex-deleted copy with a 3-coloring of the other copy with its selected edge deleted. Permute colors to agree at x . If the endpoint of the new joining edge remains, its color differs from that of x on the vertex-deleted side, so the joining edge is properly colored. When x is deleted, color the two remaining pieces and permute colors to give the endpoints of the joining edge different colors. This proves vertex-4-criticality.

The new joining edge creates no triangle: Its endpoints have no common neighbor after the two selected edges are removed. Within either original copy, no pair acquires an additional common neighbor, because the only new neighbor in the other copy is the opposite endpoint of the joining edge, which is not adjacent to x . A pair consisting of one exclusive vertex from each copy has at most one common neighbor. As every pair in $M(C_5)$ has at most two common neighbors, the same bound holds in the join, excluding an induced $K_{3,3}^-$.



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