



Research note**Totally positive integers with 8 partitions in real quadratic fields****Jing-Jun Yu***

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Abstract: We construct an explicit totally positive integer $\alpha_D \in \mathcal{O}_K^+$ and prove that $p_K(\alpha_D) = 8$ whenever $K = \mathbb{Q}(\sqrt{D})$ is a real quadratic field satisfying $\{\xi_D\} < 1/3$. The set of such radicands D has positive density. Hence our construction provides an explicit positive density family of real quadratic fields in which the value 8 occurs. Independently, Zindulka's general theorem implies that the exceptional set $\mathcal{D}(8)$ has natural density zero, so 8 is attained in almost all real quadratic fields. The construction applies uniformly to both congruence classes $D \equiv 2, 3 \pmod{4}$ and $D \equiv 1 \pmod{4}$. We also exhibit the complete list of 8 partitions for several square-free radicands and show that infinitely many fields satisfy the required condition.

Keywords: partitions; real quadratic fields; totally positive integers**Mathematics Subject Classification:** 11P83

1. Introduction

Let K be a totally real number field, $\mathcal{O}_K$ its ring of integers, and $\mathcal{O}_K^+$ the set of totally positive integers in K . A partition of an element $\alpha \in \mathcal{O}_K^+$ is defined to be an expression of the form

$$\alpha = \sum_{i=1}^n \alpha_i, \quad \alpha_i \in \mathcal{O}_K^+.$$

Let $p_K(\alpha)$ denote the number of partitions of α . The function $p_K(\alpha)$ is referred to as the partition function associated with K . One also defines $p_K(0) := 1$. This notion was introduced by Stern and Zindulka [1] as a natural generalization of the classical integer partition function $p(n)$ [2]. An element $\alpha \in \mathcal{O}_K^+$ is called indecomposable if it cannot be decomposed as a sum $\beta + \gamma$ for any $\beta, \gamma \in \mathcal{O}_K^+$. This means the indecomposable elements are precisely those for which the partition function takes the value $p_K(\alpha) = 1$. Dress and Scharlau [3] showed that in a real quadratic field $K = \mathbb{Q}(\sqrt{D})$, these elements are described by the continued fraction of $\sqrt{D}$.

One of the central problems raised in [1] is to determine, for a given positive integer m , for which real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ there exists an element $\alpha \in \mathcal{O}_K^+$ with $p_K(\alpha) = m$. The exceptional set $\mathcal{D}(m)$ collects those squarefree $D > 1$ for which m does *not* occur as a value of p_K . In [1] the sets $\mathcal{D}(m)$ were completely determined for $m = 1, 2, 3, 5, 7, 11$, while constructions for $m = 4$ and $m = 6$ were given that work for every real quadratic field or under a mild continued fraction condition, respectively.

Recently, Zindulka [4] proved a general theorem: For every fixed integer $m \geq 1$, almost all real quadratic fields contain an element with exactly m partitions, i.e., the exceptional set $\mathcal{D}(m)$ has natural density zero. The proof in [4] is non-constructive and relies on a rather involved analysis of the associated continued fractions. It is therefore still of interest to provide simple and explicit constructions for small values of m that are not covered by the original work [1]. The first such value is $m = 8$, since 8 does not appear in the range of the classical partition function $p(n)$.

In this note we give a uniform construction that produces an element with exactly 8 partitions in almost all real quadratic fields. Our approach simultaneously treats the two congruence classes $D \equiv 2, 3 \pmod{4}$ and $D \equiv 1 \pmod{4}$.

Throughout the paper, $D > 1$ is a squarefree integer and $K = \mathbb{Q}(\sqrt{D})$. The ring of integers is $\mathcal{O}_K = \mathbb{Z}[\omega_D]$, where

$$\omega_D := \begin{cases} \sqrt{D}, & \text{if } D \equiv 2, 3 \pmod{4}, \\ \frac{1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

We continue to use the notation of Stern and Zindulka

$$\xi_D := \begin{cases} \sqrt{D}, & \text{if } D \equiv 2, 3 \pmod{4}, \\ \frac{\sqrt{D}-1}{2}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Our main result is the following.

Theorem 1. *Let $K = \mathbb{Q}(\sqrt{D})$, where $D > 1$ is a squarefree integer, and let ξ_D be as above. If the fractional part of ξ_D satisfies $\{\xi_D\} < \frac{1}{3}$, let $\alpha_D = (\lceil 3\xi_D \rceil + 2) + 3\omega_D$, then we have*

$$p_K(\alpha_D) = 8,$$

where $\{\xi_D\} = \xi_D - \lfloor \xi_D \rfloor$.

Our theorem applies to every D with $\{\xi_D\} < 1/3$. Since this set has positive density, it gives an explicit positive density family of real quadratic fields for which $p_K(\alpha_D) = 8$. Independently, Zindulka's general theorem [4] shows that for every fixed m , the exceptional set $\mathcal{D}(m)$ has natural density zero. In particular, $\mathcal{D}(8)$ has natural density zero, so 8 is attained in almost all real quadratic fields. Thus the positive density explicit construction and the almost all statement are distinct: The former is given by Theorem 1.1, while the latter is due to [4]. Moreover, our construction is completely explicit, and we list the full 8 partitions for several discriminants.

2. Preliminaries

For an element $\alpha = a + b\omega_D \in K$ with $a, b \in \mathbb{Z}$, the element α is totally positive, meaning $\alpha > 0$ and its conjugate $\bar{\alpha} > 0$, which we denote $\alpha > 0$. Since $\bar{\omega}_D = -\xi_D$, this is equivalent to

$$a + b\omega_D > 0 \quad \text{and} \quad a - b\xi_D > 0.$$

Stern and Zindulka [1] proved that for every totally positive integer $\alpha = a + b\omega_D$, there always exists another totally positive integer $\alpha' = a' + b'\omega_D$ with $a' \geq 1$, $b' \geq 0$ such that $p_K(\alpha) = p_K(\alpha')$, hence we may consider only α with $a \geq 1$, $b \geq 0$. It can be expressed as

$$\alpha = (\lceil b\xi_D \rceil + u) + b\omega_D \quad u, b \geq 0.$$

For $\alpha = a + b\omega_D \in K$ with $a, b \in \mathbb{Z}$, total positivity of α imposes a lower bound on a in terms of b . Namely, for $b \geq 1$, we define

$$L(b) = \min\{a \in \mathbb{Z} : a + b\omega_D \in O_K^+\}.$$

It is straightforward to verify that $L(b) = \lceil b\xi_D \rceil$. In particular, for $b = 0$, we use the convention $L(0) = 0$, which corresponds to counting the empty sum.

For $b = 0$, we adopt the convention $L(b) = 0$.

Following the classical convention, partitions are counted up to the order of summands. The partition function $p_K(\alpha)$ enumerates representations $\alpha = \sum_{i=1}^n \alpha_i = \sum_{i=1}^n (a_i + b_i\omega_D)$. If all parts have $b_i = 0$ for any i , then the corresponding contribution is counted by the classical partition function. We also write $p(\alpha)$ for the classical partition function.

3. Proof of Theorem 1

Assume that $\{\xi_D\} < \frac{1}{3}$. We write

$$m = \lfloor \xi_D \rfloor, \quad \delta = \{\xi_D\} = \xi_D - \lfloor \xi_D \rfloor \in (0, \frac{1}{3}).$$

Since $0 < \delta < \frac{1}{3}$, we have $2\delta < \frac{2}{3} < 1$ and $3\delta < 1$. Hence

$$\lceil \xi_D \rceil = m + 1, \quad \lceil 2\xi_D \rceil = 2m + 1, \quad \lceil 3\xi_D \rceil = 3m + 1. \quad (3.1)$$

Consequently,

$$L(0) = 0, \quad L(1) = m + 1, \quad L(2) = 2m + 1, \quad L(3) = 3m + 1.$$

And so

$$\alpha_D = (\lceil 3\xi_D \rceil + 2) + 3\omega_D = (3m + 3) + 3\omega_D.$$

Its total positivity follows from $\bar{\alpha}_D = (3m + 3) - 3\xi_D = 3(m + 1 - \xi_D) = 3(1 - \delta) > 0$.

Now consider an arbitrary partition $\alpha_D = \sum_i \alpha_i$ with $\alpha_i = a_i + b_i\omega_D \in O_K^+$. Comparing coefficients gives

$$\sum_i b_i = 3, \quad \sum_i a_i = 3m + 3. \quad (3.2)$$

Under the above conditions, we first prove that all the b_i are nonnegative.

Suppose that for some $b_i = -k < 0$ with $k \geq 1$. Since $\alpha_i = a_i - k\omega_D > 0$, we have $a_i > k\omega_D$. Taking conjugates and using $\bar{\omega}_D = -\xi_D$ yields

$$\bar{\alpha}_i = a_i + k\xi_D > k(\omega_D + \xi_D).$$

Now we estimate the lower bound of $\omega_D + \xi_D$ from below. If $D \equiv 2, 3 \pmod{4}$, then $\omega_D = \sqrt{D}$ and $\xi_D = \sqrt{D}$, hence $\omega_D + \xi_D = 2\sqrt{D}$. If $D \equiv 1 \pmod{4}$, then $\omega_D = \frac{1+\sqrt{D}}{2}$ and $\xi_D = \frac{\sqrt{D}-1}{2}$, which gives $\omega_D + \xi_D = \sqrt{D}$.

Since we assume that $\{\xi_D\} < \frac{1}{3}$, the minimal possible squarefree radicands are $D = 10$ (for $D \equiv 2, 3 \pmod{4}$) and $D = 13$ (for $D \equiv 1 \pmod{4}$). Thus in either case, $\omega_D + \xi_D \geq \min\{2\sqrt{10}, \sqrt{13}\} > 3$. Consequently,

$$\bar{\alpha}_i > k \cdot 3 \geq 3.$$

On the other hand, the conjugate of $\alpha_D = (3m+3) + 3\omega_D$ is

$$\bar{\alpha}_D = 3m+3 - 3\xi_D = 3(m+1 - \xi_D) = 3(1 - \delta) < 3.$$

We therefore obtain

$$\bar{\alpha}_i > 3.$$

Notice that, $\bar{\alpha}_D = \sum_j \bar{\alpha}_j$, and every summand $\bar{\alpha}_j$ is strictly positive. Since

$$\bar{\alpha}_D = 3(1 - \delta) < 3,$$

each summand satisfies

$$0 < \bar{\alpha}_i \leq \bar{\alpha}_D < 3,$$

and hence in particular $\bar{\alpha}_i < 3$. This contradicts $\bar{\alpha}_i > 3$. Therefore no coefficient b_i can be negative.

Having established that all $b_i \geq 0$, total positivity now imposes $a_i \geq L(b_i)$ for each i . Since $\sum_i b_i = 3$, the only options are $b_i \in \{0, 1, 2, 3\}$. We now enumerate the partitions arising from each case.

Case A: one $b_i = 3$, the rest 0

Without loss of generality, let the $b_1 = 3$ part be $a_1 + 3\omega_D$, then $a_1 = L(3) + u = 3m + 1 + u$, $u \geq 0$. The remaining parts are rational positive integers summing to $S = 3m + 3 - (3m + 1 + u) = 2 - u$. Since $S \geq 0$, we have $u \in \{0, 1, 2\}$. For each u , the number of ways to write S as a sum of positive integers is $p(S)$, with the convention $p(0) = 1$ corresponding to the empty sum. Then

u	$S = 2 - u$	$p(S)$
0	2	$p(2) = 2$
1	1	$p(1) = 1$
2	0	$p(0) = 1$

Hence they give the following four distinct partitions:

$$(3m+1+3\omega_D)+2, \quad (3m+1+3\omega_D)+1+1, \quad (3m+2+3\omega_D)+1, \quad 3m+3+3\omega_D.$$

Case B: one $b_i = 2$, one $b_j = 1$, the rest 0

Similarly, let the $b_2 = 2$ part be as $a_2 + 2\omega_D$ with $a_2 = 2m + 1 + u_2$, and let the $b = 1$ part be as $a_1 + \omega_D$ with $a_1 = m + 1 + u_1$, where $u_2, u_1 \geq 0$. Let the remaining integer sum be $c \geq 0$. Then we have

$$(2m+1+u_2) + (m+1+u_1) + c = 3m+3 \implies u_2 + u_1 + c = 1.$$

The nonnegative integer solutions are exactly three:

$$(u_2, u_1, c) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

They give the following three distinct partitions:

$$\begin{aligned} (2m + 2 + 2\omega_D) + (m + 1 + \omega_D), \quad (2m + 1 + 2\omega_D) + (m + 2 + \omega_D), \\ (2m + 1 + 2\omega_D) + (m + 1 + \omega_D) + 1. \end{aligned}$$

Case C: three $b_i = 1$, the rest 0

Let the three $b_i = 1$ parts be $(m + 1 + v_i) + \omega_D$ with $v_i \geq 0$, $i = 1, 2, 3$, and let the remaining rational integer sum be $c \geq 0$. Then

$$3(m + 1) + (v_1 + v_2 + v_3) + c = 3m + 3 \implies v_1 + v_2 + v_3 + c = 0.$$

Thus $v_1 = v_2 = v_3 = 0$ and $c = 0$. The unique partition is

$$(m + 1 + \omega_D) + (m + 1 + \omega_D) + (m + 1 + \omega_D).$$

Combining the above three cases, we obtain $p_K(\alpha_D) = 4 + 3 + 1 = 8$. This completes the proof of Theorem 1.

4. Examples

We now illustrate the theorem with several squarefree radicands covering both congruence classes modulo 4.

4.1. Squarefree radicands with $D \equiv 2, 3 \pmod{4}$

For $D \equiv 2, 3 \pmod{4}$, we have $\xi_D = \sqrt{D}$. The condition $\{\sqrt{D}\} < \frac{1}{3}$ is easy to check. Table 1 gives a few values.

Table 1. Examples for $D \equiv 2, 3 \pmod{4}$.

D	$m = \lfloor \sqrt{D} \rfloor$	α_D	Partitions
10	3	$12 + 3\sqrt{10}$	$12 + 3\sqrt{10}$ (trivial); $11 + 3\sqrt{10} + 1$; $10 + 3\sqrt{10} + 2, 10 + 3\sqrt{10} + 1 + 1$; $(8 + 2\sqrt{10}) + (4 + \sqrt{10})$; $(7 + 2\sqrt{10}) + (5 + \sqrt{10})$; $(7 + 2\sqrt{10}) + (4 + \sqrt{10}) + 1$; $(4 + \sqrt{10}) + (4 + \sqrt{10}) + (4 + \sqrt{10})$.
			$12 + 3\sqrt{11}$ (trivial); $11 + 3\sqrt{11} + 1$; $10 + 3\sqrt{11} + 2, 10 + 3\sqrt{11} + 1 + 1$; $(8 + 2\sqrt{11}) + (4 + \sqrt{11})$; $(7 + 2\sqrt{11}) + (5 + \sqrt{11})$; $(7 + 2\sqrt{11}) + (4 + \sqrt{11}) + 1$; $(4 + \sqrt{11}) + (4 + \sqrt{11}) + (4 + \sqrt{11})$.
26	5	$18 + 3\sqrt{26}$	$18 + 3\sqrt{26}$ (trivial); $17 + 3\sqrt{26} + 1$; $16 + 3\sqrt{26} + 2, 16 + 3\sqrt{26} + 1 + 1$; $(12 + 2\sqrt{26}) + (6 + \sqrt{26})$; $(11 + 2\sqrt{26}) + (7 + \sqrt{26})$; $(11 + 2\sqrt{26}) + (6 + \sqrt{26}) + 1$; $(6 + \sqrt{26}) + (6 + \sqrt{26}) + (6 + \sqrt{26})$.

4.2. Squarefree radicands with $D \equiv 1 \pmod{4}$

If $D \equiv 1 \pmod{4}$, then $\omega_D = \frac{1+\sqrt{D}}{2}$ and $\xi_D = \frac{\sqrt{D}-1}{2}$. The condition becomes $\{\frac{\sqrt{D}-1}{2}\} < \frac{1}{3}$. Table 2 presents two such examples.

Table 2. Examples for $D \equiv 1 \pmod{4}$.

D	$m = \lfloor \xi_D \rfloor$	α_D	Partitions
13	1	$6 + 3\omega_{13} = 6 + 3\frac{1+\sqrt{13}}{2}$	$6 + 3\omega_{13}$ (trivial);
			$5 + 3\omega_{13} + 1$;
			$4 + 3\omega_{13} + 2, 4 + 3\omega_{13} + 1 + 1$;
			$(4 + 2\omega_{13}) + (2 + \omega_{13})$;
			$(3 + 2\omega_{13}) + (3 + \omega_{13})$;
			$(3 + 2\omega_{13}) + (2 + \omega_{13}) + 1$;
			$(2 + \omega_{13}) + (2 + \omega_{13}) + (2 + \omega_{13})$.
29	2	$9 + 3\omega_{29} = 9 + 3\frac{1+\sqrt{29}}{2}$	$9 + 3\omega_{29}$ (trivial);
			$8 + 3\omega_{29} + 1$;
			$7 + 3\omega_{29} + 2, 7 + 3\omega_{29} + 1 + 1$;
			$(6 + 2\omega_{29}) + (3 + \omega_{29})$;
			$(5 + 2\omega_{29}) + (4 + \omega_{29})$;
			$(5 + 2\omega_{29}) + (3 + \omega_{29}) + 1$;
			$(3 + \omega_{29}) + (3 + \omega_{29}) + (3 + \omega_{29})$.

In the table, we write the elements in the basis $(1, \omega_D)$. For instance, for $D = 13$, $\omega_{13} = (1 + \sqrt{13})/2$, so $2 + \omega_{13} = (5 + \sqrt{13})/2$, which is totally positive because its conjugate is $(5 - \sqrt{13})/2 > 0$. All other parts are verified similarly.

5. Infinitely many fields

It remains to show that the condition $\{\xi_D\} < \frac{1}{3}$ holds for infinitely many squarefree D .

For $D \equiv 2, 3 \pmod{4}$, we have $\xi_D = \sqrt{D}$. To ensure that D lies in this congruence class, take

$$D = (2t + 1)^2 + 1 \quad (t \geq 1).$$

Since $(2t + 1)^2 \equiv 1 \pmod{4}$, then $D \equiv 2 \pmod{4}$, so $\xi_D = \sqrt{D}$, and

$$\{\xi_D\} = \left\{ \sqrt{(2t + 1)^2 + 1} \right\} = \sqrt{(2t + 1)^2 + 1} - (2t + 1) = \frac{1}{\sqrt{(2t + 1)^2 + 1} + (2t + 1)} < \frac{1}{3}.$$

The inequality is clear for all $t \geq 1$. Estermann [6] proved that $n^2 + 1$ takes infinitely many squarefree values. This was generalized by Ricci [7, Teorema A], which implies in particular that $n^2 + 1$ is squarefree for infinitely many odd n . Hence we obtain infinitely many such D in this congruence class.

For $D \equiv 1 \pmod{4}$, we have $\xi_D = \frac{\sqrt{D}-1}{2}$. Take

$$D = (2t + 1)^2 + 4 \quad (t \geq 1).$$

Since $(2t + 1)^2 \equiv 1 \pmod{4}$, then $D \equiv 5 \equiv 1 \pmod{4}$, so $\xi_D = \frac{\sqrt{D}-1}{2}$, and

$$\{\xi_D\} = \left\{ \frac{\sqrt{(2t + 1)^2 + 4} - 1}{2} \right\}.$$

Since $\sqrt{(2t+1)^2+4} = (2t+1) + \varepsilon$ with $0 < \varepsilon < 1$, we have

$$\frac{\sqrt{(2t+1)^2+4}-1}{2} = t + \frac{\varepsilon}{2},$$

so

$$\{\xi_D\} = \frac{\varepsilon}{2} = \frac{\sqrt{(2t+1)^2+4} - (2t+1)}{2} = \frac{2}{\sqrt{(2t+1)^2+4} + (2t+1)} < \frac{1}{3}.$$

The inequality holds for all $t \geq 1$. By Ricci's Teorema A [7], $n^2 + 4$ is squarefree for infinitely many odd n . Hence there are infinitely many squarefree $D \equiv 1 \pmod{4}$ satisfying the condition. Thus Theorem 1.1 applies to infinitely many fields in each congruence class modulo 4.

6. Concluding remarks

We have given a simple, uniform construction of a totally positive element $\alpha_D \in O_K^+$ with $p_K(\alpha_D) = 8$ in real quadratic fields. Our result complements the program initiated by Stern and Zindulka by providing an explicit family of fields for which such an element exists. The constructive method, based on elementary combinatorics of the coefficients b_i , is flexible and could be extended to other small values of m for which explicit formulas are desired. Zindulka [4] proved that the exceptional set $\mathcal{D}(8)$ has natural density zero, though it may still be infinite. It would be interesting to determine whether $\mathcal{D}(8)$ is in fact finite, or to extend the construction to other values of m .

The author confirms sole responsibility for the study, results, and manuscript preparation.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

References

1. D. Stern, M. Zindulka, Partitions in real quadratic fields, *Math. Nachr.*, **298** (2025), 548–566. <https://doi.org/10.1002/mana.202300480>
2. G. E. Andrews, K. Eriksson, *Integer partitions*, Cambridge University Press, Cambridge, 2004. <https://doi.org/10.1017/CBO9781139167239>
3. A. Dress, R. Scharlau, Indecomposable totally positive numbers in quadratic orders, *J. Number Theory*, **14** (1982), 292–306. [https://doi.org/10.1016/0022-314X\(82\)90064-6](https://doi.org/10.1016/0022-314X(82)90064-6)

4. M. Zindulka, Totally positive elements with m partitions exist in almost all real quadratic fields, *Proc. Edinb. Math. Soc.*, **69** (2026), 701–734. <https://doi.org/10.1017/s0013091525101259>
5. D. A. Cox, *Primes of the form $x^2 + ny^2$* , 2nd edn, Hoboken, NJ, 2013. <https://doi.org/10.1090/chel/387>
6. T. Estermann, Einige Sätze über quadratfreie Zahlen, *Math. Ann.*, **105** (1931), 653–662. <https://doi.org/10.1007/BF01455836>
7. G. Ricci, Ricerche aritmetiche sui polinomi, *Rend. Circ. Mat. Palermo*, **57** (1933), 433–475. <https://doi.org/10.1007/BF03017586>



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