



*Research article***On Hermite-Hadamard-Mercer type fractional integral inequalities via a new integral identity****Sinan ASLAN***

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Abstract: This study establishes new fractional integral inequalities associated with a certain class of differentiable functions. A composite fractional integral identity is derived by applying the fractional integral identity from Lemma 2.1 to the subintervals of a uniform partition. This identity serves as the main analytical tool in the subsequent analysis. By systematically combining this identity with Jensen-Mercer, Hölder, power-mean, and Young-type inequalities, several refined Hermite-Hadamard (HH)-type estimates are established. The obtained results extend and unify several existing inequalities reported in the literature while yielding a variety of particular cases through appropriate choices of the parameters.

Keywords: Jensen-Mercer; fractional integrals; Hermite-Hadamard; Young inequalities; Hölder inequalities

Mathematics Subject Classification: 26A33, 26D15

1. Introduction

Integral inequalities associated with convex functions are an important topic in mathematical analysis, with applications in numerical integration, optimization, approximation theory, and fractional calculus. Fundamental inequalities such as Jensen's and Hermite-Hadamard (HH) inequalities have motivated numerous extensions in both classical and fractional settings.

Fractional calculus provides a useful framework for extending classical integral inequalities and obtaining more general estimates. In particular, HH-type inequalities involving Riemann-Liouville (RL) fractional integrals have been extensively studied under various generalized convexity assumptions [1]. Furthermore, the Jensen-Mercer (JM) inequality has been successfully incorporated into fractional analysis, leading to a broad class of Hermite-Hadamard-Mercer (HHM) inequalities.

Despite these developments, many existing fractional integral identities used for HHM inequalities rely mainly on endpoint information and single-interval formulations. Thus, the influence of interior

points is not fully reflected in such estimates.

Motivated by this gap, we establish a new fractional integral identity based on a systematic subdivision of the underlying interval. Unlike conventional formulations, the proposed identity incorporates multiple interior nodes and provides a more flexible representation of the associated fractional integral expressions. To the best of our knowledge, this particular construction has not been explicitly considered in the existing literature.

Using the proposed identity, we derive several new HHM-type fractional integral inequalities for differentiable functions whose derivatives satisfy convexity, power-convexity, and concavity conditions. The obtained results include several existing inequalities as special cases and provide new estimates arising from the multi-node structure. The proposed framework also offers potential directions for applications in mathematical modeling, optimization, engineering, and data-driven analysis, while extensions to other fractional operators and generalized convexity classes constitute promising topics for future research. For further details and recent developments on this topic, the reader is referred to the recent studies [2, 3].

2. Preliminaries

Let $0 < \xi_1 \leq \xi_2 \leq \dots \leq \xi_\delta$ and consider nonnegative weights $(\zeta_1, \zeta_2, \dots, \zeta_\delta)$ satisfying $\sum_{j=1}^\delta \zeta_j = 1$. A classical result in convex analysis, known as the Jensen inequality, asserts that for a convex function G defined on an interval containing ξ_δ , the following relation holds [4]:

$$G\left(\sum_{j=1}^\delta \zeta_j \xi_j\right) \leq \sum_{j=1}^\delta \zeta_j G(\xi_j).$$

Another fundamental pair of estimates in the theory of convex functions is given by the HH inequality. Specifically, if $G : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is convex on an interval I and $\eta, \mu \in I$ with $\eta < \mu$, then one has the double inequality

$$G\left(\frac{\eta + \mu}{2}\right) \leq \frac{1}{\mu - \eta} \int_\eta^\mu G(\xi) d\xi \leq \frac{G(\eta) + G(\mu)}{2}.$$

These inequalities play a central role in estimating integral averages of convex mappings and form a basis for many generalizations in both classical and fractional frameworks.

In the case where G is concave, the direction of both inequalities is reversed [5].

A significant refinement of the classical Jensen inequality was put forward by Mercer [6], who introduced an alternative formulation that extends its applicability.

Theorem 2.1. [6] *Let $G : [\eta, \mu] \rightarrow \mathbb{R}$ be a convex function. For $\xi_j \in [\eta, \mu]$ and weights $\zeta_j \in [0, 1]$ ($j = 1 \dots \delta$) satisfying $\sum_{j=1}^\delta \zeta_j = 1$, one has*

$$G\left(\eta + \mu - \sum_{j=1}^\delta \zeta_j \xi_j\right) \leq G(\eta) + G(\mu) - \sum_{j=1}^\delta \zeta_j G(\xi_j).$$

Recent literature contains extensive developments and generalizations pertaining to the Jensen-Mercer (JM) inequality across various functional and operator settings. From the perspective of classical convex analysis and means, generalized integral Mercer-type inequalities and extensions for

convex functions were established in [7, 8]. In the framework of Hilbert space operators, operator variants and refinements of the JM inequality were investigated by Matković et al. [9] and Kian and Moslehian [10]. With the emergence of fractional calculus, significant efforts have been dedicated to extending JM-type inequalities to diverse fractional operators. In particular, Hermite-Jensen-Mercer-type inequalities via k -fractional and generalized fractional operators were obtained in [11, 12], while Butt et al. [13] established related JM results within the fractal domain. Furthermore, Mercer-type inequalities incorporating ψ -Hilfer and ψ_k -Raina fractional integrals were developed in [14], and corresponding majorized discrete variants for Raina operators were analyzed in [15]. Complementary to these integral variants, new inclusions and JM-type inequalities for higher-order differentiable functions, as well as fractional Milne-type counterparts, have been explored in [16–18].

Fractional calculus, also known as fractional analysis, constitutes a well-developed and continuously advancing branch of mathematical analysis that extends the classical concepts of differentiation and integration to operators of arbitrary (non-integer) order. This extension provides a versatile and unified framework capable of describing complex dynamics and nonlocal phenomena, thereby enhancing both theoretical analysis and applied modeling. The conceptual foundations of this theory can be traced back to the late seventeenth century, through the correspondence between Gottfried Wilhelm Leibniz and Guillaume de l'Hôpital, where the notion of derivatives of non-integer order was first proposed. Although initially regarded as a purely formal or speculative idea, this concept has progressively evolved into a rigorous and systematically developed field, supported by a solid theoretical structure and a broad range of modern applications.

Significant progress was achieved in the nineteenth century through the contributions of Joseph Liouville and Bernhard Riemann, who established analytical formulations for fractional integrals and derivatives [19–21]. Their work laid the foundation for what is now known as the RL fractional operators. In the twentieth century, the definition introduced by Caputo [22] provided an alternative formulation that is particularly suitable for modeling real-world problems, as it allows the incorporation of physically interpretable initial conditions.

Over the past few decades, fractional calculus has attracted substantial and growing interest owing to its effectiveness in capturing memory effects and hereditary characteristics that naturally arise in a wide range of complex dynamical systems. This feature makes it especially valuable in diverse fields such as physics, engineering, biology, and finance. Applications involving viscoelastic materials, anomalous diffusion processes, and control systems demonstrate that fractional-order models often yield more accurate and flexible descriptions compared to their classical integer-order counterparts.

More recently, novel fractional operators with non-singular kernels have been introduced by Atangana and Baleanu [23]. These operators, commonly referred to as Atangana-Baleanu fractional operators, overcome certain limitations of classical definitions and have opened new directions for research, particularly in the study of integral inequalities, numerical methods, and optimization theory.

Motivated by these developments, the present study focuses on deriving new integral inequalities associated with fractional operators under various convexity assumptions. The results derived herein not only generalize and refine several existing inequalities reported in the literature, but also offer novel perspectives and broaden the scope of potential applications within the setting of fractional analysis.

After summarizing the importance and historical development of fractional analysis, the definition of the RL fractional integral operators, which serve as important analytical tools forming the basis of the developments carried out in this study, is written as follows.

Definition 2.1. [20] Let $G \in L[\eta, \mu]$. The RL integrals $J_{\eta^+}^\alpha G$ and $J_{\mu^-}^\alpha G$ of order $0 < \alpha$ and $0 \leq \eta$ are defined by

$$J_{\eta^+}^\alpha G(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{\xi} (\xi - \tau)^{\alpha-1} G(\tau) d\tau, \quad \xi > \eta,$$

and

$$J_{\mu^-}^\alpha G(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\xi}^{\mu} (\tau - \xi)^{\alpha-1} G(\tau) d\tau, \quad \xi < \mu,$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $J_{\eta^+}^0 G(\xi) = J_{\mu^-}^0 G(\xi) = G(\xi)$.

For more details, see [19, 21, 24].

Recent progress concerning fractional integral inequalities has been extensively investigated from various perspectives. For instance, Hermite-Hadamard and related midpoint-type inequalities via fractional operators were explored in [25–27]. Furthermore, generalizations involving higher-order differentiable convex mappings, s -convex functions, and coordinate-convex settings were studied in [28–30]. In addition, new parametrizations and discrete integral inequalities under diverse fractional frameworks have been addressed in [31–33].

Throughout this paper, all fractional integrals are understood in the RL sense as defined in Definition 2.1.

Lemma 2.1. [34] Let $G : [\eta, \mu] \rightarrow \mathbb{R}$ be a function that is differentiable on the open interval (η, μ) , where $\eta < \mu$. Assume further that its derivative G' is integrable on $[\eta, \mu]$; that is, $G' \in L[\eta, \mu]$. Under these conditions, the following identity involving fractional integrals is satisfied:

$$\begin{aligned} & \frac{G(\eta + \mu - \xi) + G(\eta + \mu - \nu)}{2} - \frac{\Gamma(\alpha + 1)}{2(\nu - \xi)^\alpha} \left[J_{(\eta + \mu - \nu)^+}^\alpha G(\eta + \mu - \xi) + J_{(\eta + \mu - \xi)^-}^\alpha G(\eta + \mu - \nu) \right] \\ &= \frac{\nu - \xi}{2} \int_0^1 (\tau^\alpha - (1 - \tau)^\alpha) G'(\eta + \mu - (\tau\xi + (1 - \tau)\nu)) d\tau \end{aligned}$$

for all $\xi, \nu \in [\eta, \mu]$, $\xi < \nu$, and $\alpha > 0$.

Motivated by the integral identities established by Dragomir and Agarwal [35] and the fractional integral approach introduced by Sarıkaya et al. [36] and by the fractional HHM-type identities obtained by Sarıkaya and Ögülmüş [34], we derive the following generalized fractional integral identity via subdivision of the interval.

Unlike the previously established HHM-type fractional integral identities, which are generally derived on a single interval structure, the identity obtained in this paper is constructed via a systematic subdivision of the interval into finitely many subintervals. This approach enables the derivation of multi-node fractional estimates and provides refined bounds involving intermediate evaluation points. Consequently, the obtained inequalities cannot be recovered directly from existing results by simple parameter selections and lead to new families of HHM-type inequalities as special consequences.

3. Main results

The following integral identity has been established by making a novel partition-based modification of Lemma 2.1.

For $\delta \in N, \delta \geq 1$, we define $I_\delta(G, \varsigma, \varrho)$ as the following composite fractional integral expression associated with the uniform partition of $[b_1, b_2]$ into δ subintervals.

Lemma 3.1. Let $G : [\varsigma, \varrho] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[\varsigma, \varrho]$, where $\varsigma < \varrho$. Let $b_1, b_2 \in [\varsigma, \varrho]$ satisfy $b_1 < b_2$ and $G \in AC[\varsigma, \varrho]$. Assume that $\delta \in N, \delta \geq 1, \lambda = 0, 1, \dots, \delta - 1$, and $\alpha > 0$. Then the following fractional integral identity holds:

$$\begin{aligned} I_\delta(G, \varsigma, \varrho) &= \sum_{\lambda=0}^{\delta-1} \left[G\left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right) + G\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right) \right] \\ &\quad - \frac{\delta^\alpha \Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} \left[\sum_{\lambda=0}^{\delta-1} J_{\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right)^-}^\alpha G\left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right) \right. \\ &\quad \left. + \sum_{\lambda=0}^{\delta-1} J_{\left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right)^+}^\alpha G\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right) \right] \\ &= \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{\delta} \int_0^1 ((1 - t)^\alpha - t^\alpha) \\ &\quad \times G'\left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right)\right) dt. \end{aligned}$$

Proof. If we take $\delta \in N$ arbitrarily, then for $\lambda \in \{0, 1, 2, \dots, \delta - 1\}$, we have

$$\begin{aligned} K &= \int_0^1 ((1 - t)^\alpha - t^\alpha) \times G'\left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right)\right) dt \\ &= \frac{\delta((1 - t)^\alpha - t^\alpha) G\left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right)\right)}{b_1 - b_2} \Big|_0^1 \\ &\quad - \frac{\delta\alpha}{b_2 - b_1} \int_0^1 ((1 - t)^{\alpha-1} + t^{\alpha-1}) \\ &\quad \times G\left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right)\right) dt. \end{aligned}$$

By making use of the substitution $x = \varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right)$,

$$\begin{aligned} K &= \frac{\delta}{b_2 - b_1} \left[G\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right) + G\left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right) \right] \\ &\quad - \frac{\delta^{\alpha+1} \alpha \Gamma(\alpha)}{(b_2 - b_1)^{\alpha+1} \Gamma(\alpha)} \int_{\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}}^{\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}} \left\{ \left(x - \left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right)\right)^{\alpha-1} \right. \\ &\quad \left. + \left(\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right) - x\right)^{\alpha-1} \right\} G(x) dx \\ &= \frac{\delta}{b_2 - b_1} \left[G\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right) + G\left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right) \right] \end{aligned}$$

$$-\frac{\delta^{\alpha+1}\Gamma(\alpha+1)}{(b_2-b_1)^{\alpha+1}}\left\{J_{\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)^-}^{\alpha}G\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)+J_{\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)^+}^{\alpha}G\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)\right\}.$$

Multiplying both sides by $\frac{b_2-b_1}{\delta}$, we have

$$\begin{aligned}\frac{b_2-b_1}{\delta}K &= G\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)+G\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right) \\ &\quad -\frac{\delta^{\alpha}\Gamma(\alpha+1)}{(b_2-b_1)^{\alpha}}\left[J_{\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)^-}^{\alpha}G\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)\right. \\ &\quad \left.+J_{\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)^+}^{\alpha}G\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)\right].\end{aligned}$$

Finally, summing over λ from 0 to $\delta-1$, we have

$$\begin{aligned}\sum_{\lambda=0}^{\delta-1}\frac{b_2-b_1}{\delta}K &= \sum_{\lambda=0}^{\delta-1}\left[G\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)+G\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)\right] \\ &\quad -\frac{\delta^{\alpha}\Gamma(\alpha+1)}{(b_2-b_1)^{\alpha}}\left[\sum_{\lambda=0}^{\delta-1}J_{\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)^-}^{\alpha}G\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)\right. \\ &\quad \left.+\sum_{\lambda=0}^{\delta-1}J_{\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)^+}^{\alpha}G\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)\right].\end{aligned}$$

Lemma 3.1 provides a composite fractional integral identity obtained by applying the fractional integral identity in Lemma 2.1 to the subintervals of a uniform partition. This identity serves as the main analytical tool for establishing the subsequent inequalities presented in the remainder of the paper.

Remark 3.1. [34, Lemma 3.1] If we choose $\delta = 1$ in Lemma 3.1, and multiply both sides by $\frac{1}{2}$, we obtain the following:

$$\begin{aligned}&\frac{G(\varsigma+\varrho-b_2)+G(\varsigma+\varrho-b_1)}{2}-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\left[J_{(\varsigma+\varrho-b_1)^-}^{\alpha}G(\varsigma+\varrho-b_2)+J_{(\varsigma+\varrho-b_2)^+}^{\alpha}G(\varsigma+\varrho-b_1)\right] \\ &= \frac{b_2-b_1}{2}\int_0^1((1-t)^{\alpha}-t^{\alpha})G'(\varsigma+\varrho-(tb_2+(1-t)b_1))dt.\end{aligned}$$

Remark 3.2. [37, Lemma 1] If we choose $\alpha = 1$ in Lemma 3.1 and multiply both sides by $\frac{1}{2\delta}$, we obtain the following:

$$\begin{aligned}&\sum_{\lambda=0}^{\delta-1}\frac{1}{2\delta}\left[G\left(\varsigma+\varrho-\frac{(\delta-\lambda)b_1+\lambda b_2}{\delta}\right)+G\left(\varsigma+\varrho-\frac{(\delta-\lambda-1)b_1+(\lambda+1)b_2}{\delta}\right)\right] \\ &\quad -\frac{1}{b_2-b_1}\int_{\varsigma+\varrho-b_2}^{\varsigma+\varrho-b_1}G(x)dx\end{aligned}$$

$$= \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{2\delta^2} \times \int_0^1 (1-2t) G' \left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1-t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) \right) dt.$$

Remark 3.3. [34, Corollary 3.2] Taking $\alpha = 1$ and $\delta = 1$ in Lemma 3.1, the corresponding expression reduces to the following form:

$$\begin{aligned} & \frac{G(\varsigma + \varrho - b_2) + G(\varsigma + \varrho - b_1)}{2} - \frac{1}{b_2 - b_1} \int_{\varsigma + \varrho - b_2}^{\varsigma + \varrho - b_1} G(x) dx \\ &= \frac{(b_2 - b_1)}{2} \int_0^1 (1-2t) G'(\varsigma + \varrho - (tb_2 + (1-t)b_1)) dt. \end{aligned}$$

Remark 3.4. [36, Lemma 2] If we choose $\delta = 1$, $b_2 = \varrho$ and $b_1 = \varsigma$ in Lemma 3.1, then we obtain

$$\begin{aligned} & \frac{G(\varsigma) + G(\varrho)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varrho - \varsigma)^\alpha} \left[J_{\varrho^-}^\alpha G(\varsigma) + J_{\varsigma^+}^\alpha G(\varrho) \right] \\ &= \frac{\varrho - \varsigma}{2} \int_0^1 ((1-t)^\alpha - t^\alpha) G'(t\varsigma + (1-t)\varrho) dt. \end{aligned}$$

Since $\varsigma < \varrho$, the substitution $b_2 = \varrho$ and $b_1 = \varsigma$ preserves the orientation of the interval. Hence, the resulting expressions are taken over the interval $[\varsigma, \varrho]$, and no reversal of the integration direction occurs.

Remark 3.5. [35, Lemma 2.1] If we choose $\alpha = \delta = 1$, $b_2 = \varrho$ and $b_1 = \varsigma$ in Lemma 3.1, then we obtain

$$\frac{G(\varsigma) + G(\varrho)}{2} - \frac{1}{\varrho - \varsigma} \int_{\varsigma}^{\varrho} G(x) dx = \frac{\varrho - \varsigma}{2} \int_0^1 (1-2t) G'(t\varsigma + (1-t)\varrho) dt.$$

Corollary 3.1. If we choose $\delta = 2$ in Lemma 3.1, we have

$$\begin{aligned} & G(\varsigma + \varrho - b_1) + 2G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + G(\varsigma + \varrho - b_2) \\ & - \frac{2^\alpha \Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} \left[J_{(\varsigma + \varrho - b_1)^-}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^+}^\alpha G(\varsigma + \varrho - b_1) \right. \\ & \left. + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^-}^\alpha G(\varsigma + \varrho - b_2) + J_{(\varsigma + \varrho - b_2)^+}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) \right] \\ &= \frac{b_2 - b_1}{2} \left\{ \int_0^1 ((1-t)^\alpha - t^\alpha) G'\left(\varsigma + \varrho - \left(t \frac{b_1 + b_2}{2} + (1-t)b_1\right)\right) dt \right. \\ & \left. + \int_0^1 ((1-t)^\alpha - t^\alpha) G'\left(\varsigma + \varrho - \left(tb_2 + (1-t) \frac{b_1 + b_2}{2}\right)\right) dt \right\}. \end{aligned}$$

Theorem 3.1. Let $G : [\varsigma, \varrho] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[\varsigma, \varrho]$, where $\varsigma < \varrho$. Let $b_1, b_2 \in [\varsigma, \varrho]$ satisfy $b_1 < b_2$. Assume that $|G'|$ is convex on $[\varsigma, \varrho]$. Furthermore, let $\alpha > 0, \delta \in \mathbb{N}, \delta \geq 1$, and $\lambda = 0, 1, \dots, \delta - 1$. Then the following inequality holds:

$$|I_\delta(G, \varsigma, \varrho)| = \left| \sum_{\lambda=0}^{\delta-1} \left\{ G\left(\varsigma + \varrho - \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta}\right) + G\left(\varsigma + \varrho - \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta}\right) \right\} \right|$$

$$\begin{aligned}
& - \sum_{\lambda=0}^{\delta-1} \frac{\delta^\alpha \Gamma(\alpha+1)}{(b_2 - b_1)^\alpha} \left[J_{\left(\varsigma + \varrho - \frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta}\right)^-}^\alpha G\left(\varsigma + \varrho - \frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta}\right) \right. \\
& \quad \left. + J_{\left(\varsigma + \varrho - \frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta}\right)^+}^\alpha G\left(\varsigma + \varrho - \frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta}\right) \right] \\
& \leq \frac{(b_2 - b_1)(2^\alpha - 1)}{2^{\alpha-1}(\alpha+1)} \sum_{\lambda=0}^{\delta-1} \frac{1}{\delta} \\
& \quad \times \left\{ |G'(\varsigma)| + |G'(\varrho)| - \frac{\left| G'\left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta}\right) \right| + \left| G'\left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta}\right) \right|}{2} \right\}.
\end{aligned}$$

Proof. Integrating Lemma 3.1 and the JM inequality, we establish

$$\begin{aligned}
|I_\delta(G, \varsigma, \varrho)| & \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \int_0^1 |(1-t)^\alpha - t^\alpha| \right. \\
& \quad \times \left| G'\left(\varsigma + \varrho - \left(t \frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} + (1-t) \frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta}\right)\right) \right| dt \Big\} \\
& \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left[\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha) \right] \left[|G'(\varsigma)| + |G'(\varrho)| \right. \right. \\
& \quad \left. \left. - \left(t \left| G'\left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta}\right) \right| + (1-t) \left| G'\left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta}\right) \right| \right) \right] dt \Big\} \\
& = \frac{(b_2 - b_1)(2^\alpha - 1)}{2^{\alpha-1}(\alpha+1)} \sum_{\lambda=0}^{\delta-1} \frac{1}{\delta} \\
& \quad \times \left(|G'(\varsigma)| + |G'(\varrho)| - \frac{\left| G'\left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta}\right) \right| + \left| G'\left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta}\right) \right|}{2} \right).
\end{aligned}$$

Using the symmetry of the weight function $|(1-t)^\alpha - t^\alpha|$ with respect to $t = \frac{1}{2}$, we obtain

$$\begin{aligned}
\int_0^1 t |(1-t)^\alpha - t^\alpha| dt &= \int_0^1 (1-t) |(1-t)^\alpha - t^\alpha| dt \\
&= \frac{1}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| dt,
\end{aligned}$$

where

$$\int_0^1 |(1-t)^\alpha - t^\alpha| dt = \frac{2^\alpha - 1}{2^{\alpha-1}(\alpha+1)}.$$

Theorem 3.1 yields a family of composite HHM-type fractional integral inequalities parameterized by δ . Different choices of δ produce different composite estimates, and several known results are recovered as particular cases.

Remark 3.6. [37, Corollary 6] Setting $\delta = 1$ and $\alpha = 1$ in Theorem 3.1 yields the following result:

$$\begin{aligned} & \left| \frac{G(\varsigma + \varrho - b_2) + G(\varsigma + \varrho - b_1)}{2} - \frac{1}{b_2 - b_1} \int_{\varsigma + \varrho - b_2}^{\varsigma + \varrho - b_1} G(x) dx \right| \\ & \leq \frac{b_2 - b_1}{4} \left\{ |G'(\varsigma)| + |G'(\varrho)| - \frac{|G'(b_2)| + |G'(b_1)|}{2} \right\}. \end{aligned}$$

Remark 3.7. [35, Theorem 2.2] If we choose $\alpha = \delta = 1$, $b_2 = \varrho$, and $b_1 = \varsigma$ in Theorem 3.1, then we obtain

$$\left| \frac{G(\varsigma) + G(\varrho)}{2} - \frac{1}{\varrho - \varsigma} \int_{\varsigma}^{\varrho} G(x) dx \right| \leq \frac{(\varrho - \varsigma) (|G'(b_2)| + |G'(b_1)|)}{8}.$$

Remark 3.8. [34, Theorem 3.6] If we choose $\delta = 1$ in Theorem 3.1, then we obtain

$$\begin{aligned} & \left| \frac{G(\varsigma + \varrho - b_2) + G(\varsigma + \varrho - b_1)}{2} - \frac{\Gamma(\alpha + 1)}{2(b_2 - b_1)^\alpha} \left[J_{(\varsigma + \varrho - b_1)^-}^\alpha G(\varsigma + \varrho - b_2) + J_{(\varsigma + \varrho - b_2)^+}^\alpha G(\varsigma + \varrho - b_1) \right] \right| \\ & \leq \frac{(b_2 - b_1)(2^\alpha - 1)}{2^\alpha(\alpha + 1)} \left\{ |G'(\varsigma)| + |G'(\varrho)| - \frac{|G'(b_2)| + |G'(b_1)|}{2} \right\}. \end{aligned}$$

Remark 3.9. [36] Setting $\delta = 1$, $b_2 = \varrho$ and $b_1 = \varsigma$ in Theorem 3.1 yields the following result:

$$\left| \frac{G(\varsigma) + G(\varrho)}{2} - \frac{\Gamma(\alpha + 1)}{2(\varrho - \varsigma)^\alpha} J_{\varrho^-}^\alpha G(\varsigma) + J_{\varsigma^+}^\alpha G(\varrho) \right| \leq \frac{(\varrho - \varsigma)(2^\alpha - 1)}{2^\alpha(\alpha + 1)} \left\{ \frac{|G'(\varsigma)| + |G'(\varrho)|}{2} \right\}.$$

Corollary 3.2. Setting $\delta = 2$ in Theorem 3.1 yields the following result:

$$\begin{aligned} & \left| G(\varsigma + \varrho - b_1) + 2G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + G(\varsigma + \varrho - b_2) \right. \\ & \quad - \frac{2^\alpha \Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} \left[J_{(\varsigma + \varrho - b_1)^-}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^-}^\alpha G(\varsigma + \varrho - b_2) \right. \\ & \quad \left. \left. + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^+}^\alpha G(\varsigma + \varrho - b_1) + J_{(\varsigma + \varrho - b_2)^+}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) \right] \right| \\ & \leq \frac{(b_2 - b_1)(2^\alpha - 1)}{2^{\alpha-1}(\alpha + 1)} \left\{ |G'(\varsigma)| + |G'(\varrho)| - \frac{|G'(b_1)| + 2\left|G'\left(\frac{b_1 + b_2}{2}\right)\right| + |G'(b_2)|}{4} \right\}. \end{aligned}$$

Corollary 3.3. Setting $b_2 = \varrho$ and $b_1 = \varsigma$ in Theorem 3.1 yields the following result:

$$\begin{aligned} & \left| \sum_{\lambda=0}^{\delta-1} \left\{ G\left(\frac{(\delta - \lambda - 1)\varrho + (\lambda + 1)\varsigma}{\delta}\right) + G\left(\frac{(\delta - \lambda)\varrho + \lambda\varsigma}{\delta}\right) \right\} \right. \\ & \quad \left. - \sum_{\lambda=0}^{\delta-1} \frac{\delta^\alpha \Gamma(\alpha + 1)}{(\varrho - \varsigma)^\alpha} \left[J_{\left(\frac{(\delta - \lambda)\varrho + \lambda\varsigma}{\delta}\right)^-}^\alpha G\left(\frac{(\delta - \lambda - 1)\varrho + (\lambda + 1)\varsigma}{\delta}\right) + J_{\left(\frac{(\delta - \lambda - 1)\varrho + (\lambda + 1)\varsigma}{\delta}\right)^+}^\alpha G\left(\frac{(\delta - \lambda)\varrho + \lambda\varsigma}{\delta}\right) \right] \right| \\ & \leq \frac{(\varrho - \varsigma)(2^\alpha - 1)}{2^{\alpha-1}(\alpha + 1)} \sum_{\lambda=0}^{\delta-1} \frac{1}{\delta} \times \left(|G'(\varsigma)| + |G'(\varrho)| - \frac{|G'\left(\frac{(\delta - \lambda - 1)\varrho + (\lambda + 1)\varsigma}{\delta}\right)| + |G'\left(\frac{(\delta - \lambda)\varrho + \lambda\varsigma}{\delta}\right)|}{2} \right). \end{aligned}$$

Since $\varsigma < \varrho$ and $0 < \lambda \leq \delta - 1$, all the nodes appearing in the summation belong to the interval $[\varsigma, \varrho]$.

Theorem 3.2. Let $G : [\varsigma, \varrho] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[\varsigma, \varrho]$, where $\varsigma < \varrho$. Let $b_1, b_2 \in [\varsigma, \varrho]$ satisfy $b_1 < b_2$. Assume that $|G'|^q$ is convex on $[\varsigma, \varrho]$. Furthermore, let $\alpha > 0, \delta \in \mathbb{N}, \delta \geq 1$, and $\lambda = 0, 1, \dots, \delta - 1$. Then the following inequality holds:

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left(\frac{2^{p\alpha} - 1}{2^{p\alpha-1}(p\alpha + 1)} \right)^{\frac{1}{p}} \\ \times \left(|G'(\varsigma)|^q + |G'(\varrho)|^q - \frac{\left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q}{2} \right)^{\frac{1}{q}},$$

where $1 < p, 1 < q$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 3.1 together with Hölder's inequality, we establish

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \right. \\ \left. \times \left(\int_0^1 \left| G' \left(\varsigma + \varrho - \left(t \frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} + (1-t) \frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q dt \right)^{\frac{1}{q}} \right\}.$$

Using the JM inequality, we have

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)^p dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)^p dt \right)^{\frac{1}{p}} \right. \\ \times \int_0^1 (|G'(\varsigma)|^q + |G'(\varrho)|^q \\ \left. - \left(t \left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + (1-t) \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right) dt \right)^{\frac{1}{q}} \right\}.$$

Since for any $0 \leq \varsigma < \varrho$ in $\mathbb{R}$ and a fixed $1 \leq p$, $(\varrho - \varsigma)^p \leq \varrho^p - \varsigma^p$, we have

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left(\int_0^{\frac{1}{2}} ((1-t)^{p\alpha} - t^{p\alpha}) dt + \int_{\frac{1}{2}}^1 (t^{p\alpha} - (1-t)^{p\alpha}) dt \right)^{\frac{1}{p}} \right. \\ \times \int_0^1 (|G'(\varsigma)|^q + |G'(\varrho)|^q \\ \left. - \left(t \left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + (1-t) \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right) dt \right)^{\frac{1}{q}} \right\} \\ = \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left(\frac{2^{p\alpha} - 1}{2^{p\alpha-1}(p\alpha + 1)} \right)^{\frac{1}{p}}$$

$$\times \left(|G'(\varsigma)|^q + |G'(\varrho)|^q - \frac{\left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q}{2} \right)^{\frac{1}{q}}.$$

Using the symmetry of the weight function $|(1-t)^\alpha - t^\alpha|$ with respect to $t = \frac{1}{2}$, we obtain

$$\int_0^1 t|(1-t)^\alpha - t^\alpha| dt = \int_0^1 (1-t)|(1-t)^\alpha - t^\alpha| dt = \frac{1}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| dt.$$

Theorem 3.3. Let $G : [\varsigma, \varrho] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[\varsigma, \varrho]$, where $\varsigma < \varrho$. Let $b_1, b_2 \in [\varsigma, \varrho]$ satisfy $b_1 < b_2$. Assume that $|G'|^q$ is convex on $[\varsigma, \varrho]$. Furthermore, let $\alpha > 0, \delta \in \mathbb{N}, \delta \geq 1$, and $\lambda = 0, 1, \dots, \delta - 1$. Then the following inequality holds:

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{\delta} \left\{ \frac{2^{p\alpha} - 1}{2^{p\alpha-1} p(p\alpha + 1)} \right. \\ \left. + \frac{2 \left(|G'(\varsigma)|^q + |G'(\varrho)|^q \right) - \left(\left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right)}{2q} \right\},$$

where $1 < p, 1 < q$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 3.1 together with Young's inequality, we establish

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \frac{1}{p} \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right) \right. \\ \left. + \frac{1}{q} \left(\int_0^1 \left| G' \left(\varsigma + \varrho - \left(t \frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} + (1-t) \frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q dt \right) \right\}.$$

Using the JM inequality, we have

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \frac{1}{p} \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)^p dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)^p dt \right) \right. \\ \left. + \frac{1}{q} \int_0^1 \left(|G'(\varsigma)|^q + |G'(\varrho)|^q \right. \right. \\ \left. \left. - \left(t \left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + (1-t) \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right) dt \right) \right\}.$$

Since for any $0 \leq \varsigma < \varrho$ in $\mathbb{R}$ and a fixed $1 \leq p$, $(\varrho - \varsigma)^p \leq \varrho^p - \varsigma^p$, we have

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \frac{1}{p} \left(\int_0^{\frac{1}{2}} ((1-t)^{p\alpha} - t^{p\alpha}) dt + \int_{\frac{1}{2}}^1 (t^{p\alpha} - (1-t)^{p\alpha}) dt \right) \right. \\ \left. + \frac{1}{q} \int_0^1 \left(|G'(\varsigma)|^q + |G'(\varrho)|^q \right) \right.$$

$$\begin{aligned}
& - \left(t \left| G' \left(\frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} \right) \right|^q + (1 - t) \left| G' \left(\frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right) dt \Bigg\} \\
& = \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{\delta} \left\{ \frac{2^{p\alpha} - 1}{2^{p\alpha-1} p (p\alpha + 1)} \right. \\
& \quad \left. + \frac{2 \left(|G'(\varsigma)|^q + |G'(\varrho)|^q \right) - \left(\left| G' \left(\frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} \right) \right|^q + \left| G' \left(\frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right)}{2q} \right\}.
\end{aligned}$$

Corollary 3.4. Setting $\delta = 2$ in Theorem 3.3 yields the following result:

$$\begin{aligned}
& \left| G(\varsigma + \varrho - b_1) + 2G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + G(\varsigma + \varrho - b_2) \right. \\
& \quad - \frac{2^\alpha \Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} \left[J_{(\varsigma + \varrho - b_1)^-}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^-}^\alpha G(\varsigma + \varrho - b_2) \right. \\
& \quad \left. \left. + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^+}^\alpha G(\varsigma + \varrho - b_1) + J_{(\varsigma + \varrho - b_2)^+}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) \right] \right| \\
& \leq \frac{b_2 - b_1}{2} \left\{ \frac{2(2^{p\alpha} - 1)}{2^{p\alpha-1} p (p\alpha + 1)} \right. \\
& \quad \left. + \frac{4 \left(|G'(\varsigma)|^q + |G'(\varrho)|^q \right) - \left(|G'(b_1)|^q + 2 \left| G' \left(\frac{b_1 + b_2}{2} \right) \right|^q + |G'(b_2)|^q \right)}{2q} \right\}.
\end{aligned}$$

Theorem 3.4. Let $G : [\varsigma, \varrho] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[\varsigma, \varrho]$, where $\varsigma < \varrho$. Let $b_1, b_2 \in [\varsigma, \varrho]$ satisfy $b_1 < b_2$. Assume that $|G'|^q$ is convex on $[\varsigma, \varrho]$. Furthermore, let $\alpha > 0, \delta \in \mathbb{N}, \delta \geq 1$, and $\lambda = 0, 1, \dots, \delta - 1$. Then the following inequality holds:

$$\begin{aligned}
|I_\delta(G, \varsigma, \varrho)| & \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)(2^\alpha - 1)}{\delta 2^{\alpha-1}(\alpha + 1)} \\
& \quad \times \left(\left(|G'(\varsigma)|^q + |G'(\varrho)|^q \right) - \left(\frac{\left| G' \left(\frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q + \left| G' \left(\frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} \right) \right|^q}{2} \right) \right)^{\frac{1}{q}},
\end{aligned}$$

where $q > 1$.

Proof. Applying Lemma 3.1 together with the power-mean inequality and taking absolute values, we establish

$$\begin{aligned}
|I_\delta(G, \varsigma, \varrho)| & \leq \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{\delta} \left\{ \left(\int_0^1 |(1 - t)^\alpha - t^\alpha| dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 |(1 - t)^\alpha - t^\alpha| \right. \right. \\
& \quad \left. \left| G' \left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Using the JM inequality, we have

$$\begin{aligned}
 |I_\delta(G, \varsigma, \varrho)| &\leq \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{\delta} \left\{ \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha) dt \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left(\left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha) \right) (|G'(\varsigma)|^q + |G'(\varrho)|^q \right. \\
 &\quad \left. \left. - \left(t \left| G' \left(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} \right) \right|^q + (1-t) \left| G' \left(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q \right) dt \right)^{\frac{1}{q}} \right\} \\
 &\leq \sum_{\lambda=0}^{\delta-1} \frac{b_2 - b_1}{\delta} \left\{ \left(\frac{2^\alpha - 1}{2^{\alpha-1}(\alpha+1)} \right)^{1-\frac{1}{q}} \left(\frac{2^\alpha - 1}{2^{\alpha-1}(\alpha+1)} \right. \right. \\
 &\quad \times \left. \left. \left(|G'(\varsigma)|^q + |G'(\varrho)|^q - \left(\frac{|G'(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta})|^q + |G'(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta})|^q}{2} \right) \right) \right)^{\frac{1}{q}} \right\} \\
 &= \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)(2^\alpha - 1)}{\delta 2^{\alpha-1}(\alpha+1)} \\
 &\quad \times \left((|G'(\varsigma)|^q + |G'(\varrho)|^q) - \left(\frac{|G'(\frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta})|^q + |G'(\frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta})|^q}{2} \right) \right)^{\frac{1}{q}}.
 \end{aligned}$$

Using the symmetry of the weight function $|(1-t)^\alpha - t^\alpha|$ with respect to $t = \frac{1}{2}$, we obtain

$$\int_0^1 t |(1-t)^\alpha - t^\alpha| dt = \int_0^1 (1-t) |(1-t)^\alpha - t^\alpha| dt = \frac{1}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| dt.$$

Theorem 3.5. Let $G : [\varsigma, \varrho] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[\varsigma, \varrho]$, where $\varsigma < \varrho$. Let $b_1, b_2 \in [\varsigma, \varrho]$ satisfy $b_1 < b_2$. Assume that $|G'|^q$ is concave on $[\varsigma, \varrho]$. Furthermore, let $\alpha > 0, \delta \in \mathbb{N}, \delta \geq 1$, and $\lambda = 0, 1, \dots, \delta-1$. Then the following inequality holds:

$$|I_\delta(G, \varsigma, \varrho)| \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left(\frac{2^{p\alpha} - 1}{2^{p\alpha-1}(p\alpha+1)} \right)^{\frac{1}{p}} \left(\left| G' \left(\varsigma + \varrho - \frac{(2\delta - 2\lambda - 1)b_1 + (2\lambda + 1)b_2}{2\delta} \right) \right|^q \right)^{\frac{1}{q}},$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Applying Lemma 3.1 together with Hölder's inequality and taking absolute values, we establish

$$\begin{aligned}
 |I_\delta(G, \varsigma, \varrho)| &\leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \right. \\
 &\quad \times \left. \left(\int_0^1 \left| G' \left(\varsigma + \varrho - \left(t \frac{(\delta-\lambda-1)b_1 + (\lambda+1)b_2}{\delta} + (1-t) \frac{(\delta-\lambda)b_1 + \lambda b_2}{\delta} \right) \right|^q dt \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Since $|G'|^q$ is concave, by Jensen's integral inequality, we have

$$\begin{aligned} & \int_0^1 \left| G' \left(\varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) \right) \right|^q dt \\ & \leq \left| G' \left(\int_0^1 \varsigma + \varrho - \left(t \frac{(\delta - \lambda - 1)b_1 + (\lambda + 1)b_2}{\delta} + (1 - t) \frac{(\delta - \lambda)b_1 + \lambda b_2}{\delta} \right) dt \right) \right|^q \\ & = \left| G' \left(\varsigma + \varrho - \frac{(2\delta - 2\lambda - 1)b_1 + (2\lambda + 1)b_2}{2\delta} \right) \right|^q. \end{aligned}$$

$$\begin{aligned} |I_\delta(G, \varsigma, \varrho)| & \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left(\int_0^{\frac{1}{2}} ((1 - t)^\alpha - t^\alpha)^p dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1 - t)^\alpha)^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \times \left. \left(\left| G' \left(\varsigma + \varrho - \frac{(2\delta - 2\lambda - 1)b_1 + (2\lambda + 1)b_2}{2\delta} \right) \right|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since for any $0 \leq \varsigma < \varrho$ in $\mathbb{R}$ and a fixed $1 \leq p$, $(\varrho - \varsigma)^p \leq \varrho^p - \varsigma^p$, we have

$$\begin{aligned} |I_\delta(G, \varsigma, \varrho)| & \leq \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left\{ \left(\int_0^{\frac{1}{2}} ((1 - t)^{p\alpha} - t^{p\alpha}) dt + \int_{\frac{1}{2}}^1 (t^{p\alpha} - (1 - t)^{p\alpha}) dt \right)^{\frac{1}{p}} \right. \\ & \quad \times \left. \left(\left| G' \left(\varsigma + \varrho - \frac{(2\delta - 2\lambda - 1)b_1 + (2\lambda + 1)b_2}{2\delta} \right) \right|^q \right)^{\frac{1}{q}} \right\} \\ & = \sum_{\lambda=0}^{\delta-1} \frac{(b_2 - b_1)}{\delta} \left(\frac{2^{p\alpha} - 1}{2^{p\alpha-1}(p\alpha + 1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\left| G' \left(\varsigma + \varrho - \frac{(2\delta - 2\lambda - 1)b_1 + (2\lambda + 1)b_2}{2\delta} \right) \right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 3.5. *Setting $\delta = 2$ in Theorem 3.5 yields the following result:*

$$\begin{aligned} & \left| G(\varsigma + \varrho - b_1) + 2G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + G(\varsigma + \varrho - b_2) \right. \\ & \quad - \frac{2^\alpha \Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} \left[J_{(\varsigma + \varrho - b_1)^-}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^-}^\alpha G(\varsigma + \varrho - b_2) \right. \\ & \quad \left. \left. + J_{(\varsigma + \varrho - \frac{b_1 + b_2}{2})^+}^\alpha G(\varsigma + \varrho - b_1) + J_{(\varsigma + \varrho - b_2)^+}^\alpha G\left(\varsigma + \varrho - \frac{b_1 + b_2}{2}\right) \right] \right| \\ & \leq \frac{b_2 - b_1}{2} \left(\frac{2^{p\alpha} - 1}{2^{p\alpha-1}(p\alpha + 1)} \right)^{\frac{1}{p}} \times \left\{ \left(\left| G' \left(\varsigma + \varrho - \frac{3b_1 + b_2}{4} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| G' \left(\varsigma + \varrho - \frac{b_1 + 3b_2}{4} \right) \right|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

4. Comparison with existing composite quadrature rules

The case $\delta = 1$ corresponds to the single-interval formulation, whereas $\delta > 1$ leads to a composite representation involving additional interior nodes. Therefore, increasing δ does not necessarily imply a uniformly smaller error bound for every function and parameter choice. The main advantage of the composite formulation is that it provides multi-node estimates and allows the contribution of intermediate points to be incorporated into the resulting fractional integral inequalities. The numerical behavior of the bounds for different values of δ is illustrated in Examples 4.1 and 4.2.

Example 4.1. Consider the function $G(x) = x^2$ on the interval $[0, 1]$. Taking $\alpha = 1$, $\varsigma = 0$, $\varrho = 1$, $b_1 = 0$, and $b_2 = 1$, all the assumptions of Theorem 3.1 are satisfied since $G'(x) = 2x$ is convex on $[0, 1]$.

Solution 1. The left-hand side is computed as

$$|I_\delta(G, 0, 1)| = \left| \sum_{\lambda=0}^{\delta-1} \left\{ G\left(1 - \frac{\lambda+1}{\delta}\right) + G\left(1 - \frac{\lambda}{\delta}\right) \right\} - \delta \sum_{\lambda=0}^{\delta-1} \left[J_{(1-\frac{\lambda}{\delta})^-} G\left(1 - \frac{\lambda+1}{\delta}\right) + J_{(1-\frac{\lambda+1}{\delta})^+} G\left(1 - \frac{\lambda}{\delta}\right) \right] \right|.$$

The right-hand side is computed as

$$RHS = \frac{1}{2} \sum_{\lambda=0}^{\delta-1} \frac{1}{\delta} \left\{ |G'(0)| + |G'(1)| - \frac{|G'(\frac{\lambda+1}{\delta})| + |G'(\frac{\lambda}{\delta})|}{2} \right\}.$$

From this, we obtain Table 1 shown below.

Table 1. Numerical verification of the inequality for $G(x) = x^2$.

δ	$ I_\delta(G, 0, 1) $	RHS	$RHS - I_\delta(G, 0, 1) $
1	1/3	1/2	1/6
2	1/6	1/2	1/3
3	1/9	1/2	7/18
4	1/12	1/2	5/12
5	1/15	1/2	13/30
...	...	...	...
δ	$1/3\delta$	1/2	$(1/2)-(1/3\delta)$

Table 1 clearly demonstrates that the established inequality holds for each value of δ . Moreover, as the partition parameter δ increases, the left-hand side $|I_\delta(G, 0, 1)|$ decreases toward zero while remaining strictly bounded by the right-hand side, which confirms the theoretical validity and consistency of our result.

Example 4.2. Consider the function $G(x) = e^x$ on the interval $[0, 1]$. Taking $\alpha = 1$, $\varsigma = 0$, $\varrho = 1$, $b_1 = 0$, and $b_2 = 1$, all the assumptions of Theorem 3.1 are satisfied since $G'(x) = e^x$ is convex on $[0, 1]$.

Solution 2. The left-hand side is computed as

$$|I_\delta(G, 0, 1)| = \left| \sum_{\lambda=0}^{\delta-1} \left\{ G\left(1 - \frac{\lambda+1}{\delta}\right) + G\left(1 - \frac{\lambda}{\delta}\right) \right\} - \sum_{\lambda=0}^{\delta-1} \delta \left[J_{(1-\frac{\lambda}{\delta})^-} G\left(1 - \frac{\lambda+1}{\delta}\right) + J_{(1-\frac{\lambda+1}{\delta})^+} G\left(1 - \frac{\lambda}{\delta}\right) \right] \right|.$$

The right-hand side is computed as

$$RHS = \frac{1}{2} \sum_{\lambda=0}^{\delta-1} \frac{1}{\delta} \times \left\{ |G'(0)| + |G'(1)| - \frac{|G'(\frac{\lambda+1}{\delta})| + |G'(\frac{\lambda}{\delta})|}{2} \right\}.$$

From this, we obtain Table 2 shown below.

Table 2. Numerical verification of the inequality for $G(x) = e^x$.

δ	$ I_\delta(G, 0, 1) $	RHS	$RHS - I_\delta(G, 0, 1) $
1	0.281718	0.929570	0.647852
2	0.142597	0.982175	0.839578
3	0.095283	0.992059	0.896776
4	0.071520	0.995529	0.924009
5	0.057237	0.997138	0.939901

The numerical outcomes presented in Table 2 clearly validate the analytical correctness of Theorem 3.1 for the convex test function $G(x) = e^x$. It is readily observed that the left-hand side $|I_\delta(G, 0, 1)|$ decreases monotonically toward zero as the partition parameter δ increases from 1 to 5. Simultaneously, the right-hand side RHS steadily approaches 1, ensuring that the difference $RHS - |I_\delta(G, 0, 1)|$ remains strictly positive and expands across all refinements. This monotonic decay and persistent bounding behavior numerically corroborate the sharpness and stability of the established fractional integral inequality.

5. Special cases and connections

The derived inequalities recover various classical HH-type and fractional integral inequalities as special cases via appropriate parameter choices. In particular, setting $\delta = 1$ recovers identities consistent with previously established results in the literature, whereas choosing $\delta > 1$ yields composite Hermite-Hadamard-Mercer-type fractional integral inequalities associated with uniform partitions. These results provide a unified framework that includes several existing inequalities as particular cases. Moreover, several corollaries can be obtained by specifying the parameters suitably, thereby illustrating the flexibility and applicability of the proposed framework. Consequently, the present results unify and extend several earlier contributions within a single general setting, thereby broadening the applicability of existing Hermite-Hadamard-Mercer-type fractional integral inequalities.

6. Conclusions

In this paper, we established several Hermite-Hadamard-Mercer-type fractional integral inequalities for differentiable functions under generalized convexity assumptions. The results are based on a composite fractional integral identity obtained from Lemma 2.1 over a uniform partition, together with Jensen-Mercer, Hölder, power-mean, and Young-type inequalities.

The obtained inequalities extend several existing Hermite-Hadamard-Mercer-type fractional integral inequalities and recover various known results as special cases. The composite partition framework also provides greater flexibility by incorporating information from intermediate points of the underlying interval, which constitutes the main novelty of the proposed approach.

The applicability of the obtained results depends on the regularity, integrability, and convexity assumptions imposed on the functions and their derivatives. Therefore, the inequalities may not directly apply when these conditions are not satisfied. This limitation can be addressed by considering broader function classes and alternative fractional operators.

The proposed framework may also have potential applications in mathematical modeling, optimization, engineering analysis, and data-driven methods where fractional integral estimates are relevant. Future research may focus on extending the present results to other fractional operators, multidimensional and coordinate settings, and broader generalized convexity classes. Numerical investigations and application-oriented studies may also provide further insight into the effectiveness and scope of the proposed inequalities.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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