



*Research article***Invariant analysis, power series solutions, and conservation laws of the space–time fractional Gerdjikov–Ivanov equation****Jicheng Yu***

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Abstract: In this paper, the Lie symmetry analysis method is extended to investigate the space–time fractional Gerdjikov–Ivanov equation, a generalized derivative nonlinear Schrödinger equation that arises in various branches of nonlinear physics, including nonlinear optics, fluid mechanics, and plasma physics. By applying the Lie symmetry approach, we derive the corresponding symmetry groups for both the space–time fractional equation and its special cases—the time-fractional and the integer-order Gerdjikov–Ivanov equations. The obtained Lie symmetries are then utilized to reduce the fractional partial differential equations with Riemann–Liouville fractional derivative to systems of fractional ordinary differential equations involving the Erdélyi–Kober fractional derivative. Subsequently, the power series method is employed to construct explicit power series solutions for the reduced systems, and the convergence of these solutions is rigorously analyzed with numerical simulations provided to support the analytical results. Furthermore, the conservation laws for the studied equations are systematically established by developing a generalized version of Noether’s operators and applying the new conservation theorem. The results presented in this work not only extend the existing Lie symmetry analysis of derivative nonlinear Schrödinger-type equations but also provide a comprehensive analytical framework for studying fractional nonlinear evolution equations.

Keywords: Lie symmetry analysis; space–time fractional Gerdjikov–Ivanov equation; Riemann–Liouville fractional derivative; Erdélyi–Kober fractional derivative; conservation laws

Mathematics Subject Classification: 34K37, 35G50, 37C79, 70H33, 76M60

1. Introduction

The Gerdjikov–Ivanov equation is one of nonlinear Schrödinger equations (NLSEs), also known as derivative nonlinear Schrödinger equation type III (DNLSIII), in which the derivative items are nonlinear. As we all know, the NLSEs are widely used in the fields of nonlinear optics,

nonlinear quantum field theory, fluid mechanics, solid physics, nuclear physics, plasma physics, thermodynamics, chemistry, and so on. Recently, various generalizations of the Gerdjikov–Ivanov equation have attracted considerable attention. For instance, Yan et al. [1] investigated quasiperiodic type optical solitons in the fully nonlinear complex structured perturbed Gerdjikov–Ivanov equation, and Liu and Zeng [2] examined the discontinuous initial value problem and Whitham modulation for the generalized Gerdjikov–Ivanov equation. These studies demonstrate the ongoing interest in this equation and motivate further investigation into its fractional extensions.

As a generalization of the classical calculus, fractional calculus can be traced back to the letter written by l'Hôpital to Leibniz in 1695. Since then, it has gradually gained the attention of mathematicians. Especially in recent decades, it has developed rapidly and been successfully applied in many fields of science and technology [3–6]. Among the various approaches to fractional differential equations, both analytical and numerical methods have attracted considerable interest. On the analytical side, various techniques have been developed to construct exact solutions of fractional equations, including truncated M-fractional and conformable derivative approaches [7, 8]; on the numerical side, effective schemes such as Chebyshev collocation methods and adaptive finite element methods have been proposed for solving fractional partial differential equations (PDEs) and interface problems [9–11]. In addition, fractional models have been widely applied in fluid dynamics, heat transfer, nonlinear wave propagation, and other complex dynamical systems [12–15]. These diverse methodologies and applications highlight the richness and challenge of studying fractional differential equations. As a generalization of the NLSEs, the space–time fractional Schrödinger equations (STFSEs) have more extensive applications. Among them, Yu and Feng in [16] studied the space–time fractional cubic Schrödinger equation (STFCSE) by using Lie symmetry analysis method.

Lie symmetry analysis method was founded by Norwegian mathematician Sophus Lie at the end of the nineteenth century and then further developed by some other mathematicians, such as Ovsiannikov [17], Olver [18], Ibragimov [19–21] and so on. As a modern method among many analytic techniques, Lie symmetry analysis has been extended to fractional differential equations (FDEs) by Gazizov et al. [22] in 2007. It was then effectively applied to various models of the FDEs occurring in different areas of applied science (see [23–25]).

In this paper, Lie symmetry analysis method is extended to the following space–time fractional Gerdjikov–Ivanov equation (STFGIE):

$$iD_t^\alpha z + D_x^{2\beta} z - iz^2 \overline{D_x^\beta z} + \frac{1}{2} |z|^4 z = 0, \quad (1.1)$$

where $z(t, x) = u(t, x) + iv(t, x)$, $0 < \alpha, \beta \leq 1$, and $\overline{D_x^\beta z}$ is a complex conjugate of $D_x^\beta z$.

When $\beta = 1$, Eq (1.1) becomes time fractional Gerdjikov–Ivanov equation as follows:

$$iD_t^\alpha z + z_{xx} - iz^2 \overline{z_x} + \frac{1}{2} |z|^4 z = 0. \quad (1.2)$$

When $\alpha = \beta = 1$, Eq (1.1) becomes an integer-order Gerdjikov–Ivanov equation as follows:

$$iz_t + z_{xx} - iz^2 \overline{z_x} + \frac{1}{2} |z|^4 z = 0, \quad (1.3)$$

which is studied by Vinita and Ray in [26] with the Lie symmetry analysis method.

Although the classical integer-order Gerdjikov–Ivanov equation has been widely applied to describe nonlinear wave propagation in various physical media, it is fundamentally based on the assumption of local, homogeneous, and ideal conditions. In reality, many physical systems exhibit anomalous diffusion, long-range interactions, and memory effects, which cannot be adequately captured by integer-order derivatives. In such systems, fractional calculus provides a more accurate description of the underlying dynamics. The introduction of time-fractional derivatives accounts for memory effects and nonlocal temporal responses, and space-fractional derivatives describe long-range spatial interactions and anomalous diffusion. The space–time fractional Gerdjikov–Ivanov equation considered in this paper therefore offers a more realistic model for wave propagation in complex media, such as inhomogeneous optical fibers, turbulent plasmas, and non-Newtonian fluids, where both temporal memory and spatial nonlocality play significant roles.

The main work of this paper is the extension of the classical integer-order Gerdjikov–Ivanov equation to the space–time fractional setting, building upon the foundation established in [26]. By adopting and adapting the methodological framework developed in our previous work [16] for the space–time fractional cubic Schrödinger equation, this paper investigates the more complex fractional Gerdjikov–Ivanov equation with derivative-type nonlinearities. Specifically, this paper presents all Lie symmetries of Eqs (1.1)–(1.3) and derives the reduced equations, power series solutions, and conservation laws for Eqs (1.1) and (1.2). In addition, for some power series solutions, the convergence analysis and numerical simulation are given in this paper.

It should be noted that this paper adopts the Riemann–Liouville fractional derivative defined by [3–6],

$${}_a D_t^\alpha f(t, x) = D_t^n {}_a I_t^{n-\alpha} f(t, x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s, x)}{(t-s)^{\alpha-n+1}} ds, & n-1 < \alpha < n, n \in \mathbb{N}, \\ D_t^n f(t, x), & \alpha = n \in \mathbb{N} \end{cases}$$

for $t > a$. We denote the operator ${}_0 D_t^\alpha$ as D_t^α throughout this paper.

This paper is organized as follows. In Sections 2 and 3, Lie symmetry analysis and power series solutions of Eqs (1.1) and (1.2) are studied, respectively. In Section 4, a Lie symmetry analysis of Eq (1.3) is presented. The conservation laws of Eqs (1.1) and (1.2) are obtained in Section 5 and the conclusion is given in the last section.

2. Lie symmetry analysis and power series solutions of Eq (1.1)

Consider the space–time fractional Gerdjikov–Ivanov Eq (1.1). Because of $z(t, x) = u(t, x) + iv(t, x)$, it becomes

$$\begin{cases} D_t^\alpha u + D_x^{2\beta} v - (u^2 - v^2) D_x^\beta u - 2uv D_x^\beta v + \frac{1}{2} v(u^2 + v^2)^2 = 0, \\ D_t^\alpha v - D_x^{2\beta} u + (u^2 - v^2) D_x^\beta v - 2uv D_x^\beta u - \frac{1}{2} u(u^2 + v^2)^2 = 0. \end{cases} \quad (2.1)$$

We assume that the system (2.1) is invariant under the one-parameter (ϵ) Lie group of continuous point transformations, that is,

$$t^* = t + \epsilon \tau(t, x, u, v) + o(\epsilon),$$

$$x^* = x + \epsilon \xi(t, x, u, v) + o(\epsilon),$$

$$u^* = u + \epsilon \eta(t, x, u, v) + o(\epsilon),$$

$$v^* = v + \epsilon \zeta(t, x, u, v) + o(\epsilon),$$

$$\begin{aligned}
D_t^\alpha u^* &= D_t^\alpha u + \epsilon \eta^{\alpha,t} + o(\epsilon), \\
D_t^\alpha v^* &= D_t^\alpha v + \epsilon \zeta^{\alpha,t} + o(\epsilon), \\
D_x^\beta u^* &= D_x^\beta u + \epsilon \eta^{\beta,x} + o(\epsilon), \\
D_x^\beta v^* &= D_x^\beta v + \epsilon \zeta^{\beta,x} + o(\epsilon), \\
D_x^{2\beta} u^* &= D_x^{2\beta} u + \epsilon \eta^{2\beta,x} + o(\epsilon), \\
D_x^{2\beta} v^* &= D_x^{2\beta} v + \epsilon \zeta^{2\beta,x} + o(\epsilon),
\end{aligned} \tag{2.2}$$

where τ , ξ , η , and ζ are infinitesimals, and $\eta^{\alpha,t}$, $\zeta^{\alpha,t}$, $\eta^{\beta,x}$, $\zeta^{\beta,x}$, $\eta^{2\beta,x}$, and $\zeta^{2\beta,x}$ are the corresponding prolongations of orders α , β , and 2β , respectively.

The corresponding group generator is defined by

$$X = \tau(t, x, u, v) \frac{\partial}{\partial t} + \xi(t, x, u, v) \frac{\partial}{\partial x} + \eta(t, x, u, v) \frac{\partial}{\partial u} + \zeta(t, x, u, v) \frac{\partial}{\partial v}. \tag{2.3}$$

Therefore, the prolongation of the above group generator X has the form

$$Pr^{(\alpha, 2\beta)} X = X + \eta^{\alpha,t} \frac{\partial}{\partial u_t^\alpha} + \zeta^{\alpha,t} \frac{\partial}{\partial v_t^\alpha} + \eta^{\beta,x} \frac{\partial}{\partial u_x^\beta} + \zeta^{\beta,x} \frac{\partial}{\partial v_x^\beta} + \eta^{2\beta,x} \frac{\partial}{\partial u_x^{2\beta}} + \zeta^{2\beta,x} \frac{\partial}{\partial v_x^{2\beta}}, \tag{2.4}$$

where

$$\begin{aligned}
\eta^{\alpha,t} &= D_t^\alpha(\eta) + \tau D_t^\alpha(u_t) - D_t^\alpha(\tau u_t) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) \\
&= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + (\eta_v \frac{\partial^\alpha v}{\partial t^\alpha} - v \frac{\partial^\alpha \eta_v}{\partial t^\alpha}) \\
&\quad + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) + \mu_1 + \mu_2 \\
&\quad + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \eta_v}{\partial t^n} D_t^{\alpha-n}(v) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x),
\end{aligned} \tag{2.5}$$

with

$$\begin{aligned}
\mu_1 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{t^{n-\alpha} (-u)^r}{k! \Gamma(n+1-\alpha)} \frac{\partial^m u^{k-r}}{\partial t^m} \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial u^k}, \\
\mu_2 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{t^{n-\alpha} (-v)^r}{k! \Gamma(n+1-\alpha)} \frac{\partial^m v^{k-r}}{\partial t^m} \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial v^k}, \\
\zeta^{\alpha,t} &= \frac{\partial^\alpha \zeta}{\partial t^\alpha} + (\zeta_v - \alpha D_t(\tau)) \frac{\partial^\alpha v}{\partial t^\alpha} - v \frac{\partial^\alpha \zeta_v}{\partial t^\alpha} + (\zeta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \zeta_u}{\partial t^\alpha}) \\
&\quad + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n \zeta_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(v) + \mu_3 + \mu_4 \\
&\quad + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^n \zeta_u}{\partial t^n} D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(v_x),
\end{aligned} \tag{2.6}$$

with

$$\begin{aligned}\mu_3 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{t^{n-\alpha}(-u)^r}{k! \Gamma(n+1-\alpha)} \frac{\partial^m u^{k-r}}{\partial t^m} \frac{\partial^{n-m+k} \zeta}{\partial t^{n-m} \partial u^k}, \\ \mu_4 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{t^{n-\alpha}(-v)^r}{k! \Gamma(n+1-\alpha)} \frac{\partial^m v^{k-r}}{\partial t^m} \frac{\partial^{n-m+k} \zeta}{\partial t^{n-m} \partial v^k}, \\ \eta^{i\beta,x} &= D_x^{i\beta}(\eta) + \tau D_x^{i\beta}(u_t) - D_x^{i\beta}(\tau u_t) + \xi D_x^{i\beta}(u_x) - D_x^{i\beta}(\xi u_x) \\ &= \frac{\partial^{i\beta} \eta}{\partial x^{i\beta}} + (\eta_u - i\beta D_x(\xi)) \frac{\partial^{i\beta} u}{\partial x^{i\beta}} - u \frac{\partial^{i\beta} \eta_u}{\partial x^{i\beta}} + (\eta_v \frac{\partial^{i\beta} v}{\partial x^{i\beta}} - v \frac{\partial^{i\beta} \eta_v}{\partial x^{i\beta}}) \\ &\quad + \sum_{n=1}^{\infty} \left[\binom{i\beta}{n} \frac{\partial^n \eta_u}{\partial x^n} - \binom{i\beta}{n+1} D_x^{n+1}(\xi) \right] D_x^{i\beta-n}(u) + \mu_5 + \mu_6 \\ &\quad + \sum_{n=1}^{\infty} \binom{i\beta}{n} \frac{\partial^n \eta_v}{\partial x^n} D_x^{i\beta-n}(v) - \sum_{n=1}^{\infty} \binom{i\beta}{n} D_x^n(\tau) D_x^{i\beta-n}(u_t), \quad (i = 1, 2)\end{aligned}\tag{2.7}$$

with

$$\begin{aligned}\mu_5 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{i\beta}{n} \binom{n}{m} \binom{k}{r} \frac{x^{n-i\beta}(-u)^r}{k! \Gamma(n+1-i\beta)} \frac{\partial^m u^{k-r}}{\partial x^m} \frac{\partial^{n-m+k} \eta}{\partial x^{n-m} \partial u^k}, \\ \mu_6 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{i\beta}{n} \binom{n}{m} \binom{k}{r} \frac{x^{n-i\beta}(-v)^r}{k! \Gamma(n+1-i\beta)} \frac{\partial^m v^{k-r}}{\partial x^m} \frac{\partial^{n-m+k} \eta}{\partial x^{n-m} \partial v^k},\end{aligned}$$

and

$$\begin{aligned}\zeta^{i\beta,x} &= \frac{\partial^{i\beta} \zeta}{\partial x^{i\beta}} + (\zeta_v - i\beta D_x(\xi)) \frac{\partial^{i\beta} v}{\partial x^{i\beta}} - v \frac{\partial^{i\beta} \zeta_v}{\partial x^{i\beta}} + (\zeta_u \frac{\partial^{i\beta} u}{\partial x^{i\beta}} - u \frac{\partial^{i\beta} \zeta_u}{\partial x^{i\beta}}) \\ &\quad + \sum_{n=1}^{\infty} \left[\binom{i\beta}{n} \frac{\partial^n \zeta_v}{\partial x^n} - \binom{i\beta}{n+1} D_x^{n+1}(\xi) \right] D_x^{i\beta-n}(v) + \mu_7 + \mu_8 \\ &\quad + \sum_{n=1}^{\infty} \binom{i\beta}{n} \frac{\partial^n \zeta_u}{\partial x^n} D_x^{i\beta-n}(u) - \sum_{n=1}^{\infty} \binom{i\beta}{n} D_x^n(\tau) D_x^{i\beta-n}(v_t), \quad (i = 1, 2)\end{aligned}\tag{2.8}$$

with

$$\begin{aligned}\mu_7 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{i\beta}{n} \binom{n}{m} \binom{k}{r} \frac{x^{n-i\beta}(-u)^r}{k! \Gamma(n+1-i\beta)} \frac{\partial^m u^{k-r}}{\partial x^m} \frac{\partial^{n-m+k} \zeta}{\partial x^{n-m} \partial u^k}, \\ \mu_8 &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{i\beta}{n} \binom{n}{m} \binom{k}{r} \frac{x^{n-i\beta}(-v)^r}{k! \Gamma(n+1-i\beta)} \frac{\partial^m v^{k-r}}{\partial x^m} \frac{\partial^{n-m+k} \zeta}{\partial x^{n-m} \partial v^k}.\end{aligned}$$

When $\alpha = 1$, the prolongation operators (2.5) and (2.6) coincide with the one-order formulas as follows:

$$\begin{aligned}\eta^t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) \\ &= \eta_t + (\eta_u - \tau_t) u_t + \eta_v v_t - \tau_u u_t^2 - \tau_v v_t u_t - \xi_t u_x - \xi_u u_t u_x - \xi_v v_t u_x,\end{aligned}\tag{2.9}$$

$$\begin{aligned}\zeta^t &= D_t(\zeta) - v_t D_t(\tau) - v_x D_t(\xi) \\ &= \zeta_t + (\zeta_v - \tau_t)v_t + \zeta_u u_t - \tau_u u_t v_t - \tau_v v_t^2 - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x.\end{aligned}\quad (2.10)$$

When $\beta = 1$, the prolongation operators (2.7) and (2.8) coincide with the one- and two-order formulas as follows:

$$\begin{aligned}\eta^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) \\ &= \eta_x + (\eta_u - \xi_x)u_x + \eta_v v_x - \xi_u u_x^2 - \xi_v v_x u_x - \tau_x u_t - \tau_u u_t u_x - \tau_v v_x u_t,\end{aligned}\quad (2.11)$$

$$\begin{aligned}\zeta^x &= D_x(\zeta) - v_t D_x(\tau) - v_x D_x(\xi) \\ &= \zeta_x + (\zeta_v - \xi_x)v_x + \zeta_u u_x - \xi_u u_x v_x - \xi_v v_x^2 - \tau_x v_t - \tau_u u_x v_t - \tau_v v_t v_x,\end{aligned}\quad (2.12)$$

and

$$\begin{aligned}\eta^{xx} &= D_x^2(\eta) - u_t D_x^2(\tau) - 2u_{xt} D_x(\tau) - u_x D_x^2(\xi) - 2u_{xx} D_x(\xi) \\ &= \eta_{xx} + (2\eta_{ux} - \xi_{xx})u_x + 2\eta_{vx} v_x + (\eta_{uu} - 2\xi_{ux})u_x^2 + \eta_{vv} v_x^2 + 2(\eta_{uv} - \xi_{vx})u_x v_x \\ &\quad + (\eta_u - 2\xi_x)u_{xx} + \eta_v v_{xx} - \xi_v v_{xx} u_x - 2\xi_v u_{xx} v_x - 3\xi_u u_{xx} u_x - \xi_{vv} u_x v_x^2 \\ &\quad - \xi_{uu} u_x^3 - 2\xi_{uv} u_x^2 v_x - 2\tau_x u_{tx} - 2\tau_u u_x u_{tx} - 2\tau_v v_x u_{tx} - \tau_u u_{xx} u_t - \tau_v v_{xx} u_t \\ &\quad - \tau_{xx} u_t - 2\tau_{ux} u_x u_t - 2\tau_{vx} v_x u_t - \tau_{uu} u_x^2 u_t - \tau_{vv} v_x^2 u_t - 2\tau_{uv} u_x v_x u_t,\end{aligned}\quad (2.13)$$

$$\begin{aligned}\zeta^{xx} &= D_x^2(\zeta) - v_t D_x^2(\tau) - 2v_{xt} D_x(\tau) - v_x D_x^2(\xi) - 2v_{xx} D_x(\xi) \\ &= \zeta_{xx} + 2\zeta_{ux} u_x + (2\zeta_{vx} - \xi_{xx})v_x + \zeta_{uu} u_x^2 + (\zeta_{vv} - 2\xi_{vx})v_x^2 + 2(\zeta_{uv} - \xi_{ux})u_x v_x \\ &\quad + \zeta_u u_{xx} + (\zeta_v - 2\xi_x)v_{xx} - \xi_u u_{xx} v_x - 2\xi_u u_x v_{xx} - 3\xi_v v_{xx} v_x - \xi_{uu} u_x^2 v_x \\ &\quad - \xi_{vv} v_x^3 - 2\xi_{uv} u_x v_x^2 - 2\tau_x v_{tx} - 2\tau_u u_x v_{tx} - 2\tau_v v_x v_{tx} - \tau_u u_{xx} v_t - \tau_v v_{xx} v_t \\ &\quad - \tau_{xx} v_t - 2\tau_{ux} u_x v_t - 2\tau_{vx} v_x v_t - \tau_{uu} u_x^2 v_t - \tau_{vv} v_x^2 v_t - 2\tau_{uv} u_x v_x v_t.\end{aligned}\quad (2.14)$$

Remark 1. The infinitesimal transformations (2.2) should conserve the structure of the Riemann–Liouville fractional derivative operator, of which the lower limit in the integral is fixed. Therefore, the manifold $t = 0$, $x = 0$ should be invariant with respect to such transformations. The invariance condition arrives at

$$\tau(t, x, u, v)|_{t=0} = \xi(t, x, u, v)|_{x=0} = 0. \quad (2.15)$$

Remark 2. From the expressions of $\mu_1, \mu_2, \dots, \mu_8$, if the infinitesimals η, ζ are linear with respect to the variables u and v , then $\mu_1 = \mu_2 = \dots = \mu_8 = 0$; that is,

$$\frac{\partial^2 \eta}{\partial u^2} = \frac{\partial^2 \eta}{\partial v^2} = \frac{\partial^2 \zeta}{\partial u^2} = \frac{\partial^2 \zeta}{\partial v^2} = 0. \quad (2.16)$$

The one-parameter Lie symmetry transformations (2.2) are admitted by the system (2.1); if the following invariance criterion holds:

$$\begin{cases} Pr^{(\alpha, 2\beta)} X(D_t^\alpha u + D_x^{2\beta} v - (u^2 - v^2)D_x^\beta u - 2uvD_x^\beta v + \frac{1}{2}v(u^2 + v^2)^2)|_{(2.1)} = 0, \\ Pr^{(\alpha, 2\beta)} X(D_t^\alpha v - D_x^{2\beta} u + (u^2 - v^2)D_x^\beta v - 2uvD_x^\beta u - \frac{1}{2}u(u^2 + v^2)^2)|_{(2.1)} = 0, \end{cases} \quad (2.17)$$

which can be rewritten as

$$\begin{cases} (\eta^{\alpha, t} + \zeta^{2\beta, x} - (u^2 - v^2)\eta^{\beta, x} - 2uv\zeta^{\beta, x} - 2(uD_x^\beta u + vD_x^\beta v - u^3v - uv^3)\eta \\ \quad + (2vD_x^\beta u - 2uD_x^\beta v + \frac{1}{2}(u^4 + 6u^2v^2 + 5v^4))\zeta)|_{(2.1)} = 0, \\ (\zeta^{\alpha, t} - \eta^{2\beta, x} + (u^2 - v^2)\zeta^{\beta, x} - 2uv\eta^{\beta, x} - 2(uD_x^\beta u + vD_x^\beta v + u^3v + uv^3)\zeta \\ \quad + (2uD_x^\beta v - 2vD_x^\beta u - \frac{1}{2}(v^4 + 6u^2v^2 + 5u^4))\eta)|_{(2.1)} = 0. \end{cases} \quad (2.18)$$

Putting $\eta^{\alpha,t}, \zeta^{\alpha,t}, \eta^{\beta,x}, \zeta^{\beta,x}, \eta^{2\beta,x}, \zeta^{2\beta,x}$ into (2.18) and letting coefficients of various derivatives of u and v to be zero, we can obtain the over-determined system of differential equations as follows:

$$\tau_x = \tau_u = \tau_v = \xi_t = \xi_u = \xi_v = 0, \quad (2.19)$$

$$\eta_{vx} = \eta_{vt} = \zeta_{ux} = \zeta_{ut} = 0, \quad (2.20)$$

$$\zeta_u + \eta_v = 0, \quad (2.21)$$

$$\zeta_v - \eta_u + \alpha\tau_t - 2\beta\xi_x = 0, \quad (2.22)$$

$$\zeta_v - \eta_u - \alpha\tau_t + 2\beta\xi_x = 0, \quad (2.23)$$

$$\binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n \in N, \quad (2.24)$$

$$\binom{\alpha}{n} \frac{\partial^n \zeta_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n \in N, \quad (2.25)$$

$$\binom{i\beta}{n} \frac{\partial^n \eta_u}{\partial x^n} - \binom{i\beta}{n+1} D_x^{n+1}(\xi) = 0, \quad n \in N, \quad i = 1, 2, \quad (2.26)$$

$$\binom{i\beta}{n} \frac{\partial^n \zeta_v}{\partial x^n} - \binom{i\beta}{n+1} D_x^{n+1}(\xi) = 0, \quad n \in N, \quad i = 1, 2, \quad (2.27)$$

$$(u^2 - v^2)(\beta\xi_x - \alpha\tau_t) + 2uv(\eta_v - \zeta_u) - 2u\eta + 2v\zeta = 0, \quad (2.28)$$

$$-(u^2 - v^2)\eta_v + 2uv(\eta_u - \zeta_v + \beta\xi_x - \alpha\tau_t) - 2v\eta - 2u\zeta = 0, \quad (2.29)$$

$$(u^2 - v^2)\zeta_u + 2uv(\zeta_v - \eta_u + \beta\xi_x - \alpha\tau_t) - 2v\eta - 2u\zeta = 0, \quad (2.30)$$

$$\begin{aligned} & \left(\frac{\partial^\alpha \eta}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - v \frac{\partial^\alpha \eta_v}{\partial t^\alpha} \right) + \left(\frac{\partial^{2\beta} \zeta}{\partial x^{2\beta}} - u \frac{\partial^{2\beta} \zeta_u}{\partial x^{2\beta}} - v \frac{\partial^{2\beta} \zeta_v}{\partial x^{2\beta}} \right) - (u^2 - v^2) \left(\frac{\partial^\beta \eta}{\partial x^\beta} - \right. \\ & \left. u \frac{\partial^\beta \eta_u}{\partial x^\beta} - v \frac{\partial^\beta \eta_v}{\partial x^\beta} \right) - 2uv \left(\frac{\partial^\beta \zeta}{\partial x^\beta} - u \frac{\partial^\beta \zeta_u}{\partial x^\beta} - v \frac{\partial^\beta \zeta_v}{\partial x^\beta} \right) - \frac{1}{2} v (u^2 + v^2)^2 (\eta_u - \alpha\tau_t) \\ & + \frac{1}{2} u (u^2 + v^2)^2 \eta_v + 2(u^3 v + uv^3) \eta + \frac{1}{2} (u^4 + 6u^2 v^2 + 5v^4) \zeta = 0. \end{aligned} \quad (2.31)$$

$$\begin{aligned} & \left(\frac{\partial^\alpha \zeta}{\partial t^\alpha} - u \frac{\partial^\alpha \zeta_u}{\partial t^\alpha} - v \frac{\partial^\alpha \zeta_v}{\partial t^\alpha} \right) - \left(\frac{\partial^{2\beta} \eta}{\partial x^{2\beta}} - u \frac{\partial^{2\beta} \eta_u}{\partial x^{2\beta}} - v \frac{\partial^{2\beta} \eta_v}{\partial x^{2\beta}} \right) + (u^2 - v^2) \left(\frac{\partial^\beta \zeta}{\partial x^\beta} - \right. \\ & \left. u \frac{\partial^\beta \zeta_u}{\partial x^\beta} - v \frac{\partial^\beta \zeta_v}{\partial x^\beta} \right) - 2uv \left(\frac{\partial^\beta \eta}{\partial x^\beta} - u \frac{\partial^\beta \eta_u}{\partial x^\beta} - v \frac{\partial^\beta \eta_v}{\partial x^\beta} \right) + \frac{1}{2} u (u^2 + v^2)^2 (\zeta_v - \alpha\tau_t) \\ & - \frac{1}{2} v (u^2 + v^2)^2 \zeta_u - 2(u^3 v + uv^3) \zeta - \frac{1}{2} (v^4 + 6u^2 v^2 + 5u^4) \eta = 0. \end{aligned} \quad (2.32)$$

Solving these equations altogether, with the conditions (2.15) and (2.16), we can obtain infinitesimals as follows:

$$\xi = c_1 x, \quad \tau = \frac{2\beta}{\alpha} c_1 t, \quad \eta = -\frac{\beta}{2} c_1 u + c_2 v, \quad \zeta = -\frac{\beta}{2} c_1 v - c_2 u, \quad (2.33)$$

where c_1 and c_2 are arbitrary constants.

Therefore, System (2.1) admitted the two-dimensional Lie algebra spanned by

$$X_1 = x \frac{\partial}{\partial x} + \frac{2\beta}{\alpha} t \frac{\partial}{\partial t} - \frac{\beta}{2} u \frac{\partial}{\partial u} - \frac{\beta}{2} v \frac{\partial}{\partial v}, \quad X_2 = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}, \quad (2.34)$$

with $[X_1, X_2] = 0$.

The characteristic equation corresponding to the group generator X_1 is

$$\frac{dx}{x} = \frac{\alpha dt}{2\beta t} = \frac{2du}{-\beta u} = \frac{2dv}{-\beta v}, \quad (2.35)$$

from which, we obtain the similarity variables $xt^{-\frac{\alpha}{2\beta}}$, $ut^{\frac{\alpha}{4}}$ and $vt^{\frac{\alpha}{4}}$. Therefore, we get the invariant solutions of the system (2.1) as follows:

$$u(t, x) = t^{-\frac{\alpha}{4}} f(\omega), \quad v(t, x) = t^{-\frac{\alpha}{4}} g(\omega), \quad (2.36)$$

with $\omega = xt^{-\frac{\alpha}{2\beta}}$.

Theorem 2.1. The similarity transformations $u(t, x) = t^{-\frac{\alpha}{4}} f(\omega)$, $v(t, x) = t^{-\frac{\alpha}{4}} g(\omega)$ with the similarity variable $\omega = xt^{-\frac{\alpha}{2\beta}}$ reduce System (2.1) to the system of fractional ordinary differential equations given by

$$\begin{cases} (\mathcal{P}_{\frac{2\beta}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} f)(\omega) = -\omega^{-2\beta} (\mathcal{D}_1^{-2\beta, 2\beta} g)(\omega) + \omega^{-\beta} (f^2(\omega) - g^2(\omega)) (\mathcal{D}_1^{-\beta, \beta} f)(\omega) \\ \quad + 2\omega^{-\beta} f(\omega) g(\omega) (\mathcal{D}_1^{-\beta, \beta} g)(\omega) - \frac{1}{2} g(\omega) (f^2(\omega) + g^2(\omega))^2, \\ (\mathcal{P}_{\frac{2\beta}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} g)(\omega) = \omega^{-2\beta} (\mathcal{D}_1^{-2\beta, 2\beta} f)(\omega) - \omega^{-\beta} (f^2(\omega) - g^2(\omega)) (\mathcal{D}_1^{-\beta, \beta} g)(\omega) \\ \quad + 2\omega^{-\beta} f(\omega) g(\omega) (\mathcal{D}_1^{-\beta, \beta} f)(\omega) + \frac{1}{2} f(\omega) (f^2(\omega) + g^2(\omega))^2, \end{cases} \quad (2.37)$$

where $(\mathcal{P}_{\delta}^{\iota, \kappa})$ is the left-hand Erdélyi–Kober fractional differential operator defined by

$$(\mathcal{P}_{\delta}^{\iota, \kappa} \psi)(\omega) := \prod_{j=0}^{m-1} \left(\iota + j - \frac{1}{\delta} \omega \frac{d}{d\omega} \right) (\mathcal{K}_{\delta}^{\iota+\kappa, m-\kappa} \psi)(\omega), \quad \omega > 0, \delta > 0, \kappa > 0, \quad (2.38)$$

$$m = \begin{cases} [\kappa] + 1, & \text{if } \kappa \notin \mathbb{N}, \\ \kappa, & \text{if } \kappa \in \mathbb{N}, \end{cases}$$

where

$$(\mathcal{K}_{\delta}^{\iota, \kappa} \psi)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)} \int_1^{\infty} (s-1)^{\kappa-1} s^{-(\iota+\kappa)} \psi(\omega s^{\frac{1}{\delta}}) ds, & \text{if } \kappa > 0, \\ \psi(\omega), & \text{if } \kappa = 0, \end{cases} \quad (2.39)$$

is the left-hand Erdélyi–Kober fractional integral operator. Furthermore, $(\mathcal{D}_{\delta}^{\iota, \kappa})$ is the right-hand Erdélyi–Kober fractional differential operator defined as follows:

$$(\mathcal{D}_{\delta}^{\iota, \kappa} \psi)(\omega) := \prod_{j=1}^m \left(\iota + j + \frac{1}{\delta} \omega \frac{d}{d\omega} \right) (\mathcal{I}_{\delta}^{\iota+\kappa, m-\kappa} \psi)(\omega), \quad \omega > 0, \delta > 0, \kappa > 0, \quad (2.40)$$

$$m = \begin{cases} [\kappa] + 1, & \text{if } \kappa \notin \mathbb{N}, \\ \kappa, & \text{if } \kappa \in \mathbb{N}, \end{cases}$$

where

$$(\mathcal{I}_{\delta}^{\iota, \kappa} \psi)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)} \int_0^1 (1-s)^{\kappa-1} s^{\iota} \psi(\omega s^{\frac{1}{\delta}}) ds, & \text{if } \kappa > 0, \\ \psi(\omega), & \text{if } \kappa = 0, \end{cases} \quad (2.41)$$

is the right-hand Erdélyi–Kober fractional integral operator.

Proof: For $0 < \alpha < 1$, the Riemann–Liouville time fractional derivative of $u(t, x)$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha} (t^{-\frac{\alpha}{4}} f(\omega)) = \frac{\partial}{\partial t} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{-\frac{\alpha}{4}} f(xs^{-\frac{\alpha}{2\beta}}) ds \right].$$

Assuming $r = t/s$, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t} \left[\frac{t^{1-\frac{5\alpha}{4}}}{\Gamma(1-\alpha)} \int_1^\infty (r-1)^{-\alpha} r^{\frac{5\alpha}{4}-2} f(\omega r^{\frac{\alpha}{2\beta}}) dr \right] = \frac{\partial}{\partial t} \left[t^{1-\frac{5\alpha}{4}} (\mathcal{K}_{\frac{2\beta}{\alpha}}^{1-\frac{\alpha}{4}, 1-\alpha} f)(\omega) \right].$$

Because of $\omega = xt^{-\frac{\alpha}{2\beta}}$, the following relation holds:

$$t \frac{\partial}{\partial t} \psi(\omega) = tx \left(-\frac{\alpha}{2\beta} \right) t^{-\frac{\alpha}{2\beta}-1} \psi'(\omega) = -\frac{\alpha}{2\beta} \omega \frac{d}{d\omega} \psi(\omega).$$

Hence, we arrive at

$$\frac{\partial}{\partial t} \left[t^{1-\frac{5\alpha}{4}} (\mathcal{K}_{\frac{2\beta}{\alpha}}^{1-\frac{\alpha}{4}, 1-\alpha} f)(\omega) \right] = t^{-\frac{5\alpha}{4}} \left[\left(1 - \frac{5\alpha}{4} - \frac{\alpha}{2\beta} \omega \frac{d}{d\omega} \right) (\mathcal{K}_{\frac{2\beta}{\alpha}}^{1-\frac{\alpha}{4}, 1-\alpha} f)(\omega) \right] = t^{-\frac{5\alpha}{4}} (\mathcal{P}_{\frac{2\beta}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} f)(\omega).$$

Similarly, the Riemann–Liouville time fractional derivative of $v(t, x)$ is

$$\frac{\partial^\alpha v}{\partial t^\alpha} = t^{-\frac{5\alpha}{4}} (\mathcal{P}_{\frac{2\beta}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} g)(\omega).$$

For $0 < \beta < 1$ and $\beta \neq 1/2$, the Riemann–Liouville space fractional derivative of $u(t, x)$ becomes

$$\frac{\partial^{2\beta} u}{\partial x^{2\beta}} = \frac{\partial^n}{\partial x^n} \left[\frac{1}{\Gamma(n-2\beta)} \int_0^x (x-s)^{n-2\beta-1} t^{-\frac{\alpha}{4}} f(st^{-\frac{\alpha}{2\beta}}) ds \right], \quad n-1 < 2\beta < n \quad (n = 1, 2).$$

Letting $r = s/x$, we get

$$\frac{\partial^{2\beta} u}{\partial x^{2\beta}} = t^{-\frac{\alpha}{4}} \frac{\partial^n}{\partial x^n} \left[\frac{x^{n-2\beta}}{\Gamma(n-2\beta)} \int_0^1 (1-r)^{n-2\beta-1} f(\omega r) dr \right] = t^{-\frac{\alpha}{4}} \frac{\partial^n}{\partial x^n} [x^{n-2\beta} (\mathcal{I}_1^{0, n-2\beta} f)(\omega)].$$

Because of $(\mathcal{I}_1^{1,0} \psi)(\omega) = \psi(\omega)$, if $2\beta = n = 1, 2$, the above equation still holds. Taking into account the relation

$$x \frac{\partial}{\partial x} \psi(\omega) = xt^{-\frac{\alpha}{2\beta}} \psi'(\omega) = \omega \frac{d}{d\omega} \psi(\omega),$$

we arrive at

$$\begin{aligned} t^{-\frac{\alpha}{4}} \frac{\partial^n}{\partial x^n} [x^{n-2\beta} (\mathcal{I}_1^{0, n-2\beta} f)(\omega)] &= t^{-\frac{\alpha}{4}} \frac{\partial^{n-1}}{\partial x^{n-1}} \left[x^{n-2\beta-1} (n-2\beta + \omega \frac{d}{d\omega}) (\mathcal{I}_1^{0, n-2\beta} f)(\omega) \right] \\ &= \dots = t^{-\frac{\alpha}{4}} x^{-2\beta} \prod_{j=1}^n \left[(-2\beta + j + \omega \frac{d}{d\omega}) (\mathcal{I}_1^{0, n-2\beta} f)(\omega) \right] \\ &= t^{-\frac{\alpha}{4}} x^{-2\beta} (\mathcal{D}_1^{-2\beta, 2\beta} f)(\omega) = t^{-\frac{5\alpha}{4}} \omega^{-2\beta} (\mathcal{D}_1^{-2\beta, 2\beta} f)(\omega). \end{aligned}$$

Similarly, we can get

$$\frac{\partial^{2\beta} v}{\partial x^{2\beta}} = t^{-\frac{5\alpha}{4}} \omega^{-2\beta} (\mathcal{D}_1^{-2\beta, 2\beta} g)(\omega), \quad \frac{\partial^\beta u}{\partial x^\beta} = t^{-\frac{3\alpha}{4}} \omega^{-\beta} (\mathcal{D}_1^{-\beta, \beta} f)(\omega), \quad \frac{\partial^\beta v}{\partial x^\beta} = t^{-\frac{3\alpha}{4}} \omega^{-\beta} (\mathcal{D}_1^{-\beta, \beta} g)(\omega).$$

Then,

$$\begin{aligned}(u^2 - v^2) \frac{\partial^\beta u}{\partial x^\beta} &= t^{\frac{-5\alpha}{4}} \omega^{-\beta} (f^2(\omega) - g^2(\omega)) (\mathcal{D}_1^{-\beta, \beta} f)(\omega), \\ (u^2 - v^2) \frac{\partial^\beta v}{\partial x^\beta} &= t^{\frac{-5\alpha}{4}} \omega^{-\beta} (f^2(\omega) - g^2(\omega)) (\mathcal{D}_1^{-\beta, \beta} g)(\omega), \\ uv \frac{\partial^\beta u}{\partial x^\beta} &= t^{\frac{-5\alpha}{4}} \omega^{-\beta} f(\omega) g(\omega) (\mathcal{D}_1^{-\beta, \beta} f)(\omega), \quad uv \frac{\partial^\beta v}{\partial x^\beta} = t^{\frac{-5\alpha}{4}} \omega^{-\beta} f(\omega) g(\omega) (\mathcal{D}_1^{-\beta, \beta} g)(\omega), \\ u(u^2 + v^2)^2 &= t^{\frac{-5\alpha}{4}} f(\omega) (f^2(\omega) + g^2(\omega))^2, \quad v(u^2 + v^2)^2 = t^{\frac{-5\alpha}{4}} g(\omega) (f^2(\omega) + g^2(\omega))^2.\end{aligned}$$

This completes the proof. $\square$

Next, we use the power series method to derive the power series solutions of the reduced Eq (2.37). Let us assume that the solutions have the following form:

$$f(\omega) = \sum_{k=0}^{\infty} a_k \omega^k, \quad g(\omega) = \sum_{k=0}^{\infty} b_k \omega^k, \quad (2.42)$$

where a_k and b_k are determined later. Then,

$$f'(\omega) = \sum_{k=0}^{\infty} (k+1) a_{k+1} \omega^k, \quad f''(\omega) = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} \omega^k. \quad (2.43)$$

$$g'(\omega) = \sum_{k=0}^{\infty} (k+1) b_{k+1} \omega^k, \quad g''(\omega) = \sum_{k=0}^{\infty} (k+2)(k+1) b_{k+2} \omega^k. \quad (2.44)$$

From the definition of the left-hand Erdélyi–Kober fractional integral operator, we get

$$\begin{aligned}(\mathcal{K}_{\frac{2\beta}{\alpha}}^{1-\frac{\alpha}{4}, m-\alpha} f)(\omega) &= \frac{1}{\Gamma(m-\alpha)} \int_1^{\infty} (s-1)^{m-\alpha-1} s^{-(1-\frac{5\alpha}{4}+m)} f(\omega s^{\frac{\alpha}{2\beta}}) ds \\ &= \frac{1}{\Gamma(m-\alpha)} \int_1^{\infty} (s-1)^{m-\alpha-1} s^{-(1-\frac{5\alpha}{4}+m)} \sum_{k=0}^{\infty} (a_k \omega^k s^{\frac{k\alpha}{2\beta}}) ds \\ &= \sum_{k=0}^{\infty} a_k \omega^k \left[\frac{1}{\Gamma(m-\alpha)} \int_1^{\infty} (s-1)^{m-\alpha-1} s^{-(1-\frac{5\alpha}{4}-\frac{k\alpha}{2\beta}+m)} ds \right].\end{aligned}$$

Because of the beta function $B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$, assuming $t = 1/x$, we have

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \int_1^{\infty} (t-1)^{q-1} t^{-(p+q)} dt.$$

Therefore,

$$(\mathcal{K}_{\frac{2\beta}{\alpha}}^{1-\frac{\alpha}{4}, m-\alpha} f)(\omega) = \sum_{k=0}^{\infty} a_k \omega^k \frac{B(1-\frac{\alpha}{4}-\frac{k\alpha}{2\beta}, m-\alpha)}{\Gamma(m-\alpha)} = \sum_{k=0}^{\infty} \frac{\Gamma(1-\frac{\alpha}{4}-\frac{k\alpha}{2\beta})}{\Gamma(1-\frac{5\alpha}{4}-\frac{k\alpha}{2\beta}+m)} a_k \omega^k.$$

With $m = [\alpha] + 1 = 1$, we get

$$\begin{aligned} (\mathcal{P}_{\frac{2\beta}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} f)(\omega) &= \prod_{j=0}^{m-1} \left(1 - \frac{5\alpha}{4} + j - \frac{\alpha}{2\beta} \omega \frac{d}{d\omega}\right) (\mathcal{K}_{\frac{2\beta}{\alpha}}^{1-\frac{\alpha}{4},m-\alpha} f)(\omega) \\ &= \left(1 - \frac{5\alpha}{4} - \frac{\alpha}{2\beta} \omega \frac{d}{d\omega}\right) \left(\sum_{k=0}^{\infty} \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(2 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} a_k \omega^k\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(1 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} a_k \omega^k. \end{aligned} \quad (2.45)$$

Similarly,

$$(\mathcal{P}_{\frac{2\beta}{\alpha}}^{1-\frac{5\alpha}{4},\alpha} g)(\omega) = \sum_{k=0}^{\infty} \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(1 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} b_k \omega^k. \quad (2.46)$$

It should be noted that the gamma functions appearing in Eqs (2.45) and (2.46) may have poles when their arguments are nonpositive integers. To ensure the validity of the power series solution, we require that

$$\frac{\alpha}{4} + \frac{k\alpha}{2\beta} \notin \mathbb{Z}^+, \quad \frac{5\alpha}{4} + \frac{k\alpha}{2\beta} \notin \mathbb{Z}^+,$$

where $\mathbb{Z}^+$ denotes the set of positive integers.

From the definition of the right-hand Erdélyi–Kober fractional integral operator, we get

$$\begin{aligned} (\mathcal{I}_1^{0,n-2\beta} f)(\omega) &= \frac{1}{\Gamma(n-2\beta)} \int_0^1 (1-s)^{n-2\beta-1} f(\omega s) ds \\ &= \frac{1}{\Gamma(n-2\beta)} \int_0^1 (1-s)^{n-2\beta-1} \sum_{k=0}^{\infty} (a_k \omega^k s^k) ds \\ &= \sum_{k=0}^{\infty} a_k \omega^k \left[\frac{1}{\Gamma(n-2\beta)} \int_0^1 (1-s)^{n-2\beta-1} s^k ds \right] \\ &= \sum_{k=0}^{\infty} a_k \omega^k \frac{B(k+1, n-2\beta)}{\Gamma(n-2\beta)} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+n-2\beta)} a_k \omega^k. \end{aligned}$$

With $n = [2\beta] + 1 = 1, 2$, assuming $n = 1$, we get

$$\begin{aligned} (\mathcal{D}_1^{-2\beta,2\beta} f)(\omega) &= \prod_{j=1}^n \left(-2\beta + j + \omega \frac{d}{d\omega}\right) (\mathcal{I}_1^{0,n-2\beta} f)(\omega) \\ &= (-2\beta + 1 + \omega \frac{d}{d\omega}) \left(\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+2-2\beta)} a_k \omega^k\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-2\beta)} a_k \omega^k. \end{aligned} \quad (2.47)$$

Similarly,

$$(\mathcal{D}_1^{-2\beta,2\beta} g)(\omega) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-2\beta)} b_k \omega^k, \quad (2.48)$$

and

$$(\mathcal{D}_1^{-\beta,\beta}g)(\omega) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} a_k \omega^k, \quad (\mathcal{D}_1^{-\beta,\beta}g)(\omega) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\beta)} b_k \omega^k. \quad (2.49)$$

Substituting (2.42)–(2.49) into the system (2.37) arrives at the following equations:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(1 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} a_k \omega^k &= -\omega^{-2\beta} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-2\beta)} b_k \omega^k + \omega^{-\beta} \sum_{k=0}^{\infty} \omega^k \left[\sum_{l+m+n=k} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} \right. \\ &\quad \left. (a_l a_m a_n - b_l b_m a_n + 2a_l b_m b_n) - \frac{1}{2} \sum_{l+m+n+p+q=k} (a_l a_m a_n a_p b_q + 2a_l a_m b_n b_p b_q + b_l b_m b_n b_p b_q) \right], \\ \sum_{k=0}^{\infty} \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(1 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} b_k \omega^k &= \omega^{-2\beta} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-2\beta)} a_k \omega^k - \omega^{-\beta} \sum_{k=0}^{\infty} \omega^k \left[\sum_{l+m+n=k} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} \right. \\ &\quad \left. (a_l a_m b_n - b_l b_m b_n - 2a_l b_m a_n) + \frac{1}{2} \sum_{l+m+n+p+q=k} (a_l a_m a_n a_p a_q + 2a_l a_m a_n b_p b_q + a_l b_m b_n b_p b_q) \right]. \end{aligned}$$

Therefore, a_k and b_k satisfy the following expressions:

$$\begin{cases} \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(1 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} a_k = -\omega^{-2\beta} \frac{\Gamma(k+1)}{\Gamma(k+1-2\beta)} b_k + \omega^{-\beta} \left(\sum_{l+m+n=k} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} (a_l a_m a_n - b_l b_m a_n \right. \\ \quad \left. + 2a_l b_m b_n) - \frac{1}{2} \sum_{l+m+n+p+q=k} (a_l a_m a_n a_p b_q + 2a_l a_m b_n b_p b_q + b_l b_m b_n b_p b_q) \right), \\ \frac{\Gamma(1 - \frac{\alpha}{4} - \frac{k\alpha}{2\beta})}{\Gamma(1 - \frac{5\alpha}{4} - \frac{k\alpha}{2\beta})} b_k = \omega^{-2\beta} \frac{\Gamma(k+1)}{\Gamma(k+1-2\beta)} a_k - \omega^{-\beta} \left(\sum_{l+m+n=k} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} (a_l a_m b_n - b_l b_m b_n \right. \\ \quad \left. - 2a_l b_m a_n) + \frac{1}{2} \sum_{l+m+n+p+q=k} (a_l a_m a_n a_p a_q + 2a_l a_m a_n b_p b_q + a_l b_m b_n b_p b_q) \right), \end{cases} \quad (2.50)$$

from which, however, it is hard to obtain the explicit expressions of a_k and b_k .

3. Lie symmetry analysis and power series solutions of Eq (1.2)

The time-fractional Gerdjikov–Ivanov equation (1.2) can be rewritten as the following system:

$$\begin{cases} D_t^\alpha u + v_{xx} - (u^2 - v^2)u_x - 2uvv_x + \frac{1}{2}v(u^2 + v^2)^2 = 0, \\ D_t^\alpha v - u_{xx} + (u^2 - v^2)v_x - 2uvu_x - \frac{1}{2}u(u^2 + v^2)^2 = 0. \end{cases} \quad (3.1)$$

The corresponding invariance criterion of System (3.1) under the group of transformations (2.2) is

$$\begin{cases} Pr^{(\alpha,2)}X(D_t^\alpha u + v_{xx} - (u^2 - v^2)u_x - 2uvv_x + \frac{1}{2}v(u^2 + v^2)^2)|_{(3.1)} = 0, \\ Pr^{(\alpha,2)}X(D_t^\alpha v - u_{xx} + (u^2 - v^2)v_x - 2uvu_x - \frac{1}{2}u(u^2 + v^2)^2)|_{(3.1)} = 0, \end{cases} \quad (3.2)$$

where

$$Pr^{(\alpha,2)}X = X + \eta^{\alpha,t} \frac{\partial}{\partial u_t^\alpha} + \zeta^{\alpha,t} \frac{\partial}{\partial v_t^\alpha} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \zeta^{xx} \frac{\partial}{\partial v_{xx}} + \eta^x \frac{\partial}{\partial u_x} + \zeta^x \frac{\partial}{\partial v_x}. \quad (3.3)$$

Therefore, the system (3.2) can be rewritten as

$$\begin{cases} (\eta^{\alpha,t} + \zeta^{xx} - (u^2 - v^2)\eta^x - 2uv\zeta^x - 2(uu_x + vv_x - u^3v - uv^3)\eta \\ \quad + (2vu_x - 2uvv_x + \frac{1}{2}(u^4 + 6u^2v^2 + 5v^4))\zeta)|_{(3.1)} = 0, \\ (\zeta^{\alpha,t} - \eta^{xx} + (u^2 - v^2)\zeta^x - 2uv\eta^x - 2(uu_x + vv_x + u^3v + uv^3)\zeta \\ \quad + (2uvv_x - 2vu_x - \frac{1}{2}(v^4 + 6u^2v^2 + 5u^4))\eta)|_{(3.1)} = 0. \end{cases} \quad (3.4)$$

Putting $\eta^{\alpha,t}, \zeta^{\alpha,t}, \eta^{xx}, \zeta^{xx}, \eta^x, \zeta^x$ into (3.4) and solving the obtained determining equations, we can get the infinitesimals as follows:

$$\xi = c_1x + c_3, \quad \tau = \frac{2}{\alpha}c_1t, \quad \eta = -\frac{1}{2}c_1u + c_2v, \quad \zeta = -\frac{1}{2}c_1v - c_2u, \quad (3.5)$$

where c_1, c_2 , and c_3 are arbitrary constants. The system (3.1) then admits the three-dimensional Lie algebra spanned by

$$X_1 = x\frac{\partial}{\partial x} + \frac{2}{\alpha}t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v}, \quad X_2 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial x}, \quad (3.6)$$

with $[X_1, X_2] = [X_2, X_3] = 0, [X_1, X_3] = -X_3$.

Case 1: X_1

From the characteristic equation corresponding to the group generator X_1 , that is,

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{2du}{-u} = \frac{2dv}{-v}, \quad (3.7)$$

we obtain the similarity variables $xt^{-\frac{\alpha}{2}}, ut^{\frac{\alpha}{4}}$ and $vt^{\frac{\alpha}{4}}$. Therefore, we get the invariant solutions of the system (3.1) as follows:

$$u(t, x) = t^{-\frac{\alpha}{4}}f(\omega), \quad v(t, x) = t^{-\frac{\alpha}{4}}g(\omega), \quad (3.8)$$

with $\omega = xt^{-\frac{\alpha}{2}}$.

Theorem 3.1. The similarity transformations $u(t, x) = t^{-\frac{\alpha}{4}}f(\omega), v(t, x) = t^{-\frac{\alpha}{4}}g(\omega)$ with the similarity variable $\omega = xt^{-\frac{\alpha}{2}}$ reduce the system (3.1) to the system of fractional ordinary differential equations given by

$$\begin{cases} (\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} f)(\omega) = -g''(\omega) + (f^2(\omega) - g^2(\omega))f'(\omega) \\ \quad + 2f(\omega)g(\omega)g'(\omega) - \frac{1}{2}g(\omega)(f^2(\omega) + g^2(\omega))^2, \\ (\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{5\alpha}{4}, \alpha} g)(\omega) = f''(\omega) - (f^2(\omega) - g^2(\omega))g'(\omega) \\ \quad + 2f(\omega)g(\omega)f'(\omega) + \frac{1}{2}f(\omega)(f^2(\omega) + g^2(\omega))^2. \end{cases} \quad (3.9)$$

Proof: Similar to the proof of Theorem (2.1). $\square$

The system (3.9) is consistent with (2.37) when β approaches 1. Following the same procedure as in (2.37), the power series solutions of (3.9) are derived as follows.

Substituting (2.42)–(2.46) into the system (3.9) and equating the coefficients of different powers of ω arrives at the following system:

$$\begin{cases} \frac{\Gamma(1-\frac{(2k+1)\alpha}{4})}{\Gamma(1-\frac{(2k+5)\alpha}{4})}a_k = -(k+2)(k+1)b_{k+2} + \sum_{l+m+n=k}(n+1)(a_l a_m a_{n+1} - b_l b_m a_{n+1} \\ \quad + 2a_l b_m b_{n+1}) - \frac{1}{2} \sum_{l+m+n+p+q=k}(a_l a_m a_n a_p b_q + 2a_l a_m b_n b_p b_q + b_l b_m b_n b_p b_q), \\ \frac{\Gamma(1-\frac{(2k+1)\alpha}{4})}{\Gamma(1-\frac{(2k+5)\alpha}{4})}b_k = (k+2)(k+1)a_{k+2} + \sum_{l+m+n=k}(n+1)(b_l b_m b_{n+1} - a_l a_m b_{n+1} \\ \quad + 2a_l b_m a_{n+1}) + \frac{1}{2} \sum_{l+m+n+p+q=k}(a_l a_m a_n a_p a_q + 2a_l a_m a_n b_p b_q + a_l b_m b_n b_p b_q), \end{cases} \quad (3.10)$$

from which, we can obtain the explicit expressions of a_k and b_k . For $k = 0$, we have

$$\begin{cases} a_2 = \frac{1}{2} \left[\frac{\Gamma(1-\frac{\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4})} b_0 - (b_0^2 b_1 - a_0^2 b_1 + 2a_0 b_0 a_1) - \frac{1}{2}(a_0^5 + 2a_0^3 b_0^2 + a_0 b_0^4) \right], \\ b_2 = -\frac{1}{2} \left[\frac{\Gamma(1-\frac{\alpha}{4})}{\Gamma(1-\frac{5\alpha}{4})} a_0 - (a_0^2 a_1 - b_0^2 a_1 + 2a_0 b_0 b_1) + \frac{1}{2}(b_0^5 + 2b_0^3 a_0^2 + b_0 a_0^4) \right]. \end{cases} \quad (3.11)$$

For $k = 1$, we have

$$\begin{cases} a_3 = \frac{1}{6} \left[\frac{\Gamma(1-\frac{3\alpha}{4})}{\Gamma(1-\frac{7\alpha}{4})} b_1 - 2(b_0 b_1^2 + b_0^2 b_2 - a_0^2 b_2 + b_0 a_1^2 + a_0 b_0 a_2) \right. \\ \quad \left. - \frac{1}{2} (4b_1 b_0^3 a_0 + b_0^4 a_1 + 4b_1 b_0 a_0^3 + 6a_0^2 b_0^2 a_1 + 5a_0^4 a_1) \right], \\ b_3 = -\frac{1}{6} \left[\frac{\Gamma(1-\frac{3\alpha}{4})}{\Gamma(1-\frac{7\alpha}{4})} a_1 - 2(a_0 a_1^2 + a_0^2 a_2 - b_0^2 a_2 + a_0 b_1^2 + a_0 b_0 b_2) \right. \\ \quad \left. + \frac{1}{2} (4a_1 a_0^3 b_0 + a_0^4 b_1 + 4a_1 a_0 b_0^3 + 6a_0^2 b_0^2 b_1 + 5b_0^4 b_1) \right]. \end{cases} \quad (3.12)$$

For $k \geq 2$, we have

$$\begin{cases} a_{k+2} = \frac{1}{(k+2)(k+1)} \left[\frac{\Gamma(1-\frac{(2k+1)\alpha}{4})}{\Gamma(1-\frac{(2k+5)\alpha}{4})} b_k - \sum_{l+m+n=k} (n+1)(b_l b_m b_{n+1} - a_l a_m b_{n+1} \right. \\ \quad \left. + 2a_l b_m a_{n+1}) - \frac{1}{2} \sum_{l+m+n+p+q=k} (a_l a_m a_n a_p a_q + 2a_l a_m a_n b_p b_q + a_l b_m b_n b_p b_q) \right], \\ b_{k+2} = \frac{-1}{(k+2)(k+1)} \left[\frac{\Gamma(1-\frac{(2k+1)\alpha}{4})}{\Gamma(1-\frac{(2k+5)\alpha}{4})} a_k - \sum_{l+m+n=k} (n+1)(a_l a_m a_{n+1} - b_l b_m a_{n+1} \right. \\ \quad \left. + 2a_l b_m b_{n+1}) + \frac{1}{2} \sum_{l+m+n+p+q=k} (a_l a_m a_n a_p b_q + 2a_l a_m b_n b_p b_q + b_l b_m b_n b_p b_q) \right]. \end{cases} \quad (3.13)$$

Next, the convergence analysis of the power series solutions for the system (3.9) is presented by using the method in [26]. From (3.11)–(3.13), we get

$$\begin{cases} |a_{k+2}| \leq 2 \left[|b_k| + \sum_{l+m+n=k} (|b_l| |b_m| |b_{n+1}| + |a_l| |a_m| |b_{n+1}| + 2|a_l| |b_m| |a_{n+1}|) \right. \\ \quad \left. + \sum_{l+m+n+p+q=k} (|a_l| |a_m| |a_n| |a_p| |a_q| + 2|a_l| |a_m| |a_n| |b_p| |b_q| + |a_l| |b_m| |b_n| |b_p| |b_q|) \right], \\ |b_{k+2}| \leq 2 \left[|a_k| + \sum_{l+m+n=k} (|a_l| |a_m| |a_{n+1}| + |b_l| |b_m| |a_{n+1}| + 2|a_l| |b_m| |b_{n+1}|) \right. \\ \quad \left. + \sum_{l+m+n+p+q=k} (|a_l| |a_m| |a_n| |a_p| |b_q| + 2|a_l| |a_m| |b_n| |b_p| |b_q| + |b_l| |b_m| |b_n| |b_p| |b_q|) \right]. \end{cases} \quad (3.14)$$

Assuming

$$C(\theta) = \sum_{k=0}^{\infty} c_k \theta^k, \quad D(\theta) = \sum_{k=0}^{\infty} d_k \theta^k, \quad (3.15)$$

where $c_0 = |a_0|$, $d_0 = |b_0|$, $c_1 = |a_1|$, $d_1 = |b_1|$, and

$$\begin{cases} c_{k+2} = 2 \left[d_k + \sum_{l+m+n=k} (d_l d_m d_{n+1} + c_l c_m d_{n+1} + 2c_l d_m c_{n+1}) \right. \\ \quad \left. + \sum_{l+m+n+p+q=k} (c_l c_m c_n c_p c_q + 2c_l c_m c_n d_p d_q + c_l d_m d_n d_p d_q) \right], \quad k = 0, 1, 2, \dots, \\ d_{k+2} = 2 \left[c_k + \sum_{l+m+n=k} (c_l c_m c_{n+1} + d_l d_m c_{n+1} + 2c_l d_m d_{n+1}) \right. \\ \quad \left. + \sum_{l+m+n+p+q=k} (c_l c_m c_n c_p d_q + 2c_l c_m d_n d_p d_q + d_l d_m d_n d_p d_q) \right], \quad k = 0, 1, 2, \dots, \end{cases} \quad (3.16)$$

that is, $|a_k| \leq c_k$, $|b_k| \leq d_k$, $k = 0, 1, 2, \dots$, we can get

$$\begin{aligned} C(\theta) &= c_0 + c_1 \theta + \sum_{k=0}^{\infty} c_{k+2} \theta^{k+2} = c_0 + c_1 \theta + 2 \sum_{k=0}^{\infty} \left[d_k + \sum_{l+m+n=k} (d_l d_m d_{n+1} + c_l c_m d_{n+1} \right. \\ &\quad \left. + 2c_l d_m c_{n+1}) + \sum_{l+m+n+p+q=k} (c_l c_m c_n c_p c_q + 2c_l c_m c_n d_p d_q + c_l d_m d_n d_p d_q) \right] \theta^{k+2} \\ &= c_0 + c_1 \theta + 2 \left[D(\theta) + C^5(\theta) + 2C^3(\theta) D^2(\theta) + C(\theta) D^4(\theta) \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \sum_{l+m+n=k} (d_l d_m d_{n+1} + c_l c_m d_{n+1} + 2c_l d_m c_{n+1}) \theta^k \right] \theta^2, \end{aligned} \quad (3.17)$$

$$\begin{aligned}
D(\theta) &= d_0 + d_1\theta + \sum_{k=0}^{\infty} d_{k+2}\theta^{k+2} = d_0 + d_1\theta + 2 \sum_{k=0}^{\infty} \left[c_k + \sum_{l+m+n=k} (c_l c_m c_{n+1} + d_l d_m c_{n+1} \right. \\
&\quad \left. + 2c_l d_m d_{n+1}) + \sum_{l+m+n+p+q=k} (c_l c_m c_n c_p d_q + 2c_l c_m d_n d_p d_q + d_l d_m d_n d_p d_q) \right] \theta^{k+2} \\
&= d_0 + d_1\theta + 2 \left[C(\theta) + D^5(\theta) + 2C^2(\theta)D^3(\theta) + C^4(\theta)D(\theta) \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \sum_{l+m+n=k} (c_l c_m c_{n+1} + d_l d_m c_{n+1} + 2c_l d_m d_{n+1}) \theta^k \right] \theta^2.
\end{aligned} \tag{3.18}$$

Consider the implicit functional system with respect to the independent variable θ ,

$$\begin{aligned}
H(\theta, C, D) &= C - c_0 - c_1\theta - 2 \left[D + C^5 + 2C^3D^2 + CD^4 \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \sum_{l+m+n=k} (d_l d_m d_{n+1} + c_l c_m d_{n+1} + 2c_l d_m c_{n+1}) \theta^k \right] \theta^2,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
I(\theta, C, D) &= D - d_0 - d_1\theta - 2 \left[C + D^5 + 2C^2D^3 + C^4D \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \sum_{l+m+n=k} (c_l c_m c_{n+1} + d_l d_m c_{n+1} + 2c_l d_m d_{n+1}) \theta^k \right] \theta^2,
\end{aligned} \tag{3.20}$$

from which, H and I are analytic in the neighborhood of point $(0, c_0, d_0)$ and $H(0, c_0, d_0) = 0$, $I(0, c_0, d_0) = 0$. The Jacobian determinant is

$$J = \frac{\partial(H, I)}{\partial(C, D)} = 1 \neq 0. \tag{3.21}$$

Then, the two series $C = C(\theta)$ and $D = D(\theta)$ are analytic in the neighborhood of $(0, c_0, d_0)$ with positive radius by implicit function theorem; that is, the series $f(\omega)$ and $g(\omega)$ are convergent in the neighborhood of $(0, c_0, d_0)$.

Therefore, the power series solutions of time-fractional Gerdjikov–Ivanov equation (1.2) have the form

$$u(t, x) = t^{-\frac{\alpha}{4}} f(\omega) = \sum_{k=0}^{\infty} a_k x^k t^{-\frac{(2k+1)\alpha}{4}}, \quad v(t, x) = t^{-\frac{\alpha}{4}} g(\omega) = \sum_{k=0}^{\infty} b_k x^k t^{-\frac{(2k+1)\alpha}{4}}, \tag{3.22}$$

where a_k and b_k are defined by (3.11)–(3.13) with the initial conditions $f(0) = a_0$, $g(0) = b_0$, $f'(0) = a_1$, and $g'(0) = b_1$.

Figures 1 and 2 illustrate the power series solutions $u(t, x)$ and $v(t, x)$ with different fractional orders. As shown in Figure 1 ($\alpha = 0.4$), both components exhibit smooth and localized wave profiles. In Figure 2 ($\alpha = 0.8$), the wave structures become more oscillatory and spread over a wider domain. This indicates that smaller values of α correspond to stronger memory effects, leading to more confined waveforms, whereas larger α yields weaker nonlocal effects and more extended patterns.

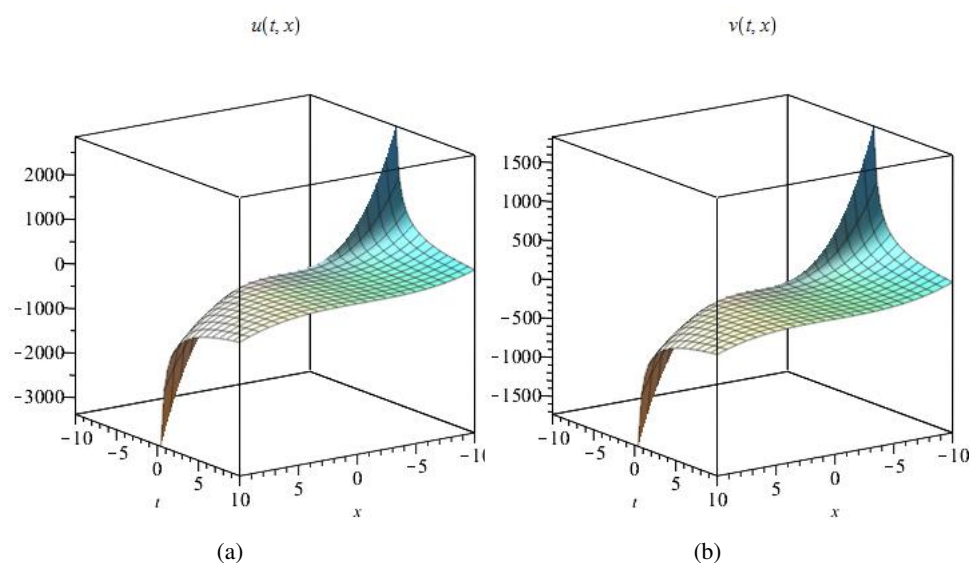


Figure 1. Numerical simulation of the power series solutions (3.22) with $a_0 = b_0 = a_1 = b_1 = 1$, $a_2 = -1.698545409$, $b_2 = 0.3014545912$, $a_3 = -1.694834108$, $b_3 = -0.9718325579$, and $\alpha = 0.4$.

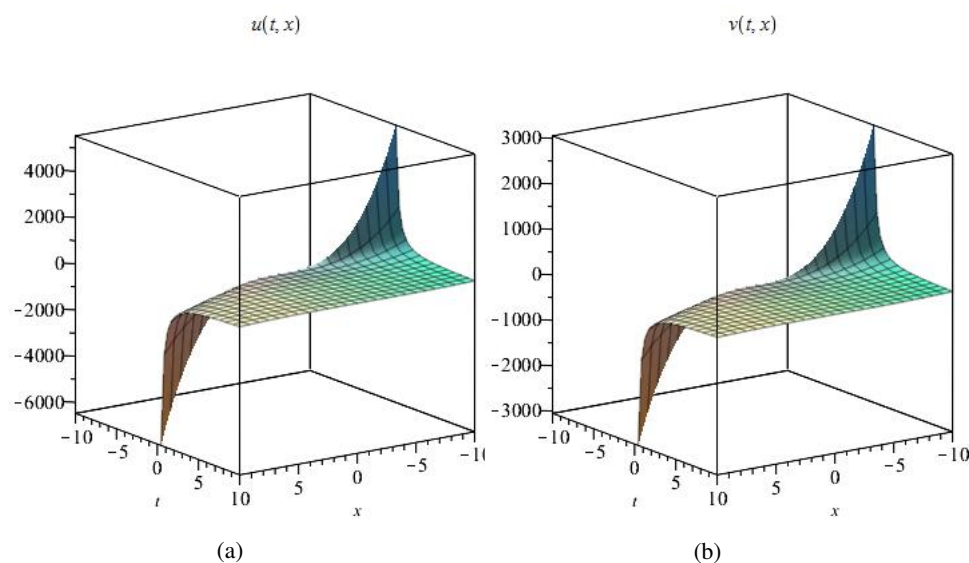


Figure 2. Numerical simulation of the power series solutions (3.22) with $a_0 = b_0 = a_1 = b_1 = 1$, $a_2 = -2$, $b_2 = 0$, $a_3 = -1.765967011$, $b_3 = -0.9006996558$, and $\alpha = 0.8$.

Case 2: $X_2 + X_3$

From the characteristic equation corresponding to the group generator $X_2 + X_3$, that is,

$$\frac{dx}{1} = \frac{\alpha dt}{0} = \frac{du}{v} = \frac{dv}{-u}, \quad (3.23)$$

we obtain the similarity variables t and ze^{ix} . Therefore, the invariant solutions of Eq (1.2) are

$$z(t, x) = (f(t) + ig(t))e^{-ix}, \quad (3.24)$$

where $f(t)$ and $g(t)$ are differentiable functions to be determined. Substituting (3.24) into Eq (1.2), we can obtain the following reduced system of time fractional ordinary differential equations:

$$\begin{cases} D_t^\alpha f(t) = g(t) - g(t)(f^2(t) + g^2(t)) - \frac{1}{2}g(t)(f^2(t) + g^2(t))^2, \\ D_t^\alpha g(t) = -f(t) + f(t)(f^2(t) + g^2(t)) + \frac{1}{2}f(t)(f^2(t) + g^2(t))^2. \end{cases} \quad (3.25)$$

4. Lie symmetry analysis of Eq (1.3)

The integer-order Gerdjikov–Ivanov equation (1.3) can be rewritten as the following system:

$$\begin{cases} u_t + v_{xx} - (u^2 - v^2)u_x - 2uvv_x + \frac{1}{2}v(u^2 + v^2)^2 = 0, \\ v_t - u_{xx} + (u^2 - v^2)v_x - 2uvu_x - \frac{1}{2}u(u^2 + v^2)^2 = 0. \end{cases} \quad (4.1)$$

Similar to the above discussions, we can obtain the corresponding determining equations as follows:

$$\begin{cases} (\eta^t + \zeta^{xx} - (u^2 - v^2)\eta^x - 2uv\zeta^x - 2(uu_x + vv_x - u^3v - uv^3)\eta \\ + (2vu_x - 2uv_x + \frac{1}{2}(u^4 + 6u^2v^2 + 5v^4))\zeta)|_{(4.1)} = 0, \\ (\zeta^t - \eta^{xx} + (u^2 - v^2)\zeta^x - 2uv\eta^x - 2(uu_x + vv_x + u^3v + uv^3)\zeta \\ + (2uv_x - 2vu_x - \frac{1}{2}(v^4 + 6u^2v^2 + 5u^4))\eta)|_{(4.1)} = 0. \end{cases} \quad (4.2)$$

Putting $\eta^t, \zeta^t, \eta^{xx}, \zeta^{xx}, \eta^x, \zeta^x$ into (4.2) and solving the obtained determining equations, we can get the infinitesimals as follows:

$$\xi = c_1x + c_3, \quad \tau = 2c_1t + c_4, \quad \eta = -\frac{1}{2}c_1u + c_2v, \quad \zeta = -\frac{1}{2}c_1v - c_2u, \quad (4.3)$$

where c_1, c_2, c_3 , and c_4 are arbitrary constants.

System (4.1) then admits the four-dimensional Lie algebra spanned by

$$X_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v}, \quad X_2 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial t}, \quad (4.4)$$

with $[X_1, X_2] = [X_2, X_3] = [X_2, X_4] = [X_3, X_4] = 0, [X_1, X_3] = -X_3, [X_1, X_4] = -2X_4$.

Note: The four group generators are consistent with those in [26], where the authors have obtained Lie symmetry reductions, power series solutions, and conservation laws of the integer-order Gerdjikov–Ivanov equation (1.3).

5. Conservation laws of Eqs (1.1) and (1.2)

In this section, the conservation laws of Eqs (1.1) and (1.2) are constructed by using the generalization of the Noether operators and the new conservation theorem [27, 28].

5.1. Conservation laws of Eq (1.1)

The system (2.1) is denoted as

$$\begin{cases} F_1 = D_t^\alpha u + D_x^{2\beta} v - (u^2 - v^2)D_x^\beta u - 2uvD_x^\beta v + \frac{1}{2}v(u^2 + v^2)^2 = 0, \\ F_2 = D_t^\alpha v - D_x^{2\beta} u + (u^2 - v^2)D_x^\beta v - 2uvD_x^\beta u - \frac{1}{2}u(u^2 + v^2)^2 = 0, \end{cases} \quad (5.1)$$

of which the formal Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= p(t, x)F_1 + q(t, x)F_2 \\ &= p(t, x)(D_t^\alpha u + D_x^{2\beta} v - (u^2 - v^2)D_x^\beta u - 2uvD_x^\beta v + \frac{1}{2}v(u^2 + v^2)^2) \\ &\quad + q(t, x)(D_t^\alpha v - D_x^{2\beta} u + (u^2 - v^2)D_x^\beta v - 2uvD_x^\beta u - \frac{1}{2}u(u^2 + v^2)^2), \end{aligned} \quad (5.2)$$

where $p(t, x)$ and $q(t, x)$ are new dependent variables. The Euler-Lagrange operators are

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha u)} + (D_x^\beta)^* \frac{\partial}{\partial (D_x^\beta u)} + (D_x^{2\beta})^* \frac{\partial}{\partial (D_x^{2\beta} u)} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha v)} + (D_x^\beta)^* \frac{\partial}{\partial (D_x^\beta v)} + (D_x^{2\beta})^* \frac{\partial}{\partial (D_x^{2\beta} v)} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v_{i_1 \cdots i_s}}, \end{aligned}$$

where $(D_t^\alpha)^*$, $(D_x^\beta)^*$, and $(D_x^{2\beta})^*$ are the adjoint operators of D_t^α , D_x^β , and $D_x^{2\beta}$, respectively. They are defined by

$$(D_t^\alpha)^* \equiv {}^c D_T^\alpha, \quad (D_x^\beta)^* \equiv {}^c D_X^\beta, \quad (D_x^{2\beta})^* \equiv {}^c D_X^{2\beta}, \quad (5.3)$$

where ${}^c D_T^\alpha$, ${}^c D_X^\beta$, and ${}^c D_X^{2\beta}$ are the right-sides of the Caputo fractional derivative. The system of adjoint equations to (2.1) is given by

$$\begin{cases} F_1^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^\alpha)^* p - (D_x^{2\beta})^* q - (D_x^\beta)^* ((u^2 - v^2)p + 2uvq) - 2(up + vq) \\ \quad \cdot D_x^\beta u - 2(vp - uq)D_x^\beta v + 2uv(u^2 + v^2)p - \frac{1}{2}(u^2 + v^2)(5u^2 + v^2) = 0, \\ F_2^* = \frac{\delta \mathcal{L}}{\delta v} = (D_t^\alpha)^* q + (D_x^{2\beta})^* p + (D_x^\beta)^* ((u^2 - v^2)q - 2uvp) - 2(up + vq) \\ \quad \cdot D_x^\beta v + 2(vp - uq)D_x^\beta u - 2uv(u^2 + v^2)q + \frac{1}{2}(u^2 + v^2)(u^2 + 5v^2) = 0. \end{cases} \quad (5.4)$$

Next, we use the above adjoint equation and the new conservation theorem to construct conservation laws of Eq (1.1). From the classical definition of the conservation laws, a vector $\mathbf{C} = (C^t, C^x)$ is called a conserved vector for the governing equation if it satisfies the conservation equation $[D_t C^t + D_x C^x]_{F_1, F_2=0} = 0$. By using Noether theorem, the components of a conserved vector can be obtained.

To begin, from the fundamental operator identity, that is,

$$Pr^{(\alpha, 2\beta)} X + D_t \tau \cdot \mathcal{I} + D_x \xi \cdot \mathcal{I} = W^u \cdot \frac{\delta}{\delta u} + W^v \cdot \frac{\delta}{\delta v} + D_t \mathcal{N}^t + D_x \mathcal{N}^x, \quad (5.5)$$

where $Pr^{(\alpha, 2\beta)} X$ is mentioned in (2.4), $\mathcal{I}$ is the identity operator, and $W^u = \eta - \tau u_t - \xi u_x$, $W^v = \zeta - \tau v_t - \xi v_x$ are the characteristics for group generator X , we can get the Noether operators as follows:

$$\begin{aligned} \mathcal{N}^t &= \tau \mathcal{I} + \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k} (W^u) D_t^k \frac{\partial}{\partial (D_t^\alpha u)} - (-1)^n J_1(W^u, D_t^n \frac{\partial}{\partial (D_t^\alpha u)}) \\ &\quad + \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k} (W^v) D_t^k \frac{\partial}{\partial (D_t^\alpha v)} - (-1)^n J_1(W^v, D_t^n \frac{\partial}{\partial (D_t^\alpha v)}), \end{aligned} \quad (5.6)$$

$$\begin{aligned}
\mathcal{N}^x = & \xi \mathcal{I} + \sum_{k=0}^{m-1} (-1)^k D_x^{2\beta-1-k} (W^u) D_x^k \frac{\partial}{\partial (D_x^{2\beta} u)} - (-1)^m J_2(W^u, D_x^m \frac{\partial}{\partial (D_x^{2\beta} u)}) \\
& + \sum_{k=0}^{m-1} (-1)^k D_x^{2\beta-1-k} (W^v) D_x^k \frac{\partial}{\partial (D_x^{2\beta} v)} - (-1)^m J_2(W^v, D_x^m \frac{\partial}{\partial (D_x^{2\beta} v)}) \\
& + \sum_{k=0}^{l-1} (-1)^k D_x^{\beta-1-k} (W^u) D_x^k \frac{\partial}{\partial (D_x^{\beta} u)} - (-1)^l J_3(W^u, D_x^l \frac{\partial}{\partial (D_x^{\beta} u)}) \\
& + \sum_{k=0}^{l-1} (-1)^k D_x^{\beta-1-k} (W^v) D_x^k \frac{\partial}{\partial (D_x^{\beta} v)} - (-1)^l J_3(W^v, D_x^l \frac{\partial}{\partial (D_x^{\beta} v)}),
\end{aligned} \tag{5.7}$$

where $n = [\alpha] + 1$, $m = [2\beta] + 1$, $l = [\beta] + 1$, and J_1, J_2, J_3 are given by

$$J_1(f, g) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \int_t^T \frac{f(\tau, x) g(\theta, x)}{(\theta - \tau)^{\alpha+1-n}} d\theta d\tau, \tag{5.8}$$

$$J_2(f, g) = \frac{1}{\Gamma(m - 2\beta)} \int_0^x \int_x^X \frac{f(t, \xi) g(t, \theta)}{(\theta - \xi)^{2\beta+1-m}} d\theta d\xi. \tag{5.9}$$

$$J_3(f, g) = \frac{1}{\Gamma(m - \beta)} \int_0^x \int_x^X \frac{f(t, \xi) g(t, \theta)}{(\theta - \xi)^{\beta+1-m}} d\theta d\xi. \tag{5.10}$$

The components of conserved vector are defined by $C^t = \mathcal{N}^t \mathcal{L}$, $C^x = \mathcal{N}^x \mathcal{L}$.

Case 1: $X_1 = x \frac{\partial}{\partial x} + \frac{2\beta}{\alpha} t \frac{\partial}{\partial t} - \frac{\beta}{2} u \frac{\partial}{\partial u} - \frac{\beta}{2} v \frac{\partial}{\partial v}$

The characteristics of X_1 are

$$W^u = -\frac{\beta}{2} u - \frac{2\beta}{\alpha} t u_t - x u_x, \quad W^v = -\frac{\beta}{2} v - \frac{2\beta}{\alpha} t v_t - x v_x. \tag{5.11}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = p D_t^{\alpha-1} (W^u) + J_1(W^u, p_t) + q D_t^{\alpha-1} (W^v) + J_1(W^v, q_t). \tag{5.12}$$

For $0 < \beta < 1/2$,

$$\begin{aligned}
C^x = & -q D_x^{2\beta-1} (W^u) - J_2(W^u, q_x) + p D_x^{2\beta-1} (W^v) + J_2(W^v, p_x) \\
& - ((u^2 - v^2)p + 2uvq) D_x^{\beta-1} (W^u) - J_3(W^u, ((u^2 - v^2)p + 2uvq)_x) \\
& + ((u^2 - v^2)q - 2uvp) D_x^{\beta-1} (W^v) + J_3(W^v, ((u^2 - v^2)q - 2uvp)_x).
\end{aligned} \tag{5.13}$$

For $1/2 \leq \beta < 1$,

$$\begin{aligned}
C^x = & -q D_x^{2\beta-1} (W^u) + q_x D_x^{2\beta-2} (W^u) + J_2(W^u, q_{xx}) \\
& + p D_x^{2\beta-1} (W^v) - p_x D_x^{2\beta-2} (W^v) - J_2(W^v, p_{xx}) \\
& - ((u^2 - v^2)p + 2uvq) D_x^{\beta-1} (W^u) - J_3(W^u, ((u^2 - v^2)p + 2uvq)_x) \\
& + ((u^2 - v^2)q - 2uvp) D_x^{\beta-1} (W^v) + J_3(W^v, ((u^2 - v^2)q - 2uvp)_x).
\end{aligned} \tag{5.14}$$

Case 2: $X_2 = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}$

The characteristics of X_2 are

$$W^u = v, \quad W^v = -u. \quad (5.15)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = p D_t^{\alpha-1} v + J_1(v, p_t) - q D_t^{\alpha-1} u - J_1(u, q_t). \quad (5.16)$$

For $0 < \beta < 1/2$,

$$\begin{aligned} C^x = & -q D_x^{2\beta-1} v - J_2(v, q_x) - p D_x^{2\beta-1} u - J_2(u, p_x) \\ & - ((u^2 - v^2)p + 2uvq) D_x^{\beta-1} v - J_3(v, ((u^2 - v^2)p + 2uvq)_x) \\ & - ((u^2 - v^2)q - 2uvp) D_x^{\beta-1} u - J_3(u, ((u^2 - v^2)q - 2uvp)_x). \end{aligned} \quad (5.17)$$

For $1/2 \leq \beta < 1$,

$$\begin{aligned} C^x = & -q D_x^{2\beta-1} v + q_x D_x^{2\beta-2} v + J_2(v, q_{xx}) \\ & - p D_x^{2\beta-1} u + p_x D_x^{2\beta-2} u + J_2(u, p_{xx}) \\ & - ((u^2 - v^2)p + 2uvq) D_x^{\beta-1} v - J_3(v, ((u^2 - v^2)p + 2uvq)_x) \\ & - ((u^2 - v^2)q - 2uvp) D_x^{\beta-1} u - J_3(u, ((u^2 - v^2)q - 2uvp)_x). \end{aligned} \quad (5.18)$$

5.2. Conservation laws of Eq (1.2)

The system (3.1) is denoted as

$$\begin{cases} F_1 = D_t^\alpha u + v_{xx} - (u^2 - v^2)u_x - 2uvv_x + \frac{1}{2}v(u^2 + v^2)^2 = 0, \\ F_2 = D_t^\alpha v - u_{xx} + (u^2 - v^2)v_x - 2uvu_x - \frac{1}{2}u(u^2 + v^2)^2 = 0, \end{cases} \quad (5.19)$$

of which the formal Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & p(t, x)F_1 + q(t, x)F_2 \\ = & p(t, x)(D_t^\alpha u + v_{xx} - (u^2 - v^2)u_x - 2uvv_x + \frac{1}{2}v(u^2 + v^2)^2) \\ & + q(t, x)(D_t^\alpha v - u_{xx} + (u^2 - v^2)v_x - 2uvu_x - \frac{1}{2}u(u^2 + v^2)^2), \end{aligned} \quad (5.20)$$

The system of adjoint equations to (5.19) is therefore given by

$$\begin{cases} F_1^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^\alpha)^* p - q_{xx} + ((u^2 - v^2)p + 2uvq)_x - 2(up + vq)u_x \\ \quad - 2(vp - uq)v_x + 2uv(u^2 + v^2)p - \frac{1}{2}(u^2 + v^2)(5u^2 + v^2) = 0, \\ F_2^* = \frac{\delta \mathcal{L}}{\delta v} = (D_t^\alpha)^* q + p_{xx} - ((u^2 - v^2)q - 2uvp)_x - 2(up + vq)v_x \\ \quad + 2(vp - uq)u_x - 2uv(u^2 + v^2)q + \frac{1}{2}(u^2 + v^2)(u^2 + 5v^2) = 0. \end{cases} \quad (5.21)$$

Similar to the discussion for Eq (1.1), the components of the conserved vector for Eq (1.2) can be obtained as follows:

$$C^t = D_t^{\alpha-1}(W^u)D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha u)} + J_1(W^u, D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha u)}) + D_t^{\alpha-1}(W^v)D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha v)} + J_1(W^v, D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha v)}),$$

$$C^x = W^u \left(\frac{\partial \mathcal{L}}{\partial u_x} - D_x \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) + D_x(W^u) \frac{\partial \mathcal{L}}{\partial u_{xx}} + W^v \left(\frac{\partial \mathcal{L}}{\partial v_x} - D_x \frac{\partial \mathcal{L}}{\partial v_{xx}} \right) + D_x(W^v) \frac{\partial \mathcal{L}}{\partial v_{xx}}.$$

Case 1: $X_1 = x \frac{\partial}{\partial x} + \frac{2}{\alpha} t \frac{\partial}{\partial t} - \frac{1}{2} u \frac{\partial}{\partial u} - \frac{1}{2} v \frac{\partial}{\partial v}$

The characteristics of X_1 are

$$W^u = -\frac{1}{2}u - \frac{2}{\alpha}tu_t - xu_x, \quad W^v = -\frac{1}{2}v - \frac{2}{\alpha}tv_t - xv_x. \quad (5.22)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = pD_t^{\alpha-1}(W^u) + J_1(W^u, p_t) + qD_t^{\alpha-1}(W^v) + J_1(W^v, q_t), \quad (5.23)$$

$$C^x = -(((u^2 - v^2)p + 2uvq) - q_x)W^u - qD_x(W^u) + (((u^2 - v^2)q - 2uvp) - p_x)W^v + pD_x(W^v). \quad (5.24)$$

Case 2: $X_2 = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}$

The characteristics of X_2 are

$$W^u = v, \quad W^v = -u. \quad (5.25)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = pD_t^{\alpha-1}v + J_1(v, p_t) - qD_t^{\alpha-1}u - J_1(u, q_t), \quad (5.26)$$

$$C^x = -(((u^2 - v^2)p + 2uvq) - q_x)v - qv_x - (((u^2 - v^2)q - 2uvp) - p_x)u - pu_x. \quad (5.27)$$

Case 3: $X_3 = \frac{\partial}{\partial x}$

The characteristics of X_3 are

$$W^u = -u_x, \quad W^v = -v_x. \quad (5.28)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -pD_t^{\alpha-1}(u_x) - J_1(u_x, p_t) - qD_t^{\alpha-1}(v_x) - J_1(v_x, q_t), \quad (5.29)$$

$$C^x = (((u^2 - v^2)p + 2uvq) - q_x)u_x + qu_{xx} - (((u^2 - v^2)q - 2uvp) - p_x)v_x - pv_{xx}. \quad (5.30)$$

6. Conclusions

This paper manifests the effectiveness of the Lie symmetry analysis method in studying an important model in mathematical physics, namely the space–time fractional Gerdjikov–Ivanov equation:

$$iD_t^\alpha z + D_x^{2\beta} z - iz^2 \overline{D_x^\beta z} + \frac{1}{2}|z|^4 z = 0.$$

For $0 < \alpha, \beta \leq 1$, all Lie symmetries of the equation are obtained, and the corresponding system of fractional partial differential equations is reduced to a system of fractional ordinary differential equations. The power series solutions for the reduced systems are derived, and the conservation laws for every Lie symmetry are established. The results show that the Lie symmetry analysis method, together with the power series method, constitutes a direct and powerful mathematical framework applicable to other fractional differential equations in mathematical physics.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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