



Research article**A new logarithmic transformation for generating flexible probability distributions: Statistical properties, inference, and applications****Aadil Ahmad Mir^{1,*}, A. A. Bhat², S. P. Ahmad¹, Badr S. Alnssyan^{3,*} and Abdelaziz Alsubie⁴**¹ Department of Statistics, University of Kashmir, Srinagar 190006, India² Department of Mathematical Sciences, Islamic University of Science and Technology, Awantipora 192122, India³ Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 52571, Saudi Arabia⁴ Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh, Saudi Arabia*** Correspondence:** Email: aadilmir.stscholar@kashmiruniversity.net; b.alnssyan@qu.edu.sa.

Abstract: Real-world lifetime data often exhibit complex hazard-rate patterns that conventional models, such as the Weibull distribution, may not adequately capture. This study introduces a new three-parameter New Log-G (NLG) family of distributions, constructed through a logarithmic transformation of a baseline cumulative distribution function with an additional shape-controlling parameter α . Using the Weibull distribution as the baseline, we derive the New Log-Weibull (NLW) distribution, which accommodates increasing, decreasing, bathtub-shaped, and unimodal hazard-rate patterns. The principal theoretical properties of the proposed family are derived, including distributional functions, moments, entropy measures, and residual lifetime characteristics. Six classical estimation methods are evaluated through Monte Carlo simulations across varying sample sizes and parameter configurations. The practical usefulness of the NLW distribution is demonstrated using three real datasets on radiation susceptibility, glass fiber tensile strength, and COVID-19 mortality. Across these applications, the NLW model consistently outperforms seven established competing distributions in terms of goodness of fit. These results demonstrate the flexibility and practical utility of the proposed NLG family as a general framework for extending lifetime distributions, with the NLW model offering broad applicability in reliability engineering, materials science, and epidemiological modeling.

Keywords: New Log-Weibull distribution; reliability assessment; COVID-19 mortality modeling; classical estimation methods; simulation; goodness-of-fit evaluation

Mathematics Subject Classification: 60E05, 62E10, 62N05, 62F10, 94A17

1. Introduction

The need for adaptable probability models is particularly evident when evaluating complex datasets in survival analysis, reliability theory, actuarial modeling, and physics. Intricate underlying mechanisms in these fields frequently produce data with heavy tails, asymmetric distributions, and non-monotonic hazard dynamics that conventional models cannot adequately fit. To overcome such analytical bottlenecks, modern researchers construct expanded distribution families via functional operators, structural compounding, and parameter expansion.

Although probability distributions remain central to statistical inference, no single standard model fits the full spectrum of practical data shapes. Driven by this demand for enhanced empirical accuracy, ongoing research continuously generates flexible model classes built upon well-established baseline distributions.

The pioneering contributions to generalized model construction began with [1], who integrated an extra shape parameter into baseline distributions to enhance overall curve flexibility. Subsequently, Marshall and Olkin [2] broadened these modeling capabilities by directly scaling the survival function, a technique that proved crucial for regulating tail weight and survival trajectories in lifetime data modeling.

The alpha power transformation (APT), introduced by Mahdavi and Kundu [3], applies an exponential-type transformation to the baseline cumulative distribution function (CDF), providing additional control over distributional shape while retaining several classical models as special cases.

In recent years, the Aadil-Shamshad-Parvaiz (ASP) family was introduced by Mir et al. [4], who used a trigonometric sine transformation to efficiently capture asymmetry and heavy-tailed behavior in engineering and biological applications. In addition to this method, Bhat et al. [5] developed the novel alpha transformation, which maintains closed-form analytical tractability while increasing curve adaptability in both the central and tail domains.

Building upon these transformation techniques, Mir et al. [6] developed the Aadil-Ajaz-Parvaiz-X (AAP-X) family. By embedding additional transformation layers, this generator gives researchers exact control over overall geometric structure and tail weight across a variety of datasets.

Recent literature highlights further advancements in functional generators, such as the hyperbolic tangent class developed by Mohammad and Mendoza [7], the log tangent-U framework formulated by Alsolmi [8] for reliability analysis, and the trigonometric generator engineered by Odhah et al. [9] for medical data. Additional flexible generators such as the sine alpha power-G [10], the sine modified-Kies generalized class [11], and logarithmic based models [12, 13] demonstrate the continuous growth of adaptable distribution families designed for biomedical, actuarial, and engineering problems. A novel log-power family of distributions was introduced by Mir and Ahmad [14], and its statistical properties and applications were subsequently investigated.

Transformation-based generators alter baseline distributions through essentially different mathematical techniques. For example, an exponentiated family [1] improves the baseline CDF via a power transformation, whereas Marshall and Olkin [2] added a new parameter to the survival structure. Similar to this, the alpha power transformation (APT) family [3] uses an exponential-type modification of the baseline CDF to add an extra shape component. On the other hand, a logarithmic generator using the converted survival component of the baseline distribution is used to create the suggested New Log-G (NLG) family. Therefore, rather than directly exponentiating or exponentially transforming the

baseline CDF, the new parameter in the NLG family enters through a fundamentally different nonlinear mechanism. Thus, an alternative method for regulating the density shape, tail behavior, and hazard-rate geometry of the Weibull baseline is offered by the New Log–Weibull (NLW) specialization. This distinction constitutes the principal mathematical novelty of the proposed family. From a practical perspective, the logarithmic construction is especially useful for lifespan data where the baseline Weibull or traditional transformation methods are unable to sufficiently capture skewness, tail behavior, and failure-rate patterns. The ensuing extra flexibility may enhance the depiction of heterogeneous failure mechanisms in reliability and survival applications by supporting more intricate hazard-rate structures, including non-monotone behavior. However, in this paper, these benefits are evaluated empirically using simulation trials, goodness-of-fit metrics, information criteria, and applications to actual datasets with comparisons versus well-known flexible models. Therefore, rather than being just another parameter-augmented extension, the proposed NLG family should be viewed as a structurally distinct and empirically testable substitute for current generators such as APT.

The NLW distribution is flexible, but it has several drawbacks. Although the constraint $\alpha > 0, \alpha \neq 1$ necessitates careful numerical handling around $\alpha = 1$, the additional shape parameter increases estimates and processing cost, especially for small samples. Furthermore, when data are sufficiently characterized by simpler models, the additional flexibility could not yield a significant benefit and might increase parameter uncertainty. Therefore, datasets with complicated skewness, tail behavior, or non-monotone hazard patterns are best suited for NLW, but simpler models might be better when parsimony and interpretability are more crucial.

Building upon these properties, this work formalizes the theoretical foundations of both the general NLG generator and its NLW specialization. We evaluate multiple parameter estimation methods via Monte Carlo simulations and assess the empirical performance of the NLW model against established flexible competitors, including the exponentiated Weibull and gamma Weibull distributions. Real-world lifetime applications confirm that this logarithmic mechanism provides a highly competitive framework for modeling complex failure structures.

The rest of this paper is structured as follows. Section 2 defines the general NLG family and its structural features, and Section 3 provides an analytical series expansion for property derivations. Section 4 focuses on the NLW special case, detailing its core functions and statistical properties. Section 5 presents the classical parameter estimation procedures, accompanied by a Monte Carlo simulation study. Practical applications to radiation, reliability, and COVID-19 datasets are presented in Section 6. Section 7 concludes with a summary of findings, limitations, and future research directions.

2. Model formulation and its properties

Let X be a continuous random variable with a baseline CDF $F(x; \zeta)$, where ζ represents the vector of parameters associated with the baseline distribution. The proposed NLG family of distributions is defined by applying a logarithmic transformation to the baseline CDF, and its CDF takes the form

$$F(x; \zeta) = 1 - \frac{1}{\log \alpha} \int_1^{(1+\alpha-\alpha^{G(x;\zeta)})} \frac{1}{w} dw, \quad x \in \mathbb{R}, \alpha > 0, \alpha \neq 1, \zeta \in \Omega. \quad (2.1)$$

$$F(x; \zeta) = \frac{1}{\log \alpha} \left[\log \left(\frac{\alpha}{1 + \alpha - \alpha^{G(x; \zeta)}} \right) \right], \quad x \in \mathbb{R}, \alpha > 0, \alpha \neq 1, \zeta \in \Omega. \quad (2.2)$$

Here, α is an additional shape-controlling parameter that governs the flexibility of the model, influencing its tail behavior and skewness. The corresponding probability density function (PDF) is obtained by differentiating the CDF in Eq (2.2) as

$$f(x; \zeta) = \frac{\alpha^{G(x; \zeta)} g(x; \zeta)}{1 + \alpha - \alpha^{G(x; \zeta)}}, \quad (2.3)$$

where $g(x; \zeta)$ denotes the PDF of the baseline distribution.

The following lemma must be demonstrated in order to evaluate the validity of Eq (2.2):

Lemma 1. The CDF $F(x; \zeta)$ obtained in Eq (2.2) holds in closed form, if

$$\lim_{x \rightarrow -\infty} F(x; \zeta) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} F(x; \zeta) = 1.$$

Proof. We begin by considering the following:

$$\begin{aligned} \lim_{x \rightarrow -\infty} F(x; \zeta) &= \lim_{x \rightarrow -\infty} \frac{1}{\log \alpha} \left[\log \left(\frac{\alpha}{1 + \alpha - \alpha^{G(x; \zeta)}} \right) \right] \\ &= \frac{1}{\log \alpha} \left[\log \left(\frac{\alpha}{1 + \alpha - \alpha^{G(-\infty)}} \right) \right] = 0. \end{aligned}$$

Also,

$$\begin{aligned} \lim_{x \rightarrow \infty} F(x; \zeta) &= \lim_{x \rightarrow \infty} \frac{1}{\log \alpha} \left[\log \left(\frac{\alpha}{1 + \alpha - \alpha^{G(x; \zeta)}} \right) \right] \\ &= \frac{1}{\log \alpha} \left[\log \left(\frac{\alpha}{1 + \alpha - \alpha^{G(\infty)}} \right) \right] = 1. \end{aligned}$$

Lemma 2. To verify that Eq (2.3) represents a valid PDF, we must show that

$$\int_{-\infty}^{\infty} f(x; \zeta) dx = 1.$$

We proceed by evaluating the integral:

$$\int_{-\infty}^{\infty} f(x; \zeta) dx = \int_{-\infty}^{\infty} \frac{\alpha^{G(x; \zeta)} g(x; \zeta)}{1 + \alpha - \alpha^{G(x; \zeta)}} dx.$$

Letting $y = 1 + \alpha - \alpha^{G(x; \zeta)}$, so that $\alpha < y < 1$, we obtain

$$\int_{-\infty}^{\infty} f(x; \zeta) dx = -\frac{1}{\log \alpha} \int_{\alpha}^1 \frac{1}{y} dy = -\frac{1}{\log \alpha} [\log 1 - \log \alpha] = 1.$$

The reliability functions derived from the known generator, being crucial in probability and other sciences, are usually expressed in a standard form. These functions are defined as follows: $R(x; \zeta)$ represents the reliability function, and $H(x; \zeta)$ represents the cumulative hazard rate function. The hazard rate function represented by $h(x; \zeta)$ may be obtained by differentiating the cumulative hazard rate function. Additionally, $r(x; \zeta)$ represents the reverse hazard rate function.

$$R(x; \zeta) = \bar{F}(x; \zeta) = \frac{\log(1 + \alpha - \alpha^{G(x; \zeta)})}{\log \alpha}. \quad (2.4)$$

$$H(x; \zeta) = -\log[\bar{F}(x; \zeta)] = -\log \left\{ \frac{\log(1 + \alpha - \alpha^{G(x; \zeta)})}{\log \alpha} \right\}. \quad (2.5)$$

$$h(x; \zeta) = \frac{f(x; \zeta)}{R(x; \zeta)} = \frac{\alpha^{G(x)} g(x; \zeta) \log \alpha}{(1 + \alpha - \alpha^{G(x; \zeta)}) \log(1 + \alpha - \alpha^{G(x; \zeta)})}. \quad (2.6)$$

$$r(x; \zeta) = \frac{f(x; \zeta)}{F(x; \zeta)} = \frac{\alpha^{G(x; \zeta)} g(x; \zeta) \log \alpha}{(1 + \alpha - \alpha^{G(x; \zeta)}) \log \left(\frac{\alpha}{1 + \alpha - \alpha^{G(x; \zeta)}} \right)}. \quad (2.7)$$

Lemma 3. The limiting behavior of Eqs (2.3)–(2.7) as $x \rightarrow -\infty$ are given by

$$\begin{aligned} f(x; \zeta) &\rightarrow \frac{g(x; \zeta)}{\alpha}, & R(x; \zeta) &\rightarrow 1, & H(x; \zeta) &\rightarrow 0, \\ h(x; \zeta) &\rightarrow \frac{g(x; \zeta)}{\alpha}, & r(x; \zeta) &\rightarrow \infty & \text{as } x &\rightarrow -\infty. \end{aligned}$$

Lemma 4. The limiting behavior of Eqs (2.3)–(2.7) as $x \rightarrow \infty$ are given by

$$\begin{aligned} f(x; \zeta) &\rightarrow \alpha g(x; \zeta), & R(x; \zeta) &\rightarrow 0, & H(x; \zeta) &\rightarrow \infty, \\ h(x; \zeta) &\rightarrow \infty, & r(x; \zeta) &\rightarrow \alpha g(x; \zeta) & \text{as } x &\rightarrow \infty. \end{aligned}$$

For the NLG family, we require the inversion function of $G(x; \zeta)$ in order to calculate the quantile function $Q_X(u)$. Therefore,

$$Q_X(u) = G^{-1} \left(\frac{\log(1 + \alpha - \alpha^{1-u})}{\log \alpha} \right), \quad 0 < u < 1. \quad (2.8)$$

For a given baseline CDF, random variates from the NLG family can therefore be produced by applying Eq (2.8).

3. Expansion form of the PDF

Several key statistical properties of the proposed distribution can be conveniently derived using series expansions of standard functions. Specifically, the PDF can be expressed as a series representation using the following identities:

$$(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1. \quad (3.1)$$

$$a^x = \sum_{p=0}^{\infty} \frac{(x \ln a)^p}{p!}, \quad \text{for } a > 0. \quad (3.2)$$

Using Eq (3.1) in Eq (2.3), we obtain the following series representation of the PDF of the NLG family:

$$f(x; \zeta) = \sum_{n=0}^{\infty} \left(\frac{1}{\alpha + 1} \right)^{n+1} g(x; \zeta) \alpha^{(n+1)G(x; \zeta)}. \quad (3.3)$$

Applying the series expansion in Eq (3.2) to the term $\alpha^{(n+1)G(x; \zeta)}$, we arrive at the following double-series representation:

$$f(x; \zeta) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Delta_{n,p} [G(x; \zeta)]^p g(x; \zeta), \quad (3.4)$$

where

$$\Delta_{n,p} = \left(\frac{1}{\alpha + 1} \right)^{n+1} \frac{(\ln \alpha)^p}{p!} (n+1)^p. \quad (3.5)$$

These series representations provide a useful basis for deriving analytical expressions for various distributional characteristics of the proposed model.

4. The NLW distribution

This section provides the introduction of the NLW distribution, a versatile three-parameter generalization of the traditional Weibull model. The NLW is proposed with the aim of enhancing modeling capability for various lifetime and reliability datasets that are typically found in engineering, biostatistics, and environmental applications. We provide explicit expressions of the PDF, CDF, and reliability functions of the NLW model, laying the groundwork for understanding its behavior.

We obtain important analytical characteristics such as moments, the moment generating function, and measures of skewness and kurtosis, which together offer complete statistical description. A closed-form expression for the quantile function is also found, enabling efficient random variate generation and useful simulation studies.

4.1. Motivation and model formulation

In this work, we introduce a more flexible three-parameter extension of the Weibull distribution, termed the NLW distribution. This model incorporates an additional shape parameter α , which enhances its ability to characterize real-world reliability and lifetime data. The increased flexibility of the NLW distribution allows it to capture a wide array of hazard rate behaviors, including increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped hazard functions commonly observed in engineering, medical, and environmental studies.

The NLW distribution is built upon the classical Weibull distribution, whose CDF serves as the baseline model. By applying a logarithmic transformation parameterized by α to the Weibull CDF, the resulting NLW distribution gains the capacity to model complex data patterns that are not well-represented by the standard Weibull model. This makes the NLW an excellent candidate for applications requiring versatile and robust lifetime modeling. The CDF of the baseline Weibull model is presented as:

$$G(x; \beta, \lambda) = 1 - \exp \left[- \left(\frac{x}{\lambda} \right)^\beta \right]; \quad x > 0, \lambda, \beta > 0,$$

and the corresponding PDF as:

$$g(x; \beta, \lambda) = \frac{\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]; \quad x > 0, \lambda, \beta > 0,$$

where $\lambda > 0$ is the scale parameter, and $\beta > 0$ is the shape parameter. For $x \leq 0$, we define $g(x) = 0$ and $G(x) = 0$. By applying a logarithmic transformation as defined in Eq (2.2) to the baseline Weibull model based on a shape-enhancing parameter $\alpha > 0$, we obtain the CDF of the NLW distribution as follows:

$$F_{\text{NLW}}(x; \alpha, \beta, \lambda) = \frac{1}{\log \alpha} \log \left(\frac{\alpha}{1 + \alpha - \alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]}} \right); \quad x > 0, \alpha, \beta, \lambda > 0; \alpha \neq 1. \quad (4.1)$$

This three-parameter distribution offers enhanced shape flexibility, making it especially effective for modeling diverse reliability data such as radiation exposure outcomes, mechanical component failure times, and other datasets which exhibit complex hazard patterns such as non-monotonic or bathtub-shaped hazard functions. In the subsequent subsections, we present related distributional functions including the PDF, reliability function, hazard rate function, and quantile function, each fundamental to the characterization and application of the NLW distribution to real-world reliability scenarios.

4.2. Related functions

By differentiating the CDF presented in Eq (4.1) with respect to x , the corresponding PDF of the NLW distribution is derived as:

$$f_{\text{NLW}}(x; \alpha, \beta, \lambda) = \frac{\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \cdot \frac{\alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]} \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]}{1 + \alpha - \alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]}}. \quad (4.2)$$

The reliability function (RF), which gives the probability that the time-to-event is greater than x , is given by:

$$R(x; \alpha, \beta, \lambda) = \frac{\log \left(1 + \alpha - \alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]} \right)}{\log \alpha}$$

and

$$R(x; \alpha, \beta, \lambda) = 1, \quad \text{for } x \leq 0.$$

The hazard rate function (HRF), which defines the instantaneous failure rate at time x , is provided as:

$$h(x; \alpha, \beta, \lambda) = \frac{\alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]} \cdot \frac{\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right] \log \alpha}{\left(1 + \alpha - \alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]} \right) \cdot \log \left(1 + \alpha - \alpha^{1 - \exp\left[-\left(\frac{x}{\lambda}\right)^\beta\right]} \right)}$$

and

$$h(x; \alpha, \beta, \lambda) = 0, \quad \text{for } x \leq 0.$$

Figures 1 and 2 show the flexibility of the PDF and HRF for different values of the parameters α , β , and λ . The PDF can take various shapes including unimodal, decreasing, right-skewed, left-skewed, and reversed J-shaped forms. The HRF can exhibit decreasing, increasing, combined decreasing-increasing, and bathtub-shaped patterns. This wide range of shapes is uncommon in standard three-parameter lifetime distributions, demonstrating the strength of the proposed NLW model in capturing complex behaviors in lifetime and reliability data, such as those in radiation and engineering applications.

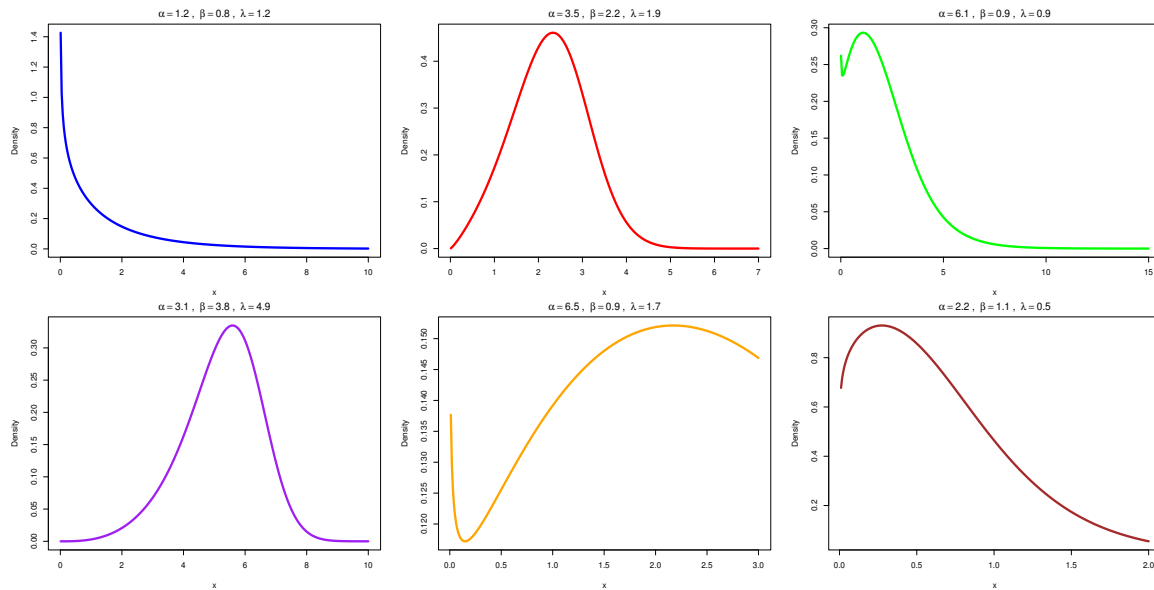


Figure 1. Graphical illustration of the PDF curves of the NLW distribution for varying values of the parameters α , β , and λ .

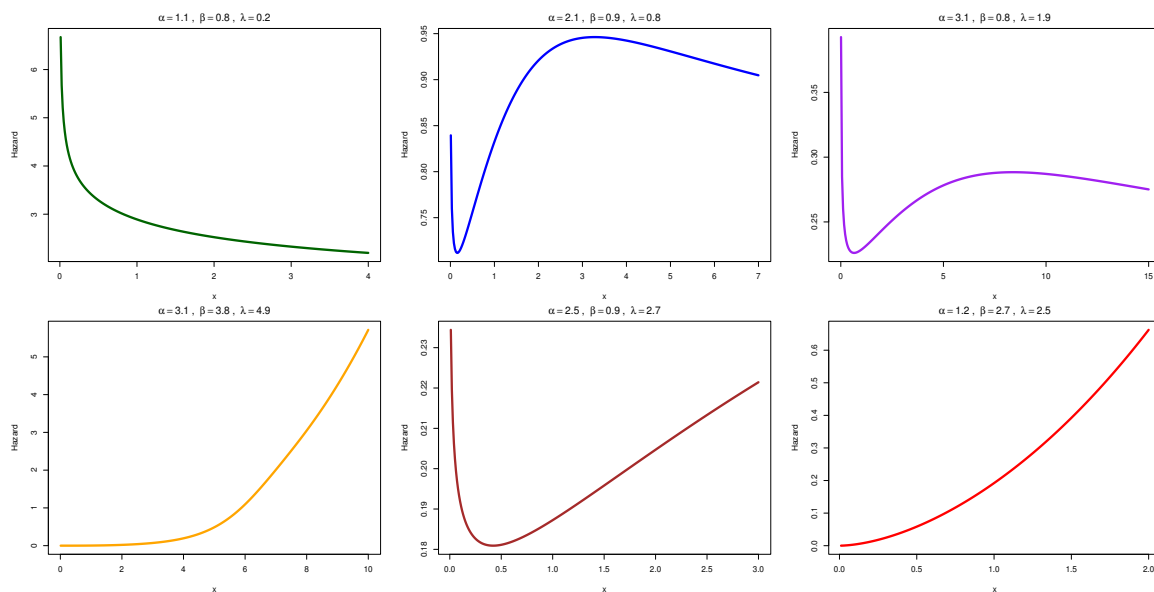


Figure 2. Graphical illustration of the HRF curves of the NLW distribution for varying parameter values α , β , and λ .

4.3. Quantile function and behavior of the hazard rate

The quantile function of the NLW model is given by:

$$Q_X(u) = \lambda \left\{ -\log \left(1 - \frac{\log(1 + \alpha - \alpha^{1-u})}{\log \alpha} \right) \right\}^{1/\beta}; \quad 0 < u < 1. \quad (4.3)$$

Furthermore, the behavior of the hazard function $h(x)$ at the boundaries is given by:

$$h(0) = \begin{cases} \infty, & \text{for } 0 < \beta < 1, \\ \frac{1}{\alpha\lambda}, & \text{for } \beta = 1, \\ 0, & \text{for } \beta > 1, \end{cases} \quad \text{and} \quad h(\infty) = \begin{cases} 0, & \text{for } 0 < \beta \leq 1, \\ \infty, & \text{for } \beta > 1. \end{cases}$$

The versatility of the NLW distribution is demonstrated through its seven important special cases, as summarized in Table 1.

Table 1. Submodels of the NLW distribution.

S.No	α	λ	β	Reduced model
1	—	1	—	NL one-parameter Weibull distribution
2	1	—	—	Two-parameter Weibull distribution
3	1	1	—	One-parameter Weibull distribution
4	—	—	2	NL Rayleigh distribution
5	1	—	2	Rayleigh distribution
6	—	—	1	NL exponential distribution
7	1	—	1	Exponential distribution

4.4. Series expansion form

Substituting Eq (4.1) into Eq (3.4), we get

$$f(x; \alpha, \beta, \lambda) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Delta_{n,p} \left\{ 1 - \exp \left[- \left(\frac{x}{\lambda} \right)^{\beta} \right] \right\}^p \frac{\beta}{\lambda} \left(\frac{x}{\lambda} \right)^{\beta-1} \exp \left[- \left(\frac{x}{\lambda} \right)^{\beta} \right], \quad (4.4)$$

where,

$$\Delta_{n,p} = \left(\frac{1}{\alpha + 1} \right)^{n+1} \frac{(\ln \alpha)^p}{p!} (n+1)^p. \quad (4.5)$$

Using the expansion form of $\{1 - \exp[-(x/\lambda)^{\beta}]\}^p = \sum_{k=0}^{\infty} (-1)^k \binom{p}{k} e^{-k(x/\lambda)^{\beta}}$, we obtain the expanded form of the PDF of the NLW distribution as follows:

$$f(x; \alpha, \beta, \lambda) = \sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\beta}{\lambda} \left(\frac{x}{\lambda} \right)^{\beta-1} e^{-(k+1)(\frac{x}{\lambda})^{\beta}}. \quad (4.6)$$

4.5. Moments and moment generating function

For the NLW model, the ordinary sth moment is calculated as

$$\mu_s' = \int_0^\infty x^s f(x, \alpha, \beta, \lambda) dx. \quad (4.7)$$

Substituting Eq (4.6) in Eq (4.7), we get

$$\mu_s' = \frac{\beta}{\lambda^\beta} \sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \int_0^\infty x^{s+\beta-1} e^{-(k+1)(\frac{x}{\lambda})^\beta} dx. \quad (4.8)$$

On solving, we get the sth moment of NLW model as given by

$$\mu_s' = \sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\lambda^s}{(k+1)^{\frac{s+\beta}{\beta}}} \Gamma\left(\frac{s+\beta}{\beta}\right). \quad (4.9)$$

Putting $s = 1$ in Eq (4.9), the mean of the NLW model is computed as:

$$\mu_1' = \sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\lambda}{(k+1)^{\frac{1+\beta}{\beta}}} \Gamma\left(\frac{1+\beta}{\beta}\right). \quad (4.10)$$

Using the series representation of e^{tx} , we have

$$M_X(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} E(X^s). \quad (4.11)$$

Substituting the value of Eq (4.9) in Eq (4.11), we get moment generating function (MGF) of the NLW model as:

$$M_X(t) = \sum_{n,p,k,s=0}^{\infty} (-1)^k \binom{p}{k} \frac{t^s}{s!} \Delta_{n,p} \frac{\lambda^s}{(k+1)^{\frac{s+\beta}{\beta}}} \Gamma\left(\frac{s+\beta}{\beta}\right). \quad (4.12)$$

Interpretation of the statistical properties. Table 2 presents key summary measures, including the mean, variance, coefficient of variation (CV), skewness, and kurtosis of the NLW distribution across selected combinations of α , β , and λ . The numerical results demonstrate that these parameters exert a strong joint influence on the location, dispersion, asymmetry, and tail heaviness of the distribution:

- **Effect of scale parameter (λ):** For fixed shape parameters α and β , increasing λ from 0.1 to 1.2 leads to a proportional increase in both the mean and variance. This confirms the direct scaling role of λ . Crucially, as shown in Table 2, the CV, skewness, and kurtosis are scale-invariant and remain constant across changes in λ .
- **Effect of baseline shape parameter (β):** For fixed α and λ , increasing the Weibull shape parameter β from 1.1 to 1.7 generally reduces the CV while elevating both skewness and kurtosis. This indicates that higher values of β produce a more concentrated distribution with more pronounced asymmetry and heavy-tailed characteristics.

- **Effect of transformation parameter (α):** Holding β and λ fixed, increasing α from 1.2 to 3.0 further decreases the CV and enhances skewness and kurtosis. For instance, when $\lambda = 1.2$, and $\beta = 1.7$, raising α from 1.2 to 3.0 decreases the CV from 0.5828 to 0.4826, whereas the skewness surges from 10.9805 to 15.5344, and kurtosis rises from 35.2770 to 50.7202.

Table 2. Statistical behavior of the NLW distribution under varying parameter settings of α , β , and λ .

α	β	λ	Mean	Variance	CV	Skewness	Kurtosis
1.2	1.1	0.1	0.1049	0.0084	0.8712	6.5706	23.8304
		1.2	1.2592	1.2035			
1.2	1.7	0.1	0.0947	0.0030	0.5828	10.9805	35.2770
		1.2	1.1369	0.4390			
2.5	1.1	0.1	0.1427	0.0110	0.7360	7.7955	26.1863
		1.2	1.7123	1.5882			
2.5	1.7	0.1	0.1179	0.0035	0.5005	14.4535	46.8516
		1.2	1.4152	0.5017			
3.0	1.1	0.1	0.1528	0.0117	0.7078	8.1871	27.1786
		1.8	1.8335	1.6841			
3.0	1.7	0.1	0.1238	0.0036	0.4826	15.5344	50.7202
		1.2	1.4853	0.5138			

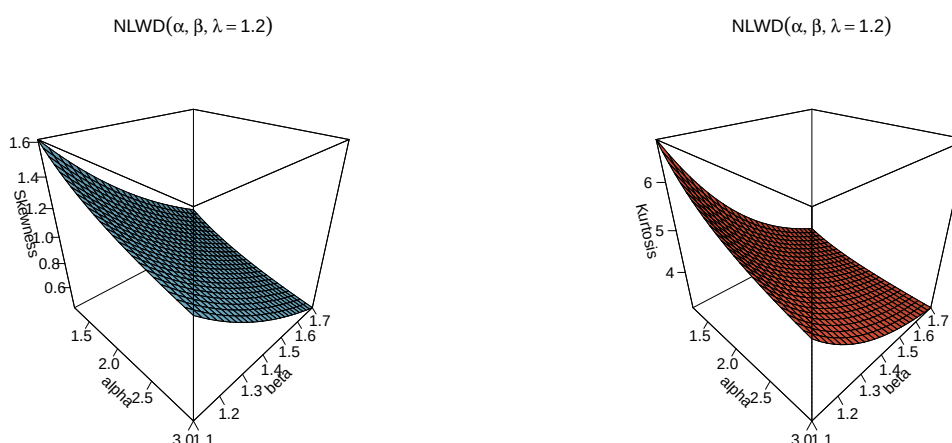


Figure 3. Three-dimensional surface plots of the skewness and kurtosis of the NLW model as functions of the parameters α and β with the parameter fixed at $\lambda = 1.2$.

These findings confirm that the proposed NLW model provides remarkable flexibility in capturing diverse distributional shapes, tail behaviors, and degrees of asymmetry. The joint influence of α and β on skewness and kurtosis is further visually emphasized by the three-dimensional surface plots in Figure 3.

Note that because CV, skewness, and kurtosis are mathematically invariant to the scale parameter λ , their values are reported once per (α, β) pair in Table 2. Consequently, empty entries in the corresponding $\lambda = 1.2$ rows reflect intentional multirow structural formatting rather than missing data.

4.6. Incomplete moment

The s th incomplete moment is defined as

$$\psi_s(z) = \int_0^z x^s f(x, \alpha, \beta, \lambda) dx. \quad (4.13)$$

Substituting Eq (4.6) into Eq (4.13), the s th incomplete moment of the NLW model is given by

$$\psi_s(z) = \sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\lambda^s}{(k+1)^{\frac{s+\beta}{\beta}}} \gamma\left(\frac{s+\beta}{\beta}, (k+1) \left(\frac{z}{\lambda}\right)^\beta\right). \quad (4.14)$$

Putting $s = 1$ in Eq (4.14), we get first incomplete moment as given by

$$\psi_1(z) = \sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\lambda}{(k+1)^{\frac{1+\beta}{\beta}}} \gamma\left(\frac{1+\beta}{\beta}, (k+1) \left(\frac{z}{\lambda}\right)^\beta\right). \quad (4.15)$$

4.7. Mean residual life (MRL) and mean waiting time (MWT)

The mean waiting time (MWT) and mean residual life (MRL) functions are crucial metrics for describing the failure characteristics and aging dynamics of the NLW distribution. Reliability engineering, preventive maintenance, replacement plans, and clinical survival analysis all depend on the MRL, which is the estimated further lifetime of a unit that has lived up to time t . On the other hand, the MWT, also called the mean inactivity time, provides supplementary insights into failure timing and risk accumulation by quantifying the expected time passed since failure given that an event occurred previous to time t . When combined, the MRL and MWT functions provide a thorough description of the aging process, greatly increasing the NLW model's usefulness in risk management, survival analysis, and reliability engineering.

The MRL for NLW distribution is given by

$$\begin{aligned} \mu(t) = & \frac{\log \alpha}{\log \left(1 + \alpha - \alpha^{1 - \exp\left[-\left(\frac{t}{\lambda}\right)^\beta\right]} \right)} \\ & \times \left[\sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\lambda}{(k+1)^{\frac{1+\beta}{\beta}}} \left(\Gamma\left(\frac{1+\beta}{\beta}\right) - \gamma\left(\frac{1+\beta}{\beta}, (k+1) \left(\frac{t}{\lambda}\right)^\beta\right) \right) \right] - t. \end{aligned}$$

The MWT for NLW distribution is given by

$$\bar{\mu}(t) = t - \frac{\sum_{n,p,k=0}^{\infty} (-1)^k \binom{p}{k} \Delta_{n,p} \frac{\lambda}{(k+1)^{\frac{1+\beta}{\beta}}} \gamma\left(\frac{1+\beta}{\beta}, (k+1)\left(\frac{t}{\lambda}\right)^{\beta}\right)}{\frac{1}{\log \alpha} \left[\log \left(\frac{\alpha}{1 + \alpha - \exp\left[-\left(\frac{t}{\lambda}\right)^{\beta}\right]} \right) \right]}.$$

The MRL and MWT functions for the NLW distribution under two scale settings ($\lambda = 1.2$ and 2.2) with fixed shape parameters ($\alpha = \beta = 1.5$) are shown numerically in Table 3. The MRL declines monotonically with increasing time t in both conditions, indicating an aging mechanism in which the estimated remaining lifetime reduces with time. On the other hand, the MWT captures the predicted accumulation of elapsed lifetime before time t and grows strictly with t . Furthermore, increasing the scale parameter from $\lambda = 1.2$ to $\lambda = 2.2$ consistently results in larger MRL values at all time periods, demonstrating the proportionate impact of λ on prolonging the survival time. Taken together, these numerical results demonstrate how well the NLW model can describe both residual and inactive lifetime dynamics.

Table 3. Numerical values of the MRL and MWT functions of the NLW distribution for selected parameter combinations.

(α, β, λ)	t	MRL(t)	MWT(t)	(α, β, λ)	t	MRL(t)	MWT(t)
(1.5, 1.5, 1.2)	0.5	0.94825	0.20389	(1.5, 1.5, 2.2)	0.5	1.94677	0.20142
	1.0	0.76817	0.42746		1.0	1.70208	0.40905
	1.5	0.64976	0.69044		1.5	1.51232	0.62821
	2.0	0.57038	1.00865		2.0	1.36194	0.86492
	2.5	0.51533	1.38605		2.5	1.24193	1.12518
	3.0	0.47504	1.81377		3.0	1.14576	1.41408
	4.0	0.41833	2.75932		4.0	1.00494	2.08866
	5.0	0.37850	3.74974		5.0	0.90861	2.88768

The MRL and MWT functions of the NLW distribution for the two chosen parameter combinations are shown in Figures 4 and 5. The MRL falls, and the MWT rises with t in both scenarios, indicating that the NLW model can describe various remaining and elapsed lifetime behaviors.

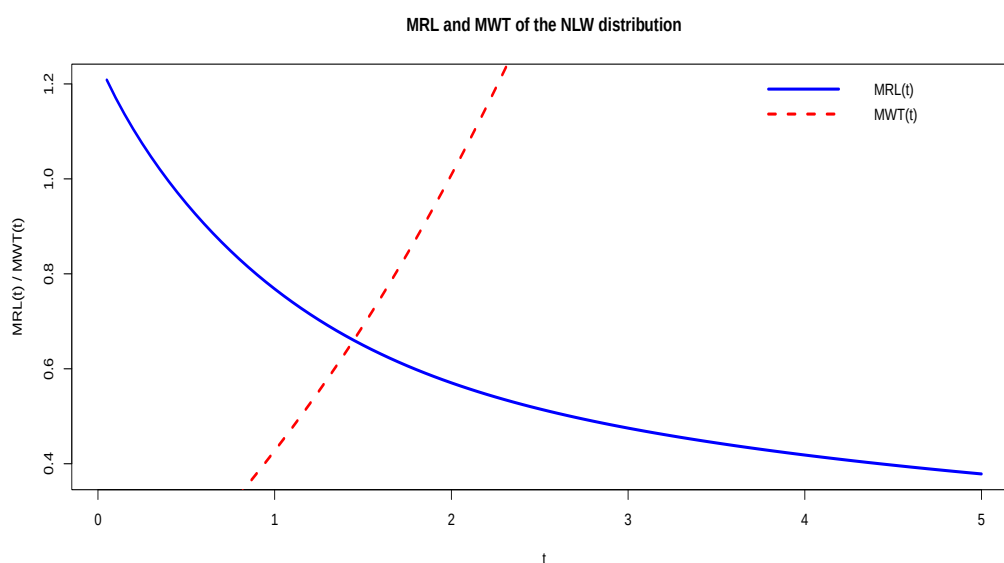


Figure 4. MRL and MWT functions of the NLW distribution for $\alpha = 1.5$, $\beta = 1.5$, and $\lambda = 1.2$.

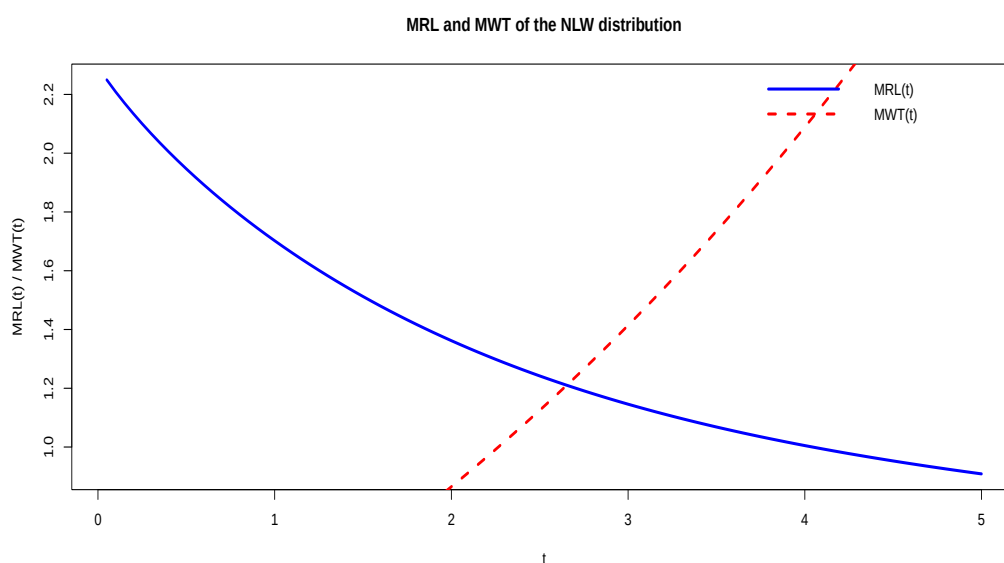


Figure 5. MRL and MWT functions of the NLW distribution for $\alpha = 1.5$, $\beta = 1.5$, and $\lambda = 2.2$.

4.8. Rényi and Tsallis entropy

The Rényi entropy of order δ corresponding for the NLW distribution is mathematically defined as

$$R_{\delta} = \frac{1}{1-\delta} \log \left[\int_0^{\infty} f^{\delta}(x; \alpha, \beta, \lambda) dx \right], \quad \delta > 0, \delta \neq 1, x \in \mathbb{R}. \quad (4.16)$$

By substituting the expression of $f(x; \alpha, \beta, \lambda)$ from Eq (4.6) into Eq (4.16), the Rényi entropy of order δ for the NLW model becomes

$$R_\delta = \frac{1}{1-\delta} \log \left[\beta^{\delta-1} \lambda^{1-\delta} \Gamma \left(\frac{\delta(\beta-1)+1}{\beta} \right) \times \sum_{n_1, p_1, k_1, \dots, n_\delta, p_\delta, k_\delta=0}^{\infty} \frac{\prod_{j=1}^{\delta} \left[(-1)^{k_j} \binom{p_j}{k_j} \Delta_{n_j, p_j} \right]}{\left[\sum_{j=1}^{\delta} (k_j + 1) \right]^{\frac{\delta(\beta-1)+1}{\beta}}} \right]. \quad (4.17)$$

By employing the expansion in Eq (4.6), the Tsallis entropy of order δ for the NLW model can be expressed as

$$Q_\delta = \frac{1}{\delta-1} \left[1 - \beta^{\delta-1} \lambda^{1-\delta} \Gamma \left(\frac{\delta(\beta-1)+1}{\beta} \right) \times \sum_{n_1, p_1, k_1, \dots, n_\delta, p_\delta, k_\delta=0}^{\infty} \frac{\prod_{j=1}^{\delta} \left[(-1)^{k_j} \binom{p_j}{k_j} \Delta_{n_j, p_j} \right]}{\left[\sum_{j=1}^{\delta} (k_j + 1) \right]^{\frac{\delta(\beta-1)+1}{\beta}}} \right]. \quad (4.18)$$

Interpretation of entropy values. Table 4 shows how the parameter combination (α, β, λ) and the entropy order δ significantly affect the Rényi and Tsallis entropy measures. For example, the Rényi and Tsallis entropy values are 5.291094 and 46.40365, respectively, when $(\alpha, \beta, \lambda) = (0.3, 0.3, 1.2)$, and $\delta = 0.35$. Such high numbers suggest that the matching NLW density has a rather high degree of dispersion or uncertainty. Conversely, a number of negative values are also noted, especially when $\delta = 1.25$. Because entropy measures related to continuous probability densities can assume negative values when the density is sufficiently concentrated over its support, these negative values do not always signify a mathematical error. Therefore, the concentration and dispersion properties of the density should be used to evaluate the sign and magnitude of the entropy.

The information uncertainty in NLW model is mostly determined by the parameters α and β . Whereas β regulates the shape and tail behavior inherited from the Weibull baseline, α controls the influence of the suggested transformation, changing the concentration and overall shape of the density. As a result, variations in these factors affect the probability mass distribution and, consequently, the associated entropy values. For instance, with $\lambda = 1.2$ and $\delta = 0.35$, raising $\alpha = \beta$ from 0.3 to 1.5 lowers the Tsallis entropy from 46.40365 to 2.193915 and the Rényi entropy from 5.291094 to 1.363480. This shows that the entropy measures accurately represent changes in the concentration and uncertainty structure of the NLW distribution and are sensitive to the shape parameters.

The effects of the shape parameters α and β on the Rényi and Tsallis entropy measures of the NLW distribution are further investigated through three-dimensional surface plots. Figures 6–8 illustrate the behavior of both entropy measures for $\lambda = 0.2, 0.4$, and 1.2 , respectively, considering the entropy orders $\delta \in (0.35, 0.65, 1.25)$ and $\alpha, \beta \in (0.3, 0.6, 0.9, 1.2, 1.5)$. These plots provide a graphical comparison of the entropy measures across different parameter combinations and entropy orders, thereby illustrating how the scale parameter λ and the shape parameters α and β influence the information characteristics of the NLW distribution.

Table 4. Numerical evaluation of Rényi and Tsallis entropy measures of orders $\delta = 0.35$, 0.65 , and 1.25 for different combinations of (α, β, λ) of the NLW distribution.

α	β	$\delta = 0.35$		$\delta = 0.65$		$\delta = 1.25$	
		Rényi	Tsallis	Rényi	Tsallis	Rényi	Tsallis
$\lambda = 0.2$							
0.3	0.3	3.49933	13.42105	0.120585	0.123166	-4.094218	-7.132278
0.6	0.6	0.828204	1.097151	-0.216453	-0.208457	-1.428446	-1.716780
0.9	0.9	0.139557	0.146083	-0.333743	-0.314988	-0.772164	-0.851730
1.2	1.2	-0.201404	-0.188778	-0.473160	-0.436057	-0.697434	-0.761929
1.5	1.5	-0.428279	-0.373834	-0.617436	-0.555279	-0.772016	-0.851551
$\lambda = 0.4$							
0.3	0.3	4.192480	21.93557	0.814794	0.942854	-3.654110	-5.972405
0.6	0.6	1.521356	2.597270	0.476808	0.518902	-0.740768	-0.813798
0.9	0.9	0.832705	1.104874	0.359410	0.382994	-0.079076	-0.079862
1.2	1.2	0.491743	0.579419	0.219989	0.228680	-0.004287	-0.004289
1.5	1.5	0.264868	0.289035	0.075711	0.076723	-0.078869	-0.079652
$\lambda = 1.2$							
0.3	0.3	5.291094	46.40365	1.914469	2.726751	-2.901412	-4.261839
0.6	0.6	2.619972	6.908144	1.575515	2.102096	0.352268	0.337202
0.9	0.9	1.931319	3.860141	1.458030	1.902309	1.019493	0.899941
1.2	1.2	1.590355	2.786978	1.318602	1.675625	1.094325	0.957398
1.5	1.5	1.363480	2.193915	1.174323	1.452415	1.019743	0.900135

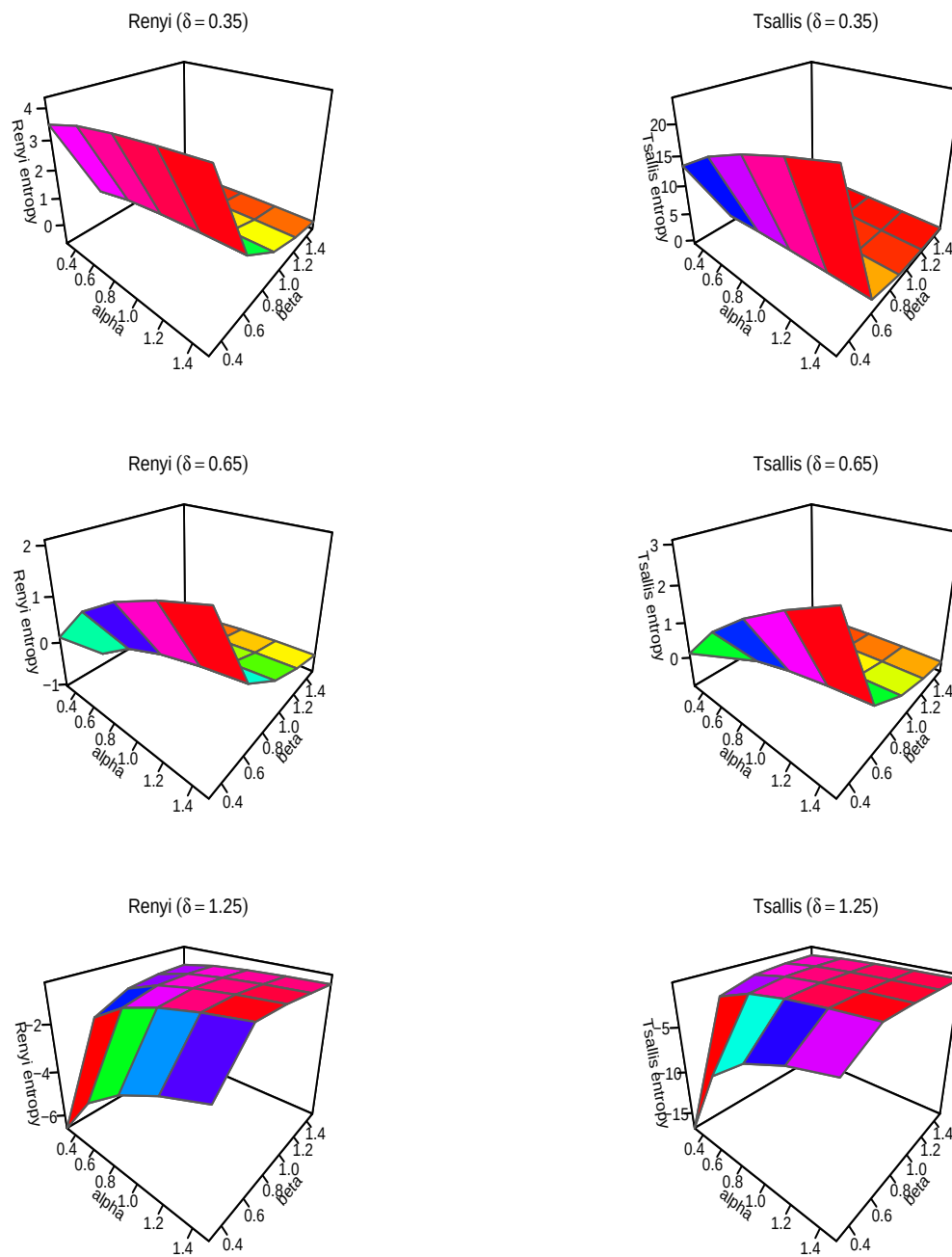


Figure 6. Three-dimensional plots of the Rényi and Tsallis entropy functions of the NLW distribution for $\lambda = 0.2$ and $\delta = (0.35, 0.65, 1.25)$ with $\alpha, \beta \in \{0.3, 0.6, 0.9, 1.2, 1.5\}$.

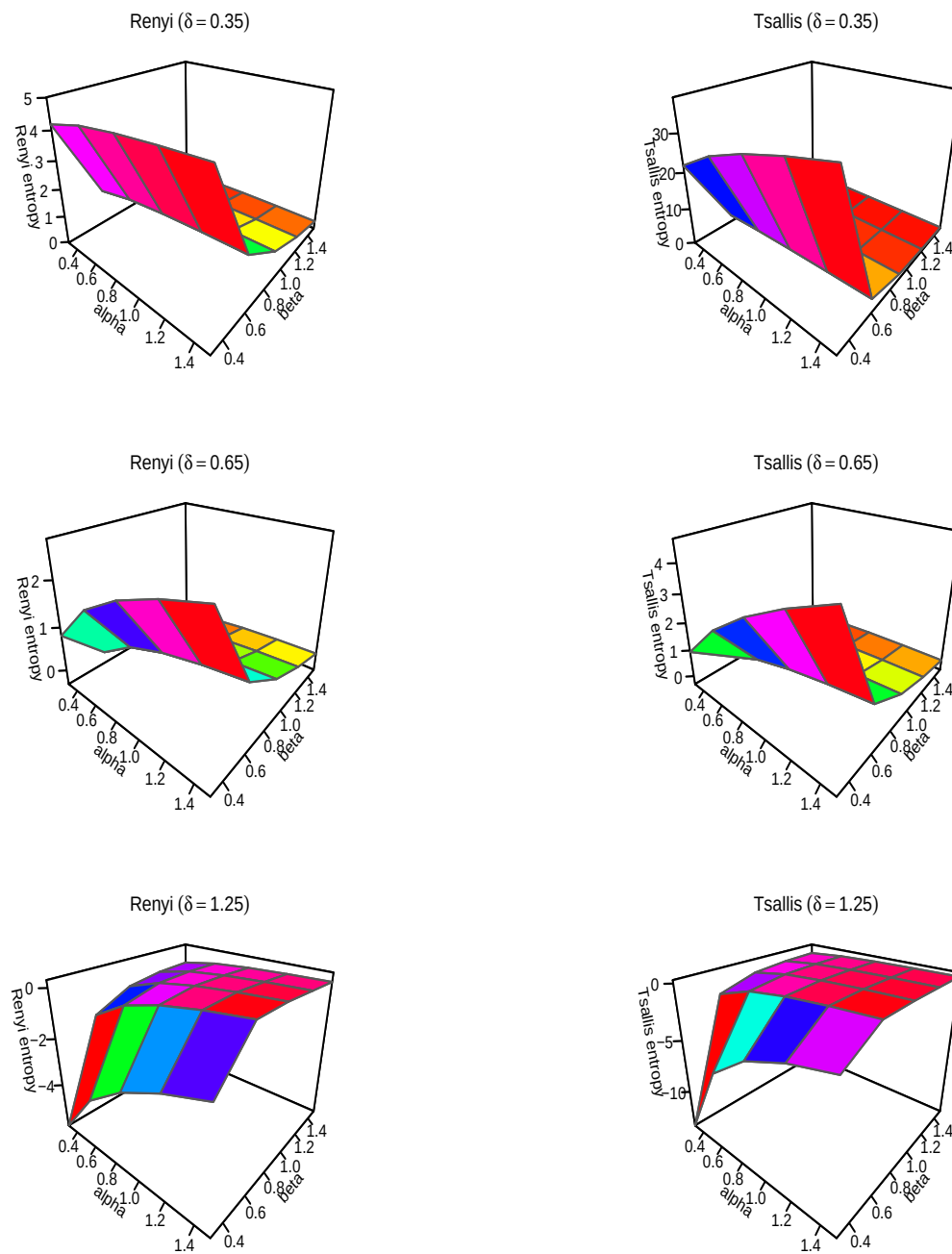


Figure 7. Three-dimensional plots of the Rényi and Tsallis entropy functions of the NLW distribution for $\lambda = 0.4$ and $\delta = 0.35, 0.65, 1.25$ with $\alpha, \beta \in \{0.3, 0.6, 0.9, 1.2, 1.5\}$.

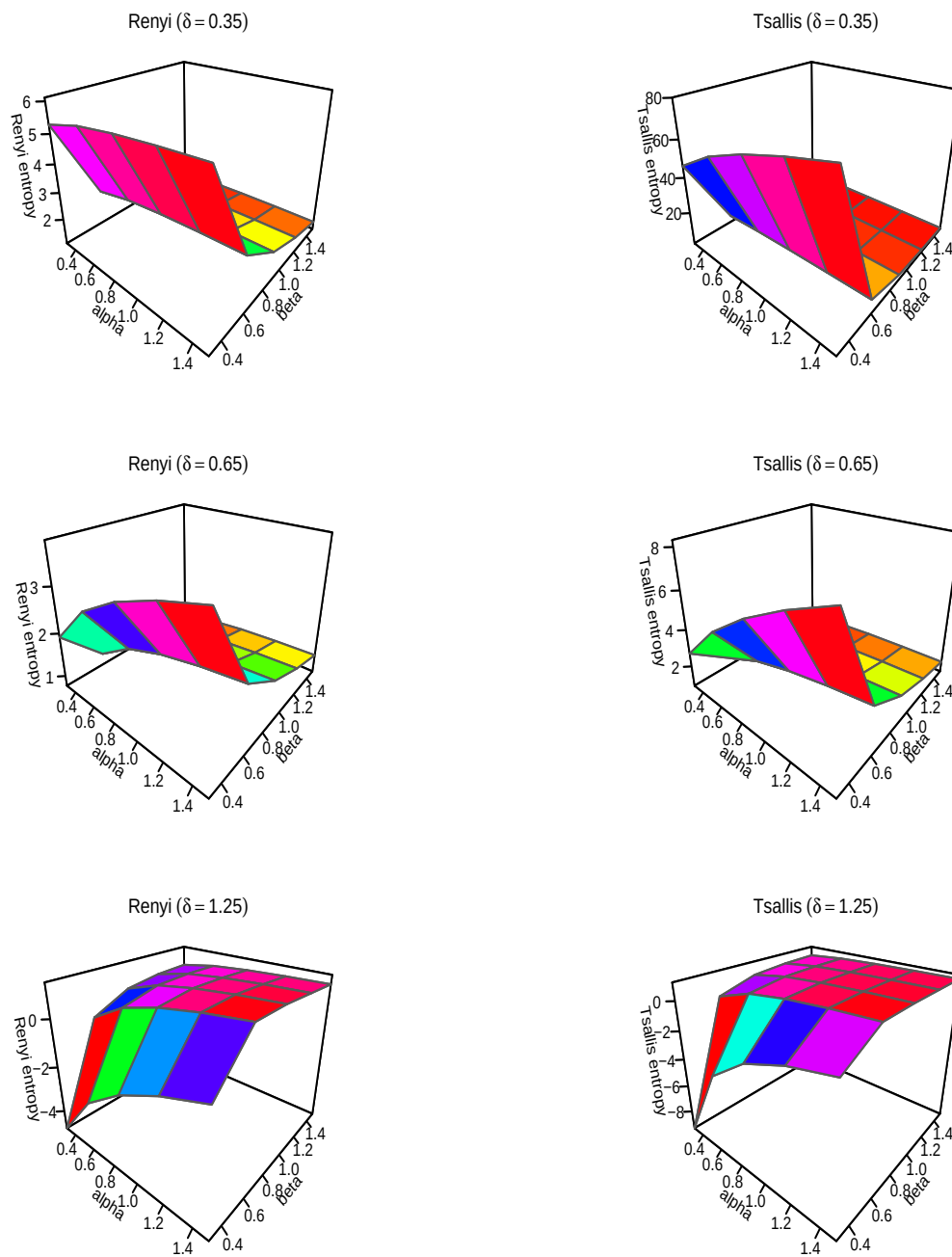


Figure 8. Three-dimensional plots of the Rényi and Tsallis entropy functions of the NLW distribution for $\lambda = 1.2$ and $\delta = 0.35, 0.65, 1.25$ with $\alpha, \beta \in \{0.3, 0.6, 0.9, 1.2, 1.5\}$.

5. Estimation methods

Because reliable parameter estimation is essential for obtaining accurate model-based inference and assessing the practical applicability of a probability distribution, parameter estimation constitutes a fundamental component of statistical distribution analysis. In this study, the performance of the proposed NLW distribution is investigated using six estimation techniques: maximum likelihood estimation (MLE) (Δ_1), Cramér–von Mises (CVM) (Δ_2), maximum product of spacings (MPS) (Δ_3), ordinary least squares (OLS) (Δ_4), weighted least squares (WLS) (Δ_5), and Anderson–Darling (AD) (Δ_6). These methods were selected because they provide complementary estimation frameworks and differ in their sensitivity to various portions of the sample distribution. MLE is a likelihood-based procedure that utilizes the probabilistic information contained in the sample and has well-established large-sample properties under standard regularity conditions. The CVM and AD methods are based on discrepancies between the empirical and fitted distribution functions, with the AD criterion assigning comparatively greater emphasis to discrepancies in the tails. The OLS and WLS procedures provide alternative distribution-function-based approaches using the ordered sample observations, whereas WLS incorporates unequal weights across the ordered observations. The MPS method provides an alternative likelihood-free estimation strategy based on the spacings between consecutive fitted probabilities and can be useful for continuous lifetime models, particularly when likelihood-based optimization encounters numerical difficulties.

The empirical justification for considering these six estimation procedures is provided through a systematic Monte Carlo simulation study. Specifically, the estimators are evaluated under different parameter configurations and sample sizes to examine their finite-sample behavior across a range of distributional settings. Their performance is assessed using bias, mean squared error (MSE), and other relevant performance measures, allowing the relative accuracy and stability of the competing estimation procedures to be examined quantitatively. The use of multiple parameter configurations and sample sizes is important because the relative performance of an estimator may depend on both the underlying parameter values and the amount of available information. In addition, the resulting estimator rankings provide an empirical basis for identifying estimation procedures that perform comparatively well across the considered scenarios. The methodology is further supported by applications to real lifetime datasets, where the fitted NLW model and its competing estimation procedures are evaluated using appropriate goodness-of-fit criteria. Thus, the selection of the six estimation methods is not based solely on theoretical considerations; rather, their suitability is empirically examined through controlled simulation experiments and real-data applications. Accordingly, the combined simulation and empirical analyses provide evidence regarding the robustness, accuracy, and practical usefulness of the proposed estimation framework. Each estimation approach for the NLW distribution is formulated and computationally implemented in the following subsections.

5.1. Method of maximum likelihood (Δ_1)

Suppose that $X_1, X_2, \dots, X_n$ denote a random sample of size n drawn from the NLW distribution. In this case, the likelihood function can be expressed as:

$$L(\alpha, \beta, \lambda) = \prod_{i=1}^n \frac{\beta}{\lambda} \left(\frac{x_i}{\lambda}\right)^{\beta-1} \frac{\alpha^{1-\exp\left\{-\left(\frac{x_i}{\lambda}\right)^\beta\right\}} \exp\left\{-\left(\frac{x_i}{\lambda}\right)^\beta\right\}}{1 + \alpha - \alpha^{1-\exp\left\{-\left(\frac{x_i}{\lambda}\right)^\beta\right\}}}.$$

Hence, the log-likelihood function is

$$\begin{aligned} \ell(\Theta) = & n \log \beta - n \log \lambda + (\beta - 1) \sum_{i=1}^n \log \left(\frac{x_i}{\lambda}\right) \\ & + \log \alpha \sum_{i=1}^n \left\{1 - \exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]\right\} - \sum_{i=1}^n \left(\frac{x_i}{\lambda}\right)^\beta \\ & - \sum_{i=1}^n \log \left\{1 + \alpha - \alpha^{1-\exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]}\right\} \end{aligned} \quad (5.1)$$

with $\Theta = (\alpha, \beta, \lambda)$. The MLEs of the parameters α, β , and λ are obtained by taking the partial derivatives of Eq (5.1) with respect to each parameter, and they are given respectively by

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \frac{1}{\alpha} \sum_{i=1}^n \left(1 - \exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]\right) - \sum_{i=1}^n \frac{1 - \left(1 - \exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]\right) \alpha^{-\exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]}}{\left(1 + \alpha - \alpha^{1-\exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]}\right)} = 0,$$

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \frac{n}{\beta} + n \log \lambda + \sum_{i=1}^n x_i + \sum_{i=1}^n \exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right] \log\left(\frac{x_i}{\lambda}\right) \left(\log \alpha - \frac{1}{\left(1 + \alpha - \alpha^{1-\exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]}\right)} \right) = 0,$$

and

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{n\beta}{\lambda} - \frac{\beta}{\lambda} \sum_{i=1}^n \exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right] \left(\frac{x_i}{\lambda}\right)^\beta \left(\log \alpha + 1 - \frac{1}{\left(1 + \alpha - \alpha^{1-\exp\left[-\left(\frac{x_i}{\lambda}\right)^\beta\right]}\right)} \right) = 0.$$

Because the resulting system of equations is highly nonlinear, the maximum likelihood estimates are obtained numerically using iterative optimization methods such as the Newton–Raphson algorithm.

5.2. Method of Cramér-von Mises (Δ_2)

The Cramér–von Mises estimation method is a goodness-of-fit-based approach that estimates the model parameters by minimizing the sum of squared differences between the empirical distribution function and the corresponding theoretical distribution function (see [15] for details):

$$C(x_i) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2.$$

5.3. Method of maximum product of spacings (Δ_3)

The maximum product spacing estimator serves as an alternative to the maximum likelihood estimator. This method estimates the parameters of the NLW distribution by maximizing the objective function given in Eq (5.2), which is based on the spacings between the values of the theoretical cumulative distribution function evaluated at the ordered sample data (see [16] for further details):

$$\delta(x_i) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log I_i(x_i), \quad (5.2)$$

where $I_i(x_i) = F(x_{i:n}) - F(x_{i-1:n})$, $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

5.4. Method of ordinary least square (Δ_4)

The ordinary least squares (OLS) estimator is a widely used method for estimating model parameters, obtained by minimizing the sum of the squared differences between observed data and the corresponding values predicted by the model (see [17] for further details):

$$S(x_i) = \sum_{k=1}^n \left[F(x_{(i)}) - \frac{i}{n+1} \right]^2.$$

5.5. Method of weighted least squares (Δ_5)

The weighted least squares (WLS) estimation method is a generalization of the ordinary least squares estimation and is particularly preferred when heteroskedasticity is present in the data (see [17]). For the NLW distribution, the WLS estimators of the parameters are obtained by minimizing the weighted sum of squared residuals, with the objective function expressed in Eq (5.3). This approach assigns higher weights to observations with lower variance, thereby improving estimation efficiency under heteroskedasticity:

$$W(x_i) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2. \quad (5.3)$$

5.6. Method of Anderson-Darling (Δ_6)

The Anderson–Darling estimation method is a widely used technique, particularly effective for small sample sizes and datasets containing outliers. Based on the goodness-of-fit statistic introduced by Anderson and Darling (see [18]), this method yields the Anderson–Darling estimators $\hat{\alpha}_{ADE}$, $\hat{\beta}_{ADE}$, and $\hat{\lambda}_{ADE}$ for the parameters α , β , and λ , respectively, by minimizing the objective function presented in Eq (5.4):

$$A(x_i) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{n-i+1:n})]. \quad (5.4)$$

5.7. Simulation study and finite-sample performance

Using the R statistical computing environment, a thorough Monte Carlo simulation analysis was carried out to assess the finite-sample behavior, numerical stability, and empirical accuracy of the

point estimators for the proposed NLW distribution. By analyzing the performance of the six estimation techniques under various sample sizes, the simulation study provides empirical support for the proposed estimation methodology. Three statistical criteria, namely absolute bias, mean squared error (MSE), and mean relative error (MRE), were employed to systematically evaluate the performance of the competing estimation methods. Bias measures the departure of the parameter estimates from their true values, whereas MSE quantifies the average squared estimation error, and MRE measures the estimation error relative to the true parameter values. Collectively, these criteria provide complementary measures of estimation accuracy and goodness of fit, thereby offering empirical evidence regarding the accuracy, stability, and reliability of the proposed estimation procedures in finite samples.

To capture diverse distributional profiles, two representative parameter configurations were investigated:

- Case I:** $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$,
Case II: $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$,
Case III: $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$.

For each iteration, the biases along with their corresponding MSE and MRE values were recorded. The numerical results presented in Tables 5–10 demonstrate a consistent improvement in estimation performance as the sample size n increases across the considered estimation techniques. This behavior provides numerical evidence of improved estimator performance and convergence with increasing sample size. Furthermore, comparative visualizations illustrating the trajectories of bias, MSE, and MRE for the parameters α , β , and λ across the six estimation methods are provided in Figures 9–17.

To systematically evaluate and compare the competing estimation methods, an overall rank-based assessment was conducted using the simulation outcomes. For each parameter (α, β, λ), sample size, and performance metric (Bias, MSE, and MRE), the estimation method yielding the smallest value was assigned rank 1, and the remaining methods were ranked sequentially in ascending order. The individual ranks were then aggregated across all simulation scenarios to obtain an overall cumulative rank for each estimation method, where a smaller cumulative rank indicates superior estimation performance.

Table 11 presents the partial and overall performance rankings of the six estimation methods under different parameter configurations and sample sizes for the proposed NLW distribution. For the first parameter configuration, $(\alpha, \beta, \lambda) = (1.25, 0.75, 2.35)$, the MLE consistently achieved the best overall performance, attaining the lowest cumulative rank across the considered sample sizes. Similarly, under the second parameter configuration $(\alpha, \beta, \lambda) = (2.50, 1.75, 0.25)$, the MLE demonstrated superior performance compared with the other estimation methods. For the third parameter configuration, $(\alpha, \beta, \lambda) = (0.25, 0.75, 1.25)$, the relative performance of the estimation methods varied across sample sizes, although the MLE generally maintained a competitive position. The remaining methods exhibited varying levels of performance, depending on the parameter configuration and sample size.

Overall, the rank-based analysis indicates that no single estimation method is uniformly optimal across all simulation scenarios. Nevertheless, the MLE exhibited the most stable and consistent performance over the considered parameter configurations and sample sizes. These findings are broadly consistent with the absolute Bias, MSE, and MRE results reported in the simulation tables and

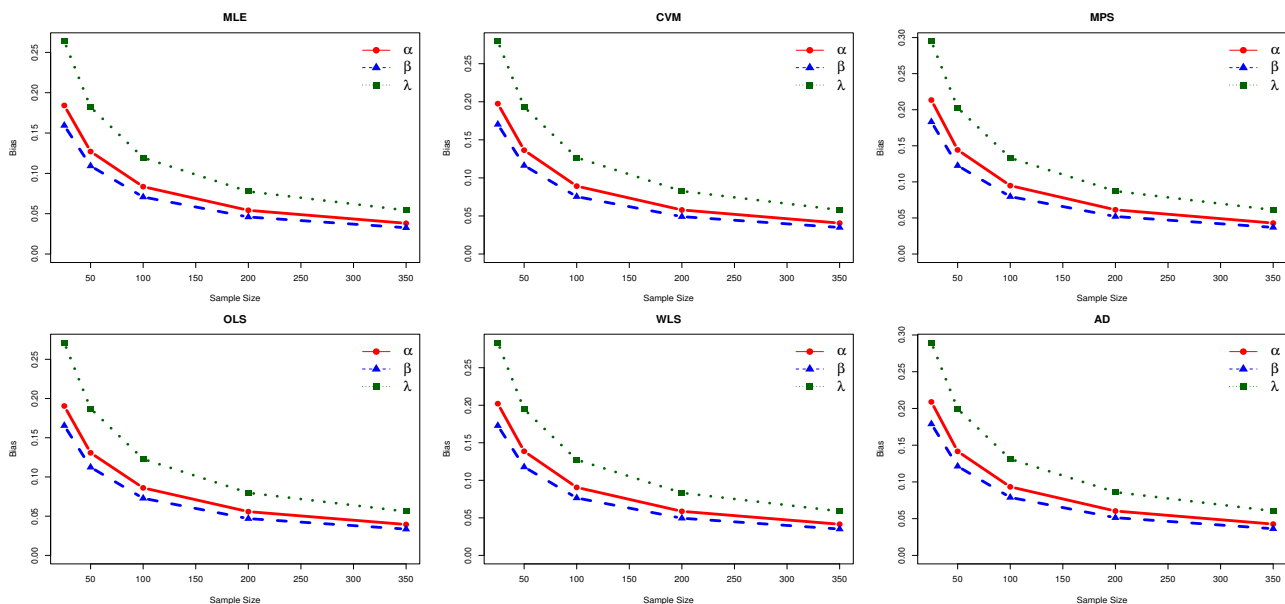
are further supported by the corresponding graphical illustrations. The results also indicate that the estimation performance generally improves with increasing sample size, providing empirical evidence of the finite-sample accuracy, stability, and reliability of the proposed estimation procedures for the NLW distribution.

Table 5. Simulation results and ranking of different estimation methods for $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$.

n	Estimate	Par.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
25	Bias	α	0.1842 ^[1]	0.1975 ^[3]	0.2134 ^[6]	0.1905 ^[2]	0.2021 ^[4]	0.2089 ^[5]
		β	0.1594 ^[1]	0.1703 ^[3]	0.1832 ^[6]	0.1655 ^[2]	0.1728 ^[4]	0.1789 ^[5]
		λ	0.2637 ^[1]	0.2794 ^[3]	0.2948 ^[6]	0.2712 ^[2]	0.2833 ^[4]	0.2895 ^[5]
	MSE	α	0.0632 ^[1]	0.0695 ^[3]	0.0753 ^[6]	0.0658 ^[2]	0.0704 ^[4]	0.0735 ^[5]
		β	0.0451 ^[1]	0.0504 ^[3]	0.0557 ^[6]	0.0473 ^[2]	0.0515 ^[4]	0.0538 ^[5]
		λ	0.1086 ^[1]	0.1193 ^[3]	0.1290 ^[6]	0.1127 ^[2]	0.1209 ^[4]	0.1261 ^[5]
	MRE	α	0.1474 ^[1]	0.1580 ^[3]	0.1707 ^[6]	0.1524 ^[2]	0.1617 ^[4]	0.1671 ^[5]
		β	0.2125 ^[1]	0.2270 ^[3]	0.2443 ^[6]	0.2206 ^[2]	0.2304 ^[4]	0.2385 ^[5]
		λ	0.1122 ^[1]	0.1189 ^[3]	0.1255 ^[6]	0.1154 ^[2]	0.1205 ^[4]	0.1232 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]
50	Bias	α	0.1271 ^[1]	0.1364 ^[3]	0.1443 ^[6]	0.1308 ^[2]	0.1387 ^[4]	0.1416 ^[5]
		β	0.1092 ^[1]	0.1163 ^[3]	0.1225 ^[6]	0.1124 ^[2]	0.1178 ^[4]	0.1213 ^[5]
		λ	0.1821 ^[1]	0.1932 ^[3]	0.2027 ^[6]	0.1873 ^[2]	0.1946 ^[4]	0.1994 ^[5]
	MSE	α	0.0361 ^[1]	0.0398 ^[3]	0.0430 ^[6]	0.0375 ^[2]	0.0407 ^[4]	0.0423 ^[5]
		β	0.0269 ^[1]	0.0301 ^[3]	0.0330 ^[6]	0.0281 ^[2]	0.0307 ^[4]	0.0324 ^[5]
		λ	0.0632 ^[1]	0.0698 ^[3]	0.0754 ^[6]	0.0658 ^[2]	0.0707 ^[4]	0.0742 ^[5]
	MRE	α	0.1017 ^[1]	0.1091 ^[3]	0.1155 ^[6]	0.1046 ^[2]	0.1109 ^[4]	0.1133 ^[5]
		β	0.1456 ^[1]	0.1550 ^[3]	0.1633 ^[6]	0.1498 ^[2]	0.1568 ^[4]	0.1618 ^[5]
		λ	0.0775 ^[1]	0.0822 ^[3]	0.0863 ^[6]	0.0797 ^[2]	0.0828 ^[4]	0.0849 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]
100	Bias	α	0.0835 ^[1]	0.0892 ^[3]	0.0948 ^[6]	0.0861 ^[2]	0.0906 ^[4]	0.0933 ^[5]
		β	0.0707 ^[1]	0.0754 ^[3]	0.0796 ^[6]	0.0728 ^[2]	0.0765 ^[4]	0.0789 ^[5]
		λ	0.1194 ^[1]	0.1267 ^[3]	0.1332 ^[6]	0.1229 ^[2]	0.1274 ^[4]	0.1312 ^[5]
	MSE	α	0.0207 ^[1]	0.0227 ^[3]	0.0248 ^[6]	0.0214 ^[2]	0.0231 ^[4]	0.0245 ^[5]
		β	0.0151 ^[1]	0.0170 ^[3]	0.0186 ^[6]	0.0158 ^[2]	0.0173 ^[4]	0.0183 ^[5]
		λ	0.0358 ^[1]	0.0396 ^[3]	0.0432 ^[6]	0.0370 ^[2]	0.0401 ^[4]	0.0423 ^[5]
	MRE	α	0.0668 ^[1]	0.0714 ^[3]	0.0759 ^[6]	0.0689 ^[2]	0.0725 ^[4]	0.0747 ^[5]
		β	0.0943 ^[1]	0.1005 ^[3]	0.1061 ^[6]	0.0970 ^[2]	0.1019 ^[4]	0.1052 ^[5]
		λ	0.0508 ^[1]	0.0539 ^[3]	0.0567 ^[6]	0.0523 ^[2]	0.0542 ^[4]	0.0558 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]

Table 6. Simulation results and ranking of different estimation methods for $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$. (Continued)

n	Estimate	Par.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
200	Bias	α	0.0542 ^[1]	0.0579 ^[3]	0.0613 ^[6]	0.0558 ^[2]	0.0587 ^[4]	0.0604 ^[5]
		β	0.0459 ^[1]	0.0491 ^[3]	0.0520 ^[6]	0.0469 ^[2]	0.0495 ^[4]	0.0513 ^[5]
		λ	0.0776 ^[1]	0.0827 ^[3]	0.0872 ^[6]	0.0798 ^[2]	0.0833 ^[4]	0.0861 ^[5]
	MSE	α	0.0113 ^[1]	0.0125 ^[3]	0.0136 ^[6]	0.0118 ^[2]	0.0127 ^[4]	0.0133 ^[5]
		β	0.0082 ^[1]	0.0093 ^[3]	0.0101 ^[6]	0.0085 ^[2]	0.0094 ^[4]	0.0100 ^[5]
		λ	0.0197 ^[1]	0.0218 ^[3]	0.0238 ^[6]	0.0204 ^[2]	0.0221 ^[4]	0.0234 ^[5]
	MRE	α	0.0433 ^[1]	0.0463 ^[3]	0.0490 ^[6]	0.0446 ^[2]	0.0469 ^[4]	0.0483 ^[5]
		β	0.0612 ^[1]	0.0654 ^[3]	0.0694 ^[6]	0.0626 ^[2]	0.0660 ^[4]	0.0684 ^[5]
		λ	0.0330 ^[1]	0.0352 ^[3]	0.0371 ^[6]	0.0340 ^[2]	0.0356 ^[4]	0.0366 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]
350	Bias	α	0.0381 ^[1]	0.0406 ^[3]	0.0429 ^[6]	0.0394 ^[2]	0.0415 ^[4]	0.0426 ^[5]
		β	0.0326 ^[1]	0.0349 ^[3]	0.0369 ^[6]	0.0335 ^[2]	0.0351 ^[4]	0.0363 ^[5]
		λ	0.0542 ^[1]	0.0579 ^[3]	0.0611 ^[6]	0.0560 ^[2]	0.0588 ^[4]	0.0604 ^[5]
	MSE	α	0.0070 ^[1]	0.0077 ^[3]	0.0083 ^[6]	0.0073 ^[2]	0.0079 ^[4]	0.0082 ^[5]
		β	0.0051 ^[1]	0.0058 ^[3]	0.0063 ^[6]	0.0053 ^[2]	0.0059 ^[4]	0.0062 ^[5]
		λ	0.0121 ^[1]	0.0134 ^[3]	0.0145 ^[6]	0.0125 ^[2]	0.0136 ^[4]	0.0143 ^[5]
	MRE	α	0.0305 ^[1]	0.0325 ^[3]	0.0343 ^[6]	0.0315 ^[2]	0.0332 ^[4]	0.0341 ^[5]
		β	0.0435 ^[1]	0.0465 ^[3]	0.0492 ^[6]	0.0446 ^[2]	0.0468 ^[4]	0.0484 ^[5]
		λ	0.0231 ^[1]	0.0246 ^[3]	0.0260 ^[6]	0.0238 ^[2]	0.0250 ^[4]	0.0257 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]

**Figure 9.** Visual comparison of bias values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$.

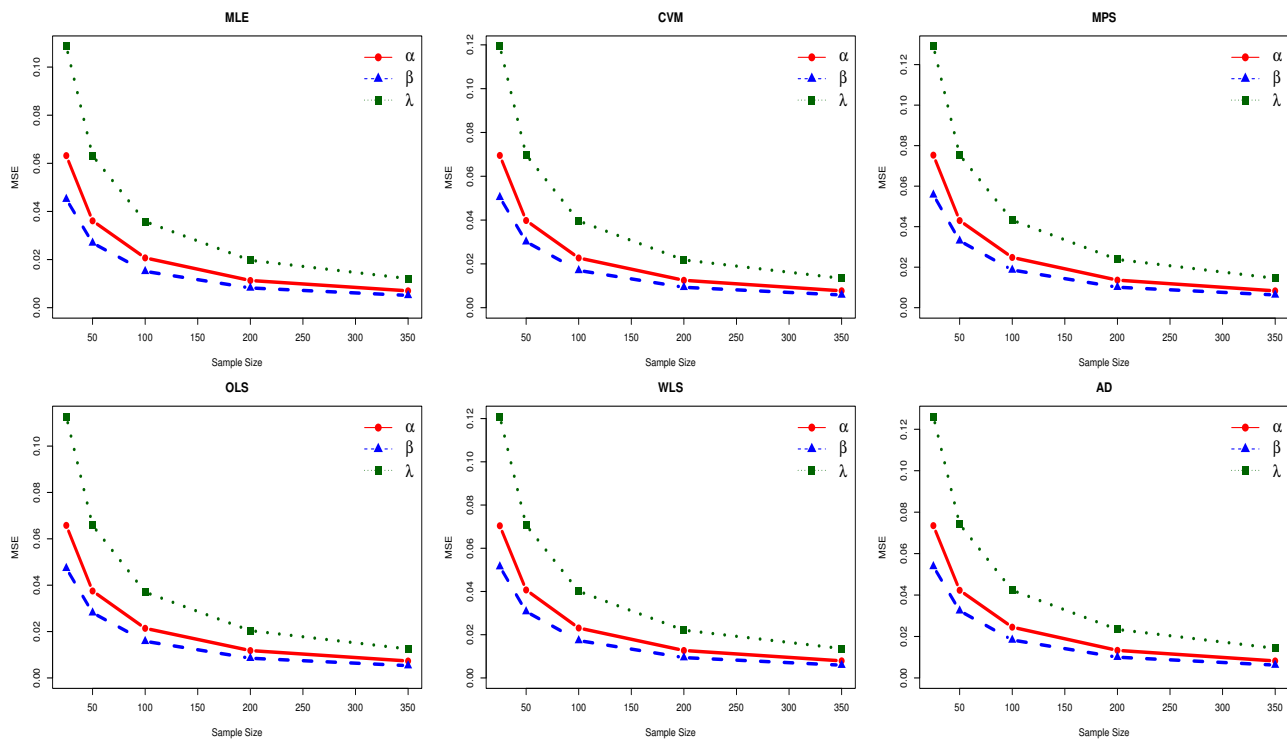


Figure 10. Visual comparison of MSE values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$.

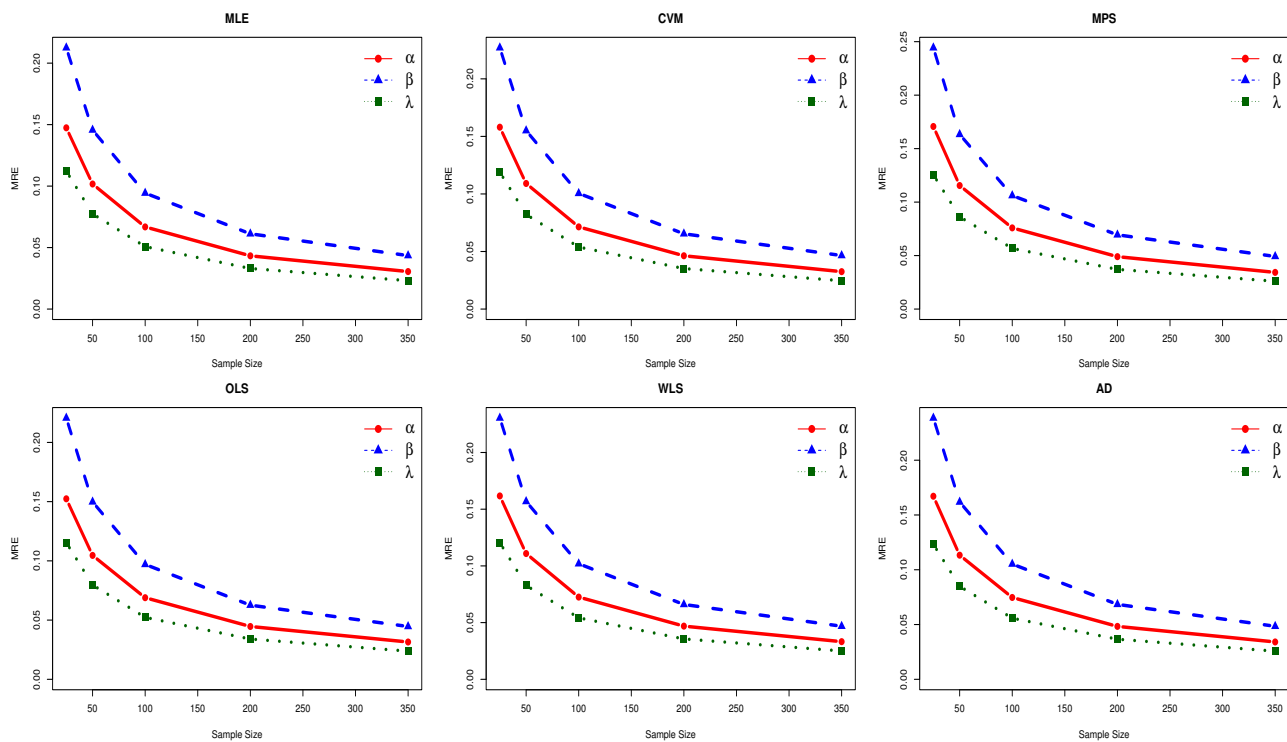


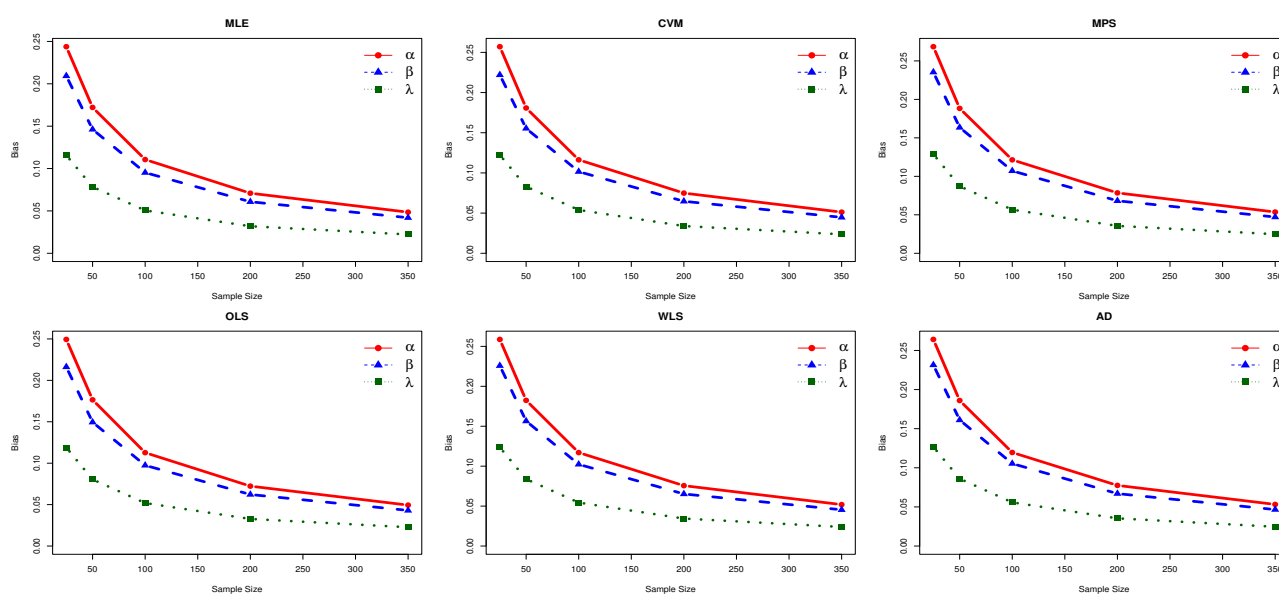
Figure 11. Visual comparison of MRE values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$.

Table 7. Simulation results and ranking of different estimation methods for $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$.

n	Estimate	Par.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
25	Bias	α	0.2438 ^[1]	0.2571 ^[3]	0.2685 ^[6]	0.2494 ^[2]	0.2587 ^[4]	0.2641 ^[5]
		β	0.2093 ^[1]	0.2220 ^[3]	0.2354 ^[6]	0.2162 ^[2]	0.2258 ^[4]	0.2315 ^[5]
		λ	0.1156 ^[1]	0.1223 ^[3]	0.1287 ^[6]	0.1189 ^[2]	0.1235 ^[4]	0.1267 ^[5]
	MSE	α	0.0791 ^[1]	0.0862 ^[3]	0.0930 ^[6]	0.0818 ^[2]	0.0873 ^[4]	0.0905 ^[5]
		β	0.0627 ^[1]	0.0690 ^[3]	0.0751 ^[6]	0.0656 ^[2]	0.0705 ^[4]	0.0737 ^[5]
		λ	0.0173 ^[1]	0.0193 ^[3]	0.0211 ^[6]	0.0181 ^[2]	0.0197 ^[4]	0.0206 ^[5]
	MRE	α	0.0975 ^[1]	0.1028 ^[3]	0.1074 ^[6]	0.0998 ^[2]	0.1035 ^[4]	0.1056 ^[5]
		β	0.1196 ^[1]	0.1269 ^[3]	0.1345 ^[6]	0.1235 ^[2]	0.1290 ^[4]	0.1323 ^[5]
		λ	0.4624 ^[1]	0.4892 ^[3]	0.5148 ^[6]	0.4756 ^[2]	0.4940 ^[4]	0.5068 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]
50	Bias	α	0.1721 ^[1]	0.1808 ^[3]	0.1883 ^[6]	0.1765 ^[2]	0.1824 ^[4]	0.1860 ^[5]
		β	0.1462 ^[1]	0.1554 ^[3]	0.1635 ^[6]	0.1495 ^[2]	0.1565 ^[4]	0.1611 ^[5]
		λ	0.0784 ^[1]	0.0830 ^[3]	0.0871 ^[6]	0.0804 ^[2]	0.0838 ^[4]	0.0861 ^[5]
	MSE	α	0.0467 ^[1]	0.0509 ^[3]	0.0549 ^[6]	0.0481 ^[2]	0.0515 ^[4]	0.0538 ^[5]
		β	0.0369 ^[1]	0.0411 ^[3]	0.0449 ^[6]	0.0383 ^[2]	0.0418 ^[4]	0.0439 ^[5]
		λ	0.0096 ^[1]	0.0107 ^[3]	0.0117 ^[6]	0.0101 ^[2]	0.0109 ^[4]	0.0115 ^[5]
	MRE	α	0.0688 ^[1]	0.0723 ^[3]	0.0753 ^[6]	0.0706 ^[2]	0.0730 ^[4]	0.0744 ^[5]
		β	0.0835 ^[1]	0.0888 ^[3]	0.0934 ^[6]	0.0854 ^[2]	0.0894 ^[4]	0.0921 ^[5]
		λ	0.3136 ^[1]	0.3321 ^[3]	0.3493 ^[6]	0.3216 ^[2]	0.3352 ^[4]	0.3444 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]
100	Bias	α	0.1106 ^[1]	0.1163 ^[3]	0.1214 ^[6]	0.1127 ^[2]	0.1170 ^[4]	0.1196 ^[5]
		β	0.0952 ^[1]	0.1016 ^[3]	0.1071 ^[6]	0.0974 ^[2]	0.1023 ^[4]	0.1054 ^[5]
		λ	0.0506 ^[1]	0.0539 ^[3]	0.0567 ^[6]	0.0519 ^[2]	0.0544 ^[4]	0.0558 ^[5]
	MSE	α	0.0261 ^[1]	0.0286 ^[3]	0.0310 ^[6]	0.0269 ^[2]	0.0290 ^[4]	0.0305 ^[5]
		β	0.0205 ^[1]	0.0229 ^[3]	0.0251 ^[6]	0.0213 ^[2]	0.0231 ^[4]	0.0244 ^[5]
		λ	0.0053 ^[1]	0.0059 ^[3]	0.0065 ^[6]	0.0055 ^[2]	0.0061 ^[4]	0.0064 ^[5]
	MRE	α	0.0442 ^[1]	0.0465 ^[3]	0.0486 ^[6]	0.0451 ^[2]	0.0468 ^[4]	0.0478 ^[5]
		β	0.0544 ^[1]	0.0581 ^[3]	0.0612 ^[6]	0.0556 ^[2]	0.0584 ^[4]	0.0602 ^[5]
		λ	0.2024 ^[1]	0.2156 ^[3]	0.2268 ^[6]	0.2076 ^[2]	0.2176 ^[4]	0.2232 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]

Table 8. Simulation results and ranking of different estimation methods for $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$. (Continued)

n	Estimate	Par.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
200	Bias	α	0.0709 ^[1]	0.0749 ^[3]	0.0785 ^[6]	0.0723 ^[2]	0.0756 ^[4]	0.0775 ^[5]
		β	0.0608 ^[1]	0.0647 ^[3]	0.0682 ^[6]	0.0622 ^[2]	0.0652 ^[4]	0.0670 ^[5]
		λ	0.0319 ^[1]	0.0339 ^[3]	0.0356 ^[6]	0.0327 ^[2]	0.0344 ^[4]	0.0354 ^[5]
	MSE	α	0.0137 ^[1]	0.0150 ^[3]	0.0162 ^[6]	0.0141 ^[2]	0.0153 ^[4]	0.0160 ^[5]
		β	0.0109 ^[1]	0.0120 ^[3]	0.0132 ^[6]	0.0113 ^[2]	0.0123 ^[4]	0.0129 ^[5]
		λ	0.0029 ^[1]	0.0032 ^[3]	0.0035 ^[6]	0.0030 ^[2]	0.0033 ^[4]	0.0035 ^[5]
	MRE	α	0.0284 ^[1]	0.0299 ^[3]	0.0314 ^[6]	0.0289 ^[2]	0.0302 ^[4]	0.0310 ^[5]
		β	0.0347 ^[1]	0.0370 ^[3]	0.0390 ^[6]	0.0355 ^[2]	0.0373 ^[4]	0.0383 ^[5]
		λ	0.1276 ^[1]	0.1356 ^[3]	0.1424 ^[6]	0.1308 ^[2]	0.1376 ^[4]	0.1416 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]
350	Bias	α	0.0485 ^[1]	0.0512 ^[3]	0.0536 ^[6]	0.0494 ^[2]	0.0519 ^[4]	0.0532 ^[5]
		β	0.0420 ^[1]	0.0447 ^[3]	0.0471 ^[6]	0.0430 ^[2]	0.0454 ^[4]	0.0466 ^[5]
		λ	0.0222 ^[1]	0.0236 ^[3]	0.0248 ^[6]	0.0227 ^[2]	0.0239 ^[4]	0.0246 ^[5]
	MSE	α	0.0080 ^[1]	0.0089 ^[3]	0.0097 ^[6]	0.0082 ^[2]	0.0090 ^[4]	0.0096 ^[5]
		β	0.0066 ^[1]	0.0073 ^[3]	0.0080 ^[6]	0.0068 ^[2]	0.0074 ^[4]	0.0079 ^[5]
		λ	0.0019 ^[1]	0.0021 ^[3]	0.0023 ^[6]	0.0020 ^[2]	0.0021 ^[4]	0.0022 ^[5]
	MRE	α	0.0194 ^[1]	0.0205 ^[3]	0.0214 ^[6]	0.0198 ^[2]	0.0208 ^[4]	0.0213 ^[5]
		β	0.0240 ^[1]	0.0256 ^[3]	0.0269 ^[6]	0.0246 ^[2]	0.0259 ^[4]	0.0266 ^[5]
		λ	0.0888 ^[1]	0.0944 ^[3]	0.0992 ^[6]	0.0908 ^[2]	0.0956 ^[4]	0.0984 ^[5]
	Σ Ranks		9 ^[1]	27 ^[3]	54 ^[6]	18 ^[2]	36 ^[4]	45 ^[5]

**Figure 12.** Visual comparison of bias values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$.

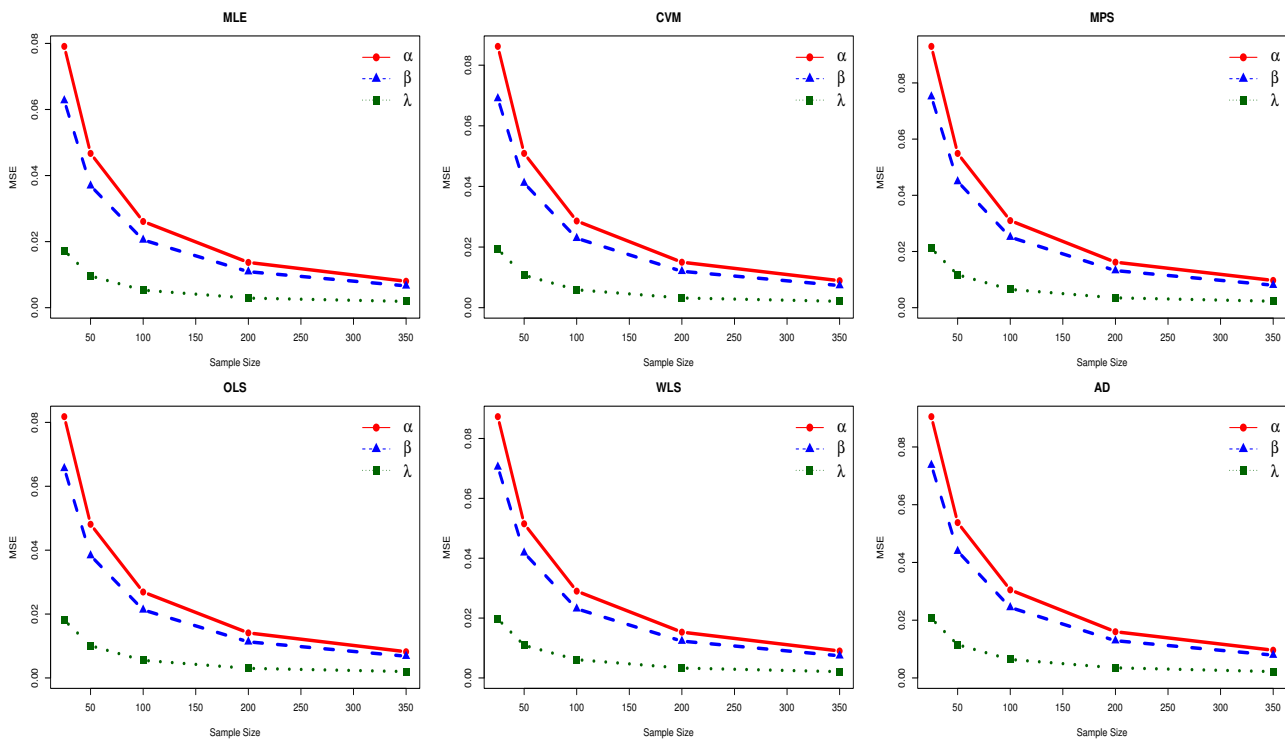


Figure 13. Visual comparison of MSE values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$.

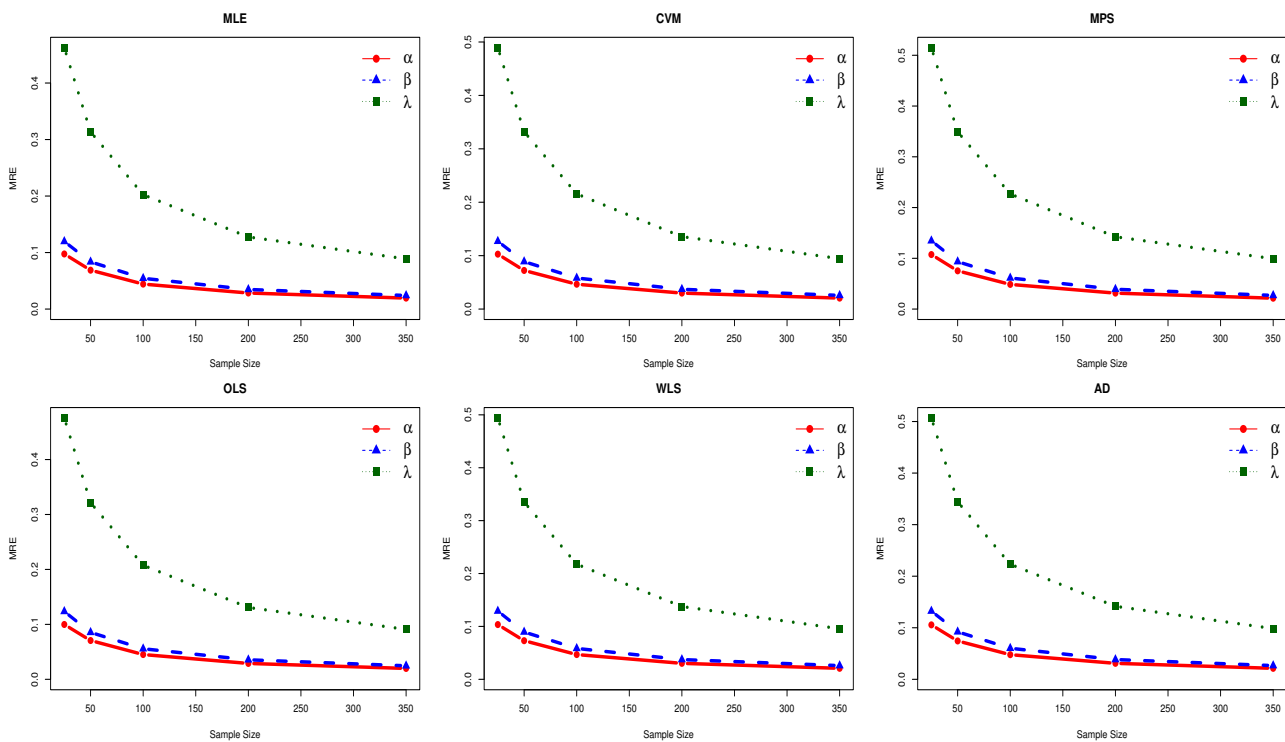


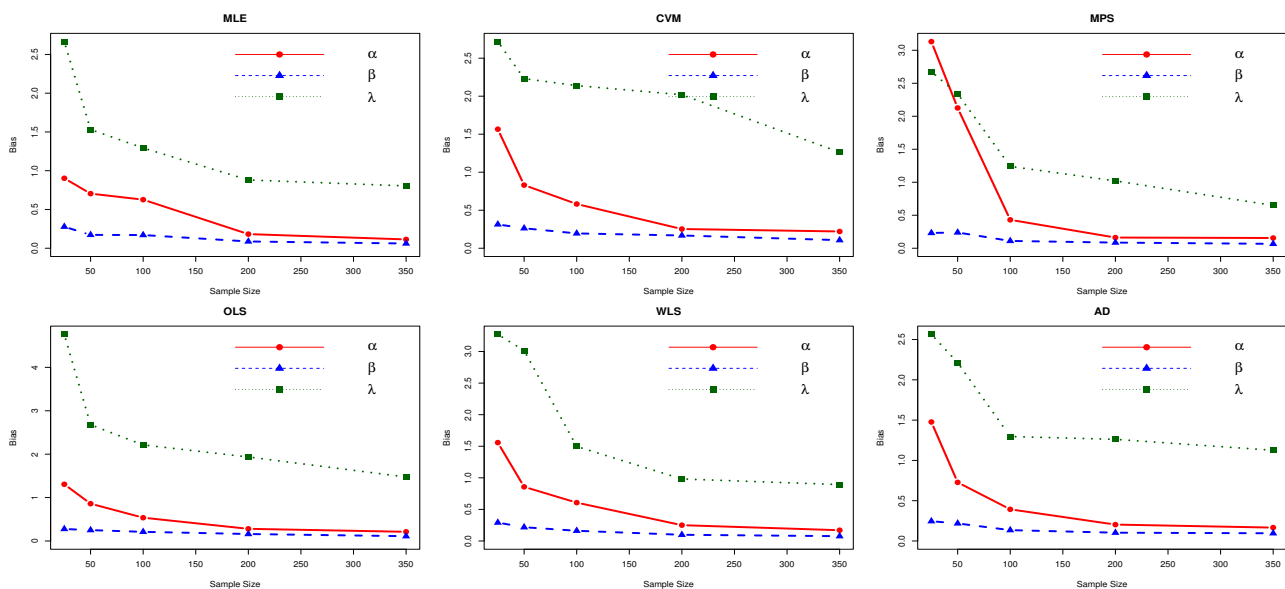
Figure 14. Visual comparison of MRE values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$.

Table 9. Simulation results and ranking of different estimation methods for $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$.

n	Estimate	Par.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
25	Bias	α	0.9023 ^[1]	1.5657 ^[5]	3.1310 ^[6]	1.3062 ^[2]	1.5578 ^[4]	1.4769 ^[3]
		β	0.2775 ^[4]	0.3134 ^[6]	0.2312 ^[1]	0.2761 ^[3]	0.2909 ^[5]	0.2453 ^[2]
		λ	2.6652 ^[2]	2.7178 ^[4]	2.6748 ^[3]	4.7601 ^[6]	3.2716 ^[5]	2.5640 ^[1]
	MSE	α	4.9255 ^[1]	6.5933 ^[2]	12.2233 ^[6]	7.6738 ^[3]	9.4880 ^[5]	8.7287 ^[4]
		β	0.1426 ^[5]	0.1431 ^[6]	0.0804 ^[1]	0.1182 ^[3]	0.1365 ^[4]	0.0947 ^[2]
		λ	27.3229 ^[3]	23.2537 ^[1]	28.2991 ^[4]	102.0321 ^[6]	37.7482 ^[5]	23.9533 ^[2]
	MRE	α	3.6093 ^[1]	6.2630 ^[5]	12.5239 ^[6]	5.2247 ^[2]	6.2312 ^[4]	5.9078 ^[3]
		β	0.3700 ^[4]	0.4179 ^[6]	0.3083 ^[1]	0.3681 ^[3]	0.3878 ^[5]	0.3270 ^[2]
		λ	2.1321 ^[2]	2.1742 ^[4]	2.1398 ^[3]	3.8081 ^[6]	2.6173 ^[5]	2.0512 ^[1]
	Σ Ranks		23 ^[2]	39 ^[5]	31 ^[3]	34 ^[4]	42 ^[6]	20 ^[1]
50	Bias	α	0.7045 ^[1]	0.8304 ^[6]	2.1257 ^[2]	0.8590 ^[5]	0.8570 ^[4]	0.7273 ^[3]
		β	0.1721 ^[1]	0.2625 ^[6]	0.2385 ^[4]	0.2503 ^[5]	0.2175 ^[2]	0.2178 ^[3]
		λ	1.5309 ^[1]	2.2310 ^[6]	2.3389 ^[2]	2.6776 ^[3]	3.0155 ^[4]	2.2126 ^[5]
	MSE	α	2.6396 ^[1]	2.8772 ^[2]	8.3671 ^[6]	2.9674 ^[4]	4.5493 ^[5]	2.8980 ^[3]
		β	0.0632 ^[1]	0.1138 ^[6]	0.0879 ^[4]	0.1022 ^[5]	0.0737 ^[3]	0.0724 ^[2]
		λ	7.2918 ^[1]	14.7397 ^[3]	14.2714 ^[2]	17.7143 ^[4]	24.1188 ^[6]	17.8823 ^[5]
	MRE	α	2.8180 ^[1]	3.3218 ^[6]	9.5030 ^[2]	3.4361 ^[5]	3.4279 ^[4]	2.9092 ^[3]
		β	0.2294 ^[1]	0.3500 ^[6]	0.3180 ^[4]	0.3337 ^[5]	0.2900 ^[2]	0.2904 ^[3]
		λ	1.2247 ^[1]	1.7848 ^[6]	1.8711 ^[2]	2.1421 ^[3]	2.4124 ^[4]	1.7701 ^[5]
	Σ Ranks		9 ^[1]	47 ^[6]	28 ^[2]	39 ^[5]	34 ^[4]	32 ^[3]
100	Bias	α	0.6265 ^[6]	0.5815 ^[4]	0.4306 ^[2]	0.5365 ^[3]	0.6072 ^[5]	0.3928 ^[1]
		β	0.1700 ^[4]	0.1948 ^[5]	0.1105 ^[1]	0.2105 ^[6]	0.1616 ^[3]	0.1345 ^[2]
		λ	1.2929 ^[2]	2.1392 ^[5]	1.2402 ^[1]	2.2133 ^[6]	1.4963 ^[4]	1.2964 ^[3]
	MSE	α	1.5637 ^[4]	1.4825 ^[3]	1.8830 ^[5]	1.1653 ^[1]	2.1770 ^[6]	1.2048 ^[2]
		β	0.0541 ^[4]	0.0702 ^[5]	0.0219 ^[1]	0.0738 ^[6]	0.0461 ^[3]	0.0352 ^[2]
		λ	5.4250 ^[1]	12.8683 ^[5]	7.1845 ^[4]	13.6284 ^[6]	6.4100 ^[3]	5.5343 ^[2]
	MRE	α	2.5060 ^[6]	2.3262 ^[4]	1.7224 ^[2]	2.1461 ^[3]	2.4287 ^[5]	1.5713 ^[1]
		β	0.2267 ^[4]	0.2597 ^[5]	0.1473 ^[1]	0.2807 ^[6]	0.2155 ^[3]	0.1794 ^[2]
		λ	1.1543 ^[3]	1.7113 ^[5]	0.9921 ^[1]	1.7706 ^[6]	1.1970 ^[4]	1.0372 ^[2]
	Σ Ranks		37 ^[4]	41 ^[5.5]	17 ^[1]	41 ^[5.5]	36 ^[3]	20 ^[2]

Table 10. Simulation results and ranking of different estimation methods for $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$. (Continued)

n	Estimate	Par.	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
200	Bias	α	0.1821 ^[2]	0.2525 ^[5]	0.1617 ^[1]	0.2800 ^[6]	0.2503 ^[4]	0.2034 ^[3]
		β	0.0871 ^[2]	0.1676 ^[6]	0.0853 ^[1]	0.1604 ^[5]	0.0979 ^[3]	0.1027 ^[4]
		λ	0.8805 ^[1]	2.0191 ^[6]	1.0203 ^[3]	1.9354 ^[5]	0.9818 ^[2]	1.2613 ^[4]
	MSE	α	0.0677 ^[2]	0.1217 ^[4]	0.0519 ^[1]	0.1731 ^[5]	0.2093 ^[6]	0.0718 ^[3]
		β	0.0141 ^[1.5]	0.0529 ^[6]	0.0141 ^[1.5]	0.0497 ^[5]	0.0166 ^[3]	0.0225 ^[4]
		λ	3.0179 ^[1]	10.5454 ^[6]	3.7888 ^[3]	10.3487 ^[5]	3.1785 ^[2]	4.7417 ^[4]
	MRE	α	0.7283 ^[2]	1.0101 ^[5]	0.6468 ^[1]	1.1199 ^[6]	1.0011 ^[4]	0.8137 ^[3]
		β	0.1161 ^[2]	0.2234 ^[6]	0.1138 ^[1]	0.2138 ^[5]	0.1305 ^[3]	0.1369 ^[4]
		λ	0.7044 ^[1]	1.6153 ^[6]	0.8162 ^[3]	1.5483 ^[5]	0.7854 ^[2]	1.0090 ^[4]
	Σ Ranks		14.5 ^[1]	50 ^[6]	15.5 ^[2]	47 ^[5]	29 ^[3]	33 ^[4]
350	Bias	α	0.1134 ^[1]	0.2205 ^[6]	0.1551 ^[2]	0.2113 ^[5]	0.1703 ^[4]	0.1660 ^[3]
		β	0.0609 ^[1]	0.1062 ^[5]	0.0664 ^[2]	0.1100 ^[6]	0.0761 ^[3]	0.0956 ^[4]
		λ	0.8037 ^[2]	1.2627 ^[5]	0.6493 ^[1]	1.4737 ^[6]	0.8933 ^[3]	1.1255 ^[4]
	MSE	α	0.0210 ^[1]	0.0926 ^[5]	0.0528 ^[2]	0.0996 ^[6]	0.0653 ^[4]	0.0540 ^[3]
		β	0.0086 ^[2]	0.0210 ^[5]	0.0072 ^[1]	0.0237 ^[6]	0.0118 ^[3]	0.0197 ^[4]
		λ	2.2552 ^[2]	4.5757 ^[5]	1.4103 ^[1]	6.3021 ^[6]	3.1211 ^[3]	3.9008 ^[4]
	MRE	α	0.4538 ^[1]	0.8820 ^[6]	0.6204 ^[2]	0.8454 ^[5]	0.6811 ^[4]	0.6641 ^[3]
		β	0.0812 ^[1]	0.1416 ^[5]	0.0885 ^[2]	0.1467 ^[6]	0.1014 ^[3]	0.1275 ^[4]
		λ	0.6430 ^[2]	1.0102 ^[5]	0.5194 ^[1]	1.1789 ^[6]	0.7147 ^[3]	0.9004 ^[4]
	Σ Ranks		13 ^[1]	47 ^[5]	14 ^[2]	52 ^[6]	30 ^[3]	33 ^[4]

**Figure 15.** Visual comparison of bias values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$.

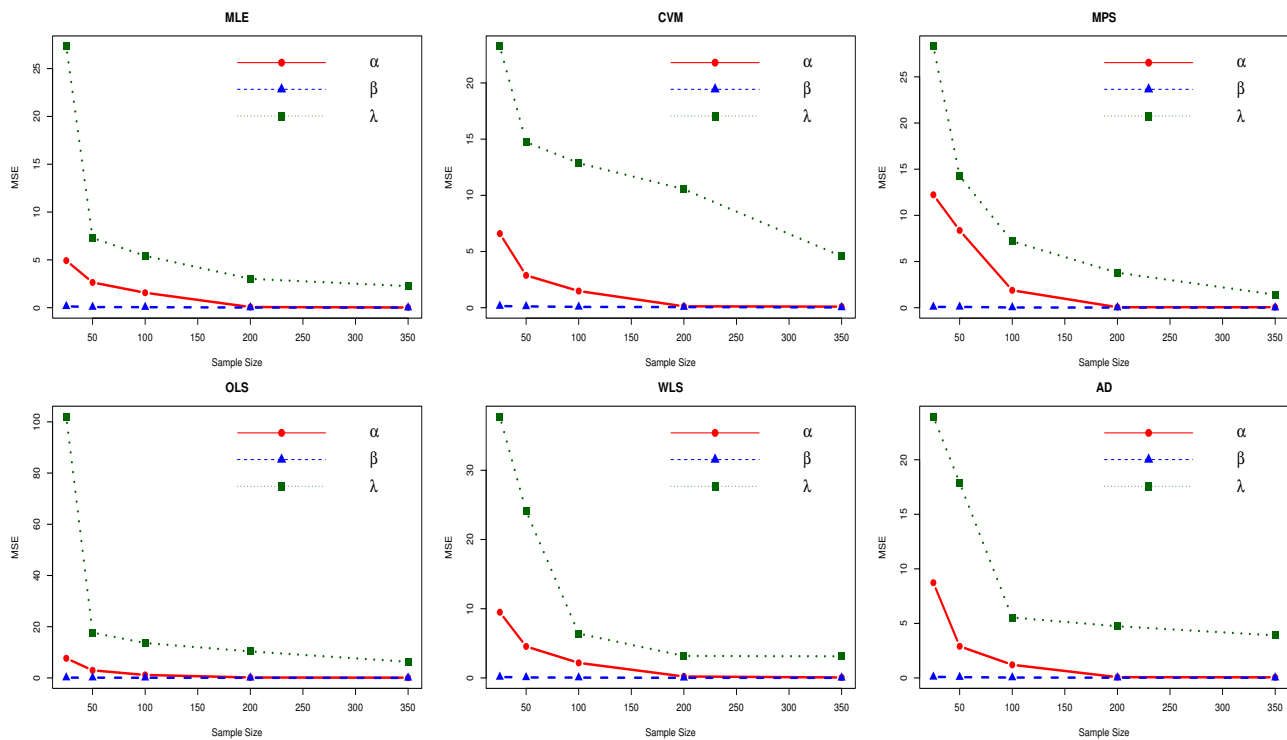


Figure 16. Visual comparison of MSE values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$.

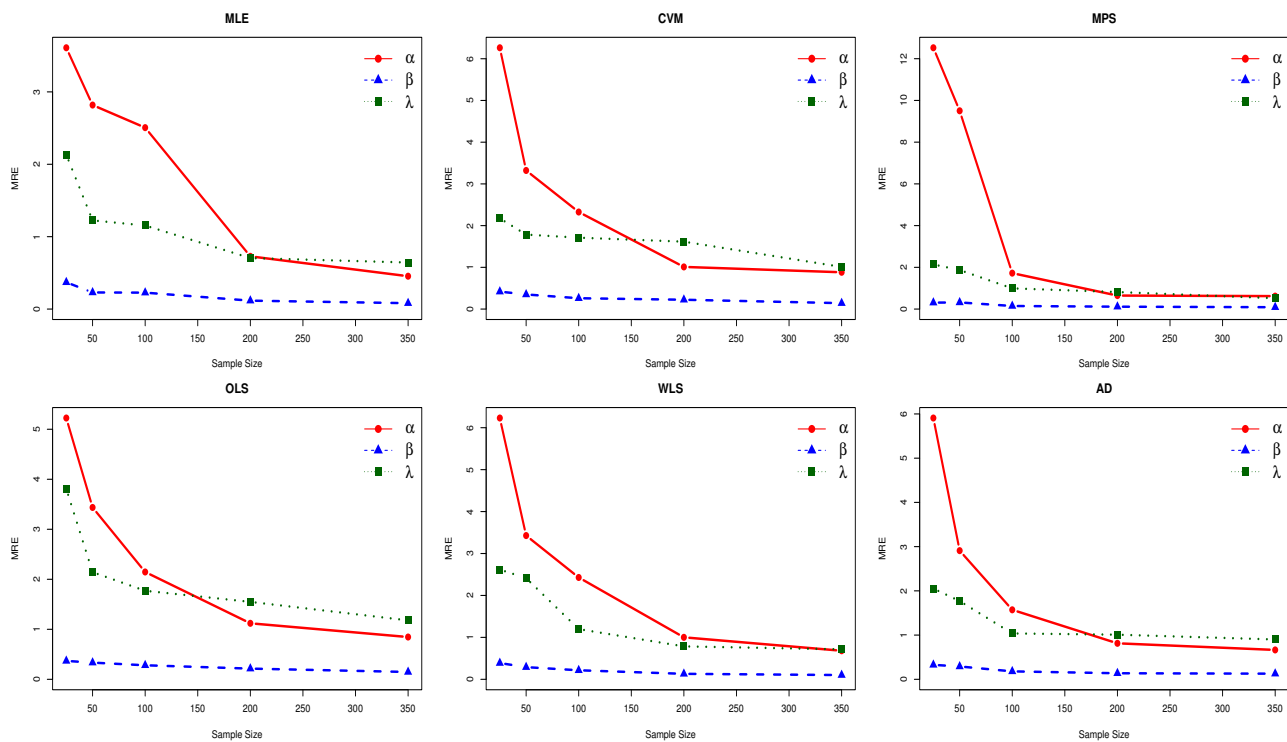
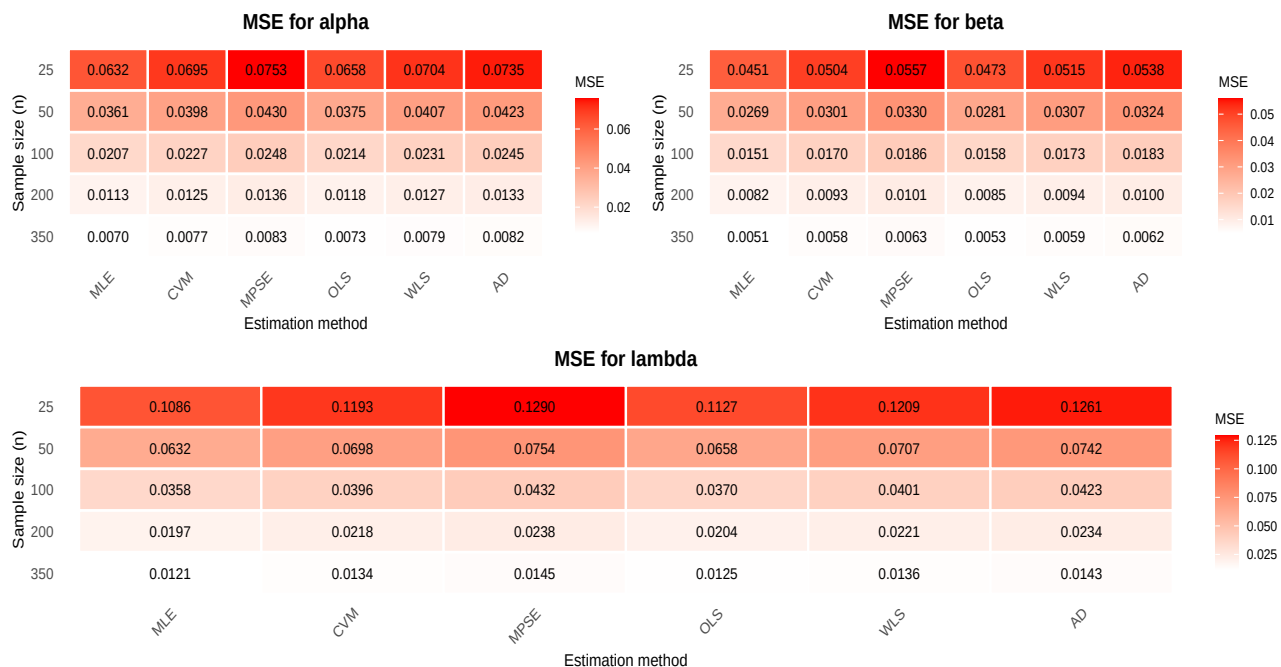


Figure 17. Visual comparison of MRE values for parameters α , β , and λ across six estimation methods under the setting $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$.

Table 11. Partial and overall ranking of estimation methods for the proposed distribution under varying model parameter settings.

(σ, β, λ)	n	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
(1.25, 0.75, 2.35)	25	1	3	6	2	4	5
	50	1	3	6	2	4	5
	100	1	3	6	2	4	5
	200	1	3	6	2	4	5
	350	1	3	6	2	4	5
	25	1	3	6	2	4	5
(2.50, 1.75, 0.25)	50	1	3	6	2	4	5
	100	1	3	6	2	4	5
	200	1	3	6	2	4	5
	350	1	3	6	2	4	5
	25	2	5	3	4	6	1
	50	1	6	2	5	4	3
(0.25, 0.75, 1.25)	100	4	5.5	1	5.5	3	2
	200	1	6	2	5	3	4
	350	1	5	2	6	3	4
	Σ Ranks	19 ^[1]	57.5 ^[3]	70 ^[6]	45.5 ^[2]	59 ^[4]	64 ^[5]

**Figure 18.** MSE heat maps with $\alpha = 1.25$, $\beta = 0.75$, and $\lambda = 2.35$ for the estimates of α , β , and λ for various sample sizes and estimation techniques.

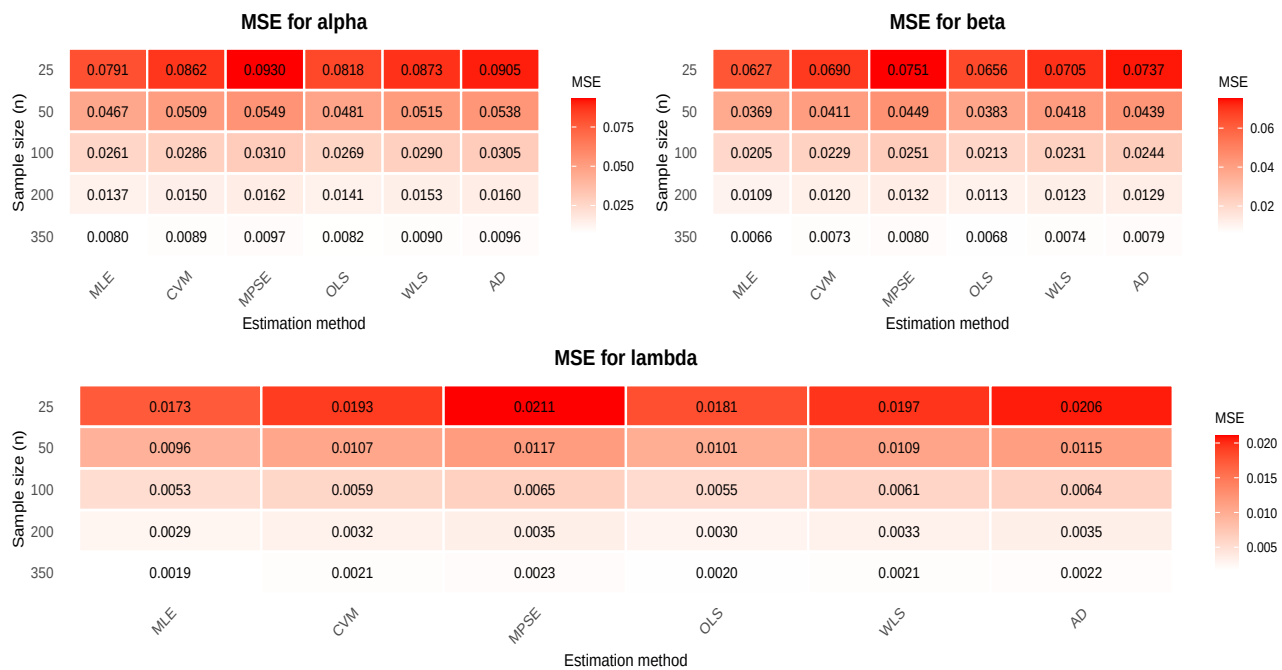


Figure 19. MSE heat maps with $\alpha = 2.50$, $\beta = 1.75$, and $\lambda = 0.25$ for the estimates of α , β , and λ for various sample sizes and estimation techniques.

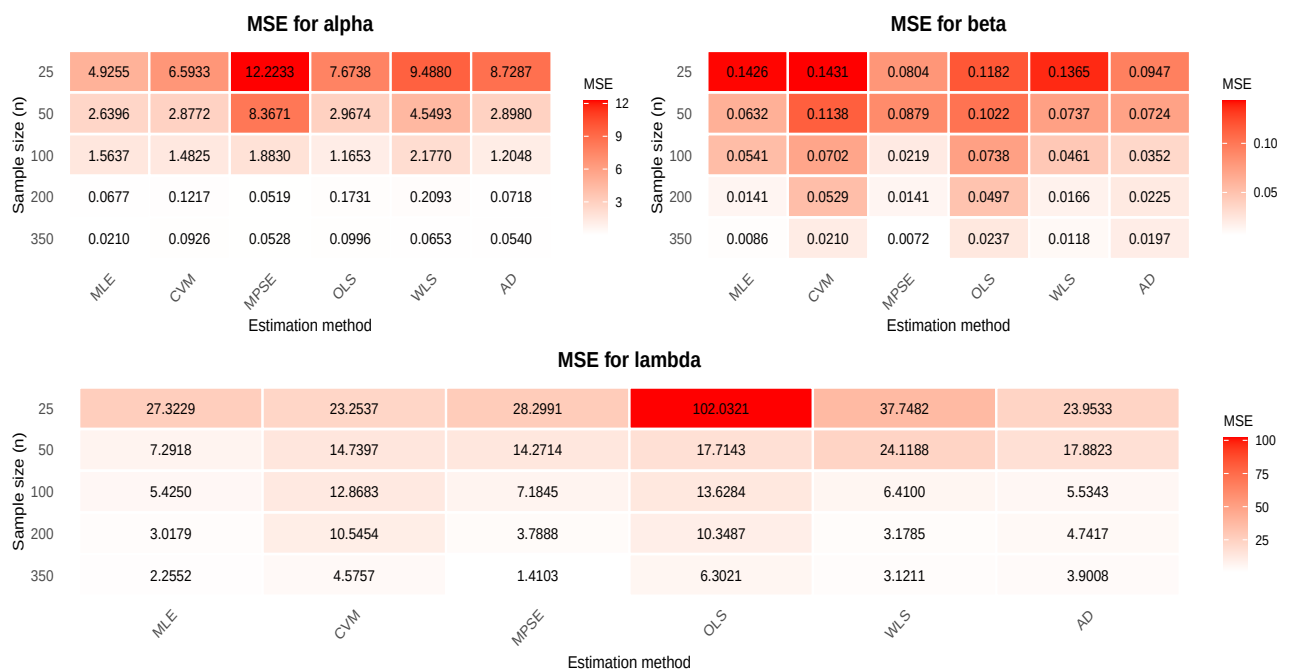


Figure 20. MSE heat maps with $\alpha = 0.25$, $\beta = 0.75$, and $\lambda = 1.25$ for the estimates of α , β , and λ for various sample sizes and estimation techniques.

Figures 18–20 show a graphic comparison of the MSEs for the estimators of α , β , and λ across the different sample sizes and estimation strategies in the two simulated situations. A clear decreasing

trend in the MSE is observed as the sample size increases from $n = 25$ to $n = 350$, supporting the estimators increasing precision and consistency. The MLE method consistently obtains the lowest MSE values for all three parameters under both simulation settings, whereas MPSE usually displays the highest MSE values. The performance of the remaining methods is mediocre: CVM, OLS, WLS, and AD. The heat maps demonstrate that MLE has the best overall finite-sample performance and that the differences between the estimating methods become less apparent as the sample size increases.

6. Applications

To illustrate the practical applicability and versatility of the NLW distribution in real-world scenarios, four distinct datasets have been analyzed.

6.1. Dataset 1: Carbon fiber tensile strength measurements

The first dataset represents the tensile strength, measured in gigapascals (GPa), of 69 carbon fibers tested under tension at gauge lengths of 20 mm. This dataset was originally reported by [19] and was recently analyzed by [8]. The dataset is as follows: 0.0312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, and 2.585.

6.2. Dataset 2: United Kingdom COVID-19 death rate data

The second dataset analyzes COVID-19 mortality rates in the United Kingdom recorded over a 76-day period from April 15 to June 30, 2020. This dataset has also been used in prior studies (see [20] and [21] for details). All observations are complete, with no censored or truncated values, and each record represents the mortality rate at a specific time point. The dataset is as follows: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019, 11.4584, 0.2751, 0.4110, 0.7193, 1.3423, 1.9844, 3.9042, and 7.4456.

6.3. Dataset 3: Glass fiber tensile strength measurements

The third dataset analyzed in this study comes from engineering science and contains 63 measurements of the tensile strengths of 1.5 cm glass fibers. The data were originally collected by researchers at the UK National Physical Laboratory and first reported by [22], and it has since been analyzed in several studies, including [23]. The data is given as follows: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64,

1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, and 2.24.

6.4. Dataset 4: Mexico COVID-19 fatality rate data

The fourth dataset consists of the COVID-19 fatality rates reported in Mexico over a period of 108 days, from March 4 to July 20, 2020. The dataset is publicly available through the World Health Organization (WHO) COVID-19 data portal and has been recently analyzed by [24, 25]. The corresponding observations are presented as follows: 8.826, 6.105, 10.383, 7.267, 13.220, 6.015, 10.855, 6.122, 10.685, 10.035, 5.242, 7.630, 14.604, 7.903, 6.327, 9.391, 14.962, 4.730, 3.215, 16.498, 11.665, 9.284, 12.878, 6.656, 3.440, 5.854, 8.813, 10.043, 7.260, 5.985, 4.424, 4.344, 5.143, 9.935, 7.840, 9.550, 6.968, 6.370, 3.537, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 3.499, 3.751, 7.486, 6.625, 6.140, 4.909, 4.661, 1.871, 2.838, 5.392, 12.042, 8.696, 6.412, 3.395, 1.815, 3.327, 5.406, 6.182, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, and 1.923.

For the four real-world datasets, the proposed NLW distribution is compared with nine competitive lifetime distributions to assess its flexibility and goodness-of-fit performance. The competing distributions considered in this study are as follows:

- **Weibull distribution (WD):**

$$f(x) = \frac{\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \exp\left(-\left(\frac{x}{\lambda}\right)^{\beta}\right).$$

- **Exponentiated Weibull distribution (EWD):**

$$f(x) = \frac{\alpha\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \left(1 - \exp\left(-\left(\frac{x}{\lambda}\right)^{\beta}\right)\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\lambda}\right)^{\beta}\right).$$

- **Exponentiated exponential distribution (EED):**

$$f(x) = \frac{\beta}{\lambda} \left(1 - \exp\left(-\frac{x}{\lambda}\right)\right)^{\beta-1} \exp\left(-\frac{x}{\lambda}\right).$$

- **Rayleigh distribution (RD):**

$$f(x) = \frac{x}{\lambda^2} \exp\left(-\frac{x^2}{2\lambda^2}\right).$$

- **Gamma distribution (GD):**

$$f(x) = \frac{1}{\Gamma(\beta)\lambda^{\beta}} x^{\beta-1} e^{-x/\lambda}.$$

- **Exponential distribution (ED):**

$$f(x) = \lambda \exp(-\lambda x).$$

• **Gamma-Weibull distribution (GWD):**

$$f(x) = \frac{\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \frac{1}{\Gamma(\alpha)} \left(\left(\frac{x}{\lambda}\right)^{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\lambda}\right)^{\beta}\right).$$

• **Alpha power exponential distribution (APED):**

$$f(x) = \frac{\log(\alpha)}{\alpha - 1} \lambda \exp(-\lambda x) \alpha^{1-\exp(-\lambda x)}.$$

• **Alpha power gamma distribution (APGD):**

$$f(x) = \frac{\log(\alpha)}{\alpha - 1} \frac{x^{\beta-1} e^{-x/\lambda}}{\Gamma(\beta) \lambda^{\beta}} \alpha^{G(x)},$$

where

$$G(x) = P\left(\beta, \frac{x}{\lambda}\right) = \frac{\gamma(\beta, x/\lambda)}{\Gamma(\beta)}.$$

The descriptive characteristics of the considered datasets, including the variance (Var), skewness, and kurtosis are summarized in Table 12. The maximum likelihood estimation (MLE) method is used to fit the suggested NLW distribution to the datasets under consideration. For each dataset, the estimated values of the model parameters are presented in Tables 13, 15, 17, and 19. To assess the performance and suitability of the fitted models, a number of commonly used information criteria and goodness-of-fit statistics are utilized. For model comparison, the Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion (CAIC), and Hannan–Quinn information criterion (HQIC) are utilized. The agreement between the fitted models and the observed data is further evaluated by taking into account the Cramér–von Mises (CVM) statistic, the Anderson–Darling (AD) statistic, and the Kolmogorov–Smirnov (K–S) statistic together with its corresponding p -value.

The corresponding goodness-of-fit results are presented in Tables 14, 16, 18, and 20. In general, a model is regarded as providing a better fit when it produces smaller values of AIC, BIC, CAIC, HQIC, K–S, AD and CVM, together with a comparatively larger K–S p -value. Thus, these criteria collectively provide a comprehensive basis for evaluating and comparing the proposed NLW distribution with the competing models. All numerical computations and statistical analyses were carried out using the R software.

The graphical assessments of the fitted NLW model are provided in Figures 22, 24, 26 and 28, which include the fitted probability density function (PDF), probability–probability (P–P) plots, quantile–quantile (Q–Q) plots, estimated survival function (SF), total time on test (TTT) transforms, and boxplots. These graphical representations complement the numerical goodness-of-fit results and provide further insight into the ability of the proposed NLW distribution to adequately describe the observed datasets. Finally, the comparative results shown through the bar plots in Figures 21, 23, 25 and 27 provide a visual assessment of the performance of the competing models across the considered datasets.

Table 12. Descriptive statistics for the carbon fiber tensile strength measurements, the United Kingdom COVID-19 death rate records, the glass fiber tensile strength measurements, and Mexico COVID-19 fatality rate data.

Dataset	n	Mean	Median	Min	Max	Range	SD	Var	Skewness	Kurtosis
Dataset 1	69	1.4470	1.4780	0.0312	2.5850	2.5538	0.5060	0.2560	-0.1639	0.3459
Dataset 2	76	2.4372	1.2484	0.0587	11.4584	11.3997	2.9360	8.6198	1.7301	2.3205
Dataset 3	63	1.4887	1.5500	0.5500	2.2400	1.6900	0.2962	0.0877	-0.4461	0.3395
Dataset 4	108	5.7581	5.1925	1.0410	16.4980	15.4570	3.2541	10.5893	0.9867	0.6813

Table 13. Maximum likelihood estimates of the model parameters for the carbon fiber tensile strength measurements.

Model	α	β	λ
NLWD	20.5740	1.7294	0.8048
WD	—	3.0394	1.6036
EWD	0.5234	4.5343	1.8721
EED	—	4.9928	0.6544
RD	—	—	1.0831
GD	—	4.9769	0.2907
ED	—	—	0.6911
APED	15222.7463	—	1.9008
APGD	221.7951	2.5141	0.3269
GWD	0.4889	1.9824	4.8232

Table 14. Log-likelihood and goodness-of-fit measures for the competing statistical models for the carbon fiber tensile strength measurements.

Model	LogL	AIC	CAIC	BIC	HQIC	AD	CVM	KS	p -value
NLWD	-50.9198	107.8396	114.2088	114.5419	110.4986	0.2145	0.0218	0.0510	0.9939
WD	-53.9576	111.9152	116.0971	116.3834	113.6879	0.4523	0.0567	0.0661	0.9239
EWD	-52.6957	111.3913	117.7606	118.0936	114.0504	0.4937	0.0634	0.0787	0.7867
EED	-67.5369	139.0738	143.2556	143.5420	140.8464	3.0047	0.4875	0.1443	0.1130
RD	-61.6799	125.3597	127.4194	127.5938	126.2461	4.1746	0.8006	0.2001	0.0080
GD	-63.1758	130.3517	134.5335	134.8199	132.1244	1.9350	0.3089	0.1265	0.2193
ED	-94.4937	190.9875	193.0472	193.2216	191.8738	13.3875	2.7566	0.3633	0.0001
APED	-58.1325	120.2651	124.4469	124.7333	122.0377	1.4230	0.2041	0.1006	0.4868
APGD	-54.6612	115.3223	121.6916	122.0246	117.9814	0.6018	0.0723	0.0673	0.9133
GWD	-52.6353	111.2705	117.6397	117.9728	113.9295	0.4857	0.0621	0.0785	0.7888

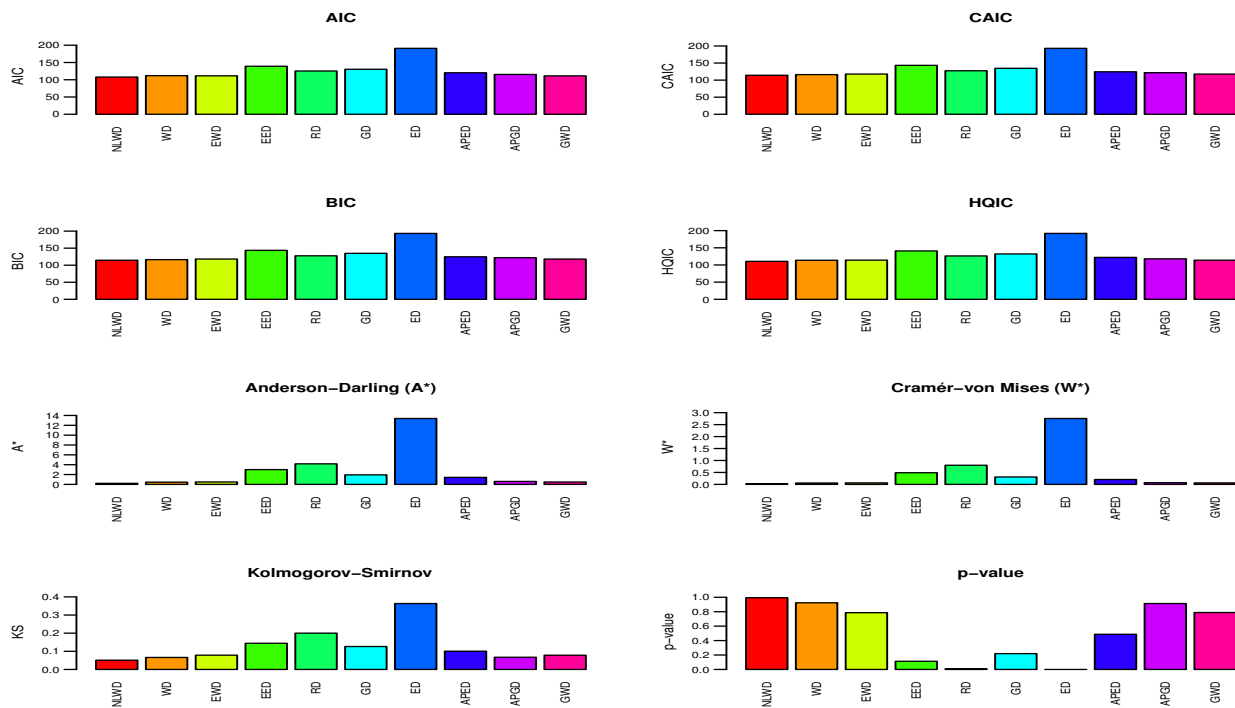


Figure 21. Comparative bar plots of goodness-of-fit statistics for the fitted probability models using the carbon fiber tensile strength measurements (Dataset 1).

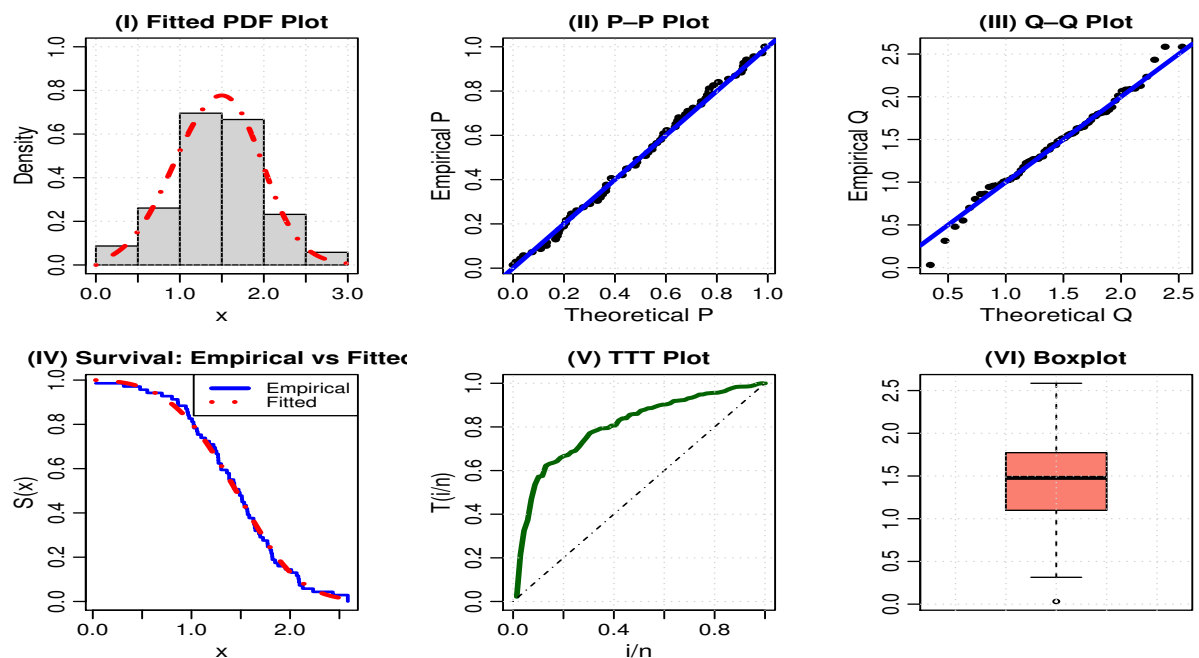


Figure 22. Comparative graphical assessment of the NLW model applied to the carbon fiber tensile strength measurements data (Dataset 1).

Table 15. Maximum likelihood estimates of the model parameters for the United Kingdom's COVID-19 daily death rate data.

Model	α	β	λ
NLWD	0.000004	2.950194	22.161512
WD	—	0.846593	2.220001
EWD	6.213639	0.373080	0.138453
EED	—	0.800938	2.832584
RD	—	—	2.687583
GD	—	0.803740	3.032307
ED	—	—	0.410309
APED	0.152081	—	0.254030
APGD	0.164096	0.976300	3.986368
GWD	14.684723	0.000002	0.202284

Table 16. Log-likelihood and goodness-of-fit measures for the competing statistical models for the United Kingdom's COVID-19 daily death rate data.

Model	LogL	AIC	CAIC	BIC	HQIC	AD	CVM	KS	p -value
NLWD	-136.6836	279.3672	285.7005	286.3594	282.1616	10.4582	1.8956	0.0514	0.9881
WD	-141.7517	287.5033	291.6677	292.1648	289.3662	11.6877	2.0787	0.0806	0.7074
EWD	-139.8146	285.6291	291.9625	292.6213	288.4236	11.2437	2.0785	0.0574	0.9638
EED	-142.5027	289.0054	293.1698	293.6669	290.8684	12.3694	2.1724	0.0985	0.4516
RD	-214.6545	431.3089	433.3630	433.6397	432.2404	72.7905	7.6747	0.4293	0.0001
GD	-142.4102	288.8205	292.9849	293.4820	290.6834	12.2772	2.1589	0.0962	0.4833
ED	-143.7041	289.4083	291.4623	291.7390	290.3397	15.1801	2.5581	0.1447	0.0830
APED	-140.8744	285.7488	289.9132	290.4103	287.6118	12.0486	2.1360	0.0811	0.6997
APGD	-140.8575	287.7150	294.0484	294.7072	290.5095	11.8300	2.1068	0.0769	0.7594
GWD	-139.6908	285.3816	291.7149	292.3738	288.1760	11.2856	2.0790	0.0590	0.9539

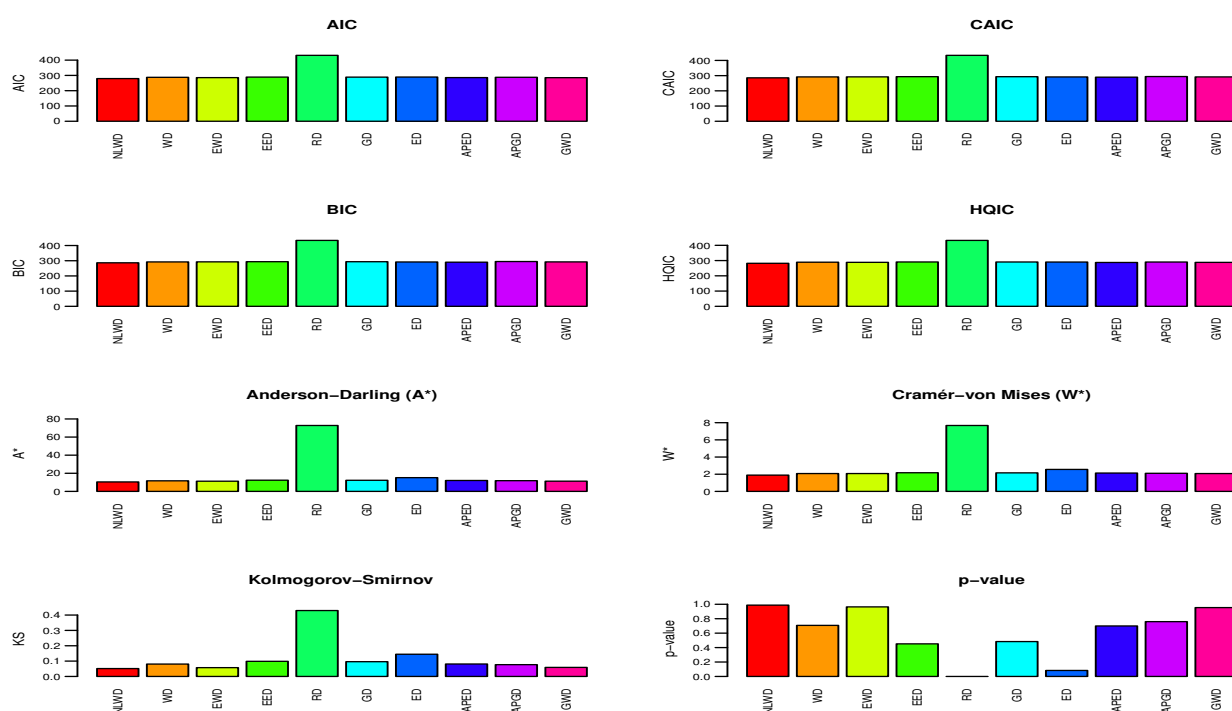


Figure 23. Comparative bar plots of goodness-of-fit statistics for the fitted probability models using the United Kingdom COVID-19 death rate (Dataset 2).

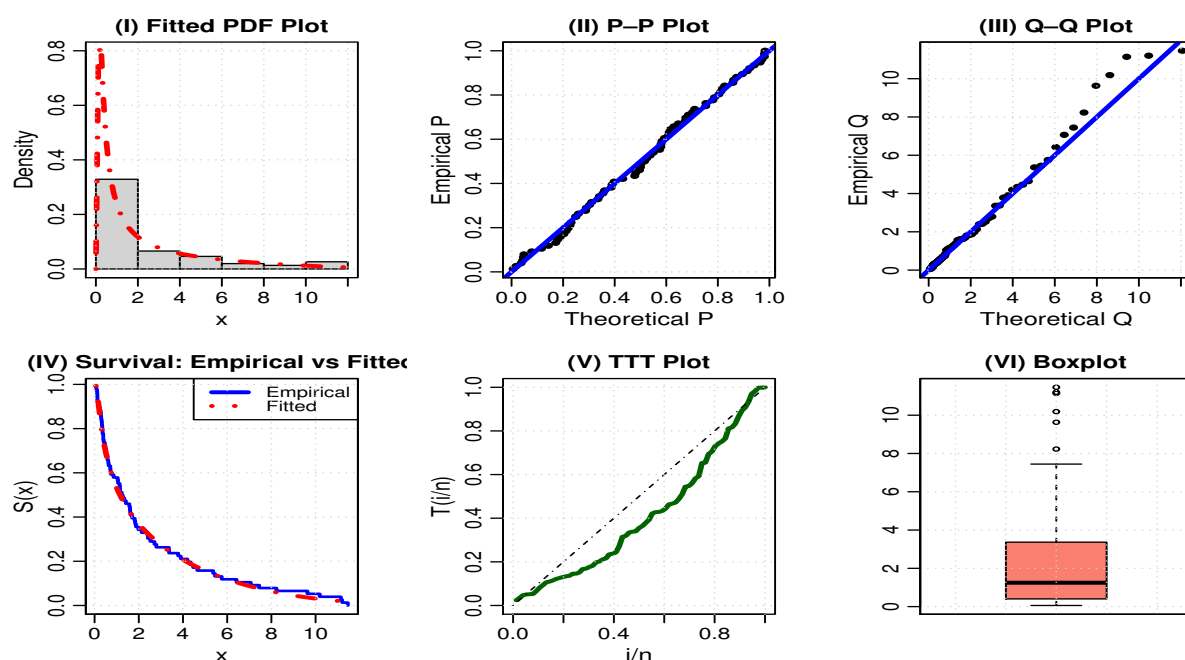


Figure 24. Comparative graphical assessment of the NLW model applied to the United Kingdom COVID-19 death rate records data (Dataset 2).

Table 17. Maximum likelihood estimates of the model parameters for the glass fiber tensile strength measurements.

Model	α	β	λ
NLWD	6.3894	4.0464	1.3166
WD	—	5.7807	1.6281
EWD	0.6712	7.2846	1.7181
EED	—	31.3489	0.3829
RD	—	—	1.0895
GD	—	17.4396	0.0864
ED	—	—	0.6636
APED	9.8070×10^{14}	—	2.6566
APGD	25.9415	14.3490	0.0874
GWD	0.6197	1.7692	7.7625

Table 18. Log-likelihood and goodness-of-fit measures for competing statistical models fitted to the glass fiber tensile strength measurements.

Model	LogL	AIC	CAIC	BIC	HQIC	AD	CVM	KS	p -value
NLWD	-13.0324	32.0647	38.4715	38.4941	34.5935	0.7552	0.1262	0.1243	0.2849
WD	-15.2068	34.4137	38.6137	38.7000	36.0995	1.2408	0.2151	0.1522	0.1078
EWD	-14.6755	35.3510	41.7578	41.7805	37.8798	1.0866	0.1976	0.1462	0.1351
EED	-31.3835	66.7669	70.9669	71.0532	68.4528	4.3371	0.7985	0.2290	0.0027
RD	-49.7909	101.5818	103.6474	103.7249	102.4247	11.4249	2.3221	0.3339	0.0001
GD	-23.9515	51.9031	56.1031	56.1894	53.5889	3.0873	0.5659	0.2164	0.0055
ED	-88.8303	179.6606	181.7262	181.8038	180.5035	18.4258	3.8622	0.4180	0.0000
APED	-30.5179	65.0358	69.2358	69.3220	66.7216	4.1595	0.7581	0.2225	0.0039
APGD	-18.7667	43.5333	49.9401	49.9627	46.0620	1.8640	0.2882	0.1638	0.0679
GWD	-14.5876	35.1752	41.5820	41.6046	37.7039	1.0675	0.1945	0.1453	0.1399

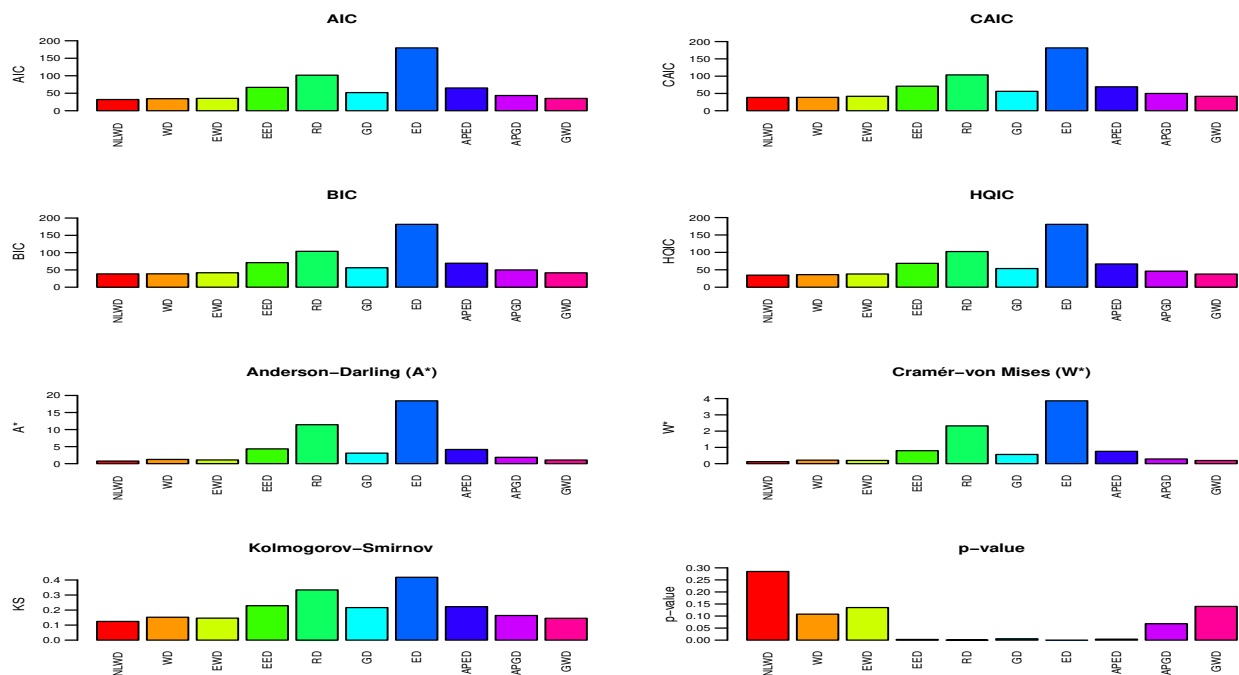


Figure 25. Comparative bar plots of goodness-of-fit statistics for the fitted probability models using the glass fiber tensile strength measurements (Dataset 3).

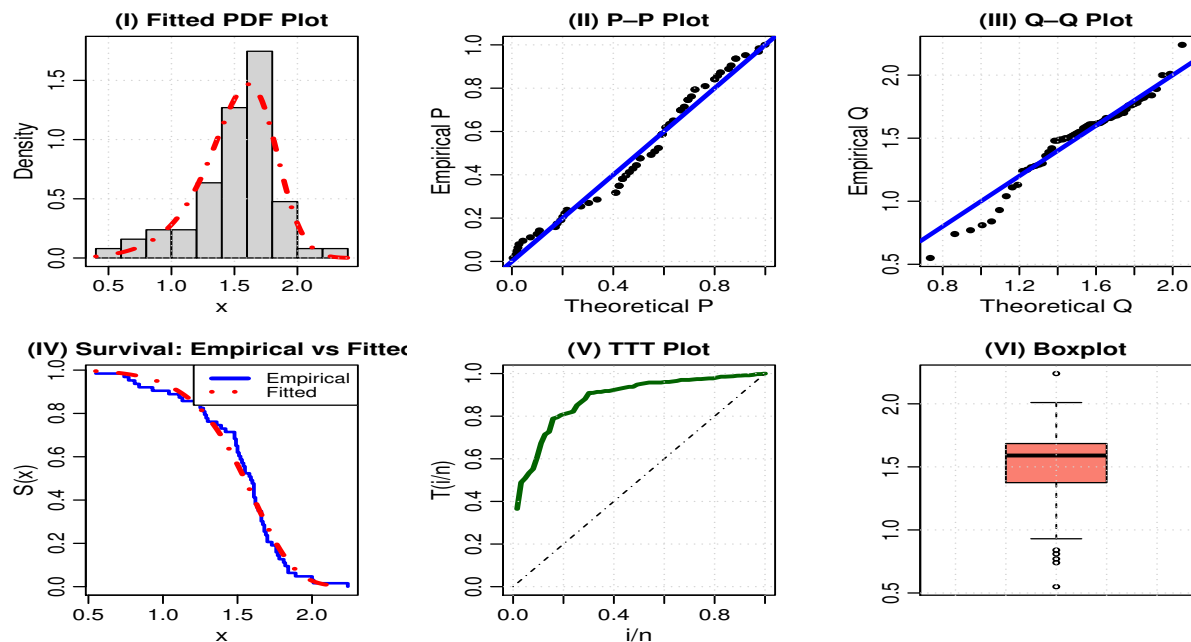


Figure 26. Comparative graphical assessment of the NLW model applied to the glass fiber tensile strength measurements data (Dataset 3).

Table 19. Maximum likelihood estimates of the model parameters for the Mexico COVID-19 fatality rate data.

Model	α	β	λ
NLWD	0.0016	4.0999	17.4294
WD	—	1.8969	6.5210
EWD	4.5396	0.9484	2.4922
EED	—	3.9981	2.7624
RD	—	—	4.6716
GD	—	3.2729	1.7593
ED	—	—	0.1737
APED	291.5378	—	0.4067
APGD	0.4903	3.4462	1.8546
GWD	10.1725	0.0792	0.5501

Table 20. Log-likelihood and goodness-of-fit measures for the competing statistical models for the Mexico COVID-19 fatality rate data.

Model	LogL	AIC	AICc	BIC	HQIC	AD	CVM	KS	p -value
NLWD	-264.9971	535.9943	536.2250	544.0407	539.2568	0.1870	0.0310	0.0475	0.9679
WD	-268.9554	541.9108	542.0250	547.2750	544.0858	0.6643	0.0970	0.0736	0.6026
EWD	-266.1626	538.3252	538.5560	546.3716	541.5877	0.3351	0.0615	0.0686	0.6897
EED	-266.1758	536.3515	536.4658	541.7158	538.5265	0.3408	0.0621	0.0706	0.6549
RD	-269.2284	540.4569	540.4946	543.1390	541.5444	0.9345	0.1633	0.0934	0.3030
GD	-266.4187	536.8375	536.9518	542.2017	539.0125	0.3786	0.0660	0.0762	0.5582
ED	-297.0655	596.1310	596.1687	598.8131	597.2185	10.0458	1.8170	0.2407	0.0001
APED	-268.1019	540.2038	540.3181	545.5680	542.3788	0.5767	0.0956	0.0850	0.4158
APGD	-266.1982	538.3964	538.6272	546.4428	541.6589	0.3425	0.0614	0.0717	0.6351
GWD	-266.0382	538.0763	538.3071	546.1227	541.3388	0.3218	0.0591	0.0680	0.6994

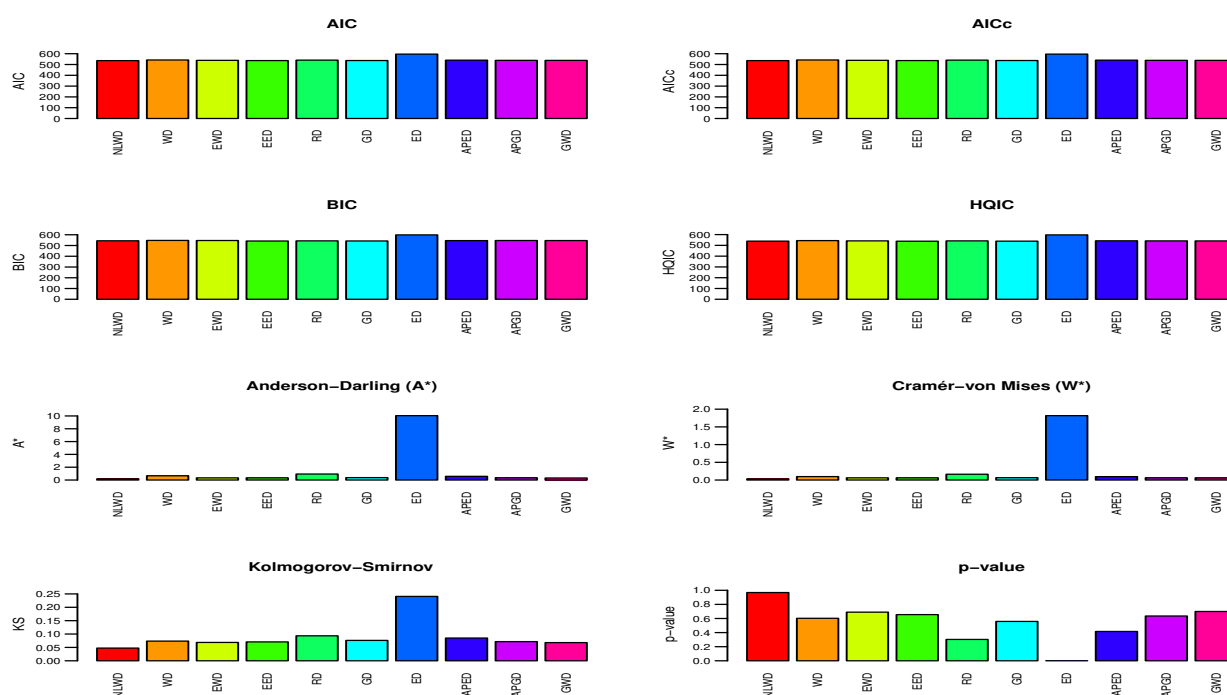


Figure 27. Comparative bar plots of goodness-of-fit statistics for the fitted probability models using the Mexico COVID-19 fatality rate data (Dataset 4).

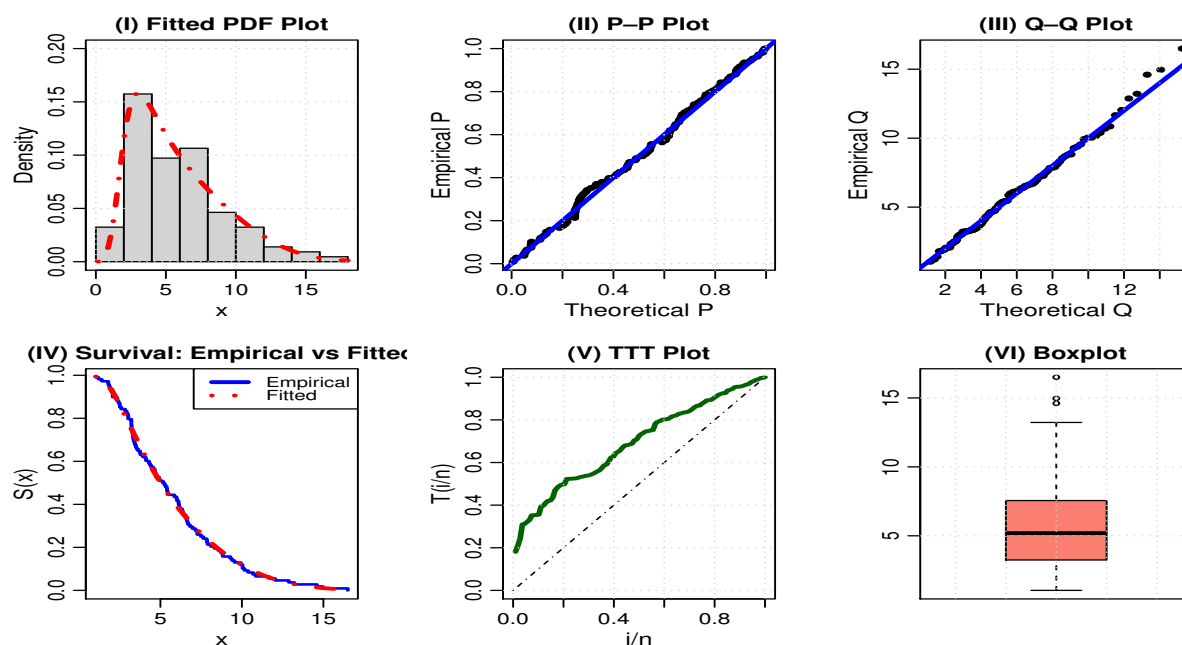


Figure 28. Comparative graphical assessment of the NLW model applied to the Mexico COVID-19 fatality rate data (Dataset 4).

Numerical optimization and profile likelihood analysis: The bounded `nlminb` technique in R was used to estimate the parameters (α, β, λ) of the NLW distribution by maximum likelihood. All parameters were limited to $[10^{-10}, 100]$, and several starting values were taken into consideration. The convergent solution with the lowest negative log-likelihood was chosen to be the joint MLE.

Whereas β and λ were re-estimated for the profile likelihood analysis, α was fixed over a logarithmically spaced grid. $\ell_p(\alpha) - \max_{\alpha} \ell_p(\alpha)$ was used to determine the relative profile log-likelihood with a 95% likelihood-ratio threshold of -1.921 . Figures 29–32 show the profiles that were produced.

The α profile estimations were 20.574 for Dataset 1, 0.000004454 for Dataset 2, 6.3894 for Dataset 3, and 0.0016361 for Dataset 4. The findings show that α is identifiable depending on the dataset, with some datasets showing stronger identification around the boundary and others showing clearer inner maxima.

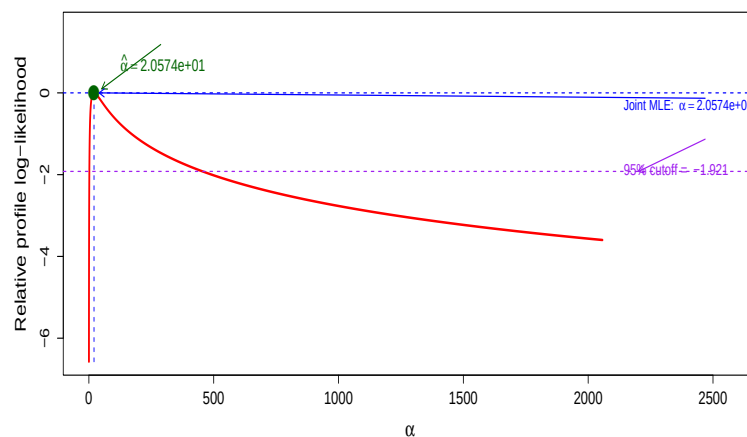


Figure 29. Profile log-likelihood of α for Dataset 1, with β and λ re-estimated for each fixed value of α .

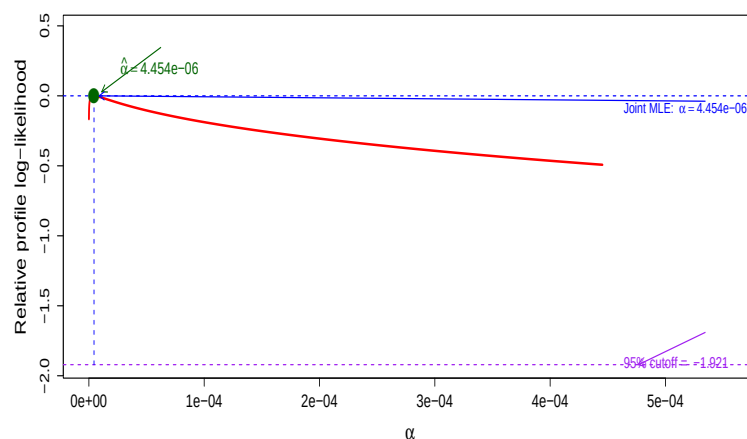


Figure 30. Profile log-likelihood of α for Dataset 2, with β and λ re-estimated for each fixed value of α .

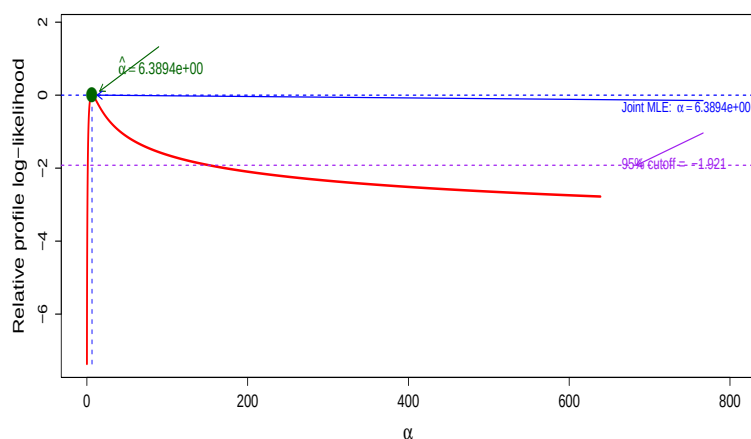


Figure 31. Profile log-likelihood of α for Dataset 3, with β and λ re-estimated for each fixed value of α .

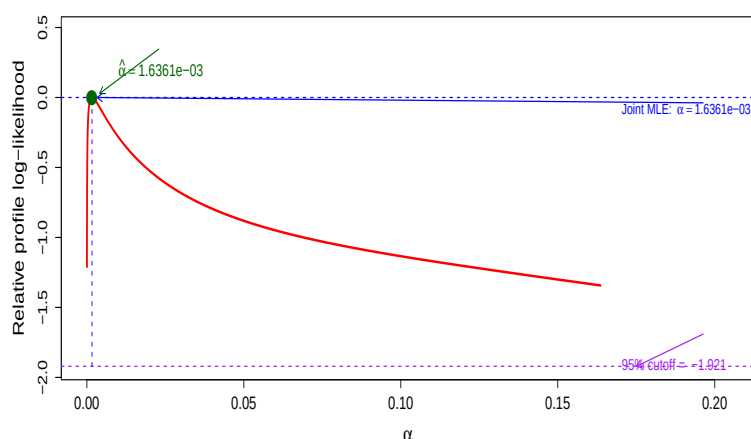


Figure 32. Profile log-likelihood of α for Dataset 4, with β and λ re-estimated for each fixed value of α .

7. Conclusions

This study introduced and examined the NLW distribution, a three-parameter extension of the Weibull model, developed to provide greater flexibility for reliability and complex survival data. The primary contributions and findings of this study are summarized as follows:

- **Theoretical and numerical contributions:** Some important structural and distributional properties of the proposed NLW distribution are derived, and the performance of six classical parameter estimation methods are evaluated by extensive Monte Carlo simulations. The simulation results provide empirical evidence of improved estimation performance with increasing sample size in terms of bias, mean squared error, and mean relative error.

- **Empirical performance:** The NLW model was applied to glass fiber strength, UK COVID-19 mortality, glass fiber strength, and Mexico fatality rate data and was found to have a competitive and in some cases better goodness-of-fit performance than the competing distributions considered. These results demonstrate the potential usefulness of NLW distribution in modeling lifetime data with various distributional and hazard-rate characteristics.
- **Practical considerations:** The extra shape parameter makes the suggested model more flexible, but it might also necessitate careful numerical optimization, especially for datasets or small samples that result in extreme parameter values. In order to achieve reliable parameter estimations, it is crucial to use numerical optimization techniques and appropriate beginning values.

7.1. Study limitations

Despite its strong structural flexibility and superior empirical performance, several methodological limitations of the proposed model should be acknowledged:

- **Numerical optimization and identifiability:** The inclusion of an additional shape parameter increases likelihood surface complexity, which can introduce parameter identifiability challenges and convergence sensitivity during numerical optimization, particularly when true parameter values approach boundary regions or in small-sample regimes.
- **Data structure constraints:** The present theoretical and empirical derivations are tailored for complete, uncensored observations. The current formulation does not explicitly incorporate censoring mechanisms (e.g., right, left, or interval censoring), truncation, competing risk structures, or covariate-dependent regression structures frequently encountered in survival analysis.
- **Theoretical asymptotics:** Although extensive Monte Carlo simulations provide strong empirical support for estimator consistency and efficiency across finite samples, formal analytical proofs of asymptotic normality and efficiency for all six non-MLE estimation frameworks remain outside the scope of this work.

7.2. Future extensions

Future research may focus on Bayesian estimation, computationally efficient optimization, and extensions of the NLW model to censored, truncated, and covariate-dependent data. Applications to larger and more diverse datasets, as well as predictive and risk-analysis frameworks, could further assess and enhance the practical utility of the proposed model.

Author contributions

Aadil Ahmad Mir: Writing—original draft, conceptualization, methodology, formal analysis, data curation, software, writing—review and editing, formal analysis; A. A. Bhat: Writing—original draft, methodology, formal analysis, visualization, writing—review and editing; S. P. Ahmad: Investigation, software, writing—original draft, writing—review and editing; Badr S. Alnssyan: Writing—original draft, investigation, software, writing—review and editing, conceptualization, formal analysis, data curation;

Abdelaziz Alsubie: Formal analysis, visualization, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (<https://www.qu.edu.sa>) for financial support (QU-APC-2026).

Conflict of interest

The authors declare no conflict of interest.

References

1. G. S. Mudholkar, D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE T. Reliab.*, **42** (1993), 299–302. <https://doi.org/10.1109/24.229504>
2. A. W. Marshall, I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika*, **84** (1997), 641–652. <https://doi.org/10.1093/biomet/84.3.641>
3. A. Mahdavi, D. Kundu, A new method for generating distributions with an application to exponential distribution, *Commun. Stat.–Theor. M.*, **46** (2017), 6543–6557. <https://doi.org/10.1080/03610926.2015.1130839>
4. A. A. Mir, S. U. Rasool, S. P. Ahmad, A. A. Bhat, T. M. Jawa, N. Sayed-Ahmed, et al., A robust framework for probability distribution generation: analyzing structural properties and applications in engineering and medicine, *Axioms*, **14** (2025), 281. <https://doi.org/10.3390/axioms14040281>
5. A. A. Bhat, A. A. Mir, S. P. Ahmad, B. S. Alnssyan, A. Alsubie, Y. S. Raghav, A novel Alpha-power X family: A flexible framework for distribution generation with focus on the half-logistic model, *Entropy*, **27** (2025), 632. <https://doi.org/10.3390/e27060632>
6. A. A. Mir, A. A. Bhat, S. P. Ahmad, B. S. Alnssyan, A. Alsubie, Y. S. Raghav, Statistical inference and goodness-of-fit assessment using the AAP-X probability framework with symmetric and asymmetric properties: Applications to medical and reliability data, *Symmetry*, **17** (2025), 863. <https://doi.org/10.3390/sym17060863>
7. S. Mohammad, I. Mendoza, A new hyperbolic tangent family of distributions: Properties and applications, *Ann. Data Sci.*, **12** (2025), 457–480. <https://doi.org/10.1007/s40745-024-00516-5>
8. M. M. Alsolmi, A new logarithmic Tangent-U family of distributions with reliability analysis in engineering data, *Comput. J. Math. Stat. Sci.*, **4** (2025), 258–282.

9. O. H. Odhah, H. M. Alshanbari, Z. Ahmad, F. Khan, A. H. El-Bagoury, A new family of distributions using a trigonometric function: Properties and applications in the healthcare sector, *Heliyon*, 2024.
10. A. S. Alghamdi, S. F. ALoufi, L. A. Baharith, The sine Alpha power-G family of distributions: Characterizations, regression modeling, and applications, *Symmetry*, **17** (2025), 468. <https://doi.org/10.3390/sym17030468>
11. V. Mulagala, V. B. V. Nagarjuna, Sine Modified-Kies generalized family of distributions: Properties and applications, *Phys. Scripta*, 2025.
12. Z. Shah, G. S. S. Abdalla, D. M. Khan, G. S. Rao, F. Jamal, E. M. Almetwally, A logarithmic exponent power half Logistic-G family of Distributions: Repetitive acceptance sampling plan and applications to engineering and radiation sectors, *J. Radiat. Res. Appl. Sc.*, **18** (2025), 101797. <https://doi.org/10.1016/j.jrras.2025.101797>
13. Z. Shah, E. M. Almetwally, D. M. Khan, F. Jamal, A novel odd Type-X family of distributions: Model, theory, and applications to medical, insurance, and engineering data sets, *J. Radiat. Res. Appl. Sc.*, **18** (2025), 101451. <https://doi.org/10.1016/j.jrras.2025.101451>
14. A. A. Mir, S. P. Ahmad, A novel and flexible statistical distribution family: Theoretical development, mathematical properties, Monte Carlo simulation and applications to engineering failure-time, composite material strength and digital advertising data, *Life Cycle Reliab. Saf. Eng.*, 2026, 1–27.
15. K. Choi, W. G. Bulgren, An estimation procedure for mixtures of distributions, *J. Roy. Stat. Soc. B*, **30** (1968), 444–460.
16. R. C. H. Cheng, N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *J. Roy. Stat. Soc. B*, **45** (1983), 394–403. <https://doi.org/10.1111/j.2517-6161.1983.tb01268.x>
17. G. A. F. Seber, A. J. Lee, *Linear regression analysis*, 2Eds., Wiley Series in Probability and Statistics, John Wiley & Sons, 2012.
18. T. W. Anderson, D. A. Darling, Asymptotic theory of certain goodness of fit criteria based on stochastic processes, *Ann. Math. Stat.*, **23** (1952), 193–212. <https://doi.org/10.1214/aoms/1177729437>
19. M. G. Bader, A. M. Priest, Statistical aspects of fibre and bundle strength in hybrid composites, *Prog. Sci. Eng. Compos.*, 1982, 1129–1136.
20. I. E. Ragab, S. Khan, N. Alsadat, F. Jamal, G. G. Hamedani, M. Elgarhy, Modeling to COVID-19 and cancer data: Using the generalized Burr-Hatke model, *J. Radiat. Res. Appl. Sc.*, **17** (2024), 100972. <https://doi.org/10.1016/j.jrras.2024.100972>
21. M. Kargbo, A. W. Gichuhi, A. K. Wanjoya, A novel extended inverse-exponential distribution and its application to COVID-19 data, *Eng. Rep.*, **6** (2024), e12828.
22. R. L. Smith, J. C. Naylor, A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *J. Roy. Stat. Soc. C–Appl.*, **36** (1987), 358–369. <https://doi.org/10.2307/2347795>

23. A. A. Al-Babtain, D. Kumar, A. M. Gemeay, S. Dey, A. Z. Afify, Modeling engineering data using extended power-Lindley distribution: Properties and estimation methods, *J. King Saud Univ. Sci.*, **33** (2021), 101582.
24. H. M. Almongy, E. M. Almetwally, H. M. Aljohani, A. S. Alghamdi, E. H. Hafez, A new extended Rayleigh distribution with applications of COVID-19 data, *Results Phys.*, **23** (2021), 104012. <https://doi.org/10.1016/j.rinp.2021.104012>
25. A. B. S. Mastor, O. Ngesa, J. Mung'atu, N. M. Alfaer, A. Z. Afify, The extended exponential Weibull distribution: Properties, inference, and applications to real-life data, *Complexity*, **2022** (2022), 4068842. <https://doi.org/10.1155/2022/4068842>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)