



Research article

The differential Hilbert operator on some analytic function spaces

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Abstract: We studied the exact norm of the differential Hilbert operator $\mathcal{H}_2$ on several classical analytic function spaces, including the space of bounded analytic functions H^∞ , the Korenblum spaces H_α^∞ , and the α -Bloch spaces $\mathcal{B}^\alpha$. We established the operator norm of $\mathcal{H}_2 : H_\alpha^\infty \rightarrow H_\beta^\infty$ and explicitly determined its exact value when $\beta = 1 + \alpha$. The norms of $\mathcal{H}_2 : H_\alpha^\infty \rightarrow \mathcal{B}^{\alpha+2}$, of $\mathcal{H}_2 : H^\infty \rightarrow H_1^\infty$, and of $\mathcal{H}_2 : H^\infty \rightarrow \mathcal{B}^2$ were also obtained.

Keywords: Korenblum space; differential Hilbert operator; Hardy space; Bloch space; norm

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc, and let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. For $0 < p \leq \infty$, let H^p denote the Hardy space consisting of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty, \quad 0 < p < \infty,$$

and $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty$. The space H^∞ is also called the bounded analytic function space. For $0 < \alpha < \infty$, the Korenblum space H_α^∞ , also called weighted type space, consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

Each H_α^∞ is a Banach space. For $0 < \alpha < \beta$, the inclusion $H_\alpha^\infty \subset H_\beta^\infty$ holds with contractive embedding. For $0 < \alpha < \infty$, the α -Bloch space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ with

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The space $\mathcal{B}^1$ is the classical Bloch space $\mathcal{B}$. For more results on these spaces mentioned above, we refer to [20].

For $x > 0$ and $y > 0$, the Beta function is defined by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$. For $0 < \alpha < 1$, we use the following identity frequently:

$$B(1 + \alpha, 1 - \alpha) = \alpha B(\alpha, 1 - \alpha) = \frac{\alpha\pi}{\sin(\pi\alpha)}.$$

The classical Hilbert matrix $\mathcal{H} = \left(\frac{1}{n+k+1}\right)_{n,k \geq 0}$ induces an operator on $H(\mathbb{D})$ by its action on the Taylor coefficients as follows:

$$\mathcal{H}f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n,$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Diamantopoulos and Siskakis [8] showed that for $f \in H^1$,

$$\mathcal{H}f(z) = \int_0^1 \frac{f(t)}{1-tz} dt. \quad (1.1)$$

In 2000, Diamantopoulos and Siskakis [8] initiated the study of the Hilbert operator $\mathcal{H}$ on Hardy spaces. They proved that $\mathcal{H}$ is not bounded on H^1 , nor on H^∞ , whereas $\mathcal{H} : H^p \rightarrow H^p$ is bounded for $1 < p < \infty$. In 2008, Dostanić, Jevtić, and Vukotić [7] computed the exact norm of $\mathcal{H}$ acting on H^p :

$$\|\mathcal{H}\|_{H^p \rightarrow H^p} = \frac{\pi}{\sin \frac{\pi}{p}}.$$

Diamantopoulos [5] studied the boundedness of $\mathcal{H}$ on the Bergman spaces A^p . Dostanić, Jevtić, and Vukotić [7] proved that

$$\|\mathcal{H}\|_{A^p \rightarrow A^p} = \frac{\pi}{\sin \frac{2\pi}{p}}$$

for all $p \geq 4$. Bozin and Karapetrović [1] extended this formula to the range $2 < p < 4$ (see also [2, 6, 16] for norm estimates on Bergman spaces). The exact norm of $\mathcal{H}$ on H^∞ was first obtained by Lindström, Miihkinen, and Wikman [13] for $0 < \alpha \leq 2/3$, and subsequently extended by Dai [4] to $0 < \alpha \leq \alpha_0$, where $\alpha_0 \approx 0.7125258$ is determined by $\frac{2\alpha_0-1}{2} B(\frac{1}{2}, 1-\alpha_0) = 1$.

In this paper, we study the differential Hilbert operator $\mathcal{H}_2$, obtained by replacing the kernel $(1-tz)^{-1}$ in (1.1) with its square, where

$$\mathcal{H}_2 f(z) = \int_0^1 \frac{f(t)}{(1-tz)^2} dt. \quad (1.2)$$

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, a straightforward expansion yields

$$\mathcal{H}_2 f(z) = \sum_{n=0}^{\infty} (n+1) \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n. \quad (1.3)$$

The term “differential” shows the relation $(1-tz)^{-2} = t^{-1} \frac{d}{dz} (1-tz)^{-1}$. It is easy to check that

$$\mathcal{H}_2 f = z(\mathcal{H}f)' + \mathcal{H}f.$$

Indeed,

$$(\mathcal{H}f)'(z) = \int_0^1 \frac{tf(t)}{(1-tz)^2} dt,$$

and therefore,

$$z(\mathcal{H}f)'(z) + \mathcal{H}f(z) = \int_0^1 \left(\frac{tz}{(1-tz)^2} + \frac{1}{1-tz} \right) f(t) dt = \int_0^1 \frac{f(t)}{(1-tz)^2} dt = \mathcal{H}_2 f(z).$$

The operator $\mathcal{H}_2$ also arises naturally in the following contexts.

First, $\mathcal{H}_2$ is a special case of the derivative-Hilbert operator I_{μ_α} , introduced by Ye and Zhou [18],

$$I_{\mu_\alpha} f(z) = \int_0^1 \frac{f(t)}{(1-tz)^\alpha} d\mu(t),$$

where μ is a positive Borel measure on the interval $[0, 1)$. If $\alpha = \beta + 1$ and $d\mu(t) = (1-t)^\beta dt$, then I_{μ_α} reduces to the generalized Hilbert operator $\mathcal{H}_\beta$, which was introduced in [10, 12]. See [3] for related results. Further results on derivative-Hilbert operators and on generalized Hilbert operators may be found in [11, 17, 19].

Second, $\mathcal{H}_2$ is a special case of the general Hilbert-type operator $\mathcal{H}_g$, corresponding to the symbol $g(z) = z/(1-z)$. For $g \in H(\mathbb{D})$, Galanopoulos et al. [9] introduced the operator

$$\mathcal{H}_g f(z) = \int_0^1 f(t) g'(tz) dt, \quad f \in H(\mathbb{D}).$$

The classical Hilbert operator $\mathcal{H}$ is recovered by taking $g(z) = \log \frac{1}{1-z}$. Choosing $\tilde{g}(z) = \frac{z}{1-z}$ gives $\tilde{g}'(z) = (1-z)^{-2}$, and therefore,

$$\mathcal{H}_{\tilde{g}} f(z) = \int_0^1 \frac{f(t)}{(1-tz)^2} dt = \mathcal{H}_2 f(z).$$

In [9], the boundedness of $\mathcal{H}_g$ was studied on the Hardy spaces H^p , on the weighted Bergman spaces A_α^p , and on the Dirichlet-type spaces D_α^p . Further results on $\mathcal{H}_g$ are shown in [14, 15].

Third, $\mathcal{H}_2$ is related to the Bergman projection. The Bergman kernel on $\mathbb{D}$ takes the form $K(z, w) = \frac{1}{(1-\bar{w}z)^2}$, and the Bergman projection is defined as

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^2} dA(w).$$

It is clear that the operator $\mathcal{H}_2$ and the Bergman projection P share the same kernel function $(1-\cdot z)^{-2}$, differing only in the integration domain ($[0, 1]$ with Lebesgue measure vs $\mathbb{D}$ with normalized area measure).

Finally, the behavior of $\mathcal{H}_2$ differs from that of the classical Hilbert operator in several respects, as the boundedness and norm results below will make explicit; we return to this comparison at the end of the paper.

Motivated by the above mentioned results, in this paper, we mainly study the exact norm of the differential Hilbert operator $\mathcal{H}_2$ on some function spaces, such as H^∞ , Korenblum spaces H_α^∞ , and α -Bloch spaces $\mathcal{B}^\alpha$. Let $0 < \alpha, \beta < \infty$. Set

$$G_{\alpha,\beta}(r) = (1+r)^\beta(1-r)^{\beta-1-\alpha} \int_0^1 \frac{(1-r+tr)^{2\alpha}}{(1-t)^\alpha(1+t-r+tr)^\alpha} dt, \quad 0 < r < 1. \quad (1.4)$$

Our main results are stated as follows:

Theorem 1.1. *Let $0 < \alpha, \beta < \infty$. The operator $\mathcal{H}_2: H_\alpha^\infty \rightarrow H_\beta^\infty$ is bounded if and only if $0 < \alpha < 1$ and $\beta \geq 1 + \alpha$. Moreover,*

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_\beta^\infty} = \sup_{0 \leq r < 1} G_{\alpha,\beta}(r). \quad (1.5)$$

Theorem 1.2. *Let $0 < \alpha < 1$ satisfy that $G_{\alpha,1+\alpha}(r)$ is increasing on $(0, 1)$. Then*

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_{1+\alpha}^\infty} = \frac{2\alpha\pi}{\sin(\pi\alpha)}. \quad (1.6)$$

Theorem 1.3. *For $0 < \alpha < 1$,*

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow \mathcal{B}^{\alpha+2}} = \frac{1}{2}B\left(\frac{1}{2}, 1-\alpha\right) + \frac{4\alpha(\alpha+1)\pi}{\sin(\alpha\pi)}.$$

Theorem 1.4. *The operator $\mathcal{H}_2$ maps H^∞ boundedly into H_1^∞ and $\mathcal{B}^2$. Moreover,*

$$\|\mathcal{H}_2\|_{H^\infty \rightarrow H_1^\infty} = 2, \quad \|\mathcal{H}_2\|_{H^\infty \rightarrow \mathcal{B}^2} = 5.$$

Two remarks are given as follows:

Remark 1.1. The boundedness assertion in Theorem 1.1 can be deduced by specializing [15, Theorem 4.10] to the symbol $g(z) = z/(1-z)$. For completeness, we present an independent self-contained proof here. The key point is that in this proof, we obtain the norm formula (1.5). Here, we need to emphasize that our main purpose of this paper is to study norms. Tang [15] provided a full characterization for boundedness and compactness, but the operator norms are not covered. Passing from such a criterion to an exact numerical value requires a different, quantitative analysis, which is carried out in the present paper. Consequently, using formula (1.5), Theorems 1.2–1.4 cannot be deduced from the results established in [15].

Remark 1.2. The monotonicity hypothesis on $G_{\alpha,1+\alpha}$ in Theorem 1.2 is used only to identify the supremum in (1.5) with the limit as $r \rightarrow 1^-$. In Corollary 3.1 below, we need the hypothesis for $\frac{1}{2} \leq \alpha < 1$. For $0 < \alpha < \frac{1}{2}$ the monotonicity of $G_{\alpha,1+\alpha}$ on $(0, 1)$ is not known. The argument of Corollary 3.1 breaks down in this range, because the exponent $2\alpha - 1$ becomes negative, and the integrand in the formula for $G'_{\alpha,1+\alpha}(r)$ no longer has a fixed sign. Deciding whether $G_{\alpha,1+\alpha}$ is increasing on $(0, 1)$ for $0 < \alpha < \frac{1}{2}$ is an interesting open question.

The paper is organized as follows: In Section 2, we establish necessary and sufficient conditions for the boundedness of $\mathcal{H}_2: H_\alpha^\infty \rightarrow H_\beta^\infty$ and the corresponding norm formula. In Section 3, we determine the exact norm of $\mathcal{H}_2$ in the borderline case $\beta = 1 + \alpha$. In Section 4, we compute the norm of $\mathcal{H}_2: H_\alpha^\infty \rightarrow \mathcal{B}^{\alpha+2}$. In Section 5, we compute the norm of $\mathcal{H}_2: H^\infty \rightarrow H_1^\infty$ and $\mathcal{H}_2: H^\infty \rightarrow \mathcal{B}^2$.

In this paper, we write $X \asymp Y$ if there exists a constant $C > 0$ satisfying $X \leq CY$ and $Y \leq CX$.

2. Boundedness criterion

In this section, we prove Theorem 1.1, establishing necessary and sufficient conditions for the boundedness of $\mathcal{H}_2: H_\alpha^\infty \rightarrow H_\beta^\infty$ and give the exact norm of $\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_\beta^\infty}$.

Proof of Theorem 1.1. Necessity. Consider the test function

$$f_\alpha(z) = (1 - z^2)^{-\alpha}, \quad z \in \mathbb{D}.$$

A standard computation shows that $\|f_\alpha\|_{H_\alpha^\infty} = 1$. Suppose $\mathcal{H}_2: H_\alpha^\infty \rightarrow H_\beta^\infty$ is bounded. Then there exists a constant $C > 0$ such that

$$(1 - r^2)^\beta |\mathcal{H}_2 f_\alpha(r)| \leq C, \quad 0 \leq r < 1.$$

By using a change of variable, we obtain

$$\mathcal{H}_2 f(z) = \frac{1}{1 - z} \int_0^1 f(\varphi_t(z)) dt, \quad (2.1)$$

where $\varphi_t(z) = t/(1 - (1 - t)z)$. Restricting (2.1) to $z = r \in [0, 1)$, we get

$$|\mathcal{H}_2 f_\alpha(r)| = \frac{1}{1 - r} \int_0^1 (1 - \varphi_t(r)^2)^{-\alpha} dt.$$

A direct computation gives

$$\begin{aligned} 1 - \varphi_t(r)^2 &= 1 - \frac{t^2}{(1 - (1 - t)r)^2} = \frac{(1 - (1 - t)r)^2 - t^2}{(1 - (1 - t)r)^2} \\ &= \frac{[(1 - (1 - t)r) - t][(1 - (1 - t)r) + t]}{(1 - (1 - t)r)^2} \\ &= \frac{(1 - t)(1 - r)(1 + t - r + tr)}{(1 - r + tr)^2}, \end{aligned}$$

which implies that

$$|\mathcal{H}_2 f_\alpha(r)| = \frac{1}{1 - r} \int_0^1 \frac{(1 - r + tr)^{2\alpha}}{(1 - t)^\alpha (1 - r)^\alpha (1 + t - r + tr)^\alpha} dt. \quad (2.2)$$

Now we assume $\alpha \geq 1$. Fix $z \in \mathbb{D}$. As $t \rightarrow 1^-$,

$$\frac{|f_\alpha(t)|}{|1 - tz|^2} = \frac{1}{|1 - tz|^2(1 - t^2)^\alpha} \asymp \frac{1}{|1 - z|^2 2^\alpha (1 - t)^\alpha}.$$

Since

$$\int_0^1 (1 - t)^{-\alpha} dt = \infty \quad \text{for } \alpha \geq 1,$$

the integral in (1.2) defining $\mathcal{H}_2 f_\alpha(z)$ fails to converge absolutely at any point $z \in \mathbb{D}$. Hence, $\mathcal{H}_2 f_\alpha$ is not defined for a function $f_\alpha \in H_\alpha^\infty$ of norm 1, and consequently, $\mathcal{H}_2$ cannot be a bounded operator from H_α^∞ into H_β^∞ . Thus, $\alpha < 1$ is necessary, and from now on, we assume $0 < \alpha < 1$.

As $r \rightarrow 1^-$,

$$(1 - \varphi_t(r)^2)^{-\alpha} \asymp \frac{t^\alpha}{2^\alpha(1-t)^\alpha(1-r)^\alpha}.$$

Therefore,

$$|\mathcal{H}_2 f_\alpha(r)| \asymp \frac{1}{2^\alpha(1-r)^{1+\alpha}} \int_0^1 \frac{t^\alpha}{(1-t)^\alpha} dt = \frac{B(1+\alpha, 1-\alpha)}{2^\alpha(1-r)^{1+\alpha}}.$$

Consequently,

$$(1-r^2)^\beta |\mathcal{H}_2 f_\alpha(r)| \asymp \frac{B(1+\alpha, 1-\alpha)}{2^\alpha} (1-r)^{\beta-1-\alpha} \cdot 2^\beta, \quad r \rightarrow 1^-. \quad (2.3)$$

If $\beta < 1 + \alpha$, then $\beta - 1 - \alpha < 0$, and the right-hand side of (2.3) diverges as $r \rightarrow 1^-$, contradicting boundedness. Hence, $\beta \geq 1 + \alpha$ is necessary.

Sufficiency. Assume $0 < \alpha < 1$ and $\beta \geq 1 + \alpha$. Let $f \in H_\alpha^\infty$ with $\|f\|_{H_\alpha^\infty} \leq 1$. Then

$$|f(z)| \leq (1 - |z|^2)^{-\alpha}, \quad z \in \mathbb{D}. \quad (2.4)$$

For any $z \in \mathbb{D}$, by (2.1) and (2.4), we have

$$|\mathcal{H}_2 f(z)| \leq \frac{1}{|1-z|} \int_0^1 (1 - |\varphi_t(z)|^2)^{-\alpha} dt.$$

It suffices to estimate the supremum along the radius $z = r \in [0, 1)$, since for $z = re^{i\theta}$, one has $1 - |\varphi_t(z)|^2 \geq 1 - \varphi_t(r)^2$, so the radial case provides the largest contribution.

For $r \in [0, 1)$,

$$(1-r^2)^\beta |\mathcal{H}_2 f(r)| \leq (1+r)^\beta (1-r)^{\beta-1-\alpha} \int_0^1 \frac{(1-r+tr)^{2\alpha}}{(1-t)^\alpha(1+t-r+tr)^\alpha} dt.$$

It suffices to prove that $\sup_{0 \leq r < 1} G_{\alpha, \beta}(r) < \infty$.

For $r \in [0, \frac{1}{2}]$, the factor $(1-r)^{\beta-1-\alpha}$ is bounded by 1 when $\beta \geq 1 + \alpha$. The integral is bounded uniformly on this interval: For $0 \leq r \leq \frac{1}{2}$ and $0 \leq t \leq 1$, one has $1-r+tr \leq 1$ and $1+t-r+tr \geq \frac{1}{2}$, so the integrand is dominated by $2^\alpha(1-t)^{-\alpha}$, which is integrable since $\alpha < 1$. Hence, $\sup_{0 \leq r \leq 1/2} G_{\alpha, \beta}(r) < \infty$.

For $r \in [\frac{1}{2}, 1)$, set $s = 1 - t$ in the integral of (1.4),

$$\begin{aligned} G_{\alpha, \beta}(r) &= (1+r)^\beta (1-r)^{\beta-1-\alpha} \int_0^1 \frac{(1-rs)^{2\alpha}}{s^\alpha(2-s-r+rs)^\alpha} ds \\ &= (1+r)^\beta (1-r)^{\beta-1-\alpha} \int_0^1 \frac{(1-rs)^{2\alpha}}{s^\alpha(2-s(1+r))^\alpha} ds. \end{aligned}$$

Now, we split the integral at $s = \frac{1}{2}$. On $[0, \frac{1}{2}]$, the integrand is bounded by $s^{-\alpha}$ and

$$\int_0^{1/2} \frac{(1-rs)^{2\alpha}}{s^\alpha(2-s(1+r))^\alpha} ds \leq \int_0^{1/2} s^{-\alpha} ds = \frac{(\frac{1}{2})^{1-\alpha}}{1-\alpha},$$

which is finite uniformly in r because $0 < \alpha < 1$. On $[\frac{1}{2}, 1)$, put $A = 1 - rs, B = 2 - s(1 + r)$. Since $B = A + (1 - s) \geq A$, it follows that

$$\frac{(1 - rs)^{2\alpha}}{s^\alpha [2 - s(1 + r)]^\alpha} = \frac{1}{s^\alpha} \left(\frac{A^2}{B} \right)^\alpha \leq \frac{A^\alpha}{s^\alpha} \leq 2^\alpha, \quad \frac{1}{2} \leq s \leq 1.$$

On $[0, \frac{1}{2}]$, one has $B \geq 1$, and hence the integrand is bounded by $s^{-\alpha}$, which is integrable because $0 < \alpha < 1$. This gives the required estimate uniformly in r and completes the sufficiency argument.

It remains to establish the norm formula. The upper bound is contained in the estimate just proved. For the lower bound, we evaluate $\mathcal{H}_2$ on the test function f_α . Since $\|f_\alpha\|_{H_\alpha^\infty} = 1$, identity (2.2) gives, for every $r \in [0, 1)$,

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_\beta^\infty} \geq \|\mathcal{H}_2 f_\alpha\|_{H_\beta^\infty} \geq (1 - r^2)^\beta |\mathcal{H}_2 f_\alpha(r)| = G_{\alpha, \beta}(r).$$

Taking the supremum over r yields the reverse inequality in (1.5). The proof is complete.

3. Norm in the borderline case $\beta = 1 + \alpha$

Throughout this section, we assume $0 < \alpha < 1$ and $\beta = 1 + \alpha$. Then function $G_{\alpha, 1+\alpha}$ from (1.4) simplifies to

$$G_{\alpha, 1+\alpha}(r) = (1 + r)^{1+\alpha} \int_0^1 \frac{(1 - r + tr)^{2\alpha}}{(1 - t)^\alpha (1 + t - r + tr)^\alpha} dt. \quad (3.1)$$

Set $t = \frac{u(1 - r)}{1 - ru}$. A direct computation yields

$$dt = \frac{1 - r}{(1 - ru)^2} du, \quad 1 - t = \frac{1 - u}{1 - ru}, \quad 1 - r + tr = \frac{1 - r}{1 - ru},$$

and $1 + t - r + tr = \frac{(1 - r)(1 + u)}{1 - ru}$. Substituting these into the integrand of (3.1), we obtain

$$G_{\alpha, 1+\alpha}(r) = (1 - r^2)^{1+\alpha} \int_0^1 \frac{du}{(1 - u^2)^\alpha (1 - ru)^2}. \quad (3.2)$$

The integral $G_{\alpha, 1+\alpha}(r)$ converges for each fixed $r \in (0, 1)$. Near $u = 1$, the asymptotic relation $(1 - u^2)^{-\alpha} \asymp (1 - u)^{-\alpha}$ holds, and the integrability follows from $\alpha < 1$. Moreover, $(1 - ru)^{-2} \leq (1 - r)^{-2}$ remains bounded away from this singularity. Thus, $G_{\alpha, 1+\alpha}$ is well-defined on $(0, 1)$.

Proof of Theorem 1.2. By (1.5) and the assumption that $G_{\alpha, 1+\alpha}$ is increasing in $(0, 1)$,

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_{1+\alpha}^\infty} = \sup_{0 \leq r < 1} G_{\alpha, 1+\alpha}(r) = \lim_{r \rightarrow 1^-} G_{\alpha, 1+\alpha}(r). \quad (3.3)$$

Now we need to only prove that

$$\lim_{r \rightarrow 1^-} G_{\alpha, 1+\alpha}(r) = \frac{2\alpha\pi}{\sin(\pi\alpha)}.$$

Set $u = 1 - y(1 - r)$. Then, $du = -(1 - r) dy$, $1 - u^2 = y(1 - r)(2 - y(1 - r))$, and $1 - ru = (1 - r)(1 + ry)$. We have

$$G_{\alpha, 1+\alpha}(r) = (1 + r)^{1+\alpha} \int_0^{\frac{1}{1-r}} \frac{dy}{y^\alpha (2 - y(1 - r))^\alpha (1 + ry)^2}. \quad (3.4)$$

Using the form (3.4) with $s = 1 - r$,

$$G_{\alpha,1+\alpha}(r) = (2-s)^{1+\alpha} \int_0^{1/s} \frac{dy}{y^\alpha(2-ys)^\alpha(1+(1-s)y)^2}.$$

As $r \rightarrow 1^-$, the integrand converges pointwise to $y^{-\alpha} 2^{-\alpha} (1+y)^{-2}$. For $r \in [\frac{1}{2}, 1)$, the integrand in (3.4) is bounded by

$$\frac{1}{y^\alpha \cdot 1^\alpha \cdot (1 + \frac{1}{2}y)^2} \leq \frac{4}{y^\alpha(2+y)^2},$$

which is integrable on $(0, \infty)$ for $0 < \alpha < 1$. We obtain

$$\lim_{r \rightarrow 1^-} G_{\alpha,1+\alpha}(r) = 2^{1+\alpha} \cdot 2^{-\alpha} \int_0^\infty \frac{dy}{y^\alpha(1+y)^2} = \frac{2\alpha\pi}{\sin(\pi\alpha)}.$$

This completes the proof of Theorem 1.2.

Corollary 3.1. For $1/2 \leq \alpha < 1$,

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_{1+\alpha}^\infty} = \frac{2\alpha\pi}{\sin(\pi\alpha)}.$$

Proof. By Theorem 1.2, we need to only prove that $G_{\alpha,1+\alpha}(r)$ is strictly increasing on $(0, 1)$.

Split the integral at $u = r$ and apply the substitutions

$$\begin{aligned} \text{on } [r, 1] : \quad & v = \frac{u-r}{1-ru}, \quad u = \frac{v+r}{1+vr}, \quad du = \frac{1-r^2}{(1+vr)^2} dv, \\ \text{on } [0, r] : \quad & v = \frac{r-u}{1-ru}, \quad u = \frac{r-v}{1-vr}, \quad du = -\frac{1-r^2}{(1-vr)^2} dv. \end{aligned}$$

We arrive at

$$G_{\alpha,1+\alpha}(r) = \int_0^r (1-rv)^{2\alpha}(1-v^2)^{-\alpha} dv + \int_0^1 (1+rv)^{2\alpha}(1-v^2)^{-\alpha} dv.$$

After a calculation,

$$\begin{aligned} G'_{\alpha,1+\alpha}(r) &= (1-r^2)^\alpha + 2\alpha \int_0^r v[(1+rv)^{2\alpha-1} - (1-rv)^{2\alpha-1}](1-v^2)^{-\alpha} dv \\ &\quad + 2\alpha \int_r^1 v(1+rv)^{2\alpha-1}(1-v^2)^{-\alpha} dv. \end{aligned}$$

By the assumption that $\alpha \geq \frac{1}{2}$, we get $2\alpha - 1 \geq 0$. For $0 < v < r$, we have $1+rv > 1-rv$, and hence

$$(1+rv)^{2\alpha-1} \geq (1-rv)^{2\alpha-1}.$$

Consequently, $G'_{\alpha,1+\alpha}(r) > 0$ for all $r \in (0, 1)$, which implies that $G_{\alpha,1+\alpha}$ is strictly increasing on $(0, 1)$. This completes the proof.

4. Norm of $\mathcal{H}_2$ from H_α^∞ to $\mathcal{B}^{\alpha+2}$

In this section, we compute the exact norm of $\mathcal{H}_2$ from H_α^∞ to $\mathcal{B}^{\alpha+2}$.

Lemma 4.1. *For every $0 < \alpha < 1$, the function*

$$F(r) = (1 - r^2)^{\alpha+2} \int_0^1 \frac{t \, dt}{(1 - tr)^3 (1 - t^2)^\alpha}, \quad 0 < r < 1 \quad (4.1)$$

is strictly increasing on $(0, 1)$.

Proof. The integrand is nonnegative, and the integral converges. It is clear that

$$(1 - tr)^{-3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} (tr)^n, \quad 0 \leq r < 1, \quad 0 \leq t \leq 1.$$

For each fixed $r < 1$, it gives

$$F(r) = (1 - r^2)^{\alpha+2} \Phi(r), \quad \Phi(r) = \sum_{n=0}^{\infty} c_n r^n,$$

where

$$c_n = \frac{(n+1)(n+2)}{2} \int_0^1 t^{n+1} (1 - t^2)^{-\alpha} dt = \frac{(n+1)(n+2)}{4} B\left(\frac{n+2}{2}, 1 - \alpha\right) > 0. \quad (4.2)$$

After a calculation, we get

$$F'(r) = (1 - r^2)^{\alpha+1} [\Phi'(r)(1 - r^2) - 2(\alpha + 2)r\Phi(r)] = (1 - r^2)^{\alpha+1} \Psi(r),$$

where

$$\Psi(r) = \Phi'(r)(1 - r^2) - 2(\alpha + 2)r\Phi(r).$$

Since $(1 - r^2)^{\alpha+1} > 0$, it suffices to prove $\Psi(r) > 0$. Write

$$\Psi(r) = \sum_{m=0}^{\infty} d_m r^m,$$

where $d_0 = c_1$, $d_1 = 2c_2 - 2(\alpha + 2)c_0$, and

$$d_m = (m+1)c_{m+1} - (m+2\alpha+3)c_{m-1} \quad (m \geq 2).$$

We claim that $d_m > 0$ for every $m \geq 0$. It is easy to check that $d_0 > 0$ and $d_1 > 0$. Finally, let $m \geq 2$. Again by the Beta recurrence,

$$B\left(\frac{m+3}{2}, 1 - \alpha\right) = \frac{m+1}{m+3-2\alpha} B\left(\frac{m+1}{2}, 1 - \alpha\right),$$

and consequently,

$$d_m = \frac{m+1}{4} B\left(\frac{m+1}{2}, 1 - \alpha\right) \frac{(m+1)(m+2)(m+3) - m(m+2\alpha+3)(m+3-2\alpha)}{m+3-2\alpha}.$$

It is easy to see that

$$(m+1)(m+2)(m+3) - m(m+2\alpha+3)(m+3-2\alpha) = 2m(1+2\alpha^2) + 6 > 0.$$

Hence, $d_m > 0$ for all $m \geq 2$. Thus, $\Psi(r) > 0$ for $r \in (0, 1)$. It follows that $F'(r) > 0$ on $(0, 1)$, and F is strictly increasing.

Proof of Theorem 1.3. Upper bound. Let $f \in H_\alpha^\infty$ with $\|f\|_{H_\alpha^\infty} = 1$. From the definition (1.2), it is easy to see that

$$(\mathcal{H}_2 f)'(z) = 2 \int_0^1 \frac{t f(t)}{(1-tz)^3} dt, \quad z \in \mathbb{D}, \quad (4.3)$$

and

$$\mathcal{H}_2 f(0) = \int_0^1 f(t) dt. \quad (4.4)$$

By (2.4) and (4.4),

$$|\mathcal{H}_2 f(0)| = \left| \int_0^1 f(t) dt \right| \leq \int_0^1 \frac{dt}{(1-t^2)^\alpha} = \frac{1}{2} \int_0^1 u^{-1/2} (1-u)^{-\alpha} du = \frac{1}{2} B\left(\frac{1}{2}, 1-\alpha\right).$$

Using (4.3) and the pointwise estimate $|f(t)| \leq (1-t^2)^{-\alpha}$,

$$\begin{aligned} |(\mathcal{H}_2 f)'(z)| &\leq 2 \int_0^1 \frac{t}{(1-t|z|)^3} \cdot \frac{dt}{(1-t^2)^\alpha} \\ &= 2 \int_0^1 \frac{t dt}{(1-tr)^3 (1-t^2)^\alpha}, \quad r = |z|. \end{aligned}$$

Hence,

$$\sup_{z \in \mathbb{D}} (1-|z|^2)^{\alpha+2} |(\mathcal{H}_2 f)'(z)| \leq 2 \sup_{0 \leq r < 1} (1-r^2)^{\alpha+2} \int_0^1 \frac{t dt}{(1-tr)^3 (1-t^2)^\alpha}. \quad (4.5)$$

Set $\varepsilon = 1-r$ and substitute $s = 1-t$, so $t = 1-s$. Then

$$\int_0^1 \frac{t dt}{(1-tr)^3 (1-t^2)^\alpha} = \int_0^1 \frac{1-s}{(\varepsilon + (1-\varepsilon)s)^3 (2s-s^2)^\alpha} ds.$$

Set $s = \varepsilon u$. Then

$$\begin{aligned} \int_0^1 \frac{t dt}{(1-tr)^3 (1-t^2)^\alpha} &= \varepsilon \int_0^{1/\varepsilon} \frac{1-\varepsilon u}{(\varepsilon + \varepsilon(1-\varepsilon)u)^3 (2\varepsilon u - (\varepsilon u)^2)^\alpha} du \\ &= \varepsilon^{-2-\alpha} \int_0^{1/\varepsilon} \frac{1-\varepsilon u}{(1 + (1-\varepsilon)u)^3 (2u - \varepsilon u^2)^\alpha} du. \end{aligned}$$

As $\varepsilon \rightarrow 0^+$, the integrand converges pointwise to $u^{-\alpha} 2^{-\alpha} (1+u)^{-3}$ for each fixed $u > 0$. When $0 < \varepsilon \leq 1/2$ and $0 < u < 1/\varepsilon$, one has

$$2u - \varepsilon u^2 \geq u, \quad 1 + (1-\varepsilon)u \geq 1 + \frac{u}{2}.$$

Thus, the integrand is bounded by $Cu^{-\alpha}(1+u)^{-3}$, which belongs to $L^1(0, \infty)$ for $0 < \alpha < 1$. The dominated convergence theorem gives

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2+\alpha} \int_0^1 \frac{t \, dt}{(1-tr)^3(1-t^2)^\alpha} = 2^{-\alpha} \int_0^\infty \frac{u^{-\alpha}}{(1+u)^3} \, du = \frac{(\alpha+1)\alpha\pi}{2^{\alpha+1} \sin(\alpha\pi)}.$$

Returning to (4.5), multiply by $2(1-r^2)^{\alpha+2}$. We get

$$\lim_{r \rightarrow 1^-} 2(1-r^2)^{\alpha+2} \int_0^1 \frac{t \, dt}{(1-tr)^3(1-t^2)^\alpha} = \frac{4\alpha(\alpha+1)\pi}{\sin(\alpha\pi)}.$$

By Lemma 4.1, $F(r)$ is increasing on $[0, 1)$. Thus, the supremum equals the limit. This yields

$$\sup_{z \in \mathbb{D}} (1-|z|^2)^{\alpha+2} |(\mathcal{H}_2 f)'(z)| \leq \frac{4\alpha(\alpha+1)\pi}{\sin(\alpha\pi)}.$$

Therefore,

$$\|\mathcal{H}_2 f\|_{\mathcal{B}^{\alpha+2}} \leq \frac{1}{2} B\left(\frac{1}{2}, 1-\alpha\right) + \frac{4\alpha(\alpha+1)\pi}{\sin(\alpha\pi)}.$$

Lower bound. Set $f_\alpha(z) = (1-z^2)^{-\alpha}$. Then $\|f_\alpha\|_{H^\infty} = 1$. We have

$$\mathcal{H}_2 f_\alpha(0) = \int_0^1 \frac{dt}{(1-t^2)^\alpha} = \frac{1}{2} B\left(\frac{1}{2}, 1-\alpha\right)$$

and

$$(\mathcal{H}_2 f_\alpha)'(z) = 2 \int_0^1 \frac{t}{(1-tz)^3(1-t^2)^\alpha} \, dt.$$

Taking $z = r$ and letting $r \rightarrow 1^-$, the limit computed in the above proof gives

$$\|\mathcal{H}_2 f_\alpha\|_{\mathcal{B}^{\alpha+2}} \geq \frac{1}{2} B\left(\frac{1}{2}, 1-\alpha\right) + \frac{4\alpha(\alpha+1)\pi}{\sin(\alpha\pi)}.$$

The proof of this theorem is complete.

5. Norm of $\mathcal{H}_2 : H^\infty \rightarrow H_1^\infty$ and $\mathcal{H}_2 : H^\infty \rightarrow \mathcal{B}^2$

Proof of Theorem 1.4. We first consider $\|\mathcal{H}_2\|_{H^\infty \rightarrow H_1^\infty}$.

Upper bound. Let $f \in H^\infty$ with $\|f\|_{H^\infty} = 1$. From (2.1),

$$|\mathcal{H}_2 f(z)| \leq \frac{1}{1-|z|} \int_0^1 |f(\varphi_t(z))| \, dt \leq \frac{1}{1-|z|}.$$

Hence,

$$(1-|z|^2)|\mathcal{H}_2 f(z)| \leq (1+|z|) \leq 2,$$

which gives $\|\mathcal{H}_2 f\|_{H_1^\infty} \leq 2$.

Lower bound. Take $f(z) \equiv 1$. Then by (1.3) with $a_0 = 1$ and $a_k = 0$ for $k \geq 1$, we obtain

$$\mathcal{H}_2(1)(z) = \sum_{n=0}^{\infty} z^n = 1/(1-z).$$

Consequently,

$$\|\mathcal{H}_2(1)\|_{H_1^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \frac{1}{|1 - z|} = \sup_{0 \leq r < 1} \frac{1 - r^2}{1 - r} = 2.$$

Since $\|1\|_{H^\infty} = 1$, we obtain $\|\mathcal{H}_2\|_{H^\infty \rightarrow H_1^\infty} \geq 2$, as desired.

Next we compute $\|\mathcal{H}_2\|_{H^\infty \rightarrow \mathcal{B}^2}$.

Upper bound. Let $f \in H^\infty$ with $\|f\|_{H^\infty} = 1$. By (4.4),

$$|\mathcal{H}_2 f(0)| = \left| \int_0^1 f(t) dt \right| \leq \int_0^1 dt = 1.$$

For the derivative, using (4.3),

$$|(\mathcal{H}_2 f)'(z)| \leq 2 \int_0^1 \frac{t}{(1 - t|z|)^3} dt.$$

Setting $r = |z|$ and substituting $u = tr$, $t = u/r$, $dt = du/r$,

$$\int_0^1 \frac{t}{(1 - tr)^3} dt = \frac{1}{r^2} \int_0^r \frac{u}{(1 - u)^3} du.$$

Setting $v = 1 - u$,

$$\begin{aligned} \int_0^r \frac{u}{(1 - u)^3} du &= \int_{1-r}^1 \frac{v-1}{v^3} dv = \int_{1-r}^1 (v^{-2} - v^{-3}) dv \\ &= \left[-\frac{1}{v} + \frac{1}{2v^2} \right]_{1-r}^1 = -\frac{1}{2} + \frac{1}{1-r} - \frac{1}{2(1-r)^2} \\ &= \frac{r^2}{2(1-r)^2}. \end{aligned}$$

Therefore,

$$\int_0^1 \frac{t}{(1 - tr)^3} dt = \frac{1}{r^2} \cdot \frac{r^2}{2(1-r)^2} = \frac{1}{2(1-r)^2}$$

and

$$|(\mathcal{H}_2 f)'(z)| \leq \frac{1}{(1-r)^2}.$$

Thus,

$$(1 - |z|^2)^2 |(\mathcal{H}_2 f)'(z)| \leq \frac{(1 - r^2)^2}{(1 - r)^2} = (1 + r)^2 \leq 4.$$

Combining the two estimates, we obtain

$$\|\mathcal{H}_2 f\|_{\mathcal{B}^2} \leq 1 + 4 = 5.$$

Lower bound. Take $f(z) \equiv 1$, which satisfies $\|f\|_{H^\infty} = 1$. As noted above, $\mathcal{H}_2(1)(z) = 1/(1 - z)$. Then

$$\mathcal{H}_2(1)(0) = 1, \quad (\mathcal{H}_2(1))'(z) = \frac{1}{(1 - z)^2}.$$

Hence,

$$\|\mathcal{H}_2(1)\|_{\mathcal{B}^2} = 1 + \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 \frac{1}{|1 - z|^2} = 1 + \sup_{0 \leq r < 1} (1 + r)^2 = 1 + 4 = 5.$$

Therefore, $\|\mathcal{H}_2\|_{H^\infty \rightarrow \mathcal{B}^2} \geq 5$, and the desired result follows. The proof is complete.

Remark 5.1. The constant 5 in Theorem 1.4 is the limit, as $\alpha \rightarrow 0^+$, of the norm occurring in Theorem 1.3. Indeed,

$$\lim_{\alpha \rightarrow 0^+} \frac{1}{2} B\left(\frac{1}{2}, 1 - \alpha\right) = \frac{1}{2} B\left(\frac{1}{2}, 1\right) = 1$$

and

$$\lim_{\alpha \rightarrow 0^+} \frac{4\alpha(\alpha + 1)\pi}{\sin(\alpha\pi)} = \lim_{\alpha \rightarrow 0^+} \frac{4\alpha\pi}{\alpha\pi} = 4$$

give the desired result.

We conclude this paper by comparing $\mathcal{H}_2$ with $\mathcal{H}$ on Korenblum spaces. The operator $\mathcal{H}$ is bounded on H_α^∞ precisely when $0 < \alpha < 1$, while $\mathcal{H}_2 : H_\alpha^\infty \rightarrow H_\beta^\infty$ requires $\beta \geq 1 + \alpha$. The exact norm of $\mathcal{H}$ acting on H_α^∞ has been investigated by several authors. Among others, Dai [4] proved the following results:

- (i) If $0 < \alpha \leq \alpha_0$, then $\|\mathcal{H}\|_{H_\alpha^\infty \rightarrow H_\alpha^\infty} = \pi / \sin(\pi\alpha)$.
- (ii) If $\alpha_0 < \alpha < 1$, then

$$\frac{\pi}{\sin(\pi\alpha)} < \|\mathcal{H}\|_{H_\alpha^\infty \rightarrow H_\alpha^\infty} < [1 + (2\alpha - 1)(2^{1-\alpha} - 1)] \frac{\pi}{\sin(\pi\alpha)}.$$

From these, we see that

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_{1+\alpha}^\infty} = \frac{2\alpha\pi}{\sin(\pi\alpha)} = 2\alpha \|\mathcal{H}\|_{H_\alpha^\infty \rightarrow H_\alpha^\infty}$$

when $1/2 \leq \alpha < \alpha_0$. Thus, $\mathcal{H}_2$ and $\mathcal{H}$ differ on Korenblum spaces in the range of boundedness and in the size of their norms.

6. Conclusions

We studied the differential Hilbert operator $\mathcal{H}_2$ on Korenblum spaces and related spaces of analytic functions, with particular emphasis on its exact norms. For positive α and β , we proved that $\mathcal{H}_2 : H_\alpha^\infty \rightarrow H_\beta^\infty$ is bounded if and only if $0 < \alpha < 1$ and $\beta \geq 1 + \alpha$, and obtained an exact representation of its norm as the supremum of $G_{\alpha,\beta}$. The integral representation of the operator, together with radial estimates and the test function $f_\alpha(z) = (1 - z^2)^{-\alpha}$, provides upper and lower bounds. In the case $\beta = 1 + \alpha$, we established the monotonicity of $G_{\alpha,1+\alpha}$ for $1/2 \leq \alpha < 1$ and consequently obtained

$$\|\mathcal{H}_2\|_{H_\alpha^\infty \rightarrow H_{1+\alpha}^\infty} = \frac{2\alpha\pi}{\sin(\pi\alpha)}.$$

We also determined the exact norm of $\mathcal{H}_2 : H_\alpha^\infty \rightarrow \mathcal{B}^{\alpha+2}$ for the full range $0 < \alpha < 1$, as well as the norms $\|\mathcal{H}_2\|_{H^\infty \rightarrow H_1^\infty} = 2$ and $\|\mathcal{H}_2\|_{H^\infty \rightarrow \mathcal{B}^2} = 5$.

Author contributions

Xiangling Zhu: Writing—original draft; Songxiao Li: Supervision, Writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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