



*Research article***Constructing pseudo-Anosov maps from integer permutations****John Hubbard¹, Ahmad Rafiqi^{2,*} and Tom Schang³**¹ Department of Mathematics, Cornell University, 310 Malott Hall, Ithaca, 14853, New York, USA² Department of Mathematics & Statistics, American University of Sharjah, 26666, Sharjah, UAE³ Department of Mathematics, UC Berkeley, 970 Evans Hall, Berkeley, 94720, California, USA*** Correspondence:** Email: arafiqi@aus.edu; Tel: +97165154032.

Abstract: Surface homeomorphisms, particularly pseudo-Anosov maps, are central objects in topology and dynamics. We provide a characterization of orientation-preserving pseudo-Anosov homeomorphisms with orientable foliations that fix all critical trajectories, encoding them as admissible permutations of $2g + \nu - 1$ positive integers, where g is the genus and ν the number of singularities. Conversely, from any admissible permutation we construct the surface, measured foliations, and the pseudo-Anosov map. This construction realizes every Abelian differential invariant under some pseudo-Anosov homeomorphism.

Keywords: pseudo-Anosov; surface homeomorphism; permutation; entropy; algebraic integer**Mathematics Subject Classification:** 37E30, 05A05, 37B40

1. Introduction

The mapping classes of compact topological surfaces were classified by [1] into three types: periodic, reducible and pseudo-Anosov. The latter of these contains a unique representative, a PSEUDO-ANOSOV map (pA in short) of a surface S , which is specified by a pair of foliations on S invariant under f , and a real number $\lambda > 1$ called the STRETCH-FACTOR that the two foliations are respectively stretched and shrunk by, under f . The two foliations are singular, have transverse measures, and are transverse to each other; for details, we refer the reader to [1]. The stretch-factor is an important topological invariant of the mapping class, $\log(\lambda)$ being the topological entropy of f (see [2]), as well as the length of the geodesic corresponding to the mapping class $[f]$ in the moduli space of complete Riemannian metrics on S under the Teichmüller metric (see e.g., the book [3]). In [4], Fried showed that λ is an algebraic unit in the ring of algebraic integers of the number field $\mathbb{Q}(\lambda)$. Moreover, he shows that λ is BI-PERRON, meaning all its Galois conjugates μ_i satisfy $\{1/\lambda \leq |\mu_i| \leq \lambda\}$ with at most one conjugate on each boundary component.

It is an open problem to determine precisely what algebraic units occur as pseudo-Anosov stretch-factors, but this is an active area with lots of current research, see for instance [5–11]. It is this question on the precise nature of the stretch-factors that motivated the current paper. Although many constructions of pseudo-Anosov maps exist in the literature, (e.g., [12–14], to name just a few), we believe a characterization of pseudo-Anosov maps in terms of integers will lend itself more easily to finding the algebraic integers that arise as their stretch-factors.

Algebraic numbers $\lambda > 1$ whose Galois conjugates μ_i satisfy $|\mu_i| < \lambda$ are called **PERRON**. [15] showed that all Perron (and thus also bi-Perron) numbers are eigenvalues of integer **APERIODIC** matrices, that is, non-negative matrices with some strictly positive power, thus proving a converse to the classical Perron-Frobenius theorem. Using this, [2] showed that logarithms of Perron numbers correspond exactly to the topological entropies of interval maps whose critical points have finite forward orbits.

The present paper builds on work done by one of the authors in a previous paper [16] that constructed orientation preserving or reversing pA maps from certain matrices of 0's and 1's. However, unlike [16], which only constructs hyperelliptic examples with one or two singularities in each genus, our construction applies to all oriented-fixed (OF) pseudo-Anosov maps regardless of singularity structure. This generality is essential for addressing the stretch-factor realization problem.

Thurston's classification covers all compact surfaces, however we will assume our surfaces to be orientable and without boundary. Further, we restrict to orientation preserving maps, and since the pseudo-Anosov stretch-factors are our motivation, we may assume that the foliations of the map are orientable. Indeed, if this is not the case, the orientation double cover of the foliation may be taken, one with two points above every non-singular point, each with one of the two orientations of the leaf of the foliation through that point. The map then lifts to this branched double cover and the lifted map has the same stretch-factor. The advantage of doing so is that when the foliations are orientable, the induced isomorphism f_* on $H_1(S; \mathbb{Z})$ has as largest eigenvalue the stretch-factor λ of $[f]$.

The pair of invariant foliations of a pA map being orientable is equivalent to the foliations being the integral curves of the real and imaginary parts of a holomorphic 1-form ω (i.e., an Abelian differential) on a Riemann surface structure on the surface (see [3, Proposition 5.3.4]). This implies (but is not equivalent to) each singularity of the foliations having an even number of prongs. The pA homeomorphism fixes the set of zeros of ω , while it multiplies its imaginary and real parts by λ and λ^{-1} respectively. We will call the foliation whose leaves are scaled by λ under f (resp. by λ^{-1}) as the **EXPANDING** (resp. **CONTRACTING**) foliation. A **CRITICAL TRAJECTORY** of a foliation is a connected component of a singular leaf minus the singular point(s) it contains. Thus each singularity of multiplicity m has $2(m+1)$ critical trajectories, $m+1$ incoming and $m+1$ outgoing. We will be concerned with orientation-preserving homeomorphisms which fix each critical trajectory of the foliations.

Definition 1. A pseudo-Anosov homeomorphism f is *OF* if its invariant foliations are orientable, and such that all the singularities and all the critical trajectories emanating from the singularities are fixed by f .

We use a well-known decomposition of the surface carrying a pA map, known as zippered rectangles (cf. [14, 17]), albeit modified to the simpler setting of our OF maps. Given such an f , we choose a finite segment of a critical trajectory of the contracting foliation, terminating at a singularity. This choice uniquely determines a decomposition of the surface into finitely many rectangles with identifications along the boundaries as we will see below. The decomposition as well as the pA map can then be

encoded as a finite permutation σ of size $n = 2g + \nu - 1$, and a list of positive integers $\mathbf{k} = (k_1, \dots, k_n)$, where ν is the number of distinct singularities of the invariant foliations. The pair $(\sigma, \mathbf{k})$ defines an ORDERED BLOCK PERMUTATION (OBP, Definition 3), and it satisfies a condition we call ADMISSIBLE (Definition 4). Our main result (cf. Theorem 5) is the following.

Theorem 1. *A choice of a critical segment of the contracting foliation of an OF pseudo-Anosov map uniquely determines an admissible OBP. Conversely, an admissible OBP uniquely determines a closed oriented surface with a pair of transverse orientable foliations with measures unique up to scale, and an OF pseudo-Anosov map preserving the foliations.*

We note that while the OBP depends on the choice of a critical trajectory of the contracting foliation and on the length of the segment chosen, different choices of length produce finitely many different OBPs, (see §2.5). Since there are only finitely many singular trajectories to choose from, we can only obtain a finite number of different OBPs for a given OF pseudo-Anosov map.

The paper is organized as follows: In Section 2, we recall the zippered rectangle decomposition of the surface carrying a pseudo-Anosov map, modified to our setting. In Section 3, we see how an OF pA map has associated to it a permutation of a list of integers that describe the OBP. In Section 4, we study the data of an OBP purely combinatorially and without reference to a surface, and define admissibility. In Section 5, we study the general shape of the rectangular decomposition, and in Section 6, we provide the construction of the surface and the OF pA map from an admissible OBP and prove the main theorem (Theorem 5).

2. Rectangular decomposition

2.1. Invariant holomorphic 1-form of a pseudo-Anosov map

In this section, let S be a compact oriented surface (without boundary) of genus $g \geq 2$, and $f : S \rightarrow S$ be an orientation-preserving pseudo-Anosov homeomorphism of S with orientable foliations.

As is well-known from the works [17, 18], the expanding and contracting foliations of f can be thought of as the integral curves of the imaginary and real parts of a holomorphic 1-form $\omega \in \Omega^1(X)$, (also known as an ABELIAN DIFFERENTIAL), with respect to a Riemann surface structure $[\phi : S \rightarrow X]$. The pA map f preserves the zero set of ω and scales the real and imaginary parts,

$$f^*(\operatorname{Re} \omega) = \lambda \operatorname{Re} \omega \quad \text{and} \quad f^*(\operatorname{Im} \omega) = \frac{1}{\lambda} \operatorname{Im} \omega. \quad (2.1)$$

In a neighborhood U of a point $p \in X$, coordinate charts $z : U \rightarrow \mathbb{C}$ can be found such that in the coordinate z on U , $\omega_p = z^m dz$. The multiplicity m is zero except at a finite number of points of X , which correspond to the singularities. At a zero of multiplicity m , there are $2(m + 1)$ critical trajectories for each of the expanding and contracting foliations. Half of these, $m + 1$, are incoming and half outgoing. There is at least one singularity - in fact the sum of the multiplicities of the zeros of the holomorphic 1-form ω equals $2g - 2 = |\chi(S)|$, where $\chi(S)$ is the Euler characteristic.

2.2. Zippered rectangles

We now describe the rectangular decomposition of X associated to a pA map with oriented foliations. This construction appears in a lot of places in the literature, see for instance [17] or Proposition 5.3.4 in [3]. Choose a segment $\tilde{J}$ of any positive length of a critical leaf of the contracting foliation terminating at a singularity $z_0 \in X$. This is the only choice involved. Now any critical trajectory of the expanding foliation, being dense in the surface and transverse to the contracting foliation, eventually intersects $\tilde{J}$. So lead each critical trajectory of the expanding foliation from each zero of ω (including z_0) until the trajectory intersects $\tilde{J}$ and then stop. Let z_1 be the point where the final expanding critical trajectory intersects $\tilde{J}$, and let $J = [z_0, z_1]$. Call Γ the union of all these critical expanding segments. As is tradition, we will be drawing the expanding foliation horizontally and the contracting foliation vertically, oriented to the right and upwards, respectively.

We will assume (after possibly reversing the orientation of the horizontal foliation) that the final horizontal trajectory to meet J (at z_1) intersects J from the left. Continue this last horizontal trajectory past z_1 (the dashed edge in Figure 1) till it meets J again, and add it to Γ . We call this dashed edge the NON-SINGULAR EDGE, as it contains no singularity, simply going from J back to J . All other horizontal edges drawn from J lead to singularities. We make the assumption that the dashed edge is on the right of z_1 solely for brevity of exposition; the other possibility of it being on the left is very similar, see Remark 1 below. Moreover, after having chosen J , one can ensure the dashed edge is on the right side of J by reversing the orientation of the expanding foliation.

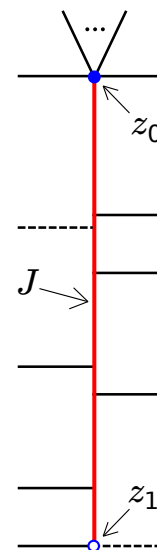


Figure 1. J in red.

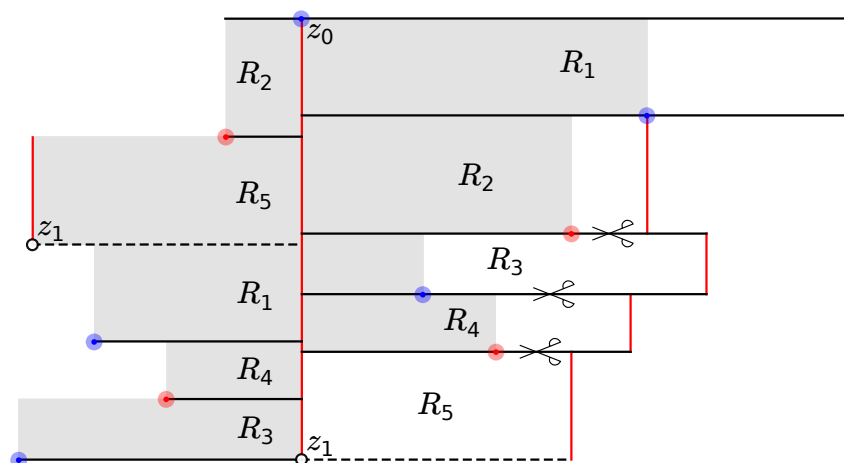


Figure 2. A genus 2 surface with two simple zeros (red and blue) has $2g + \nu - 1 = 5$ rectangles, R_1 to R_5 drawn on the right of J , which cover the whole surface. To the right of each singularity there are slits (indicated by the scissors in the picture). If instead each rectangle is shaded on both the left and right of J until the singularity on its bottom edge, the shaded region also covers the surface and contains no slits in its interior. Gluing along the boundary can be determined by the order (σ) in which the rectangles are attached on the left.

The components of the complement of $\Gamma \cup J$ in X are all metric rectangles for the Euclidean metric coming from the 1-form ω , namely the metric $dx^2 + dy^2$ in the coordinate charts where $\omega = dz = dx + idy$. The rectangles alternately have horizontal sides (on Γ) and vertical sides (on J). Let $\mathcal{R}$ be the set of these rectangles. Each horizontal edge of each rectangle contains a singularity of ω , except the bottom edge of the last rectangle to the right of J .

Number the rectangles in $\mathcal{R}$ as $R_1, \dots, R_n$ as they are ordered on the right of J . This way R_1 is the rectangle with z_0 as its top-left corner, and R_n has the non-singular point z_1 as its bottom-left corner. The rectangles cover the whole surface, $\bigcup_{i=1}^n R_i = X$, since otherwise the rectangles would form a subsurface whose boundary consists entirely of horizontal leaves, and pA foliations don't have any closed horizontal leaves.

2.3. The image under $\mathbf{f}$ of the zippered rectangles

The pseudo-Anosov f stretches each R_i horizontally by the stretch-factor λ and shrinks it vertically by the same factor. Since each R_i has its two vertical sides on J and since $f(J)$ is the subinterval $J' := [z_0, f(z_1)] \subset J$, of length $|J'| = |J|/\lambda$, the thinner longer rectangle $f(R_i)$ must begin and end on J' . Beginning with its left edge on J' , the image $f(R_i)$ thus passes end-to-end within some number of rectangles, before ending with its right vertical edge on J' again. Moreover, if $f(R_i)$ contained one of the drawn critical horizontal segments in its interior, R_i would contain that same critical horizontal segment in its interior as well, since f is a homeomorphism fixing each critical trajectory - but that is impossible since the R_i 's are the complement of the drawn critical trajectories by construction. As a result, the image rectangles $f(R_i)$, $i = 1, \dots, n$, form thinner rectangular strips, stacked within the rectangles R_i without overlap, and each strip going end-to-end in a single rectangle. We call these image sub-rectangle strands, (see Figure 3).

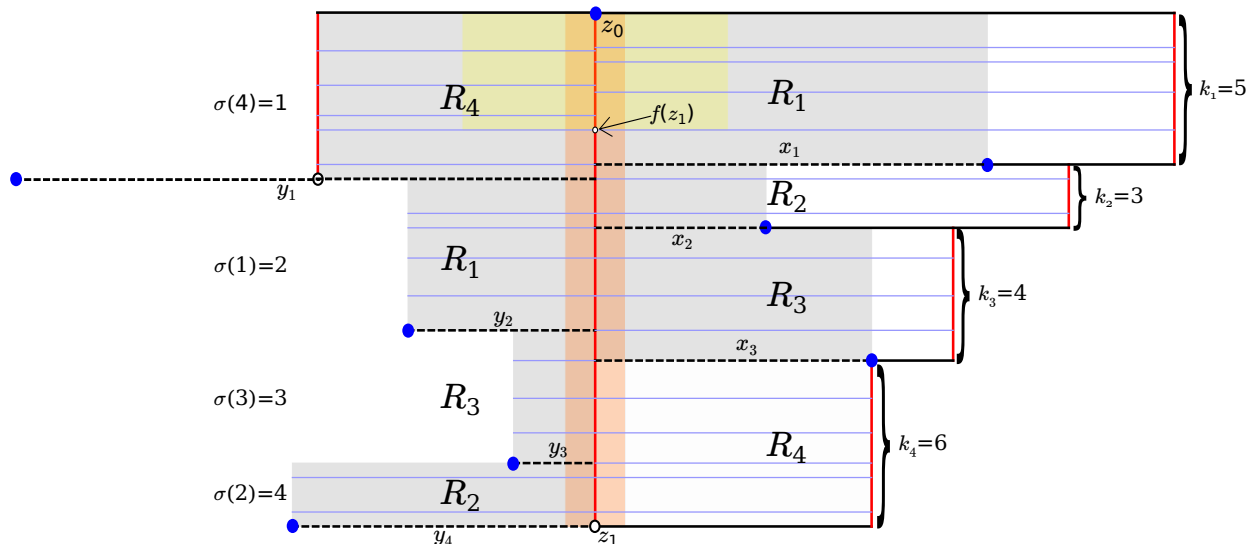


Figure 3. A genus 2 surface with one six-pronged singularity has 4 rectangles in its decomposition. The highlighted neighborhood of J has image the wider neighborhood of $f(J)$. The blue lines bound the images of the R_i , the strands. Counting the strands in each rectangle gives $\mathbf{k}$, and the order of the R_i on the left defines σ . Here $\mathbf{k} = (5, 3, 4, 6)$ so $K = 18$, and $\sigma(1, 2, 3, 4) = (2, 4, 3, 1)$.

Definition 2. In the rectangular decomposition of an oriented-fixed map using a critical vertical segment J , and critical horizontal segments Γ drawn on the surface as above, a STRAND is a connected component of $X \setminus (J \cup f(\Gamma))$.

Except for the top-most n strands, the strands to the left and right of the contracting segment J match up exactly. That is, if K is the total number of strands, for each $n < q \leq K$, the right edge of the q^{th} strand to the left of J is identified with the left edge of the q^{th} strand to the right of J , (cf. Figure 3).

We can then define an $n \times n$ matrix A of non-negative integers whose $(i, j)^{\text{th}}$ entry a_{ij} is the number of times $f(R_j)$ crosses R_i . Namely, a_{ij} is the number of strands in R_i that belong to the image $f(R_j)$. The i^{th} column of A encodes the image of rectangle R_i . Suppose R_i has horizontal length l_i and vertical height h_i , measured using the invariant $\omega = dx + idy$.

Proposition 2. The leading eigenvalue λ of A is the stretch-factor of f . The vectors

$$\text{of lengths } \mathbf{l} = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix} \quad \text{and of heights } \mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

are eigenvectors of A^T and A respectively, each with eigenvalue λ :

$$A^T \mathbf{l} = \lambda \mathbf{l} \quad \text{and} \quad A \mathbf{h} = \lambda \mathbf{h}.$$

Proof. For a fixed j , consider the image of R_j , which has length λl_j . This image is made up of strands, precisely a_{ij} strands in each R_i , which have length l_i . Hence, the sum $\sum_i a_{ij} l_i$ is exactly the length of $f(R_j)$, i.e., $\lambda l_j = \sum_i a_{ij} l_i$. Thus, $\lambda \mathbf{l} = A^T \mathbf{l}$.

Similarly, $\sum_j a_{ij} (h_j / \lambda)$ is the sum of the heights of the $f(R_j)$ crossing R_i , which has height h_i . This leads to $h_i = \sum_j a_{ij} (h_j / \lambda)$. Thus, $\lambda \mathbf{h} = A \mathbf{h}$. $\square$

2.4. The number of rectangles $\mathbf{n} = 2\mathbf{g} + \mathbf{v} - 1$

Suppose the Riemann surface X of genus $g > 1$ with an OF pA map f has been decomposed into rectangles $R_1, \dots, R_n$, after having chosen a critical vertical segment. Suppose the ν singularities of the foliations have multiplicities $m_1, \dots, m_\nu$, where each $m_i \geq 1$.

By construction, for each $i = 1, \dots, (n-1)$, the intersection $\overline{R_i} \cap \overline{R_{i+1}}$ contains the critical horizontal edge x_i . We will label these edges as INCOMING critical trajectories. Similarly, to the left of J , for each $i = 1, \dots, n$, there is a critical horizontal segment drawn at the bottom of the i^{th} rectangle. These emanate from singularities, and terminate at J , so we will call these OUTGOING critical trajectories, $y_1, \dots, y_n$. Note that the edge $y_{\sigma(n)}$ below R_n on the left of J is also the edge at the bottom of the figure.

So we have drawn $n-1$ incoming horizontal segments. On the other hand, at a zero of multiplicity m_i , there are $m_i + 1$ incoming horizontal trajectories. Since the sum of multiplicities is $m_1 + \dots + m_\nu = 2g - 2$, we see that

$$n - 1 = \sum_{i=1}^{\nu} (m_i + 1) = 2g - 2 + \nu, \quad \text{whence, } \boxed{n = 2g + \nu - 1} \quad (2.2)$$

Note that $g \geq 2$ implies $n \geq 4$. And since $1 \leq \nu \leq 2g - 2$, we obtain bounds on n and g in terms of each other

$$2g \leq n \leq 4g - 3 \quad \text{and} \quad \left\lceil \frac{n+3}{4} \right\rceil \leq g \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (2.3)$$

2.5. Finitely many OBPs

In this section, we show that the number of possible OBPs that could be associated to a given OF pseudo-Anosov map is finite. Assume an OBP has been computed using a vertical segment J , as in Figure 3.

First, suppose instead of J , we had chosen the segment $f(J)$, of length $|J|/\lambda$, to decompose the surface. The resulting decomposition would be a horizontally stretched version of the decomposition along J . Hence, it has the same OBP associated to it. Therefore, by changing the length of J continuously, there is a periodically changing list of OBPs obtained.

Next, note that the OBPs change discretely when one continuously varies the length of J : If J is made slightly shorter, the lowest critical horizontal edge meeting J is lost, and since n remains constant, some horizontal edge needs to be drawn further till it hits J again on the left. Then, J is shortened to the new lowest point of contact with a critical horizontal edge and we may also need to switch the orientation of the expanding foliation and reflect the picture across a vertical line.

Similarly, if instead of J we choose $J_2 \subset J$ to decompose the surface, where $|J|/\lambda < |J_2| < |J|$, some critical horizontal segments have to be drawn further until they hit J_2 . However, the new critical horizontal segments drawn are a subset of $f(\Gamma)$, i.e. they are part of the horizontal boundaries of the strands that occur in the decomposition with J . Since there are finitely many strands (K), we see that the length of the critical vertical segment is shortened to at most $K - n$ different points when choosing a segment J_2 of length between $|J|$ and $|J|/\lambda$. Together with the fact that OBPs change periodically with the length of J , we see that there are finitely many different OBPs associated to a fixed incoming critical trajectory.

Finally, since there are $n - 1 = 2g + \nu - 2$ incoming critical trajectories to choose from, we see that the number of different OBPs that could be associated to a given OF pseudo-Anosov map is finite.

3. The combinatorial data of an oriented-fixed pseudo-Anosov

3.1. The permutation σ and the integers $(k_1, \dots, k_n)$

We can define the permutation σ by saying that the i^{th} rectangle on the right side of J , namely R_i , is glued in position $\sigma(i)$ on the left side of J . Thus, if seen to the left of J , the top-most rectangle is $R_{\sigma^{-1}(1)}$, and R_1 is glued on the left of J in position $\sigma(1)$.

As discussed above Definition 2, the image rectangles $f(R_i)$, $i = 1, \dots, n$, form thinner rectangular strands, stacked within the rectangles R_i without overlap, and each strand going end-to-end in a single rectangle. We then define k_i to be the number of strands that pass through the rectangle R_i .

Together, the permutation σ and the ordered list of positive integers $\mathbf{k} = (k_1, \dots, k_n)$ consist of the combinatorial data that we use to encode oriented-fixed pseudo-Anosov maps.

3.2. Ordered block permutations

To work with the combinatorial data $(\sigma, \mathbf{k})$, we number the strands as they are ordered on the right side of J , from 1 to $K := k_1 + \dots + k_n$. The order in which the strands appear on the left of J defines a permutation τ of the bigger set $\mathbb{N}_K = \{1, \dots, K\}$. Figure 4 shows the numbered strands for the example in Figure 3, where for instance $\tau(1) = 7$ since the 1st strand is attached in the 7th position on the left of J . This permutation is determined by σ and $\mathbf{k} = (k_1, \dots, k_n)$, and is what we define below as the

ordered block permutation $\tau = \tau_{\sigma, \mathbf{k}}$, named so due to its structure of permuting blocks of numbers together. For its definition, we first define the “block-number” function $\beta : \mathbb{N}_K \rightarrow \mathbb{N}_n$, defining $\beta(j) = i$ if the j^{th} strand lies in the i^{th} rectangle R_i .

Definition 3. The OBP $\tau = \tau_{\sigma, \mathbf{k}}$ is a permutation of the set $\mathbb{N}_K$ defined by

$$\tau(j) := \sum_{1 \leq i < \sigma(\beta(j))} k_{\sigma^{-1}(i)} + j - \sum_{1 \leq i < \beta(j)} k_i \quad (3.1)$$

$\tau(j)$ is simply the position in which the j^{th} strand is attached to J on the left - see Figure 4, where $\tau(1) = 7$ since the first strand on the right ends up as the 7th strand on the left of J . This example, corresponding to the map in Figure 3, is fully worked out below in Example 1.

To understand (3.1), note that the j^{th} strand, $1 \leq j \leq K$, is the $(j - \sum_{1 \leq i < \beta(j)} k_i)^{\text{th}}$ strand of the rectangle $R_{\beta(j)}$. And on the left side of J , the number of strands are in the order $(k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(n)})$, so the first sum on the right in (3.1) simply counts how many strands lie above the position where R_i is placed.

Now, a strand among the first n , say strand j , with $1 \leq j \leq n$, is the first strand of the image of R_j . Its vertical right side is glued to the left side of strand $\tau(j)$. Similarly, the right side of strand $\tau(j)$ is glued to the left side of strand $\tau(\tau(j))$. This continues until some $p_j > 1$ such that $1 \leq \tau^{p_j}(j) \leq n$. The collection of the p_j strands $\{j, \dots, \tau^{(p_j-1)}(j)\}$ of the τ -orbit of j provide the full image of rectangle R_j under f . We call this ordered set of numbers $O_j := (j, \dots, \tau^{(p_j-1)}(j))$ the **ORBIT** of j , where $1 \leq j \leq n$. Namely the orbit stops as soon as the next image under τ is back in the subset $\mathbb{N}_n \subset \mathbb{N}_K$.

Note that the orbits O_j are disjoint, and that for $j \in \mathbb{N}_n$, the first return under iteration of τ to $\mathbb{N}_n$ is the same as σ . For instance, in Figures 3 & 4, the τ -orbit of 4 is $O_4 = (4, 10, 13)$ of length $p_4 = 3$, and $\tau^{p_4}(4) = 1$ same as $\sigma(4) = 1$.

4. Admissible OBPs

4.1. The combinatorial data

Here we provide a purely combinatorial way of describing the data of these rectangles, without reference to a surface or map. Let σ be a permutation of $\mathbb{N}_n = \{1, 2, \dots, n\}$. Let $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ be n positive integers, and set $K = k_1 + \dots + k_n$.

Partition the ordered set $(1, \dots, K)$ into **BLOCKS** $B_1, \dots, B_n$, where B_1 consists of the ordered set of the first k_1 integers $(1, \dots, k_1)$, B_2 contains the next k_2 elements $(k_1 + 1, \dots, k_1 + k_2)$, and in general, for $1 \leq i \leq n$ we have

$$B_i = \left(\sum_{j < i} k_j + 1, \dots, \sum_{j \leq i} k_j \right). \quad (4.1)$$

If the strand $j \in (1, \dots, K)$ belongs to block B_i , we define $\beta(j) = i$, $\beta : (1, \dots, K) \rightarrow (1, \dots, n)$. Using β we can define the OBP τ using (3.1) above, (cf. Example 1).

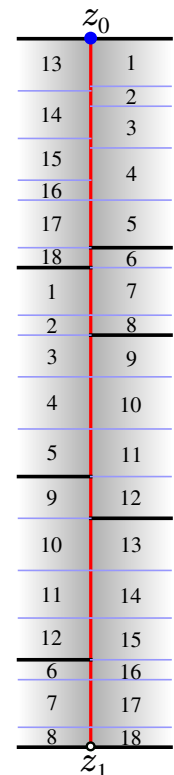


Figure 4. OBP corresponding to Figure 3.

We similarly define orbits O_i for each $1 \leq i \leq n$ as the ordered set $O_i := (i, \dots, \tau^{\circ(p_i-1)}(i))$, where $p_i > 1$ is the smallest integer such that $\tau^{\circ p_i}(i) \in \mathbb{N}_n \subset \mathbb{N}_K$. Here p_i is guaranteed to exist as τ being a permutation of $\mathbb{N}_K$ has a cycle decomposition. Define $\tau' = \tau'_{\sigma, \mathbf{k}} : \mathbb{N}_n \rightarrow \mathbb{N}_n$ by $\tau'(i) = \tau^{\circ p_i}(i)$, so τ' is the first return map under iteration of τ to the subset $(1, \dots, n) \subset (1, \dots, K)$.

Define the matrix A by setting the entry a_{ij} in the i^{th} row and the j^{th} column of A equal to the number of elements of the i^{th} block B_i that are in the j^{th} orbit O_j . That is, a_{ij} is the number of elements in the intersection:

$$a_{ij} = |B_i \cap O_j| \quad (4.2)$$

Entries in the j^{th} -column of A add up to $p_j :=$ the length of the orbit O_j , (cf. the number of strands in the image of R_j). And when the union of the orbits is all the strands, $\bigcup_i O_i = \mathbb{N}_K$, then the i^{th} -row of A adds up to $k_i =$ the size of block B_i , (cf. the number of strands that pass R_i).

4.2. Admissibility of the OBP $\tau_{\sigma, \mathbf{k}}$

With a slight abuse of notation, we will also call the smallest and largest elements of each block B_i , namely the elements $\sum_{j < i} k_j + 1$ and $\sum_{j \leq i} k_j$, as the TOP and BOTTOM STRANDS of B_i respectively.

Definition 4. An ordered block permutation $\tau = \tau_{\sigma, \mathbf{k}}$ is ADMISSIBLE if

- (I) $\tau' = \sigma$;
- (II) $\bigcup_{i=1}^n O_i = \{1, \dots, K\}$;
- (III) Each orbit O_i includes the top and bottom strands of block B_i , for $1 \leq i \leq n-1$. The orbit O_n contains the top strand of B_n but not the bottom strand. The last strand $K \in B_n$ belongs to $O_{\sigma^{-1}(n)}$;
- (IV) The matrix A defined by $a_{ij} = |B_i \cap O_j|$ is irreducible;

For instance, the OBP depicted above corresponding to Figure 3 is admissible.

Lemma 3. An ordered block permutation associated to an oriented-fixed pseudo-Anosov homeomorphism is admissible.

Proof. Suppose the OBP $\tau_{\sigma, \mathbf{k}}$ is computed from an OF pseudo-Anosov homeomorphism as above. Condition (I) expresses the fact that $f(J) = J'$, so everything you see for blocks along J has to be true of strands along J' . Condition (II) is satisfied since the union $\bigcup_i R_i$ is the whole surface, so is the union $\bigcup_i f(R_i)$. Condition (III) expresses that all the singularities, and all the leaves emanating from singularities, are fixed by f . Thus, the singularity on the bottom of R_i must also be on the bottom of $f(R_i)$, so $f(R_i)$ must contain that point as the bottom of some strand, and similarly for the top. Since the last singularity is on the left and $\lambda > 1$, $f(R_{\sigma^{-1}(n)})$ continues past z_1 and contains the bottom strand of R_n . Condition (IV) holds; in fact, the matrix A is aperiodic: A^n has strictly positive entries, (see e.g. [2]). $\square$

Remark 1. If we instead reverse the horizontal foliation, obtaining the conjugate surface $\bar{X}$, the permutation σ and the vector $\mathbf{k} = (k_i)_{i=1}^n$ of positive integers, change to σ^{-1} and $(k_{\sigma^{-1}(i)})_{i=1}^n$. The corresponding OBP $\tau_{\sigma^{-1}, \mathbf{k}_{\sigma^{-1}(\cdot)}}$ is just the inverse $(\tau_{\sigma, \mathbf{k}})^{-1}$, and satisfies the conditions of admissibility, except that in part (III), O_n contains the bottom strand of $B_{\sigma^{-1}(n)}$. The orbit of $\sigma^{-1}(i)$ continues as the

orbit of i backwards, without the i , i being it's first return under τ^{-1} . The corresponding matrix is $(A_{\sigma^{-1}(i), \sigma^{-1}(j)})_{i,j=1}^n$, it is also a non-negative matrix, in fact $A_{\tau^{-1}} = P_{\sigma} A_{\tau} P_{\sigma}^{-1}$ where P_{σ} is the permutation matrix $(\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(n)})$, with $P_{\sigma^{-1}} = P_{\sigma}^{-1} = P_{\sigma}^{\top}$.

Example 1. Here we compute the OBP for the example in Figure 3, with $n = 4$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ and $\mathbf{k} = (5, 3, 4, 6)$, so $K = 18$: The blocks are $B_1 = (1, 2, 3, 4, 5)$, $B_2 = (6, 7, 8)$, $B_3 = (9, 10, 11, 12)$, $B_4 = (13, 14, 15, 16, 17, 18)$, and the permutation τ is

$$\tau_{\sigma, \mathbf{k}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 7 & 8 & 9 & 10 & 11 & 16 & 17 & 18 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \quad (4.3)$$

Since $\sigma(4) = 1$, the fourth block $B_4 = (13, 14, 15, 16, 17, 18)$ is moved to the first place under τ , so its image is $(1, 2, 3, 4, 5, 6)$. Continuing this way, as $\sigma(1) = 2$, τ sends the first block $B_1 = (1, 2, 3, 4, 5)$ to $(7, 8, 9, 10, 11)$, and so on.

The orbits of τ are

$$\begin{aligned} O_1 &= (1, 7, 17, 5, 11, 14), [2] & O_2 &= (2, 8, 18, 6, 16), [4] \\ O_3 &= (3, 9, 12, 15), [3] & O_4 &= (4, 10, 13), [1], \end{aligned}$$

where we have colored in green the top strands and in purple the bottom strands. Each orbit O_i contains the top and bottom strand of block B_i , except O_n doesn't contain the bottom strand of B_n , strand $18 = K$, which belongs to $O_{\sigma^{-1}(n)} = O_2$, as required by admissibility.

In square brackets at the end of each orbit is the first return to $(1, 2, 3, 4)$, namely τ' , which is indeed equal to σ for this example and the union of the orbits is also all 18 strands. The matrix A defined as $a_{ij} = |B_i \cap O_j|$ is

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

which is irreducible (in fact, $A^2 > 0$), so combinatorially, $(\sigma, \mathbf{k})$ is indeed an admissible OBP. The characteristic polynomial of A is $1 - 7x + 13x^2 - 7x^3 + x^4$ and its Perron root $\lambda \sim 4.39026$ equals $\lambda = \left(7 + \sqrt{5} + \sqrt{38 + 14\sqrt{5}}\right)/4$. $\square$

Next we list some consequences of the combinatorial definition of an admissible OBP, which will be useful in what follows.

Lemma 4. Suppose σ is a permutation of $\mathbb{N}_n$ and $\mathbf{k} \in \mathbb{N}^n$ such that the ordered block permutation $\tau_{\sigma, \mathbf{k}}$ is admissible. Further, let $A = (a_{ij})$ be the matrix associated to τ , and λ the biggest eigenvalue of A in absolute value. Then,

- (i) $k_n \geq 2$, and the rest $k_i \geq 3$, $\forall i < n$. Likewise, $a_{nn} \geq 1$, while $a_{ii} \geq 2$, $\forall i < n$.
- (ii) A is aperiodic with $A^n > 0$, and $\lambda > 2$.
- (iii) $k_1 \geq n$ and $k_{\sigma^{-1}(1)} \geq n$.

(iv) $\sigma(1) \neq 1 \neq \sigma^{-1}(1)$ and $\sigma(n) \neq n \neq \sigma^{-1}(n)$. More generally $\nexists j < n$ such that $\sigma(\{1, \dots, j\}) \subseteq \{1, \dots, j\}$, and similarly for σ^{-1} .

Proof. Part (i): Suppose some $k_i = 1$. Then, the i^{th} row of A consists entirely of 0s except for a 1 in the i^{th} position, as O_i must contain the top strand of B_i . If P is the permutation matrix swapping i and n , we see that $P^{-1}AP = \begin{pmatrix} A_{ii} & * \\ \mathbf{0} & 1 \end{pmatrix}$, where A_{ii} is the submatrix obtained by deleting the i^{th} row and column of A , and $\mathbf{0}$ is a row of $n - 1$ zeros. Hence, A is reducible, contradicting part (iv) of admissibility (Definition 4).

Thus, the $k_i \geq 2$, so the top and bottom strands of the blocks B_i are distinct. Since $\forall i < n$, the orbit O_i contains the top and bottom strands of B_i , the entries $a_{ii} = |O_i \cap B_i| \geq 2$, $\forall i < n$. For $i = n$, we get $a_{nn} \geq 1$ since O_n may only contain the top strand of B_n . For $i < n$, if $k_i = 2$, then B_i consists exactly of its top and bottom strands. Both are in O_i by admissibility, so $a_{ii} = 2$ and $a_{ij} = 0$ for $j \neq i$. Thus, A has a column with a single non-zero entry, making it reducible. Hence, $k_i \geq 3$ for $i < n$.

For (ii), we repeat a well-known proof that a non-negative irreducible matrix with a positive diagonal is aperiodic: Associated to an integer non-negative $n \times n$ matrix A is an oriented graph with n vertices $v_1, \dots, v_n$, where the number of edges from v_i to v_j is exactly a_{ji} . $(A^m)_{ji}$ then counts the number of paths of length m from v_i to v_j . Irreducibility of A is equivalent to there being a path from any v_i to v_j , while aperiodicity is equivalent to there being a path of length n from any v_i to v_j . Since $a_{ii} > 0 \forall i$ (by part (i)) we know there is a loop at every vertex. Since there are only n vertices, by irreducibility there is a path of length $\leq n$ from any vertex to any other. By adding loops to a path of length less than n , we can ensure there is a directed path of length exactly n between any two vertices, hence $A^n > 0$.

Since A is aperiodic, by the classical Perron-Frobenius theorem [19, Chapter XIII, Theorems 2 and 8], it has a simple positive eigenvalue λ of maximal absolute value. Note that the k_i cannot all be equal, otherwise the blocks would line up on the left and right, and orbits would contain top and bottom strands of multiple blocks, contradicting part (iii) of admissibility. And since the k_i form the row-sums of the matrix A , we have $\min_i \{k_i\} < \lambda < \max_i \{k_i\}$ [19, Chapter XIII, Inequality 37]. As $k_i \geq 2$, we get $\lambda > 2$.

For (iii), suppose $k_1 < n$. Then, the $(k_1 + 1)^{\text{st}}$ strand, being one of the first n strands, is the first element of orbit O_{k_1+1} . But it is the top strand of B_2 , which can only belong to O_2 . So $k_1 = 1$, which contradicts part (i) above. To show $n \leq k_{\sigma^{-1}(1)}$ we consider the OBP of $(\sigma^{-1}, k_{\sigma^{-1}(i)})$ (see Remark 1) which satisfies admissibility, except that in part (iii), O_n contains the bottom strand of $B_{\sigma^{-1}(n)}$.

For (iv), note that $\sigma(\mathbb{N}_j) \subseteq \mathbb{N}_j$ iff $\sigma(\mathbb{N}_j) = \mathbb{N}_j$ iff $\sigma^{-1}(\mathbb{N}_j) \subseteq \mathbb{N}_j$, since σ is a permutation, hence bijective. If $\sigma(\mathbb{N}_j) = \mathbb{N}_j$, by part (iii) all orbits remain within the first j blocks and hence never pass through B_{j+1} , contradicting Definition 4(ii). The special cases follow by setting $j = 1$ or $j = n - 1$. $\square$

5. The general shape

5.1. Structure of zippered rectangles

In this section we will study the general structure of the zippered rectangles in our decomposition of an OF pA map. By construction, each rectangle $R_1, \dots, R_n$ has a singularity on its top edge: for R_1 it is at the top-left corner; for $R_{\sigma^{-1}(1)}$ it is at its top-right corner; and for the rest of the R_i , it is in the interior of their top edge. Also, each $R_1, \dots, R_{n-1}$ has a singularity in the interior of its bottom edge—the bottom horizontal edge of R_n doesn't contain a singularity, as Figure 5.

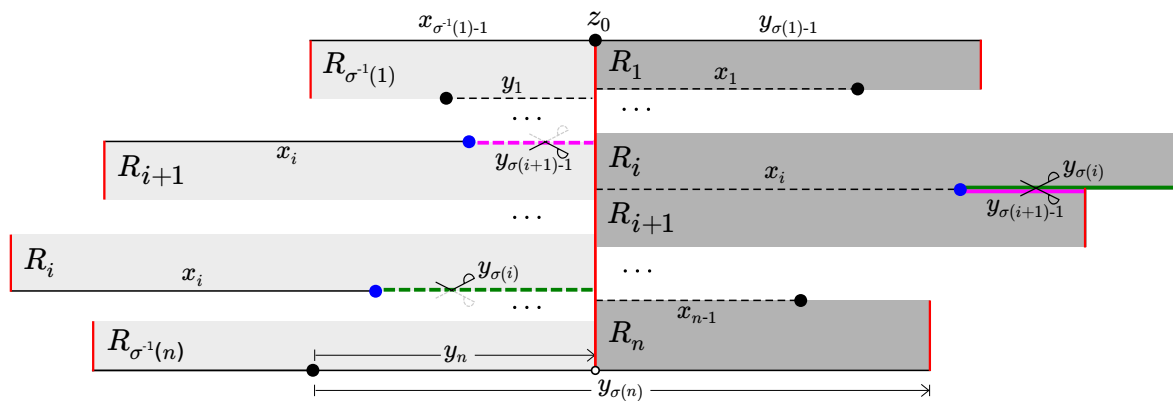


Figure 5. In general, $R_1, \dots, R_n$ are glued along edges $x_1, \dots, x_{n-1}$ until the singularities. Beyond the singularities, gluings can be determined by σ - the order of R_i on the left of J - along edges $y_1, \dots, y_n$. Since R_1 arrives in position $\sigma(1)$ on the left, its top edge is $y_{\sigma(1)-1}$, well-defined by Lemma 4(iv). Similarly, to the left of z_0 is the edge $x_{\sigma^{-1}(1)-1}$. Below R_n is the rectangle $R_{\sigma^{-1}(\sigma(n)+1)}$ whose top-right edge $y_{\sigma(n)}$ is glued first to the bottom-right of $R_{\sigma^{-1}(n)}$ along y_n , and the rest past z_1 to the bottom of R_n .

We will refer to the part of the bottom horizontal edge of R_i before (resp. after) the singularity as the BOTTOM-LEFT (resp. BOTTOM-RIGHT) edge of R_i , $\forall 1 \leq i < n$. Similarly define TOP-LEFT and TOP-RIGHT edges of R_i , $\forall 1 < i \leq n$.

For each $i \in \mathbb{N}_{n-1}$, the bottom-left of R_i is glued to the top-left of R_{i+1} along the critical edge x_i until the singularity between them. The top edge of $R_{\sigma^{-1}(1)}$, is the edge $x_{\sigma^{-1}(1)-1}$ which is amongst $x_1, \dots, x_{n-1}$, since $2 \leq \sigma^{-1}(1) \leq n$ by Lemma 4(iv). Similarly, for instance, the bottom-left edge of $R_{\sigma^{-1}(n)}$ is the edge $x_{\sigma^{-1}(n)}$, which is also amongst the x_i , since $\sigma^{-1}(n) < n$. These take care of all the identifications along incoming critical horizontal edges.

Similarly to the left of J , between the rectangles are the critical edges $y_1, \dots, y_{n-1}$. In addition y_n is the bottom-right edge of $R_{\sigma^{-1}(n)}$. For each $i \in \mathbb{N}_{n-1} \setminus \{\sigma(n)\}$, the bottom-right of $R_{\sigma^{-1}(i)}$ is glued to the top-right of $R_{\sigma^{-1}(i+1)}$ along y_i from the singularity between them until J .

Among the edges $y_1, \dots, y_{n-1}$ is the edge $y_{\sigma(n)}$, right below where R_n is attached on the left of J , which requires a little attention since R_n doesn't contain a singularity on its bottom edge (cf. Figures 7, 9). The edge $y_{\sigma(n)}$ starts from the singularity on the bottom edge of $R_{\sigma^{-1}(n)}$ and continues past the non-singular point z_1 till the end of R_n . Thus, y_n is a subset of $y_{\sigma(n)}$, the latter exceeding it by l_n , the length of R_n . This takes care of all the identifications along outgoing critical horizontal edges. The identifications along the vertical sides all happen on J and are clear.

Thus, for instance, the bottom-left of R_1 is the edge x_1 whereas its bottom-right is the edge $y_{\sigma(1)}$. The top edge of R_1 is $y_{\sigma(1)-1}$, which is among $y_1, \dots, y_{n-1}$, again by Lemma 4(iv), and it may equal $y_{\sigma(n)}$.

5.2. Minimality of the permutation σ

When an OBP is defined using an OF pA map, all the singularities of the Abelian differential have multiplicities $m \geq 1$. That is, at every singularity, the cone angle is one of $\{4\pi, 6\pi, 8\pi, \dots\}$. But when an admissible OBP is combinatorially defined, it may be that the cone angle at a “singularity” is 2π ,

so the corresponding point is not a singularity, just a fixed-point of f . The formula (2) and bounds (3) and (4) no longer hold, so we would like to avoid this possibility.

The multiplicities of the singularities depend entirely on the permutation σ , so in this section we study the properties of σ that guarantee that no singularity has multiplicity $m = 0$. Note that apart from the singularities at the very top and bottom, the neighborhood of a singularity on R_i is determined as in the Figure 6.

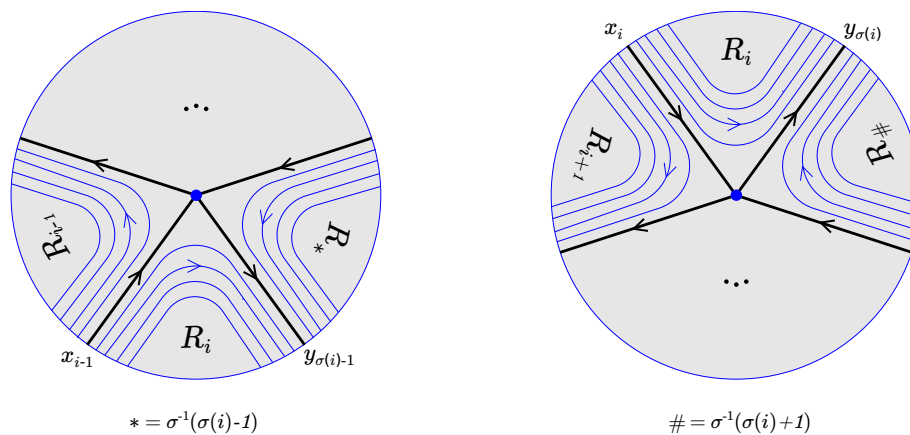


Figure 6. On the left is depicted a conformal chart centred at the zero on the top edge of R_i , for any $i \notin \{1, \sigma^{-1}(1), \sigma^{-1}(\sigma(n) + 1)\}$. On the right is the zero on the bottom edge of R_i , for every $i \notin \{\sigma^{-1}(n), \sigma^{-1}(1) - 1, \sigma^{-1}(\sigma(1) - 1)\}$. The excluded points correspond to z_0 and $R_{\sigma^{-1}(n)}$; the top and bottom.

To determine whether the singularity at the bottom of R_i has cone angle $\geq 4\pi$, one can check if $\sigma(i+1) \neq \sigma(i) + 1$, as can be seen from Figure 6. This works for all $i \in \{1, \dots, n-1\}$, except $i = \sigma^{-1}(n)$ and $i = \sigma^{-1}(1) - 1$. These exceptional values correspond to the singularity z_0 at the top of R_1 and the singularity at the bottom of $R_{\sigma^{-1}(n)}$. There are two cases that occur for these, depending on whether R_1 is directly below R_n on the left of J or not, namely whether R_1 is or isn't $R_{\sigma^{-1}(\sigma(n)+1)}$. We describe the differences as Case I and Case II below.

5.2.1. Case I

$\sigma(1) \neq \sigma(n) + 1$. The rectangle R_1 is in position $\sigma(1)$ on the left, so its top edge is always $y_{\sigma(1)-1}$. Note that $y_{\sigma(1)-1}$ is well-defined since $\sigma(1) > 1$ by Lemma 4(iv). Similarly, below R_n to the left of J is always $R_{\sigma^{-1}(\sigma(n)+1)}$. In case I, as $\sigma^{-1}(\sigma(n) + 1) \neq 1$, R_1 is not directly below R_n . Thus, turning *counter-clockwise* around z_0 from R_1 leads to $R_{\sigma^{-1}(\sigma(1)-1)}$ which is not R_n (or indeed $R_{\sigma^{-1}(n)}$ either, since then $\sigma(1) = n + 1$). On the other hand, turning *clockwise* around z_0 leads first into $R_{\sigma^{-1}(1)}$ and then past $x_{\sigma^{-1}(1)-1}$ into $R_{\sigma^{-1}(1)-1}$. The situation is depicted in Figures 7 and 8.

If z_0 were to have cone angle equal to 2π , turning either way around z_0 would lead to the same rectangle above, thus $R_{\sigma^{-1}(1)-1}$ would be the same as $R_{\sigma^{-1}(\sigma(1)-1)}$. Thus, in order to test (in case I) whether z_0 is a genuine singularity with multiplicity $m \geq 1$, we check if $\sigma^{-1}(1) - 1 \neq \sigma^{-1}(\sigma(1) - 1)$.

Similarly, turning clockwise around the singularity at the bottom of $R_{\sigma^{-1}(n)}$ leads to $R_{\sigma^{-1}(\sigma(n)+1)}$ which isn't R_1 (or indeed $R_{\sigma^{-1}(1)}$ either). Turning counter-clockwise instead leads to $R_{\sigma^{-1}(n)+1}$.

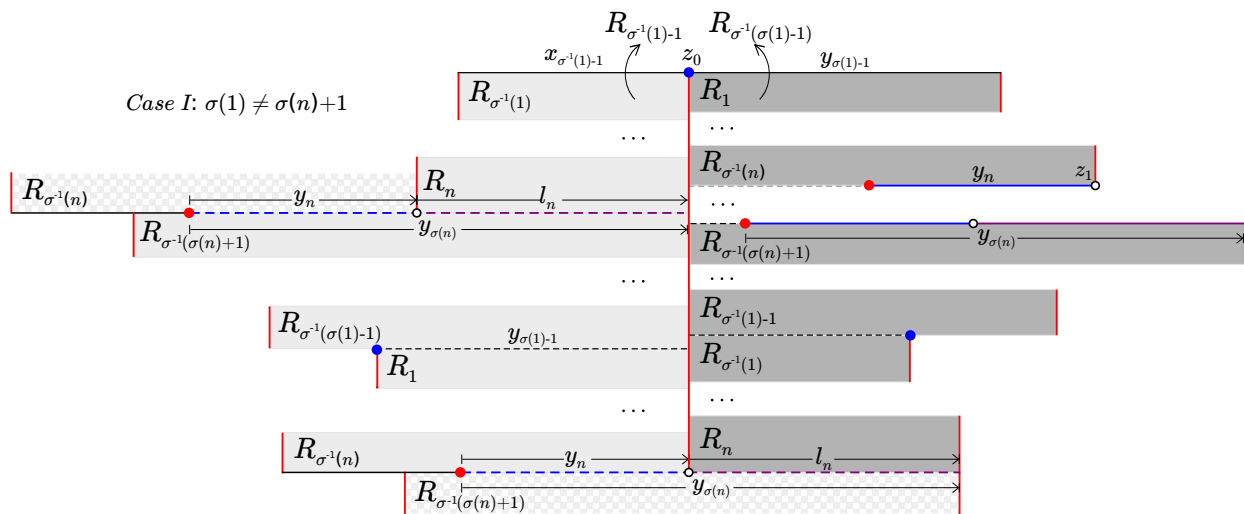


Figure 7. Case I: R_1 is not directly below R_n to the left of J , since $\sigma^{-1}(\sigma(n) + 1) \neq 1$. Thus, the rectangle above R_1 is unambiguously $R_{\sigma^{-1}(\sigma(1)-1)}$.

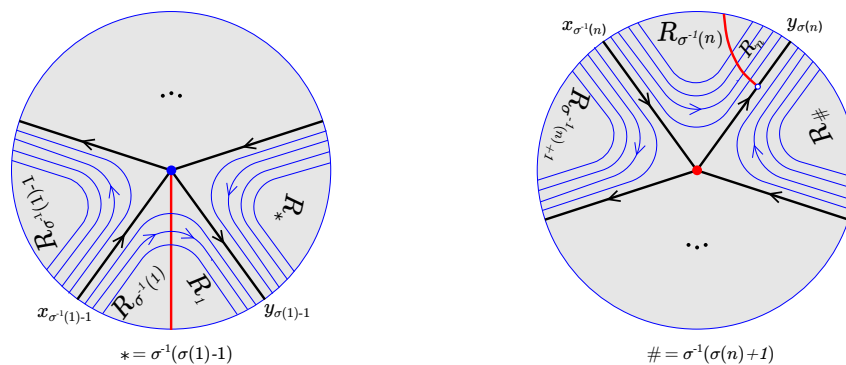


Figure 8. Conformal charts centred at the singularity at the top of R_1 (left), and at the singularity at the bottom of $R_{\sigma^{-1}(n)}$ (right), are shown for case I.

So to test whether the singularity at the bottom of $R_{\sigma^{-1}(n)}$ has cone angle $\geq 4\pi$, we check if $\sigma^{-1}(n) + 1 \neq \sigma^{-1}(\sigma(n) + 1)$.

5.2.2. Case II

$\sigma(\mathbf{1}) = \sigma(\mathbf{n}) + \mathbf{1}$. Depicted in Figures 9 and 10, in case II the top and bottom are vertically adjacent, so the two exceptional points of case I coincide. Since, $\sigma(1) = \sigma(n) + 1$, we have that the rectangle above R_1 is $R_{\sigma^{-1}(\sigma(1)-1)} = R_n$, and indeed R_n is glued above R_1 to the left of J . However, since R_n doesn't have a singularity on its bottom edge, part of the top edge of R_1 is also glued to $R_{\sigma^{-1}(n)}$ along y_n . As a result, turning counter-clockwise around z_0 from R_1 doesn't lead to R_n , as the formula from case I would suggest, but rather leads into $R_{\sigma^{-1}(n)}$.

Therefore, in case II, in order to determine if z_0 is indeed a singularity of angle $\geq 4\pi$, the condition to check is $\sigma^{-1}(1) \neq \sigma^{-1}(n) + 1$, since otherwise the incoming edges around $x_{\sigma^{-1}(1)-1}$ and $x_{\sigma^{-1}(n)}$ are the same.

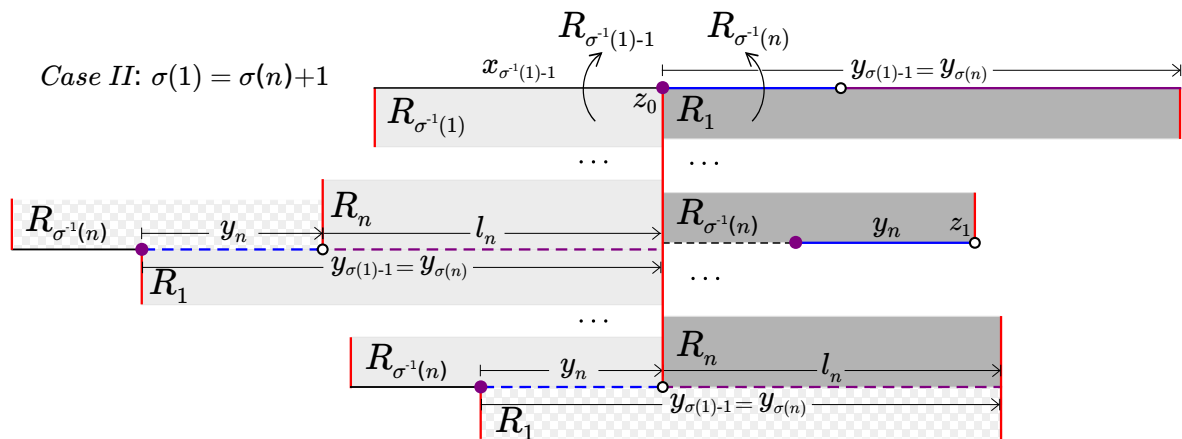


Figure 9. Case II: When R_1 is below R_n on the left of J , so the two exceptional singularities of case I coincide. The top of R_1 is first glued to $R_{\sigma^{-1}(n)}$ along y_n and the rest of the top of R_1 is glued to the bottom of R_n .

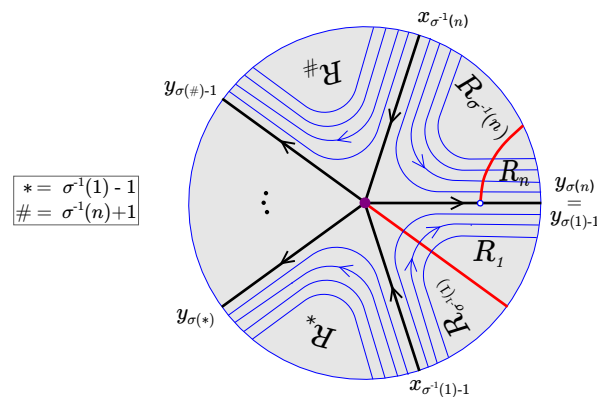


Figure 10. Case II: Even though to the left of J , the rectangle above R_1 is R_n , turning counter-clockwise around z_0 from R_1 leads into the rectangle $R_{\sigma^{-1}(n)}$.

5.2.3. Minimal σ

Putting cases I and II together we make the following definition, which guarantees that none of the singularities have angle 2π :

Definition 5. A permutation σ is called **MINIMAL** if the following conditions are satisfied:

- (i) $\sigma(i+1) \neq \sigma(i) + 1$, for each $i \in \{1, \dots, n-1\} \setminus \{\sigma^{-1}(n), \sigma^{-1}(1) - 1\}$.
- (ii) When $\sigma(1) = \sigma(n) + 1$, $\sigma^{-1}(1) \neq \sigma^{-1}(n) + 1$.
- (iii) When $\sigma(1) \neq \sigma(n) + 1$, both $\sigma^{-1}(1) - 1 \neq \sigma^{-1}(\sigma(1) - 1)$ and $\sigma^{-1}(n) + 1 \neq \sigma^{-1}(\sigma(n) + 1)$.

For instance, when $n = 4$, the following 7 of the 24 permutations of $(1, 2, 3, 4)$ are minimal: $(2, 4, 1, 3)$, $(2, 4, 3, 1)$, $(3, 1, 4, 2)$, $(3, 2, 4, 1)$, $(4, 1, 3, 2)$, $(4, 2, 1, 3)$, and $(4, 3, 2, 1)$. It is clear that when the OBP $\tau_{\sigma, \mathbf{k}}$ is computed for an OF pA map, σ is minimal by construction.

6. Proof of the main theorem

Theorem 5. *Let σ be a permutation of size $n \geq 4$, and $\mathbf{k}$ a vector of n positive integers such that $\tau_{\sigma, \mathbf{k}}$ is an admissible ordered block permutation. Let A be the corresponding $n \times n$ matrix, with leading eigenvalue λ . Choose positive λ -eigenvectors $\mathbf{l}$ for A^T and $\mathbf{h}$ for A , and let $R_1, \dots, R_n$ be rectangles with R_i of size $l_i \times h_i$, in fact identified with $[0, l_i] \times [0, h_i] \subset \mathbb{R}^2$. The rectangles fit together to give a genus g Riemann surface X with a holomorphic 1-form ω , and the maps*

$$x + iy \mapsto \lambda x + i \frac{y}{\lambda}$$

in each rectangle fit together to give an oriented-fixed pseudo-Anosov homeomorphism $f_{\sigma, \mathbf{k}} : X \rightarrow X$. When σ is minimal, the genus $g = (n + 1 - v)/2$ and is bounded as $\frac{1}{4}(n + 3) \leq g \leq \frac{1}{2}n$, where v is the number of distinct singularities.

Conversely, if $f : X \rightarrow X$ is an oriented-fixed pseudo-Anosov homeomorphism of a surface of genus g , the choice of a singularity z_0 and of a contracting segment $\tilde{J}$ of any length, emanating from z_0 , defines an admissible ordered block permutation of size $n = 2g + v - 1$, bounded $2g \leq n \leq 4g - 3$, (after possibly reversing the orientation of the expanding foliation).

Proof. ($\Leftarrow$): We have already shown the converse in Lemma 3. Given an OF pA map, we may have to reverse the orientation of the expanding foliation to ensure that the critical expanding segment furthest from z_0 along J meets J on the left. Moreover, since we have drawn critical segments from singularities of angle $\geq 4\pi$, the permutation σ is minimal. As shown in §2.5, there is a finite number of such OBPs that could be associated to a given OF pA map depending on the choice of critical contracting trajectory and the length of the segment J .

($\Rightarrow$): Due to its length, we break the proof of the direct statement into 8 comparatively short steps.

Step I: Construction of rectangles.

Given an admissible OBP $\tau_{\sigma, \mathbf{k}}$, with blocks B_i and orbits O_j computed as in Section 4.1, let $A = (a_{ij})$ be the associated matrix defined by $a_{ij} = |B_i \cap O_j|$. A is irreducible (by admissibility) and aperiodic with a leading eigenvalue $\lambda > 2$ (by Lemma 4(ii)), so by the Perron-Frobenius theorem [19, Chapter XIII, Theorem 2], A has left and right λ -eigenvectors $\mathbf{l}$ and $\mathbf{h}$ consisting of entirely positive entries. The eigenvectors $\mathbf{l}$ and $\mathbf{h}$ are unique up to scaling by a positive real, so we fix some scaling, but keep it arbitrary so as to construct every possible pair of orientable foliations carrying a pA map, (not necessarily orientation preserving or fixing the critical trajectories, see Remark 2).

Let R_i be a rectangle of height h_i and length l_i , $R_i = [0, l_i] \times [0, h_i] \subset \mathbb{C} \cong \mathbb{R} \oplus i\mathbb{R}$. Let J be a vertical segment of length equal to the L_1 norm of $\mathbf{h}$, i.e. $|J| = \sum_{i=1}^n h_i$. Arrange the rectangles $R_1, \dots, R_n$ in order along the right side of J as in Figure 5. The vertical right sides of the R_i are identified to segments of J according to the permutation σ .

Step II: Locating the singularities.

The next task is identifications along the $2n$ horizontal edges of the R_i . For this we need to uniquely determine the location of the singularities on the edges. With slight abuse of notation we will denote by x_i both the edge itself and its length, and the same for the y_i . Thus we need to find the lengths $x_1, \dots, x_{n-1}$ and $y_1, \dots, y_n$ such that $y_{\sigma(1)-1} = l_1$, $y_{\sigma(n)} = y_n + l_n$, (cf. Figures 5, 7, 9), and for each i ,

$$1 \leq i \leq n-1,$$

$$\begin{aligned} x_i + y_{\sigma(i)} &= l_i, \\ x_i + y_{\sigma(i+1)-1} &= l_{i+1}, \end{aligned} \quad (6.1)$$

and such that all the terms are positive, except in the second equation for $i = \sigma^{-1}(1) - 1$, for then we have $x_{\sigma^{-1}(1)-1} + 0 = l_{\sigma^{-1}(1)}$, (here we have to define $y_0 := 0$).

Since we are trying to define an ordered-fixed map, each edge x_i when stretched by the eigenvalue λ of A should yield an edge whose length is equal to x_i plus the sum of the lengths of the rectangles that the τ -orbit O_i passes before the bottom strand of B_i ; and similarly for the y_i . Namely, for each $i < n$, if O'_i is the part of O_i before the bottom strand $\sum_{r=1}^i k_r$ of B_i , (guaranteed to be in O_i by Definition 4(III)), and O''_i is the part after the bottom strand, so that for $1 \leq i \leq n-1$,

$$O_i = O'_i \sqcup \left\{ \sum_{r=1}^i k_r \right\} \sqcup O''_i, \quad (6.2)$$

then we must have for each $1 \leq i \leq n-1$,

$$\begin{aligned} \lambda x_i &= \sum_{j \in O'_i} l_{\beta(j)} + x_i, \\ \lambda y_{\sigma(i)} &= y_{\sigma(i)} + \sum_{j \in O''_i} l_{\beta(j)}. \end{aligned} \quad (6.3)$$

Here x_i is the bottom-left edge of R_i and $y_{\sigma(i)}$ its bottom-right edge. These equations uniquely determine the values of x_i and $y_{\sigma(i)}$ for $1 \leq i \leq n-1$, as

$$\boxed{x_i = \frac{1}{\lambda-1} \sum_{j \in O'_i} l_{\beta(j)}}, \text{ and } \boxed{y_{\sigma(i)} = \frac{1}{\lambda-1} \sum_{j \in O''_i} l_{\beta(j)}}. \quad (6.4)$$

Moreover, for $1 \leq i \leq n-1$, the sets O'_i and O''_i are never empty, as the bottom strand of B_i is never the first or the last element of O_i . This is because each orbit has its first element in B_1 and its last element in $B_{\sigma^{-1}(1)}$, since $\tau' = \sigma$, and $k_1 \geq n$ and $k_{\sigma^{-1}(1)} \geq n$ by Lemma 4(iii). For B_1 , the bottom strand k_1 isn't the first element of O_1 ; while the last strand of $O_{\sigma^{-1}(1)}$ is the top strand of $B_{\sigma^{-1}(1)}$, hence also not the bottom strand. Thus, since the $l_i > 0$ and $\lambda > 2$, we get $\forall i < n$,

$$\boxed{x_i > 0} \text{ and } \boxed{y_{\sigma(i)} > 0}. \quad (6.5)$$

Step III: Edges x_i and $y_{\sigma(i)}$ add up to the length of R_i .

By choosing $\mathbf{l}$ as the left λ -eigenvector of A , we know that $(A^\top \mathbf{l})_i = \lambda l_i$. And the left side $(A^\top \mathbf{l})_i = \sum_{j \in O_i} l_{\beta(j)}$ can be decomposed into the orbits before and after the bottom strand of B_i , thus giving us

$$\sum_{j \in O'_i} l_{\beta(j)} + l_i + \sum_{j \in O''_i} l_{\beta(j)} = \lambda l_i, \quad (6.6)$$

which in turn simplifies to, for $1 \leq i \leq n-1$,

$$\underbrace{\frac{1}{\lambda-1} \sum_{j \in O'_i} l_{\beta(j)}}_{x_i} + \underbrace{\frac{1}{\lambda-1} \sum_{j \in O''_i} l_{\beta(j)}}_{y_{\sigma(i)}} = l_i. \quad (6.7)$$

Thus, by defining x_i and y_i according to (6.4), we also get for $1 \leq i \leq n-1$,

$$\boxed{x_i + y_{\sigma(i)} = l_i} \quad (6.8)$$

We have thus uniquely determined the values of $x_1, \dots, x_{n-1}$ and $y_1, \dots, \widehat{y_{\sigma(n)}}, \dots, y_n$; the only missing length to determine is that of $y_{\sigma(n)}$, which should equal $y_n + l_n$ (cf. Figures 5, 7, and 9).

Step IV: Finding the last edge $y_{\sigma(n)}$.

We can also decompose each orbit O_i according to when the top strand is crossed. Say we denote by $O_i^{(3)}$ the first part of the orbit O_i before the top strand of B_i , after which the top strand is crossed, and the rest of the orbit as $O_i^{(4)}$. Thus,

$$O_i = O_i^{(3)} \sqcup \left\{ 1 + \sum_{r=1}^{i-1} k_r \right\} \sqcup O_i^{(4)}, \quad (6.9)$$

where $1 + \sum_{r=1}^{i-1} k_r$ is the top strand of B_i , and in O_i , $\forall i \leq n$, (Definition 4(III)). However, here $O_1^{(3)}$ and $O_{\sigma^{-1}(1)}^{(4)}$ are empty as the corresponding top strand is the first and last element of the respective orbits. For $i \notin \{1, \sigma^{-1}(1)\}$, $O_i^{(3)}$ and $O_i^{(4)}$ are all non-empty, for the same reason as above. Using (6.9), and the same derivation as for (6.4), we get for each i with $1 \leq i \leq n-1$,

$$x_i = \frac{1}{\lambda - 1} \sum_{j \in O_{i+1}^{(3)}} l_{\beta(j)}, \quad \text{and} \quad y_{\sigma(i+1)-1} = \frac{1}{\lambda - 1} \sum_{j \in O_{i+1}^{(4)}} l_{\beta(j)}, \quad (6.10)$$

where x_i is seen as the top-left edge of R_{i+1} , and $y_{\sigma(i+1)-1}$ as its top-right edge.

Step V: Adjacency of orbits.

Let us first see why the x_i defined by (6.10) and (6.4) are the same. For each $i < n$, the orbits $O_i = \{i, \tau(i), \dots, \tau^{o(p_i-1)}(i)\}$ and $O_{i+1} = \{i+1, \tau(i+1), \dots, \tau^{o(p_{i+1}-1)}(i+1)\}$ start with adjacent elements i and $i+1$ respectively, both in block B_1 (as $k_1 \geq n$, Lemma 4(iii)). Unless $\tau(i)$ is the bottom strand of B_i (and therefore $\tau(i+1) = \tau(i) + 1$ the top strand of B_{i+1}), the orbits continue to remain adjacent, and pass through the same block again. This behaviour continues by virtue of the definition of an admissible ordered block permutation, and the orbits O_i and O_{i+1} only cease being adjacent when the orbit O_i passes the bottom strand of B_i , (simultaneously as O_{i+1} passes the top strand of B_{i+1}). The lengths $l_{\beta(j)}$ are thus the same for $j \in O_i'$ and $j \in O_{i+1}^{(3)}$, and hence (6.4) and (6.10) define the same values for the x_i , $1 \leq i \leq n-1$.

Similarly, (6.10) recalculates the y_i (except $y_{\sigma(n)}$). These values include $y_1, \dots, \widehat{y_{\sigma(1)-1}}, \dots, y_{n-1}$. So for $j \notin \{n, \sigma(n)\}$, the value y_j has been twice computed, once according to the orbit $O_{\sigma^{-1}(j)}$ after the bottom strand (6.4), and once according to the orbit $O_{\sigma^{-1}(j+1)}$ after the top strand (6.10), since if $\sigma(i+1) - 1 = j$, then $O_{i+1} = O_{\sigma^{-1}(j+1)}$. The two orbits $O_{\sigma^{-1}(j)}, O_{\sigma^{-1}(j+1)}$ can be run backwards, starting with the j^{th} and $(j+1)^{\text{st}}$ strands of $B_{\sigma^{-1}(1)}$ and, just as above, remain adjacent and within the same blocks, only ceasing to be adjacent when $O_{\sigma^{-1}(j)}$ hits the bottom strand of $B_{\sigma^{-1}(j)}$, (while $O_{\sigma^{-1}(j+1)}$ simultaneously hits the top strand of $B_{\sigma^{-1}(j+1)}$.) Thus, again as above, the lengths $l_{\beta(j)}$ are the same for $j \in O_{\sigma^{-1}(j)}''$ and $j \in O_{\sigma^{-1}(j+1)}^{(4)}$, and hence (6.4) and (6.10) define the same values for the y_i , $1 \leq i \leq n-1$.

Step VI: Verifying $y_{\sigma(n)} = y_n + l_n$.

By the same argument as for the x_i in Step II, we see that $\boxed{y_{\sigma(i+1)-1} > 0}$, except when $i = \sigma^{-1}(1) - 1$, since $O_{\sigma^{-1}(1)}^{(4)} = \emptyset$. In this case, we get that the top right edge of $R_{\sigma^{-1}(1)}$ is 0, as it should be since the

singularity z_0 is at the top-right corner of $R_{\sigma^{-1}(1)}$. And again, as $\mathbf{l}$ is an eigenvector of $A^\top$, we have for $1 \leq i \leq n-1$,

$$(A^\top \mathbf{l})_{i+1} = \sum_{j \in O_{i+1}^{(3)}} l_{\beta(j)} + l_{i+1} + \sum_{j \in O_{i+1}^{(4)}} l_{\beta(j)} = \lambda l_{i+1}. \quad (6.11)$$

Thus, we also get for $1 \leq i \leq n-1$,

$$\underbrace{\frac{1}{\lambda-1} \sum_{j \in O_{i+1}^{(3)}} l_{\beta(j)}}_{x_i} + \underbrace{\frac{1}{\lambda-1} \sum_{j \in O_{i+1}^{(4)}} l_{\beta(j)}}_{y_{\sigma(i+1)-1}} = l_{i+1}, \quad (6.12)$$

Or, $\boxed{x_i + y_{\sigma(i+1)-1} = l_{i+1}}$

We have thus uniquely determined all the x_i and y_i , satisfying (6.1). The only thing left to check is $y_{\sigma(n)} = y_n + l_n$. For this, we subtract the two equations in (6.1) to get for $1 \leq i \leq n-1$,

$$y_{\sigma(i+1)-1} - y_{\sigma(i)} = l_{i+1} - l_i \quad (6.13)$$

Adding over $i = 1, \dots, n-1$ yields,

$$\left(\sum_{i=1}^{n-1} y_i - y_{\sigma(1)-1} \right) - \left(\sum_{i=1}^{n-1} y_i + y_n - y_{\sigma(n)} \right) = \sum_{i=1}^{n-1} (l_{i+1} - l_i) = l_n - l_1. \quad (6.14)$$

Simplifying and using $y_{\sigma(1)-1} = l_1$, we get:

$$\boxed{y_{\sigma(n)} = y_n + l_n}. \quad (6.15)$$

Step VII: Identifications and Riemann surface structure.

We have shown that after a choice of scaling for the left and right λ -eigenvectors $\mathbf{l}$ and $\mathbf{h}$ of A , all the x_i, y_i are uniquely determined. By equations (6.1) and (6.15), the rectangles R_i can be glued by translations along segments of their boundaries: R_i and R_{i+1} along x_i , $\forall 1 \leq i < n$; and $R_{\sigma^{-1}(i)}$ and $R_{\sigma^{-1}(i+1)}$ along y_i , $\forall 1 \leq i < n, i \neq \sigma(n)$. y_n followed by the non-singular edge (the bottom of R_n) are glued to $y_{\sigma(n)}$ (see Figure 7).

At each vertex of the rectangular decomposition, the gluings define a cyclic order of the incident sectors. A straightforward check using the admissibility conditions shows that this cyclic order is consistent around each vertex, so the resulting surface has no boundary. The minimality conditions (Definition 5) ensure that no vertex has cone angle 2π , so all vertices correspond to genuine singularities. This takes care of all the gluing, and all points have topological disks as neighborhoods, so we obtain a surface S without boundary.

J can be identified with a vertical segment in $\mathbb{C}$ and the interiors of the R_i attached to it on the left and right can be seen as charts to $\mathbb{C}$. A rectangular neighborhood of J of width $\leq \min_i \{x_i\}$ can be seen as another chart. Transition maps between small enough neighborhoods of non-singular points are translations ($z \mapsto z + c$), hence conformal. Charts around the singularities can be obtained by applying the map $z \mapsto z^{1/q}$ in a small neighborhood U of a singularity whose cone angle in the charts R_i is $2\pi q$, $q \geq 1$. Since $q \in \mathbb{N}$, $z^{1/q}$ is conformal in each $U \cap \text{Int}(R_i)$, so the transition maps are conformal and we obtain a Riemann surface X . The holomorphic 1-form dz in the charts R_i can be pulled back to

the surface to obtain an Abelian differential ω .

Step VIII: Defining the OF pA map f and checking that it is a homeomorphism.

We can define the map $f = f_{(\sigma, \mathbf{h})} : X \rightarrow X$ as follows. The collection of strands in O_i form a connected long and thin rectangle in the surface. Stretch each rectangle R_i horizontally by λ , and shrink it vertically by λ , and place it along this orbit O_i . The number of image rectangles $f(R_j)$ thus crossing R_i is $k_i = \sum_{j=1}^n a_{ij}$. Each has height h_j/λ ; hence, the total height of the rectangles passing through R_i is $\sum_j a_{ij}(h_j/\lambda)$, which is equal to h_i since $\mathbf{h}$ was chosen to be a λ -eigenvector of A .

Thus, the image rectangles fit inside the R_i without overlapping interiors, and f is clearly a homeomorphism in the interior of the R_i . The same can be seen in a small neighborhood of an edge x_i , since admissibility conditions imply that the orbits O_i and O_{i+1} remain adjacent till the bottom strand of B_i . Thus, f is well-defined on the edges x_i on the intersection of the rectangles R_i and R_{i+1} . Similarly, the orbits $O_{\sigma^{-1}(j)}$ and $O_{\sigma^{-1}(j+1)}$ have adjacent tails after the bottom strand of $B_{\sigma^{-1}(j)}$ which shows that f is well-defined on $y_{\sigma^{-1}(j)}$. The right endpoints of the x_i are the singularities, which by (6.3) are fixed points of f , and their multi-sheeted neighborhoods are simply stretched horizontally and shrunk vertically.

Hence, f is a homeomorphism. And since the horizontal and vertical foliations of the plane provide orientable singular measured foliations on X invariant under f , we have constructed a pseudo-Anosov map that leaves an Abelian differential invariant.

Finally, when σ is minimal, the singularities all have cone angle $2\pi q$, $q \geq 2$, since otherwise one of the minimality conditions would necessarily be violated. Hence, the multiplicities m_i are ≥ 1 , $\forall i \leq \nu$, so (2.2) applies, and so do the bounds in (2.3).

□

Remark 2. In this remark, we justify constructing every possible orientable foliation on a closed surface that is invariant under a pA map. Suppose S is a closed surface, and $g : S \rightarrow S$ a pseudo-Anosov map with orientable invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$, but g doesn't necessarily fix all critical trajectories, or is even an orientation-preserving homeomorphism. $g^{\circ 2}$ is orientation preserving, and some higher power $g^{\circ N}$ necessarily fixes all critical trajectories. So $g^{\circ N}$ has an associated OBP. However, taking an iterate of g neither changes the surface nor the invariant foliations. Thus, we have constructed the surface with the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ out of the data of an OBP above.

7. Conclusions

The matrices A of size $n = 2g + \nu - 1$ that one obtains for admissible OBPs as above have the property that for every eigenvalue μ , $1/\mu$ is also an eigenvalue. This can be seen as follows. Define γ_i to be the closed curve in the surface running down the middle of R_i with its right end-point connected to its initial point along J . Then, $\gamma_1, \dots, \gamma_n$ form a spanning set for the homology group $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, with rank $2g$ and $\nu - 1$ relations. Let S be the intersection matrix of the γ_i . So $S_{ii} = 0$ and if σ preserves the relative order of i and j , then $S_{ij} = S_{ji} = 0$. If σ reverses their relative order, S_{ij} is $+1$ above the diagonal and -1 below. S is the intersection form, expressed in terms of a spanning set (when $\nu > 1$) rather than a basis of the homology.

Since f is a homeomorphism, $f(\gamma_i) \cap f(\gamma_j)$ has the same intersection number as $\gamma_i \cap \gamma_j$. By the

definition of the matrix A , the image $f(\gamma_j)$ is represented in homology by $\sum_i a_{ij}\gamma_i$. Therefore,

$$S_{ij} = \sum_{p,q} a_{pi} a_{qj} S_{pq} = (A^\top S A)_{ij}.$$

Namely,

$$A^\top S A = S. \quad (7.1)$$

Thus, if μ is an eigenvalue of A with eigenvector $\vec{w}$, we get

$$(S\vec{w}) = A^\top S A \vec{w} = \mu A^\top (S\vec{w}) \quad (7.2)$$

so $A^\top$, and therefore A has eigenvalue $1/\mu$. This is a well-known property of pseudo-Anosov stretch factors and is related to the characteristic polynomial of the induced action f_* on H_1 being palindromic, meaning its coefficients are the same read forwards and backwards. [20] proves a somewhat converse statement, that admitting an invariant symplectic form is only possible if the characteristic polynomial is palindromic.

So given an aperiodic integer matrix A whose characteristic polynomial is palindromic, it makes sense to ask whether such a matrix is induced by a pseudo-Anosov map with orientable foliations. The answer to this question is not known. While we cannot answer this, we can provide a finite algorithm to decide whether the given matrix A is the matrix of an admissible OBP:

1. Let $k_1, \dots, k_n$ be the row-sums of A .
2. For each σ in the permutation group of $\mathbb{N}_n$, check if the OBP with σ and $\mathbf{k} = (k_1, \dots, k_n)$ is admissible and generates the same matrix A .

Since the group of permutations of $\mathbb{N}_n$ is finite (of size $n!$), and checking admissibility using a given σ is a finite process, the algorithm is finite.

Author contributions

This paper develops ideas in the doctoral thesis of Author², written under supervision of Author¹. Author¹ wrote the first draft of this paper, and the first working examples were found by Author³. Subsequent drafts including the final version were written by Author².

Acknowledgments

We would like to thank Ana Anusic, Hyungryul Baik, Sylvain Bonnot, André de Carvalho, Arcelino do Nascimento, Chenxi Wu and Xuan Zhang for many helpful discussions. Further, we thank the two reviewers for their many helpful corrections and suggestions. The examples were found using code written in C++, and the pictures were drawn using Inkscape and Mathematica (licensed to the American University of Sharjah). This paper represents the opinions of the authors and does not mean to represent the position or opinions of the American University of Sharjah.

Author² would like to acknowledge the support of the American University of Sharjah, Open Access Publications program. During the course of writing this paper, author² was also funded in part by the postdoctoral grant No. 2018/13688-4 by the Fundação de Amparo à Pesquisa do Estado de São

Paulo (FAPESP), and by research grant No. 2236-121C062 by the Türkiye Bilimsel ve Teknolojik Araştırma Kurumu (TÜBİTAK) with Horizon 2020 Marie Skłodowska-Curie from the European Research Council (ERC). Author³ would like to acknowledge the support of the Cornell University RTG: “Dynamics, Probability, and Partial Differential Equations in Pure and Applied Mathematics”, from the U.S. National Science Foundation (NSF) Award No. 1645643.

Use of Generative-AI tools declaration

The authors declare that they have not used any Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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