



*Research article***Fuzzy number on subset of $\mathbb{R}$ and application****Man Huang^{1,*}, Guixiang Wang², Chengjia Li¹ and Mengmeng Song³**

¹ Liberal Arts and Sciences College, Hangzhou Dianzi University Information Engineering College, No. 1 Hangdian Road, Qingshanhu Street, Hangzhou 311305, Zhejiang, China

² School of Sciences, Hangzhou Dianzi University, No. 1158 2nd Avenue, Baiyang Street, Hangzhou 310018, Zhejiang, China

³ Hangzhou Wenhui School, No. 518 Jinqiao Street, Hangzhou 310018, Zhejiang, China

* **Correspondence:** Email: 20189088@hziee.edu.cn; Tel: +15968137513.

Abstract: In this paper, some issues related to fuzzy numbers on number sets were studied. The definition of fuzzy numbers on number sets was given. Continuous fuzzy numbers and discrete fuzzy numbers are fuzzy numbers defined on the set of real numbers. The fuzzy numbers defined on the number set are sometimes neither continuous fuzzy numbers nor discrete fuzzy numbers (for example, they may be fuzzy rational numbers). Therefore, the definition of fuzzy numbers on number sets unifies the definition forms of continuous fuzzy numbers and discrete fuzzy numbers, and also expands the scope of fuzzy numbers. The representation theorem of fuzzy numbers on number sets was given, and the operation rules of fuzzy numbers on number sets were discussed. Then, a binary relation was defined based on the center of mass of fuzzy numbers on number sets. Next, specific implementation steps for sorting target objects with mixed uncertain multi-channel digital information characteristics were presented, and these steps were applied to a specific example.

Keywords: continuous fuzzy numbers; discrete fuzzy numbers; discrete set; operation closure; mixed multi-channel uncertain digital information; ranking

Mathematics Subject Classification: 03E72, 90B50, 90C70, 93C42

1. Introduction

Zadeh first proposed the concept of the fuzzy set in 1965 [1]. Then, the research on fuzzy sets appeared one after another. For example, in [2], Alshehri, Alghamdi, and Qarni proposed new distance measures applied to medical diagnoses, improving accuracy and validating the practicality and broad application potential of the methods. In [3], Kahraman proposed a fuzzy set extension model based on proportional relationships, which simplifies the determination process of multiple parameters'

membership degrees and enhances the stability and application convenience of expert judgment. In [4], Jana and Mahanta analyzed the concept of fuzzy boundaries of sets in fuzzy topological spaces, and discussed several algebraic models that use fuzzy boundaries to analyze spatial objects. In [5], Karapnar, et al. newly introduced an auxiliary function and applied it to define contractions in both metric spaces and fuzzy metric spaces. In [6], Azam, et al. studied coincidence point of L-fuzzy sets endowed with a graph. Chang and Zadeh proposed the concept of fuzzy numbers to study the properties of probability function in 1972 [7]. Then, the concept of fuzzy numbers became more and more important with the development of fuzzy number theory and application. For example, in [8], Negoita and Ralescur regarded continuous fuzzy numbers as an interval family, and then obtained the first representation theorem of continuous fuzzy numbers. In [9], Diamond and Kloeden gave the definition of the continuous fuzzy number. In [10], Wang and Wu defined the fuzzy n -cell number, which is a special multidimensional fuzzy number with a wide range of uses. In [11], Jayalakshmi, et al. provided the generalization of fuzzy real numbers by imposing the elevator's condition upon its legs. In [12], multi-channel uncertain numerical information was expressed by constructing a multi-level linear fuzzy elliptic number, and relevant applications were given. In [13], Zhang and Sun defined new possibilistic mean and variance for generalized trapezoidal intuitionistic fuzzy numbers, revealed their properties, and derived the relationships between the variances. In [14], Jin and Wu investigated the Ulam stability of two types of fuzzy number-valued functional equations in Banach spaces and explored the existence and uniqueness of their solutions. In [15], Saeed, Khan, Treanta, et al. defined a new type of upper and lower logarithmic convex fuzzy numerical mapping, derived the improved Fejer-type inequality, and verified its validity. In [16], Khan, Othman, et al. introduced the upper and lower A-convex fuzzy numerical mappings and their new variants, and established the weighted Hermit-Hadamard type fuzzy inclusion relation, thereby extending the existing inequality theory. In [17], Voxman first proposed the concept of the discrete fuzzy number in 2011. The theoretical research and application of discrete fuzzy numbers have also achieved a lot of results. For example, in [18], Wang, et al. proposed the concepts of quasi-discist fuzzy number and quasi-discist fuzzy vector. In [19,20], Gong, et al. gave the definitions of the three-dimensional discrete fuzzy number and three-dimensional generalized discrete fuzzy number and applied them to practice. In [21], Qin, et al. gave the concept of the special discrete fuzzy numbers on countable sets and obtained some conclusions of the special fuzzy number, and applied the above works in imagine fusion and supervisor evaluation aggregation.

The most results of the above discussion about fuzzy numbers are either about continuous fuzzy numbers or about discrete fuzzy numbers. We know that continuous fuzzy numbers can only describe and process continuous uncertain digital information and discrete fuzzy numbers can only describe and process discrete uncertain digital information. For uncertain digital information with mixed multi-channel characteristics (that is, continuous characteristics, discrete characteristics, and even other types of characteristics), the continued use of continuous or discrete fuzzy numbers is no longer applicable. For example, in [10], the authors only considered the working speed and the qualified rate of production products (both of which are characterized by continuity) to evaluate the working status of workers. However, in order to more accurately evaluate the working status of workers, we should also consider other types of influencing factors other than the initial work speed and product pass rate, such as whether workers are provided with opportunities for skill improvement, that is, training opportunities (with discrete characteristics), or whether workers have a clear promotion path, that is, the promotion

path (with discrete characteristics). If the above four factors are considered at the same time, we cannot continue to use the fuzzy n-cell method to evaluate the working status of workers. Therefore, it is important and meaningful to propose the concept of fuzzy number on a number set which can process continuous uncertain digital information, discrete uncertain digital information, and even other types of uncertain digital information. Next, our first task is how to define the fuzzy number on the number set. We find inspiration from the definition of continuous fuzzy number [9] and discrete fuzzy number [17]. Using this inspiration, we define fuzzy numbers on a set of numbers. On the basis of this, we not only generalize and unify the concepts of continuous fuzzy numbers and discrete fuzzy numbers, but also expand the scope of fuzzy numbers.

The structure of this paper is as follows: Section 2 reviews relevant concepts and conclusions from previous studies. Section 3 presents the definition and representation theorem of fuzzy numbers on a number set, proving that continuous fuzzy numbers are equivalent to fuzzy numbers on $\mathbb{R}$, and that a fuzzy number u on a number set is continuous if and only if its support is an interval in $\mathbb{R}$. It also defines discrete sets, derives two of their properties, and obtains a result similar to Theorem 4. Section 4 provides the algorithm for fuzzy numbers on a number set and discusses the equivalence between the operational closure of such fuzzy numbers and that of the number set. Section 5 defines a binary relation based on the centroid of fuzzy numbers on a number set, fuzzy number vectors, and centroid vectors, and completes the ranking of specific target objects through given ranking steps.

2. Preliminaries

Let X be a domain. The fuzzy set A on X refers to a mapping $A : X \rightarrow [0, 1]$.

For fuzzy set u of X , $[u]_X^r$ denotes the r -cut (or level) set, i.e., $[u]_X^r = \{x \in X : u(x) \geq r\}$ for any $r \in (0, 1]$, and denotes the closure of the support (denoted by $\text{supp}u$) of fuzzy set u , i.e., $[u]_X^0 = \overline{\text{supp}u} = \{x \in X : u(x) > 0\}$.

Definition 2.1. [9] Let $u \in F(\mathbb{R})$ (the whole of the fuzzy set on $\mathbb{R}$). If the following conditions (1)–(4) hold:

- (1) There exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
- (2) $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$;
- (3) $u(x)$ is semi-continuous;
- (4) The closure $[u]_{\mathbb{R}}^0$ of the support set of u is a tight set (i.e., a compact set in standard topology), then u is a continuous fuzzy number, and the collection of all continuous fuzzy numbers is denoted by E .

Theorem 2.1. [8] (Representation theorem of continuous fuzzy number) If $u \in E$, then

- (1) $[u]_{\mathbb{R}}^r$ is a nonempty bounded closed interval on $\mathbb{R}$ for any $r \in [0, 1]$;
- (2) $[u]_{\mathbb{R}}^{r_2} \subset [u]_{\mathbb{R}}^{r_1}$ for $0 \leq r_1 \leq r_2 \leq 1$;
- (3) if $\{r_n\}$ is a positive nondecreasing sequence and converges to $r \in (0, 1]$, then $\bigcap_{n=1}^{\infty} [u]_{\mathbb{R}}^{r_n} = [u]_{\mathbb{R}}^r$.

Conversely, if for each $r \in [0, 1]$, $A_r \subset \mathbb{R}$ satisfies the conditions (1)–(3), then there exists unique fuzzy $u \in E$ such that $[u]^r = A_r$ for each $r \in (0, 1]$ and $[u]_{\mathbb{R}}^0 = \bigcup_{0 < r \leq 1} A_r \subset A_0$.

Definition 2.2. [17] A fuzzy set $u : \mathbb{R} \rightarrow [0, 1]$ is called a discrete fuzzy number, if the support of u is finite (it is obvious $\text{supp}u = [u]^0$), i.e., there exist $x_1, x_2, \dots, x_n \in \mathbb{R}$ with $x_1 < x_2 < \dots < x_n$ such that $[u]_{\mathbb{R}}^0 = \{x_1, x_2, \dots, x_n\}$, and there exist natural numbers s, t with $1 \leq s \leq t \leq n$ such that

- (1) $u(x_i) = 1$ for any natural number i with $s \leq i \leq t$;
- (2) $u(x_i) \leq u(x_j)$ for any natural number i, j with $1 \leq i \leq j \leq s$;
- (3) $u(x_i) \geq u(x_j)$ for any natural number i, j with $t \leq i \leq j \leq n$.

We denote the collection of all discrete fuzzy numbers by D .

3. The fuzzy number on number set S

We already have the concepts of continuous fuzzy numbers and discrete fuzzy numbers. In the following, we propose the definition of fuzzy numbers on the set S ($S \subset \mathbb{R}$) to associate continuous fuzzy numbers with discrete fuzzy numbers and to generalize the concepts of continuous fuzzy numbers and discrete fuzzy numbers.

Let $A \subset \mathbb{R}$. We denote $\min A = \min \{x : x \in A\}$, $\max A = \max \{x : x \in A\}$. For $A, S \subset \mathbb{R}$, we denote $A_S (= S_A) = \{x : x \in A, x \in S\} = A \cap S$.

Definition 3.1. Let $S \subset \mathbb{R}$. u is a fuzzy number on S , if the following conditions (1)–(4) hold:

- (1) u is normal, that is, there exists $x_0 \in S$ such that $u(x_0) = 1$;
- (2) $[u]_S^0 (= \overline{\{x \in S : u(x) > 0\}})$ is a bounded set;
- (3) for any $r \in (0, 1]$, $[u]_S^r = \{x \in S : u(x) \geq r\}$ is a closed subset of S ;
- (4) $u(x)$ is nondecreasing on $[\min[u]_S^0, \min[u]_S^1]_S$, $u(x)$ is nonincreasing on $[\max[u]_S^1, \max[u]_S^0]_S$.

The collection of all the fuzzy numbers on S is denoted by F_S .

Remark 3.1. Condition (3) requires $[u]_S^r$ to be a closed set. We emphasize that this condition is formulated to be applicable to both continuous and discrete spaces. In a continuous space, this imposes a constraint, whereas in a discrete space, every subset is closed; thus, Condition (3) is trivially satisfied. This unified formulation allows our framework to seamlessly cover both settings without requiring separate definitions.

Theorem 3.1. (Representation theorem of fuzzy number on a number set) If u is a fuzzy number on subset S of $\mathbb{R}$, i.e., $u \in F_S$, then

- (1) $[u]_S^r$ is a nonempty bounded closed interval on S for any $r \in [0, 1]$;
- (2) $[u]_S^{r_2} \subset [u]_S^{r_1}$ for any $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$;
- (3) For any nondecreasing positive sequence r_n that converges to r with $r \in (0, 1]$, there are

$$\bigcap_{n=1}^{\infty} [u]_S^{r_n} = [u]_S^r.$$

Conversely, if $\{A_r \subset S : r \in [0, 1]\}$ satisfies:

- (i) A_r is a nonempty bounded closed interval on S , i.e., $A_r = [\underline{A}_r, \overline{A}_r]_S$ for any $r \in [0, 1]$;
- (ii) $A_{r_2} \subset A_{r_1}$ for any $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$;
- (iii) For any nondecreasing positive sequence r_n that converges to r with $r \in (0, 1]$, there are

$$\bigcap_{n=1}^{\infty} A_{r_n} = A_r,$$

then there exists a unique $u \in F_S$ such that $[u]_S^r = A_r$ with $r \in (0, 1]$, and $[u]_S^0 = \bigcup_{0 < r \leq 1} \overline{A_r} \subset A_0$.

Proof. The proof of the first part of the theorem is as follows:

If $u \in F_S$, then it is obvious that the conclusion (2) holds, so we only need to prove the conclusions (1) and (3).

From the condition (1) in Definition 3.1, $[u]_S^r$ is nonempty for each $r \in [0, 1]$. From the condition (2) in Definition 3.1 and $[u]_S^r \subset [u]_S^0$ for each $r \in [0, 1]$, $[u]_S^0$ is bounded. From the condition (3) in

Definition 3.1 and Definition $[u]_S^0$, $[u]_S^r$ is closed for each $r \in [0, 1]$. Therefore, for each $r \in [0, 1]$, $\min[u]_S^r, \max[u]_S^r$ exist and $\min[u]_S^r, \max[u]_S^r \in [u]_S^r \subset S$. We denote $\underline{x} = \min[u]_S^r, \bar{x} = \max[u]_S^r$. Thus, we have that $[u]_S^r = \{x \in S : u(x) \geq r\} \subset [\underline{x}, \bar{x}] \cap S = [\underline{x}, \bar{x}]_S$ ($r \in [0, 1]$). On the other hand, let $r \in [0, 1]$ and $\hat{x}$ be a normal point of u . Then $\underline{x} < \hat{x} < \bar{x}$. As $x \in [\underline{x}, \hat{x}]_S$, we have $u(x) \geq u(\hat{x}) \geq r$ since $u(x)$ is nondecreasing on $[\underline{x}, \hat{x}]_S$. So we obtain $x \in [u]_S^r$. As $x \in [\hat{x}, \bar{x}]_S$, we have $u(x) \geq u(\bar{x}) \geq r$ since $u(x)$ is nonincreasing on $[\hat{x}, \bar{x}]_S$. So we obtain $x \in [u]_S^r$. Thus, $[\underline{x}, \bar{x}]_S \subset [u]_S^r$. So $[\underline{x}, \bar{x}]_S = [u]_S^r$ for each $r \in [0, 1]$. Therefore, the conclusion (1) holds.

Let $\{r_n\}$ be a positive nondecreasing sequence that converges to $r \in (0, 1]$. By $r_n \leq r$ ($n = 1, 2, \dots$), we easily know that $[u]_S^r \subset [u]_S^{r_n}$, then $x \in [u]_S^r \subset \bigcap_{n=1}^{\infty} [u]_S^{r_n}$. If $x_0 \in \bigcap_{n=1}^{\infty} [u]_S^{r_n}$, then for any $n = 1, 2, \dots$, $u(x_0) \geq r_n$ holds. Therefore, we have $u(x_0) \geq \lim_{n \rightarrow \infty} r_n = r$, $x_0 \in [u]_S^r$. So $\bigcap_{n=1}^{\infty} [u]_S^{r_n} = [u]_S^r$. Thus, conclusion (3) holds.

The first part of the theorem has been proved. In the following, we prove the second section of the theorem:

We define:

$$u(x) = \begin{cases} \sup\{r \in [0, 1] : x \in A_r\}, & x \in A_0 \\ 0, & x \notin A_0 \end{cases}$$

We only need to prove that $[u]_S^r = A_r$ ($r \in (0, 1)$) and $[u]_S^0 = \overline{\bigcup_{0 < r \leq 1} A_r} \subset A_0$ and $u \in F_S$ (the uniqueness of u is obvious). For this reason, we first show $\{r \in [0, 1] : x \in A_r\} = [0, u(x)]$ as $x \in A_0$. As $u(x) = 0$, i.e., $\sup\{r \in [0, 1] : x \in A_r\} = 0$, then $\{r \in [0, 1] : x \in A_r\} = \{0\}$, so $\{r \in [0, 1] : x \in A_r\} = [0, u(x)]$ holds. As $u(x) > 0$, for each $r_0 \in (0, u(x))$, from the definition of the supremum, there exists $r' \in \{r \in [0, 1] : x \in A_r\}$ such that $r' > r_0$, then $x \in A_{r'} \subset A_{r_0}$, i.e., $r_0 \in \{r \in [0, 1] : x \in A_r\}$; and $0 \in \{r \in [0, 1] : x \in A_r\}$ from $x \in A_0$; in addition, by the definition of $u(x)$, we can take a positive nondecreasing sequence $\{r_n\}$ in $\{r \in [0, 1] : x \in A_r\}$ that converges to $u(x)$, then $x \in A_{r_n}$ ($n = 1, 2, \dots$); and by the condition (iii) we know that $x \in \bigcap_{n=1}^{\infty} A_{r_n} = A_{u(x)}$, i.e., $u(x) \in \{r \in [0, 1] : x \in A_r\}$. Thus, we obtain $[0, u(x)] \subset \{r \in [0, 1] : x \in A_r\}$. On the other hand, as $r_0 \in \{r \in [0, 1] : x \in A_r\}$, then $r_0 \leq \sup\{r \in [0, 1] : x \in A_r\} = u(x)$, so $r_0 \in [0, u(x)]$, thus $\{r \in [0, 1] : x \in A_r\} \subset [0, u(x)]$ also holds. Therefore, $\{r \in [0, 1] : x \in A_r\} = [0, u(x)]$ is true.

In the following, we show that $[u]_S^r = A_r$ ($r \in (0, 1)$). If $x_0 \in A_{r_0}$, then $r_0 \in \{r \in [0, 1] : x_0 \in A_r\}$, so $u(x_0) = \sup\{r \in [0, 1] : x_0 \in A_r\} \geq r_0$, thus $x_0 \in [u]_S^{r_0}$, i.e., $A_{r_0} \subset [u]_S^{r_0}$ for any $r_0 \in (0, 1]$. Conversely, if $x_0 \in [u]_S^{r_0}$, then $u(x_0) \geq r_0 > 0$, so $x_0 \in A_0 \subset S$ and $r_0 \in [0, u(x_0)] = \{r \in [0, 1] : x_0 \in A_r\}$, thus $x_0 \in A_{r_0}$, i.e., $[u]_S^{r_0} \subset A_{r_0}$. Therefore, $[u]_S^{r_0} = A_{r_0}$ for any $r_0 \in (0, 1]$.

Second, let us prove $[u]_S^0 = \overline{\bigcup_{0 < r \leq 1} A_r} \subset A_0$. $\overline{\bigcup_{0 < r \leq 1} A_r} \subset A_0$ can be guaranteed by $A_r \subset A_0$ ($\forall r \in (0, 1]$) and the closeness of A_0 . Suppose that $x_0 \in [u]_S^0$, then there exists $\{x_n\} \subset \text{supp}u$ such that $\{x_n\} \rightarrow x_0$ ($n \rightarrow \infty$). Since $\{x_n\} \subset \text{supp}u$, then $\{x_n\} \subset \bigcup_{0 < r \leq 1} A_r$, i.e., $x_0 \in \overline{\bigcup_{0 < r \leq 1} A_r}$, thus $[u]_S^0 \subset \overline{\bigcup_{0 < r \leq 1} A_r}$. Conversely, from $A_r = [u]_S^r$ ($r \in (0, 1]$), we have $\overline{\bigcup_{0 < r \leq 1} A_r} = \overline{\bigcup_{0 < r \leq 1} [u]_S^r} \subset \overline{\text{supp}u} = [u]_S^0$, so $[u]_S^0 = \overline{\bigcup_{0 < r \leq 1} A_r} \subset A_0$ holds.

Finally, let us prove $u \in F_S$, i.e., u satisfies all the conditions in Definition 3.1. From the condition (i) and $[u]_S^1 = A_1$, u is normal. So the condition (1) of Definition 3.1 is true. From $[u]_S^0 \subset A_0 \subset S$ and A_0 is bounded, $[u]_S^0 \subset S$ is bounded. So the condition (2) of Definition 3.1 is true. Since $[u]_S^r = A_r$ for each $r \in (0, 1]$ and A_r is closed, $[u]_S^r = \{x \in S : u(x) \geq r\}$ is closed. So the condition (3) of Definition 3.1 is true. From the definition of u and the condition (i) of the theorem, $[u]_S^r$ is a fuzzy set on S and $\overline{\text{supp}u}$ is a closed interval on S , and we denote it by $\overline{\text{supp}u} = [\min[u]_S^0, \max[u]_S^0]_S$. For any

$x_i, x_j \in [\min[u]_S^0, \hat{x}]_S$ and $x_i < x_j$, in the following we use the inverse method to prove $u(x_i) \leq u(x_j)$. Suppose that $u(x_i) \leq u(x_j)$ does not hold, then $u(x_i) > u(x_j)$. So $[u]_S^{u(x_i)} \subset [u]_S^{u(x_j)}$ and $x_j \notin [u]_S^{u(x_i)}$. Thus, $x_i \geq \min[u]_S^{u(x_i)} \geq x_j$, which is contradictory to $x_i < x_j$. Therefore, $u(x_i) \leq u(x_j)$. Similarly, as $x_i < x_j$ and $x_i, x_j \in [\hat{x}, \max[u]_S^0]_S$, we have $u(x_i) \geq u(x_j)$. So the condition (4) of Definition 3.1 is true. Thus, $u \in F_S$.

The theorem is proved. $\square$

Theorem 3.2. u is a continuous fuzzy number $\iff u$ is a fuzzy number on $\mathbb{R}$.

Proof. $(\Rightarrow)$ Let u be a continuous fuzzy number. Then u is defined on $\mathbb{R}$ and u satisfies the conditions (1)–(4) of definition of continuous fuzzy numbers (Definition 2.1). From the condition (1) in Definition 2.1, u satisfies the condition (1) in Definition 3.1 on $\mathbb{R}$. From the condition (4) in Definition 2.1, $[u]_{\mathbb{R}}^0 (= \{x \in \mathbb{R} : u(x) > 0\})$ is bounded and $[u]_{\mathbb{R}}^0 \subset \mathbb{R}$. Thus, u satisfies the condition (2) in Definition 3.1 on $\mathbb{R}$. The closeness of $\{x \in \mathbb{R} : u(x) \geq r\}$ can be guaranteed by the upper semi-continuity of $u(x)$ (the condition (3) in Definition 2.1). So u satisfies the condition (3) in Definition 3.1 on $\mathbb{R}$. In the following, we show that u satisfies the condition (4) in Definition 3.1 on $\mathbb{R}$. For any $x, y \in [\min[u]_{\mathbb{R}}^0, \min[u]_{\mathbb{R}}^1]_{\mathbb{R}}$ with $x < y$, from $\min[u]_{\mathbb{R}}^0 \leq x \leq y \leq \min[u]_{\mathbb{R}}^1$, there exists $s \in [0, 1]$ such that $y = sx + (1 - s)\min[u]_{\mathbb{R}}^1$. Therefore, from the fuzzy convexity of u , we see that $u(y) = u(sx + (1 - s)\min[u]_{\mathbb{R}}^1) \geq \min\{u(x), u(\min[u]_{\mathbb{R}}^1)\} = \min\{u(x), 1\} = u(x)$. So $u(x)$ is non-decreasing on $[\min[u]_{\mathbb{R}}^0, \min[u]_{\mathbb{R}}^1]_{\mathbb{R}}$. On the other hand, for any $x, y \in [\max[u]_{\mathbb{R}}^1, \max[u]_{\mathbb{R}}^0]_{\mathbb{R}}$ with $x < y$, from $\max[u]_{\mathbb{R}}^1 \leq x \leq y \leq \max[u]_{\mathbb{R}}^0$, there exists $t \in [0, 1]$ such that $x = t\max[u]_{\mathbb{R}}^1 + (1 - t)y$. Therefore, from the fuzzy convexity of u , we see that $u(x) = u(t\max[u]_{\mathbb{R}}^1 + (1 - t)y) \geq \min\{u(\max[u]_{\mathbb{R}}^1), u(y)\} = \min\{1, u(y)\} = u(y)$. $u(x)$ is nonincreasing on $[\max[u]_{\mathbb{R}}^1, \max[u]_{\mathbb{R}}^0]_{\mathbb{R}}$.

Therefore, u satisfies all the conditions in Definition 3.1 on $\mathbb{R}$. Thus, u is a fuzzy number on $\mathbb{R}$.

$(\Leftarrow)$ Since $u \in F_{\mathbb{R}}$, u satisfies the conclusions (1)–(3) of Theorem 3.1 on $\mathbb{R}$, i.e., u satisfies the following conclusions: (1) $[u]_{\mathbb{R}}^r$ is a nonempty bounded closed interval on $\mathbb{R}$ for any $r \in [0, 1]$; (2) $[u]_{\mathbb{R}}^{r_2} \subset [u]_{\mathbb{R}}^{r_1}$ for any $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$; (3) for any nondecreasing positive sequence r_n that converges to r with $r \in (0, 1]$, there are $\bigcap_{n=1}^{\infty} [u]_{\mathbb{R}}^{r_n} = [u]_{\mathbb{R}}^r$. From all of the above, thus u satisfies all the conditions in Theorem 2.1. So u is a continuous fuzzy number. $\square$

Remark 3.2. For any $u \in E$, from Definition 2.1, it is known that u should be defined at every point of $\mathbb{R}$ and satisfies all the conditions in Definition 2.1. For any $u \in F_S$ ($S \subset \mathbb{R}$), if the definition of u can be supplemented from S to the entire $\mathbb{R}$ in the way of $u(x) = 0$ for $x \in \mathbb{R} \setminus S$ such that u satisfies all the conditions of Definition 2.1 or all the conditions of Theorem 2.1, we still consider u as a continuous fuzzy number. In this case, we have the following result.

Theorem 3.3. Let $S \subset \mathbb{R}$ and $u \in F_S$. Then,

u is a continuous fuzzy number $\iff [u]_S^0$ is an interval on $\mathbb{R}$.

Proof. $\Rightarrow$ Since $u \in F_S$ and u is a continuous fuzzy number, u satisfies all the conditions in Definition 2.1 on the set S . From the condition (2) in Definition 2.1, we know that $u(tx + (1 - t)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in S$ and $t \in [0, 1]$. Assuming that $[u]_S^0$ is not an interval on $\mathbb{R}$, then there exist $x, y \in [u]_S^0$ (let $x < y$) and $k \in (0, 1)$ such that $kx + y(1 - k) \notin [u]_S^0$. So $u(kx + y(1 - k)) = 0 \leq \min\{u(x), u(y)\}$. This contradicts the condition (2) in Definition 2.1. Thus, the hypothesis is not true. So $[u]_S^0$ is an interval on $\mathbb{R}$.

$\Leftarrow$ Since $u \in F_S$, u satisfies the conclusions (1)–(3) of Theorem 3.1 on S . By $[u]_S^0$ is an interval on $\mathbb{R}$ and the conclusion (1) in Theorem 3.1, we know that $[u]_S^0$ is a nonempty bounded closed interval on $\mathbb{R}$. Since $[u]_S^1 \subset [u]_S^r (= \{x \in S : u(x) \geq r\}) \subset [u]_S^0 (= \{x \in S : u(x) > 0\})$ for any $r \in (0, 1]$ and $[u]_S^r$ is closed and $[u]_S^1$ is nonempty, $[u]_S^r$ is a nonempty bounded closed interval on $\mathbb{R}$ for any $r \in (0, 1]$. Thus, for any $r \in [0, 1]$, $[u]_S^r$ is a nonempty bounded closed interval on $\mathbb{R}$. So u satisfies the condition (1) in Theorem 2.1. By u satisfying the conclusions (2) and (3) of Theorem 3.1 on S , so u satisfies the conditions (2) and (3) in Theorem 2.1 on S . Thus, u is a continuous fuzzy number (i.e., the continuous fuzzy number in Remark 3.2). $\square$

Definition 3.2. Let $\mathbb{S} \subset \mathbb{R}$. For $x \in \mathbb{S}$, if there exists an open interval U such that $x \in U$, but $(U \setminus \{x\}) \cap \mathbb{S} = \emptyset$, then x is called the isolated point of $\mathbb{S}$. If the elements in the set $\mathbb{S}$ are all isolated points of the set $\mathbb{S}$, then $\mathbb{S}$ is called a discrete set.

Property 3.1. If A is a discrete subset of $\mathbb{R}$, then A is a closed set.

Proof. Assuming A is an open set, for each $x \in A$, then there exists $\delta > 0$ such that $(x + \delta, x - \delta) \subset A$. This contradicts that A is a discrete subset of $\mathbb{R}$. So A is a closed set. $\square$

Property 3.2. A is a discrete subset of $\mathbb{R}$. If $B \subset A$ and B is bounded, then B is a discrete and finite set.

Proof. First, since A is a discrete set, its subset B is obviously discrete as well. Next, we prove that B is a finite set. According to the definition of a discrete set, for each point $x \in B$, there exists an open neighborhood U_x such that $U_x \cap B = \{x\}$. The collection of these open neighborhoods, denoted as $\mathcal{U} = \{U_x \mid x \in B\}$, forms an open cover of B . Since B is a bounded set and a discrete set in $\mathbb{R}$ has no limit points (which implies that B is a closed set), B is a closed and bounded set. By the Heine-Borel theorem, B is a compact set. By the definition of compactness, any open cover of B admits a finite subcover. Therefore, from the open cover $\mathcal{U}$, there must exist finitely many open sets $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ whose union completely covers B . Since each of these finitely many open sets contains exactly one point of B , their union can contain at most n points. Thus, B must be a finite set. This completes the proof. $\square$

Remark 3.3. For any $u \in D$, from Definition 2.2, it is known that u should be defined at every point of $\mathbb{R}$ and satisfies all the conditions in Definition 2.2. For any $u \in F_S$ ($S \subset \mathbb{R}$), if the definition of u can be supplemented from S to the entire $\mathbb{R}$ in the way of $u(x) = 0$ for $x \in \mathbb{R} \setminus S$ such that u satisfies all the conditions of Definition 2.2, we still consider u as a discrete fuzzy number. In this case, we have the following result.

Theorem 3.4. Let $S \subset \mathbb{R}$ and $u \in F_S$. Then,

u is a discrete fuzzy number $\iff [u]_S^0$ is a discrete subset of $\mathbb{R}$.

Proof. $\Rightarrow$ Since u is a discrete fuzzy number, $[u]_S^0$ is a finite set. It is obvious that $[u]_S^0$ is a discrete subset of $\mathbb{R}$.

$\Leftarrow$ By $u \in F_S$, u satisfies all the conditions of Definition 3.1. Since $[u]_S^0 \subset S$ is bounded from the condition (2) in Definition 3.1 and $[u]_S^0$ is a discrete subset of $\mathbb{R}$, $[u]_S^0$ is finite, i.e., there exist $x_1, x_2, \dots, x_n \in S \subset \mathbb{R}$ with $x_1 < x_2 < \dots < x_n$ such that $[u]_S^0 = \{x_1, x_2, \dots, x_n\}$. There exists $x_0 \in S$ such that $u(x_0) = 1$ from the condition (1) in Definition 3.1. So u satisfies the condition (1) in Definition 2.2

on the set S . For any $x_i, x_j \in [\min[u]_S^0, \min[u]_S^1]_S$ with $x_i \leq x_j$, from $u(x)$ is nondecreasing on $[\min[u]_S^0, \min[u]_S^1]_S$, we have $u(x_i) \leq u(x_j)$. So u satisfies the condition (2) in Definition 2.2. For any $x_i, x_j \in [\max[u]_S^1, \max[u]_S^0]_S$ with $x_i \leq x_j$, from $u(x)$ is nonincreasing on $[\min[u]_S^0, \min[u]_S^1]_S$, we have $u(x_i) \geq u(x_j)$. So u satisfies the condition (3) in Definition 2.2. Thus, u satisfies all the conditions in Definition 2.2 on S . So u is a discrete fuzzy number. $\square$

Remark 3.4. In a sense, the concept of fuzzy numbers on a subset of $\mathbb{R}$ not only generalizes the concepts of continuous fuzzy numbers and discrete fuzzy numbers, but also unifies the concepts of continuous fuzzy numbers and discrete fuzzy numbers. In fact, continuous fuzzy numbers are called fuzzy real numbers and discrete fuzzy numbers are called fuzzy integers as the discrete fuzzy numbers are defined on set of the integer. Of course, the fuzzy numbers on a subset of $\mathbb{R}$ not only include continuous fuzzy numbers and discrete fuzzy numbers, but also include other types of fuzzy numbers. For example, a mapping $u: \mathbb{Q} \rightarrow [0, 1]$ is called a fuzzy number on $\mathbb{Q}$ which is defined on $\mathbb{Q}$ and satisfies all the conditions in Definition 3.1, then we also call u a fuzzy rational number. In the following, we give an example of a specific fuzzy rational number: $u: \mathbb{Q} \rightarrow [0, 1]$,

$$u(x) = \begin{cases} \frac{1}{4}, & x \in [1, 2] \cap \mathbb{Q} \\ 1, & x \in [2, 3] \cap \mathbb{Q} \\ \frac{1}{2}, & x \in [3, 4] \cap \mathbb{Q} \\ 0, & x \in (\mathbb{R} \setminus [1, 4]) \cap \mathbb{Q} \end{cases}$$

4. The operation of fuzzy number on the number set S

Let $A, B \subset \mathbb{R}$. If A, B are nonempty compact sets and $\lambda \in \mathbb{R}$, we denote:

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\lambda A = \{\lambda a : a \in A\}$$

$$AB = \{ab : a \in A, b \in B\}$$

For any $u, v \in F_S$, $\lambda \in \mathbb{R}$, we denote:

$$(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x), & \lambda \neq 0 \\ \hat{0}(x), & x = 0 \end{cases}$$

$$(uv)(x) = \sup_{yz=x} \min\{u(y), v(z)\}$$

Theorem 4.1. Let $u, v \in F_S$, $k \in \mathbb{R}$. $\forall r \in [0, 1]$, and we have:

- (1) $[u + v]_S^r = [u]_S^r + [v]_S^r$;
- (2) $[ku]_S^r = k[u]_S^r$;
- (3) $[uv]_S^r = [u]_S^r [v]_S^r$.

Proof. (1) To prove the conclusion (1), we must establish the result for both $r \in (0, 1]$ and $r = 0$ separately.

First, we establish the result for $r \in (0, 1]$. If $x_0 \in [u]_S^r + [v]_S^r$ for each $r \in (0, 1]$, then there exist $y_0 \in [u]_S^r, z_0 \in [v]_S^r$ such that $x_0 = y_0 + z_0$, i.e., $u(y_0) \geq r, v(z_0) \geq r$, so $(u+v)(x_0) = \sup_{y+z=x_0} \min\{u(y), v(z)\} \geq \min\{u(y_0), v(z_0)\} \geq r$, thus $x_0 \in [u+v]_S^r$. So $[u]_S^r + [v]_S^r \subset [u+v]_S^r$ for each $r \in (0, 1]$. If $x_0 \in [u+v]_S^r$ for each $r \in (0, 1]$, then $(u+v)(x_0) \geq r$, i.e., $\sup_{y+z=x_0} \min\{u(y), v(z)\} \geq r$. So there exist $y_m, z_m \in S$ ($m = 1, 2, \dots$) such that $y_m + z_m = x_0$ and $\min\{u(y_m), v(z_m)\} > \sup_{y+z=x_0} \min\{u(y), v(z)\} - \frac{1}{m} \geq r - \frac{1}{m}$. Since $(u)_S^0 = \bigcup_{0 < r \leq 1} [u]_S^r$ (the detailed proof is provided in Appendix A), it is obvious that $u(y_m), v(z_m) > 0$ when $m > \frac{1}{r}$, i.e., $y_m \in \bigcup_{0 < r \leq 1} [u]_S^r, z_m \in \bigcup_{0 < r \leq 1} [v]_S^r$. So $\{y_m\}$ and $\{z_m\}$ are bounded and there exist convergent subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and $\{z_{m_k}\}$ of $\{z_m\}$ such that $y_{m_k} \rightarrow y_0$ and $z_{m_k} \rightarrow z_0$, thus $y_0 + z_0 = x_0$. So $y_{m_k} \in \bigcup_{0 < r \leq 1} [u]_S^r, z_{m_k} \in \bigcup_{0 < r \leq 1} [v]_S^r$. Since $[u]_S^r$ and $[v]_S^r$ are closed for each $r \in (0, 1]$, then $y_0 \in \bigcup_{0 < r \leq 1} [u]_S^r$ and $z_0 \in \bigcup_{0 < r \leq 1} [v]_S^r$. Thus, there exist $r', r'' \in (0, 1]$ such that $y_0 \in [u]_S^{r'}, z_0 \in [v]_S^{r''}$. Let $r = \min\{r', r''\}$, then $y_0 \in [u]_S^r, z_0 \in [v]_S^r$. So $y_0 + z_0 = x_0 \in [u]_S^r + [v]_S^r$ and $[u+v]_S^r \subset [u]_S^r + [v]_S^r$. Therefore, $[u+v]_S^r = [u]_S^r + [v]_S^r$ with $r \in (0, 1]$.

Second, we establish the result for $r = 0$. Since $\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r) = \bigcup_{0 < r \leq 1} [u]_S^r + \bigcup_{0 < r \leq 1} [v]_S^r$ and $\overline{(u)_S^0 + (v)_S^0} = \overline{(u)_S^0} + \overline{(v)_S^0}$ (the detailed proof is provided in Appendix B), therefore, it follows easily that $[u+v]_S^0 = \overline{(u+v)_S^0} = \overline{\bigcup_{0 < r \leq 1} [u+v]_S^r} = \overline{\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)} = \overline{\bigcup_{0 < r \leq 1} [u]_S^r + \bigcup_{0 < r \leq 1} [v]_S^r} = \overline{(u)_S^0} + \overline{(v)_S^0} = \overline{(u)_S^0} + \overline{(v)_S^0} = [u]_S^0 + [v]_S^0$.

Thus, the conclusion (1) holds.

(2) Similar to the proof of Conclusion (1), we proceed by distinguishing between two cases for $r \in (0, 1]$ and $r : r = 0$.

For $r \in (0, 1]$, we consider k separately: If $k = 0$, then $[0u]^r = \{x \in S : \hat{0}(x) \geq r\} = \{0\} = 0[u]_S^r$; If $k \neq 0$, then $[ku]_S^r = \{x \in S : (ku)(x) \geq r\} = \{x \in S : u(x/k) \geq r\} = \{ky \in S : u(y) \geq r\} = k[y \in S : u(y) \geq r] = k[u]_S^r$.

For $r = 0$, it is clear that $[ku]_S^0 = \overline{(ku)_S^0} = \overline{\bigcup_{0 < r \leq 1} [ku]_S^r} = \overline{\bigcup_{0 < r \leq 1} k[u]_S^r} = k \overline{\bigcup_{0 < r \leq 1} [u]_S^r} = k[u]_S^0$.

Thus, the conclusion (2) holds.

The proof of the conclusion (3) is similar to that of the conclusion (1), so it is omitted. $\square$

If $u \in F_S$, by the representation theorem of fuzzy number on the subset of $\mathbb{R}$, we know $[u]_S^r$ is a closed interval on the set S for each $r \in [0, 1]$. We denote $[u]_S^r = [\underline{u}(r), \bar{u}(r)]_S$. Then we obtain the following conclusions.

Theorem 4.2. Let, $u, v \in F_S, k \in \mathbb{R}, \forall \in [0, 1]$, and we have:

- (1) $[\underline{u}(r), \bar{u}(r)]_S + [\underline{v}(r), \bar{v}(r)]_S (= [u]_S^r + [v]_S^r) = [\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)]_S$;
- (2) $k[\underline{u}(r), \bar{u}(r)]_S (= k[u]_S^r) = [k\underline{u}(r), k\bar{u}(r)]_S$ as $k \geq 0$, $k[\underline{u}(r), \bar{u}(r)]_S (= k[u]_S^r) = [k\bar{u}(r), k\underline{u}(r)]_S$ as $k < 0$;
- (3) $[\underline{u}(r), \bar{u}(r)]_S [\underline{v}(r), \bar{v}(r)]_S (= [u]_S^r [v]_S^r)$
 $= [\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}]_S$.

Proof. (1)

$$\begin{aligned} z_0 &\in [\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)]_S \\ &\Leftrightarrow \exists x_0 \in [\underline{u}(r), \bar{u}(r)]_S, y_0 \in [\underline{v}(r), \bar{v}(r)]_S, s.t. z_0 = x_0 + y_0 \\ &\Leftrightarrow z_0 \in [\underline{u}(r), \bar{u}(r)]_S + [\underline{v}(r), \bar{v}(r)]_S \end{aligned}$$

(2) As $k \geq 0$, then:

$$\begin{aligned} y_0 &\in [k\underline{u}(r), k\bar{u}(r)]_S \\ &\Leftrightarrow \exists x_0 \in [\underline{u}(r), \bar{u}(r)]_S, s.t. y_0 = kx_0 \\ &\Leftrightarrow y_0 \in k[\underline{u}(r), \bar{u}(r)]_S = k[u]_S^r \end{aligned}$$

As $k < 0$, then:

$$\begin{aligned} y_0 &\in [k\bar{u}(r), k\underline{u}(r)]_S \\ &\Leftrightarrow \exists x_0 \in [\underline{u}(r), \bar{u}(r)]_S, s.t. y_0 = kx_0 \\ &\Leftrightarrow y_0 \in k[\underline{u}(r), \bar{u}(r)]_S = k[u]_S^r \end{aligned}$$

(3) On the one hand, if $z_0 \in [\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}]_S$, then $\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\} \leq z_0 \leq \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}$. So there exist $x_0, y_0 \in S$ such that $\underline{u}(r) \leq x_0 \leq \bar{u}(r)$, $\underline{v}(r) \leq y_0 \leq \bar{v}(r)$ and $z_0 = x_0 y_0$. Thus, $z_0 \in [\underline{u}(r), \bar{u}(r)]_S [\underline{v}(r), \bar{v}(r)]_S$. Therefore, $[\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}]_S$

$$\subset [\underline{u}(r), \bar{u}(r)]_S [\underline{v}(r), \bar{v}(r)]_S.$$

On the other hand, if $z_0 \in [\underline{u}(r), \bar{u}(r)]_S [\underline{v}(r), \bar{v}(r)]_S$, then there exist $x_0 \in [\underline{u}(r), \bar{u}(r)]_S$ and $y_0 \in [\underline{v}(r), \bar{v}(r)]_S$ such that $z_0 = x_0 y_0$. Therefore,

$$z_0 \in [\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}]_S.$$

So $[\underline{u}(r), \bar{u}(r)]_S [\underline{v}(r), \bar{v}(r)]_S$

$$\subset [\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}]_S.$$

Thus, the conclusion (3) is true. □

Theorem 4.3. Let $S \subset \mathbb{R}$.

- 1) If $x + y \in S$ for any $x, y \in S$, then $u + v \in F_S$ for any $u, v \in F_S$, and we have $[u + v]_S^r = [(\underline{u} + \underline{v})(r), (\bar{u} + \bar{v})(r)]_S = [\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)]_S$;
- 2) If $kx \in S$ for any $x \in S$ and $k \in \mathbb{R}$, then $ku \in F_S$ for any $u \in F_S$ and $k \in \mathbb{R}$, and we have $[ku]_S^r = [(\underline{ku})(r), (\bar{ku})(r)]_S = [k\underline{u}(r), k\bar{u}(r)]_S$;
- 3) If $xy \in S$ for any $x, y \in S$, then $uv \in F_S$ for any $u, v \in F_S$, and we have $[uv]_S^r = [(\underline{uv})(r), (\bar{uv})(r)]_S = [\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}]_S$.

Proof. (1) For any $u, v \in F_S$ and $r \in [0, 1]$, then $[u]_S^r = [\underline{u}(r), \bar{u}(r)]_S$ and $[v]_S^r = [\underline{v}(r), \bar{v}(r)]_S$ are nonempty bounded closed intervals in S . So $\underline{u}(r), \bar{u}(r), \underline{v}(r), \bar{v}(r) \in S$ and $\underline{u}(r) \leq \bar{u}(r), \underline{v}(r) \leq \bar{v}(r)$. Since the set S is closed to the addition operation, $\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r) \in S$ and $\underline{u}(r) + \underline{v}(r) \leq \bar{u}(r) + \bar{v}(r)$. Therefore, by conclusions (1) of Theorems 4.1 and 4.2, $[u + v]_S^r = [\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)]_S$ is a nonempty bounded closed interval in S for all $r \in [0, 1]$. For any $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$, we have $[u]_S^{r_2} \subset [u]_S^{r_1}$ and $[v]_S^{r_2} \subset [v]_S^{r_1}$, i.e., $[\underline{u}(r_2), \bar{u}(r_2)]_S \subset [\underline{u}(r_1), \bar{u}(r_1)]_S$ and $[\underline{v}(r_2), \bar{v}(r_2)]_S \subset [\underline{v}(r_1), \bar{v}(r_1)]_S$. Thus, $\underline{u}(r_1) \leq \underline{u}(r_2) \leq \bar{u}(r_2) \leq \bar{u}(r_1)$ and $\underline{v}(r_1) \leq \underline{v}(r_2) \leq \bar{v}(r_2) \leq \bar{v}(r_1)$. Thus $\underline{u}(r_1) + \underline{v}(r_1) \leq \underline{u}(r_2) + \underline{v}(r_2) \leq \bar{u}(r_2) + \bar{v}(r_2) \leq \bar{u}(r_1) + \bar{v}(r_1)$. Therefore, for any $r_1, r_2 \in [0, 1]$ with $r_1 \leq r_2$, we have $[\underline{u}(r_2) + \underline{v}(r_2), \bar{u}(r_2) + \bar{v}(r_2)]_S \subset [\underline{u}(r_1) + \underline{v}(r_1), \bar{u}(r_1) + \bar{v}(r_1)]_S$, i.e., $[u + v]_S^{r_2} \subset [u + v]_S^{r_1}$. Let $\{r_n\}$ be a positive nondecreasing sequence that converges to $r \in (0, 1]$. By $r_n \leq r$ ($n = 1, 2, \dots$), we easily know that $[u + v]_S^r \subset [u + v]_S^{r_n}$. Thus, $[u + v]_S^r \subset \bigcap_{n=1}^{\infty} [u + v]_S^{r_n}$. If $x_0 \in \bigcap_{n=1}^{\infty} [u + v]_S^{r_n}$, then for any $n = 1, 2, \dots$, $(u + v)(x_0) \geq r_n$ holds. Therefore, we have $(u + v)(x_0) \geq \lim_{n \rightarrow \infty} r_n = r$, $x_0 \in [u + v]_S^r$. Thus, $\bigcap_{n=1}^{\infty} [u + v]_S^{r_n} \subset [u + v]_S^r$. Therefore, $\bigcap_{n=1}^{\infty} [u + v]_S^{r_n} = [u + v]_S^r$. Thus, the set $\{[u + v]_S^r \subset S : r \in [0, 1]\}$ satisfies the conditions (i) – (iii) of Theorem 3.1. Consequently, there exists a unique fuzzy number, denoted as $u + v \in F_S$ determined by this set. Specifically, $[u + v]_S^r = [(\underline{u} + \underline{v})(r), (\bar{u} + \bar{v})(r)]_S$. To summarize, the conclusion (1) holds.

The proofs of the conclusions (2) and (3) are similar to that of the conclusion (1), so they are omitted. □

Theorem 4.4. Let $S \subset \mathbb{R}$. We have the following conclusions:

- 1) If $u + v \in F_S$ for any $u, v \in F_S$, then $x + y \in S$ for any $x, y \in S$;
- 2) If $ku \in F_S$ for any $u \in F_S$ and $k \in \mathbb{R}$, then $kx \in S$ for any $x \in S$ and $k \in \mathbb{R}$;
- 3) If $uv \in F_S$ for any $u, v \in F_S$, then $xy \in S$ for any $x, y \in S$.

Proof. (1) For any $x, y \in S$, we define $\hat{x} : S \rightarrow [0, 1]$ and $\hat{y} : S \rightarrow [0, 1]$ as $\hat{x}(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x \end{cases}$ and $\hat{y}(t) = \begin{cases} 1, & t = y, \\ 0, & t \neq y. \end{cases}$ Then, $\hat{x}, \hat{y} \in F_S$ and $x \in [\hat{x}]_S^0, y \in [\hat{y}]_S^0$. Thus, $x + y \in [\hat{x}]_S^0 + [\hat{y}]_S^0 = [\hat{x} + \hat{y}]_S^0 \subset S$, since $\hat{x} + \hat{y} \in F_S$. Therefore, the number set S is closed to the addition operation.

The proofs of the conclusions (2) and (3) are similar to that of the conclusion (1), so they are omitted. □

Remark 4.1. From Theorems 4.3 and 4.4, it is known that the closeness of addition, scalar multiplication, and multiplication operation on number set S and F_S are consistent, respectively.

5. The application of fuzzy number on the number set S

Definition 5.1. Let $u \in F_S, S \subset \mathbb{R}$. We call

$$C(u) = \frac{\int_{\text{supp}(u)} xu(x)dx}{\int_{\text{supp}(u)} u(x)dx} \quad (5.1)$$

the center of mass of the fuzzy number on the set S .

Definition 5.2. Let $u_i \in F_{S_i} (i = 1, 2, \dots, n), S_i \subset \mathbb{R}$. We call the order array $(u_1, u_2, \dots, u_n)$ a fuzzy number vector on the set $\prod_{i=1}^n S_i$. The collection of all the fuzzy number vectors on the set $\prod_{i=1}^n S_i$ is denoted by $\prod_{i=1}^n F_{S_i}$.

Remark 5.1. (1) $\prod_{i=1}^n S_i$ is the Cartesian product of $S_1, S_2, \dots, S_n$.

(2) $\prod_{i=1}^n F_{S_i}$ is the Cartesian product of $F_{S_1}, F_{S_2}, \dots, F_{S_n}$.

Definition 5.3. Let $u = (u_1, u_2, \dots, u_n) \in \prod_{i=1}^n F_{S_i}$. We call

$$\vec{C}(u) = (C(u_1), C(u_2), \dots, C(u_n))$$

the center of mass vector of the fuzzy number vector u on the set $\prod_{i=1}^n S_i$.

Definition 5.4. Let $u, v \in \prod_{i=1}^n F_{S_i}$, $w \in (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$, $\sum_{i=1}^n w_i = 1$, and $w_i \geq 0$. We define a binary relationship $<_F^w$ on the $\prod_{i=1}^n F_{S_i}$ as follows

$$<_F^w(u, v) = \begin{cases} \frac{\sum_{i=1}^n w_i (C_i^M(u, v) - C(u_i))}{\sum_{i=1}^n w_i |C(v_i) - C(u_i)|}, & \vec{C}(u) \neq \vec{C}(v), \\ \frac{1}{2}, & \vec{C}(u) = \vec{C}(v). \end{cases} \quad (5.2)$$

where, $C_i^M(u, v) = \max\{C(u_i), C(v_i)\} (i = 1, 2, \dots, n)$. For any $u, v \in \prod_{i=1}^n F_{S_i}$, if $<_F^w(u, v) = a$, we denote: $u <_a^w v$, i.e., the confidence of the fuzzy u is less than the fuzzy v is a .

Remark 5.2. The value $u <_a^w v \in [0, 1]$ represents the degree of confidence that u is less than v . The closer the value is to 1, the stronger the evidence that $u <_a^w v$; the closer to 0, the stronger the evidence that $v <_a^w u$. To determine the ranking order, we compare the bidirectional confidence values: If $v <_a^w u > u <_a^w v$, we conclude $v <_a^w u$ (i.e., u is superior to v).

Under the preparation of the above theory, we give the construction method of fuzzy numbers on the set S . The specific steps are as follows:

Assuming that the target(O) has n different feature attributes, we use O_i to present the i -th feature attribute, in which each feature attribute corresponds to the amount of an uncertain attribute (both continuous and discrete). For the different characteristic attributes of target O , we can get the following set of data by randomly observing it many times:

$$O_1 : \begin{matrix} o_{11} & o_{12} & \cdots & o_{1m} \\ o_{21} & o_{22} & \cdots & o_{2m} \\ \vdots & \vdots & & \vdots \\ o_{n1} & o_{n2} & \cdots & o_{nm} \end{matrix}$$

where, o_{ij} is the observation value of the j th ($j = 1, 2, \dots, m$) of the i th characteristic attribute of target O . In the following, we sort the target objects according to the fuzzy numbers on the set S .

Step 1: For each feature attribute O_i ($i = 1, 2, \dots, n$), we calculate the average (denoted as μ_i) and left and right dispersion (denoted as $L\sigma_i$ and $R\sigma_i$, respectively), i.e.,

$$\mu_i = \frac{1}{m} \sum_{k=1}^m o_{ik} \quad (5.3)$$

and

$$L\sigma_i = \frac{1}{N_{L_i}} \sum_{o_{ij} \leq \mu_i} (\mu_i - o_{ij}) \quad (5.4)$$

and

$$R\sigma_i = \frac{1}{N_{R_i}} \sum_{o_{ij} > \mu_i} (\mu_i - o_{ij}) \quad (5.5)$$

where, N_{L_i} and N_{R_i} represent the number of characteristic attributes o_{ij} that satisfy $o_{ij} \leq \mu_i$ and $o_{ij} > \mu_i$, respectively.

Step 2: According to the actual situation, for each i ($i = 1, 2, \dots, n$), give O_i a suitable value range D_i and $D_i \in S_i$, and a suitable parameter γ is selected.

For each i ($i = 1, 2, \dots, n$), we construct a 1-dimensional triangular fuzzy number, i.e.,

$$u_i(x) = \begin{cases} \frac{x - (\underline{\mu}_i - \gamma L\sigma_i)}{\gamma L\sigma_i}, & x \in [\underline{\mu}_i - \gamma L\sigma_i, \underline{\mu}_i]_{S_i} \cap D_i \\ 1, & x \in [\underline{\mu}_i, \bar{\mu}_i]_{S_i} \cap D_i \\ \frac{(\bar{\mu}_i + \gamma R\sigma_i) - x}{\gamma R\sigma_i}, & x \in [\bar{\mu}_i, \bar{\mu}_i + \gamma R\sigma_i]_{S_i} \cap D_i \\ 0, & x \notin [\underline{\mu}_i - \gamma L\sigma_i, \bar{\mu}_i + \gamma R\sigma_i]_{S_i} \end{cases} \quad (5.6)$$

where, $\underline{\mu}_i = \{x \in S_i : x \leq \mu_i\}$ and $\bar{\mu}_i = \{x \in S_i : x \geq \mu_i\}$.

Step 3: From the formula (1), we calculate the center of mass value of each feature attribute O_i , and then the center of mass vector $\vec{C}(u)$ of each target object O is obtained.

Step 4: Based on the different degrees of importance of each feature attribute, we assign the appropriate weight coefficient w_i ($\sum_{i=1}^n w_i = 1$) to each feature attribute, and then obtain the dynamic fuzzy order and confidence of the two target objects from formula (2).

In the following, based on the construction method of fuzzy numbers on the set of numbers proposed above, we will demonstrate the sorting of target objects characterized by multi-channel mixing uncertain numerical information through a specific example.

Example 5.1. According to the feedback of tourists, sort the hand and love degree of scenic sports A, B, C, D, E . In order to simplify the problem, we only consider the following two factors as the basis for ranking:

Factor 1 (E_1): The cumulative number of tourist during the National Day (Unit: Ten thousand times);

Factor 2 (E_2): Satisfaction with the access to the scenic spot (Unit: Percentage).

Taking scenic spot A as an example, we randomly selected 15 questionnaires and obtained 15 sets of

data as follows:

	E_1	E_2
Questionnaire 1:	61.8	93
Questionnaire 2:	67.4	90
Questionnaire 3:	66.8	94
Questionnaire 4:	66.6	95
Questionnaire 5:	62.5	98
Questionnaire 6:	61.3	92
Questionnaire 7:	66.3	90
Questionnaire 8:	65.0	93
Questionnaire 9:	61.7	84
Questionnaire 10:	62.0	89
Questionnaire 11:	59.2	86
Questionnaire 12:	66.1	98
Questionnaire 13:	61.9	93
Questionnaire 14:	71.4	96
Questionnaire 15:	60.6	87

In the following, we construct a 2-dimensional fuzzy vector on the set $\prod_{i=1}^2 S_i (S_1 = \mathbb{R}, S_2 = \mathbb{N})$ to present scenic spot A:

Step 1: From Formulas (5.3)–(5.5), we calculate:

$$\mu_{A_1} = 64.04, L\sigma_{A_1} = 2.67, R\sigma_{A_1} = 3.04$$

$$\mu_{A_2} = 91.87, L\sigma_{A_2} = 4.2, R\sigma_{A_2} = 2.8$$

Step 2: We choose the value interval of each factor according to the actual situation: $D_1 = [0, 100]_{\mathbb{R}}, D_2 = [0, 100]_{\mathbb{N}}$, and select the parameter $\gamma = 2$.

It is easy to know that the data in E_1 is continuous and the data in E_2 is discrete. From Formula (5.6), we construct their triangular fuzzy numbers as follows:

$$u_{A_1}(x) = \begin{cases} \frac{x-57.16}{5.34}, & x \in [57.16, 62.5], \\ 1, & x \in [62.5, 65], \\ \frac{71.8-x}{6.08}, & x \in [65, 71.8], \\ 0, & x \notin [57.16, 71.8], \end{cases}$$

$$u_{A_2}(x) = \begin{cases} \frac{x-81.6}{8.4}, & x \in \{82, 83, 84, 85, 86, 87, 88, 89\}, \\ 1, & x \in \{90, 91, 92\}, \\ \frac{71.8-x}{6.08}, & x \in \{93, 94, 95, 97\}, \\ 0, & x \notin [81.6, 97.6]_{\mathbb{N}}, \end{cases}$$

Step 3: Since the data in E_1 is continuous and the data in E_2 is discrete, we calculate the center of mass values separately.

For the continuous fuzzy number u_{A_1} , applying the centroid Formula (5.1) involves evaluating the definite integrals over the support $[57.16, 71.8]$: $C(u_{A_1}) = \frac{\int_{57.16}^{62.5} \frac{x-57.16}{5.34} dx + \int_{62.5}^{65} x dx + \int_{65}^{71.8} \frac{71.8-x}{6.08} dx}{\int_{57.16}^{62.5} \frac{x-57.16}{5.34} dx + \int_{62.5}^{65} 1 dx + \int_{65}^{71.8} \frac{71.8-x}{6.08} dx}$. Evaluating

these integrals, we obtain the numerator as 577.287 and the denominator as 8.973. Thus, $C(u_{A_1}) = \frac{577.287}{8.973}$. For the discrete fuzzy number u_{A_2} , we use the equivalent form $C(u) = \frac{\sum_{i=1}^n u(x_i)x_i}{\sum_{i=1}^n u(x_i)}$ of Formula (5.1).

By substituting the discrete data points from E_2 into this summation formula, we obtain $C(u_{A_2}) = 90.13$. Therefore, we obtain that the center of mass vector of scenic spot A is $\vec{C}(u_A) = (C(u_{A_1}), C(u_{A_2})) = (64.34, 90.13)$.

Similarly, $\vec{C}(u_B), \vec{C}(u_C), \vec{C}(u_D), \vec{C}(u_E)$ can be also obtained. Assuming $\vec{C}(u_B) = (76.48, 73.21)$; $\vec{C}(u_C) = (65.21, 69.85)$; $\vec{C}(u_D) = (71.10, 69.04)$; $\vec{C}(u_E) = (84.41, 90.55)$.

Step 4: We select the weight coefficient $w = (\frac{1}{2}, \frac{1}{2})$. Taking scenic spots A and B as an example, with $\vec{C}(u_A) = (64.34, 90.13)$ and $\vec{C}(u_B) = (76.48, 73.21)$, the binary relation value of A with respect to B is calculated directly by Formula (5.2) as follows: $C_1^M(u_A, u_B) = C_1^M(u_B, u_A) = \max\{C(u_{A_1}), C(u_{B_1})\} = 76.48$, $C_2^M(u_A, u_B) = C_2^M(u_B, u_A) = \max\{C(u_{A_2}), C(u_{B_2})\} = 90.13$. Thus,

$$\begin{aligned} \angle_F^w(u_A, u_B) &= \frac{w_1[C_1^M(u_A, u_B) - C(u_{A_1})] + w_2[C_2^M(u_A, u_B) - C(u_{A_2})]}{w_1|C(u_{B_1}) - C(u_{A_1})| + w_2|C(u_{B_2}) - C(u_{A_2})|} \\ &= \frac{0.5(76.48 - 64.34) + 0.5(90.13 - 90.13)}{0.5|76.48 - 64.34| + 0.5|73.21 - 90.13|} \\ &\approx 0.418 \\ \angle_F^w(u_B, u_A) &= \frac{w_1[C_1^M(u_B, u_A) - C(u_{B_1})] + w_2[C_2^M(u_B, u_A) - C(u_{B_2})]}{w_1|C(u_{A_1}) - C(u_{B_1})| + w_2|C(u_{A_2}) - C(u_{B_2})|} \\ &= \frac{0.5(76.48 - 76.48) + 0.5(90.13 - 73.21)}{0.5|64.34 - 76.48| + 0.5|90.13 - 73.21|} \\ &\approx 0.582 \end{aligned}$$

Therefore, according to Remark 5.2, since $\angle_F^w(u_B, u_A) > \angle_F^w(u_A, u_B)$, we conclude that $\angle_F^w(u_B, u_A)$ (i.e., $u_B \angle_{0.582}^w u_A$), which means the popularity of scenic spot A is higher than that of scenic spot B.

Similarly, applying the same calculation method, we obtain the binary relation values for the other pairs of scenic spots:

$$\begin{aligned} \angle_F^w(u_A, u_C) &= 0.041 (\angle_F^w(u_C, u_A) = 0.959); \\ \angle_F^w(u_A, u_D) &= 0.243 (\angle_F^w(u_D, u_A) = 0.757); \\ \angle_F^w(u_A, u_E) &= 1.000 (\angle_F^w(u_E, u_A) = 0.000); \\ \angle_F^w(u_B, u_C) &= 0.000 (\angle_F^w(u_C, u_B) = 1.000); \\ \angle_F^w(u_B, u_D) &= 0.000 (\angle_F^w(u_D, u_B) = 1.000); \\ \angle_F^w(u_B, u_E) &= 1.000 (\angle_F^w(u_E, u_B) = 0.000); \\ \angle_F^w(u_C, u_D) &= 0.879 (\angle_F^w(u_D, u_C) = 0.121); \\ \angle_F^w(u_C, u_E) &= 1.000 (\angle_F^w(u_E, u_C) = 1.000); \\ \angle_F^w(u_D, u_E) &= 1.000 (\angle_F^w(u_E, u_D) = 0.000). \end{aligned}$$

From the above calculation results, it can be seen that:

$$u_C \prec_{0.879}^w u_D \prec_{1.000}^w u_B \prec_{0.582}^w u_A \prec_{1.000}^w u_E.$$

That is, the popularity of the five scenic spots from low to high is: C, D, B, A, E .

6. Conclusions

First, this paper gives the definition of the fuzzy number on number set and its representation theorem, the a equivalent relationship between the continuous fuzzy number and the fuzzy number on $\mathbb{R}$, the conclusion that a fuzzy number is a continuous fuzzy number if and only if its support set is an interval in $\mathbb{R}$, the definition and two properties of the discrete set, and the conclusion that a fuzzy number on a number set is a discrete fuzzy number if and only if its support set is a discrete subset of $\mathbb{R}$. Second, this paper gives linear operation and multiplication operation of the fuzzy number on the number set, and on this basis, gives the mathematical logic relationship between the closure of the linear operation and multiplication operation of fuzzy numbers on the number set and the closure of the linear operation and multiplication operation of the number set. Finally, the paper gives the definition of the center of mass of the fuzzy number on the number set, the definition of the fuzzy number vector and the definition of the center of mass of fuzzy number vector, a binary relationship, gives specific implementation steps to rank the target object with complex characteristic attributes, and a specific example to present the above ranking steps.

Author contributions

Man Huang: Conceptualization, Methodology, Writing–original draft, Writing–review & editing; Guixiang Wang: Conceptualization, Methodology, Supervision, Writing–original draft, Writing–review & editing; Chengjia Li: Conceptualization, Methodology, Supervision, Writing–original draft, Writing–review & editing; Mengmeng Song: Methodology, Writing–original draft, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

A Project Supported by Scientific Research Fund of Zhejiang Provincial Education Department(Grant No. Y202558182).

Supported by Scientific Research Fund of Hangzhou Dianzi University Information Engineering College. No. KYP0324017.

Conflict of interest

No potential conflict of interest was reported by the authors.

References

1. L. A. Zadeh, Fuzzy set, *Information and Control*, **8** (1965), 338–353. [http://doi.org/10.1016/S0019-9958\(65\)90241-X](http://doi.org/10.1016/S0019-9958(65)90241-X)
2. N. O. Alshehri, R. S. Alghamdi, N. A. A. Qarni, Development of novel distance measures for picture hesitant fuzzy sets and their application in medical diagnosis, *AIMS Math.*, **10** (2025), 270–288. <http://doi.org/10.3934/math.2025013>
3. C. Kahraman, Proportional fuzzy set extensions and imprecise proportions, *Informatica*, **35** (2024), 311–339. <https://doi.org/10.15388/24-INFOR550>
4. S. Jana, J. Mahanta, Boundary of a fuzzy set and its application in GIS: a review, *Artif. Intell. Rev.*, **56** (2023), 6477–6507. <https://doi.org/10.1007/s10462-022-10331-0>
5. E. Karapnar, J. Martínez-Moreno, N. Shahzad, A. F. Roldán López de Hierro, Extended Proinov X-contractions in metric spaces and fuzzy metric spaces satisfying the property NC by avoiding the monotone condition, *RACSAM*, **116** (2022), 140. <https://doi.org/10.1007/s13398-022-01268-8>
6. A. Azam, M. N. A. Khan, N. Mehmood, S. Radenović, T. Došenović, Coincidence point of L-fuzzy sets endowed with graph, *RACSAM*, **112** (2018), 915–931. <https://doi.org/10.1007/s13398-017-0398-3>
7. S. S. L. Chang, L. A. Zadeh, On fuzzy mapping and control, *IEEE Trans. Syst. Man Cyber.*, **2** (1972), 30–34. <https://doi.org/10.1109/TSMC.1972.5408553>
8. C. V. Negoita, D. A. Ralescu, *Applications of fuzzy sets to systems analysis*, Basel: Birkhäuser, 1975. <https://doi.org/10.1007/978-3-0348-5921-9>
9. P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Set. Syst.*, **35** (1990), 241–249. [https://doi.org/10.1016/0165-0114\(90\)90197-E](https://doi.org/10.1016/0165-0114(90)90197-E)
10. G. X. Wang, C. X. Wu, Fuzzy n-cell numbers and the differential of fuzzy -cell number value mappings, *Fuzzy Set. Syst.*, **130** (2002), 367–381. [https://doi.org/10.1016/S0165-0114\(02\)00113-6](https://doi.org/10.1016/S0165-0114(02)00113-6)
11. N. Jayalakshmi, M. M. Shanmugapriya, Some new types of generalized fuzzy real numbers, *J. Intell. Fuzzy Syst.*, **46** (2024), 2179–2192. <https://doi.org/10.3233/JIFS-224320>
12. T. F. Wang, P. Shi, Constructing multistage linear fuzzy ellipsoid number to express multi-channel uncertain digital information and application, *J. Intell. Fuzzy Syst.*, **45** (2023), 2551–2563. <https://doi.org/10.3233/JIFS-222761>
13. Q. S. Zhang, D. F. Sun, Some notes on possibilistic variances of generalized trapezoidal intuitionistic fuzzy numbers, *AIMS Math.*, **6** (2021), 3720–3740. <https://doi.org/10.3934/math.2021221>
14. Z. Y. Jin, J. R. Wu, Ulam stability of two fuzzy number-valued functional equations, *AIMS Math.*, **5** (2020), 5055–5062. <https://doi.org/10.3934/math.2020324>
15. T. Saeed, M. B. Khan, S. Treanta, H. H. Alsulami, M. S. Alhodaly, Study of log convex mappings in fuzzy aunnam calculus via fuzzy inclusion relation over fuzzy-number space, *Mathematics*, **11** (2023), 2043. <https://doi.org/10.3390/math11092043>
16. M. B. Khan, H. A. Othman, G. Santos-García, M. A. Noor, M. S. Soliman, Some new concepts in fuzzy calculus for up and down A-convex fuzzy- number valued mappings and related inequalities, *AIMS Math.*, **8** (2023), 6777–6803. <https://doi.org/10.3934/math.2023345>

17. W. Voxman, Canonical representations of discrete fuzzy numbers, *Fuzzy Set. Syst.*, **118** (2001), 457–466. [https://doi.org/10.1016/S0165-0114\(99\)00053-6](https://doi.org/10.1016/S0165-0114(99)00053-6)
18. Q. Y. Qi, G. X. Wang, C. J. Li, Y. F. Xu, Ranking multichannel uncertain integer quantity degial informatiom based on discrete qussi-fuzzy, *ICIC Express Letter*, **14** (2020), 505–514. <https://doi.org/10.24507/icicel.14.05.505>
19. Z. T. Gong, N. Qin, G. C. Zhang, Three-dimensional generalized discrete fuzzy number and applications in color mathematical morphology, *IEEE Access*, **9** (2021), 25405–25421. <https://doi.org/10.1109/ACCESS.2021.3057623>
20. N. Qin, Z. T. Gong, Color image denoising by means of three-dimensional discrete fuzzy numbers, *Vis. Comput.*, **39** (2023), 2051–2063. <https://doi.org/10.1007/s00371-022-02464-8>
21. N. Qin, Z. T. Gong, Special discrete fuzzy numbers on countable sets and their applications, *Symmetry*, **16** (2024), 264. <https://doi.org/10.3390/sym16030264>

Appendix

A. Proof of $(u)_S^0 = \bigcup_{0 < r \leq 1} [u]_S^r$

If $x_0 \in (u)_S^0$, then $x_0 \in S$ and $u(x_0) > 0$. Thus, there exists $r_0 \in (0, 1]$ such that $u(x_0) = r_0$. So $x_0 \in [u]_S^{r_0}$ and $x_0 \in \bigcup_{0 < r \leq 1} [u]_S^r$. Therefore, $(u)_S^0 \subset \bigcup_{0 < r \leq 1} [u]_S^r$.

B. Proofs of $\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r) = \bigcup_{0 < r \leq 1} [u]_S^r + \bigcup_{0 < r \leq 1} [v]_S^r$ and $\overline{\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)} = \overline{\bigcup_{0 < r \leq 1} [u]_S^r} + \overline{\bigcup_{0 < r \leq 1} [v]_S^r}$

Suppose $x \in \bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)$, then there exists $r_0 \in (0, 1]$ such that $x \in [u]_S^{r_0} + [v]_S^{r_0}$, so there exist $y \in [u]_S^{r_0} \subset \bigcup_{0 < r \leq 1} [u]_S^r$ and $z \in [v]_S^{r_0} \subset \bigcup_{0 < r \leq 1} [v]_S^r$ such that $x = y + z$, then $x \in \bigcup_{0 < r \leq 1} [u]_S^r + \bigcup_{0 < r \leq 1} [v]_S^r$. Conversely, if $x \in \bigcup_{0 < r \leq 1} [u]_S^r + \bigcup_{0 < r \leq 1} [v]_S^r$, then there exist $y \in \bigcup_{0 < r \leq 1} [u]_S^r$ and $z \in \bigcup_{0 < r \leq 1} [v]_S^r$ such that $x = y + z$, so $\exists r', r'' \in (0, 1]$ such that $y \in [u]_S^{r'}, z \in [v]_S^{r''}$. If $r_0 = \min\{r', r''\}$, then $r_0 > 0$ and $y \in [u]_S^{r_0}, z \in [v]_S^{r_0}$, so $x = y + z \in [u]_S^{r_0} + [v]_S^{r_0} \subset \bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)$. Thus, $\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r) = \bigcup_{0 < r \leq 1} [u]_S^r + \bigcup_{0 < r \leq 1} [v]_S^r$ is true. Suppose $x \in \overline{\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)}$, then there exists $\{x_m\} \subset \bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)$ such that $x_m \rightarrow x$, so there exist $\{y_m\} \subset \bigcup_{0 < r \leq 1} [u]_S^r$ and $\{z_m\} \subset \bigcup_{0 < r \leq 1} [v]_S^r$ such that $y_m + z_m = x_m \rightarrow x$. By the boundary of $\{y_m\}$ and $\{z_m\}$, we can see there exist convergent subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and $\{z_{m_k}\}$ of $\{z_m\}$ such that $y_{m_k} \rightarrow y_0$ and $z_{m_k} \rightarrow z_0$, $y_0 + z_0 = x$ and $y_0 \in \overline{\bigcup_{0 < r \leq 1} [u]_S^r}, z_0 \in \overline{\bigcup_{0 < r \leq 1} [v]_S^r}$, thus $x \in \overline{\bigcup_{0 < r \leq 1} [u]_S^r} + \overline{\bigcup_{0 < r \leq 1} [v]_S^r}$. Conversely, as $x \in \overline{\bigcup_{0 < r \leq 1} [u]_S^r} + \overline{\bigcup_{0 < r \leq 1} [v]_S^r}$, so there exist $y \in \overline{\bigcup_{0 < r \leq 1} [u]_S^r}$ and $z \in \overline{\bigcup_{0 < r \leq 1} [v]_S^r}$ such that $x = y + z$, i.e., there exist $\{y_m\} \subset \bigcup_{0 < r \leq 1} [u]_S^r$ and $\{z_m\} \subset \bigcup_{0 < r \leq 1} [v]_S^r$ such that $\{y_m\} \rightarrow y$ and $\{z_m\} \rightarrow z$, thus $y_m + z_m \in \bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)$ and $y_m + z_m \rightarrow y + z = x$, so $x \in \overline{\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)}$. Therefore, $\overline{\bigcup_{0 < r \leq 1} ([u]_S^r + [v]_S^r)} = \overline{\bigcup_{0 < r \leq 1} [u]_S^r} + \overline{\bigcup_{0 < r \leq 1} [v]_S^r}$ is true.



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)