



Research article

Preservation of relative aging orders under Cumulative-Hazard transformations

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Abstract: Preservation under a cumulative-hazard transformation is generally a sufficient-condition problem; universal preservation asks which generators work for every admissible baseline pair. We characterized this stronger property for the aging-faster (AF) and aging-faster-in-average (AFA) orders. For a proper survival generator Φ , put $\Omega = -\log\Phi$, $\omega = \Omega'$, $\eta_\omega(u) = u\omega'(u)/\omega(u)$, and $\kappa_\Omega(u) = u\omega(u)/\Omega(u)$. Among proper baselines with positive interior hazards and $X \geq_{\text{st}} Y$, universal AF preservation held if and only if ω was nondecreasing and η_ω was nonincreasing; universal AFA preservation held if and only if κ_Ω was nonincreasing. The sufficient directions were established distortion-preservation results. Our constructive necessity argument used proportional exponential baselines and smooth hazard ratios reaching one. Different-generator comparisons were followed by transitivity, and gamma thresholds recovered known AF and AFA upper-record comparisons in nonhomogeneous Poisson epoch-time form. Analytical power-threshold consequences and a family-specific eventual-failure result distinguished global conclusions from finite-horizon numerical observations. A reproducible censored component-life-test calibration illustrated model-based use of the criteria while separating fitted-model implications from evidence about the underlying populations.

Keywords: relative aging; aging faster order; aging faster-in-average order; cumulative hazard; cumulative-hazard transformation; nonhomogeneous Poisson process; epoch time; stochastic order

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1. Introduction

Relative aging concerns changes in risk between two lifetime distributions rather than only their

absolute magnitudes. We consider the aging-faster (AF) and aging-faster-in-average (AFA) orders. The hazard-ratio formulation was developed by Sengupta and Deshpande [1], while Rowell and Siegrist [2] placed related notions in a common ordering framework. Aging intensity provides a complementary cumulative-hazard description [3]. Background on lifetime aging and stochastic orders is given in [4,5]. Relative-aging comparisons also arise in coherent systems, heterogeneous structures, and dependent components [6–8].

A distortion function $h: [0,1] \rightarrow [0,1]$ is an increasing continuous distribution function on the unit interval, with $h(0) = 0$ and $h(1) = 1$. Applying h to a survival function gives another survival function. This construction is established in actuarial risk and system reliability [9–12]. The cumulative-hazard parameterization used here is exactly

$$\bar{F}_{T_{\Phi X}}(t) = \Phi(\Lambda_X(t)) = h(\bar{F}_X(t)), \quad h(p) = \Phi(-\log p).$$

Relative-aging preservation by distortion functions has been studied by Kayid and Almohsen [13] for relative hazard rates and by Kayid et al. [14] for aging intensity.

Upper-record comparisons in Kayid [15] are also relevant because upper records and nonhomogeneous Poisson process (NHPP) epoch times share a marginal gamma-threshold representation.

Stochastic comparisons of counting-process event times have a broader literature. Belzunce et al. [16] studied epoch and inter-epoch comparisons for NHPP and pure-birth processes, and Belzunce et al. [17] treated multivariate aging. Trend-renewal and generalized Pólya comparisons appear in [18,19]. These results motivate distinguishing a marginal time-change statement from a process-level dependence statement. Our gamma calculations use a power-series ratio criterion [20,21]; total positivity supplies a broader background for related monotonicity arguments [22].

The receiver operating characteristic (ROC) perspective in [23] gives another interpretation of the distortion. In Section 2, we fix notation and regularity. In Section 3, we develop the generator criteria, and in Section 4, we treat NHPP epoch times. In Section 5, we give numerical illustrations and a censored NIST life-test calibration [24]. In Section 6, we discuss scope and limitations.

Contributions summary. The main result is the constructive necessity in Theorem 3.5, which characterizes universal preservation over the stated class of proper baselines; Proposition 5.1 adds a family-specific explanation of eventual failure. The common-generator AF and AFA sufficient criteria are, respectively, the non-parenthetical branch of [13, Theorem 4] and [14, Theorem 12] (via [14, Lemma 6]); the elasticity transfer is in [14, Remarks 7 and 8]. Both parts of Theorem 4.1 recover earlier upper-record comparisons: AF via its restatement in [15, Proposition 20] and AFA via [15, Theorem 23]; broader AF conditions appear in [15, Proposition 21]. Different-generator results are followed by transitivity, while the power-threshold consequences, numerical checks, and censored-data uncertainty analysis provide interpretation and implementation. The threshold families are those of [14, Section 5.3 and Example 10]; cumulative-hazard coordinates do not define a new distortion class.

2. Relative-aging orders and cumulative-hazard time changes

Here, we collect the relative-aging orders and the cumulative-hazard time-change notation used in the proofs. The generator elasticities introduced below are the quantities that drive the preservation results in Section 3.

2.1. Basic notation and orders

Let X be a proper, nonnegative, absolutely continuous lifetime with survival function $\bar{F}_X$, density f_X , hazard λ_X , and cumulative hazard Λ_X . Throughout the theoretical comparisons, the two baselines have common support $(0, b)$, where $0 < b \leq \infty$, no atom at either endpoint, and strictly positive, locally continuously differentiable hazards on the interior. Their cumulative hazards tend to zero at the left endpoint and to infinity at b . The generator assumptions below are also imposed on positive interior arguments. All hazard and cumulative-hazard ratios are evaluated at $0 < t < b$; endpoint assertions mean one-sided limits. Positivity at $t = 0$ is not required. These are convenient sufficient regularity assumptions, not a minimal domain characterization [4,5]. We write

$$\lambda_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}, \quad \Lambda_X(t) = -\log \bar{F}_X(t).$$

Equivalently,

$$\Lambda_X(t) = \int_0^t \lambda_X(s) ds.$$

The average hazard rate is $A_X(t) = \Lambda_X(t)/t$ for $t > 0$.

For their usual, unshifted parameter ranges, Weibull, lognormal, and gamma lifetimes satisfy these conditions on $(0, \infty)$; thus, here $b = \infty$. Their hazards are positive and smooth at every interior age, even when the limit at zero vanishes or is unbounded; administrative censoring at a finite time does not truncate the underlying lifetime support. Membership in these families does not, by itself, establish the pairwise AF/AFA or stochastic-order assumptions.

Definition 2.1. (relative-aging orders) For two nonnegative absolutely continuous random variables X and Y :

1. $X \leq_{AF} Y$ if $\lambda_X(t)/\lambda_Y(t)$ is nondecreasing in t .
2. $X \leq_{AFA} Y$ if $A_X(t)/A_Y(t)$, equivalently $\Lambda_X(t)/\Lambda_Y(t)$, is nondecreasing in t .

The AFA order equals the aging-intensity (AI) order [3,14]. Indeed, for $I_X(t) = t\lambda_X(t)/\Lambda_X(t)$,

$$\frac{d}{dt} \log \frac{\Lambda_X(t)}{\Lambda_Y(t)} = \frac{I_X(t) - I_Y(t)}{t}.$$

Thus, AFA is equivalent to the pointwise inequality $I_X \geq I_Y$, not to monotonicity of I_X/I_Y . The usual stochastic order [4] is expressed by

$$X \geq_{st} Y \Leftrightarrow \bar{F}_X(t) \geq \bar{F}_Y(t) \quad \forall t \geq 0,$$

or, equivalently, $\Lambda_X(t) \leq \Lambda_Y(t)$ for every $0 < t < b$.

The next observation will be used repeatedly.

Lemma 2.1. If $X \leq_{AF} Y$, then $X \leq_{AFA} Y$.

Proof. Put $r(t) = \lambda_X(t)/\lambda_Y(t)$. Since r is nondecreasing,

$$\frac{\Lambda_X(t)}{\Lambda_Y(t)} = \frac{\int_0^t r(s) \lambda_Y(s) ds}{\int_0^t \lambda_Y(s) ds} \leq r(t).$$

Differentiation gives

$$\frac{d}{dt} \frac{\Lambda_X(t)}{\Lambda_Y(t)} = \frac{\lambda_Y(t)}{\Lambda_Y(t)} \left[r(t) - \frac{\Lambda_X(t)}{\Lambda_Y(t)} \right] \geq 0.$$

Hence, Λ_X/Λ_Y is nondecreasing.

2.2. A random cumulative-hazard threshold representation

Let $\Phi: [0, \infty) \rightarrow (0, 1]$ be a survival function satisfying $\Phi(0) = 1$ and $\Phi(u) \rightarrow 0$ as $u \rightarrow \infty$. Define

$$\Omega(u) = -\log \Phi(u), \quad \omega(u) = \Omega'(u).$$

Assume that Ω is twice continuously differentiable on $(0, \infty)$, with $\omega(u) > 0$, $\Omega(0) = 0$ by continuity, and $\Omega(u) \rightarrow \infty$. In particular, $\Omega(u) = \int_0^u \omega(v) dv$. For a baseline X satisfying Section 2.1, define the cumulative-hazard time change $T_\Phi X$ by

$$\bar{F}_{T_\Phi(X)}(t) = \Phi(\Lambda_X(t)).$$

If Z is a nonnegative random variable with survival function Φ and Λ_X is strictly increasing, then

$$T_\Phi(X) \stackrel{d}{=} \Lambda_X^{-1}(Z).$$

The transformation is the time at which the deterministic cumulative-hazard scale reaches a random threshold Z . No dependence between a baseline observation and Z is needed to define this marginal distribution; one may construct Z independently on an auxiliary probability space. No coupling assumption between the two compared populations is involved.

Remark 2.1. (distortion and ROC interpretations) The function $h(p) = \Phi(-\log p)$, continuously extended to $p = 0$, is a distortion function, whereas Φ is the threshold survival function. We use “distortion of a survival function” for $h \circ \bar{F}_X$, to keep these objects distinct. If $\bar{H}(u) = e^{-u}$, then $h(p) = \Phi\left(\bar{H}^{-1}(p)\right)$, which is the survival-quantile composition underlying ROC-based distortions [23].

This is a distributional interpretation, not an empirical ROC analysis. The same distortion framework already supports relative-aging comparisons [13, 14]. To avoid ambiguity, the argument and subscript distinguish the unit-interval distortion $h(p)$ from the hazard notation used for transformed lifetimes.

By differentiation,

$$h_{T_\Phi(X)}(t) = \lambda_X(t) \omega(\Lambda_X(t)),$$

while its cumulative hazard is

$$\Lambda_{T_\Phi(X)}(t) = \Omega(\Lambda_X(t)).$$

Two scale-free quantities associated with the generator are central to the sequel.

Definition 2.2. (generator elasticities) Define

$$\eta_\omega(u) = \frac{u\omega'(u)}{\omega(u)}$$

and

$$\kappa_\Omega(u) = \frac{u\Omega'(u)}{\Omega(u)} = \frac{u\omega(u)}{\Omega(u)}.$$

The first quantity is the derivative of $\log \omega(u)$ with respect to $\log u$, so nonincreasing η_ω

means concavity of $z \mapsto \log \omega(e^z)$. The second is the corresponding elasticity of Ω . Locally, a 1% increase in accumulated baseline hazard changes the hazard multiplier by approximately $\eta_\omega(u)\%$ and the transformed cumulative hazard by $\kappa_\Omega(u)\%$. These are dimensionless sensitivities to the cumulative-hazard level, not probabilities or derivatives with respect to calendar time. They describe how a threshold rule amplifies or attenuates accumulated risk.

Remark 2.2. (Why interior positivity matters) Omitting zero-denominator intervals is not an adequate substitute for a common positive domain. For example, let λ_Y equal 1,0,1 on $(0,1), (1,2), (2,\infty)$, respectively, and let λ_X equal 0.2,0.4,0.5 on those intervals, respectively. Both distributions are proper and $\Lambda_X \leq \Lambda_Y$. The ratio on the disconnected set where $\lambda_Y > 0$ takes the increasing values 0.2,0.5. Yet, for $t > 2$,

$$\frac{\Lambda_X(t)}{\Lambda_Y(t)} = \frac{1}{2} + \frac{1}{10(t-1)}, \quad \frac{h_{T_\Phi X}(t)}{h_{T_\Phi Y}(t)} = \frac{1}{4} + \frac{1}{20(t-1)}, \quad \Omega(u) = u^2.$$

Both ratios decrease. This demonstrates failure of the denominator-positive-region convention for Lemma 2.1 and AF preservation; the standing interior-positivity assumptions exclude this case. The piecewise hazards here illustrate the failure of that broader convention and are not examples in the revised regularity class.

3. General preservation theory

We now establish the AF and AFA preservation results. The analysis begins with a common generator, then moves to different generators and to the links between the corresponding AF and AFA conditions.

3.1. AF preservation under a common generator

The common-generator AF criterion [13, Theorem 4] is stated on the positive interior domain of Section 2. Remark 3.2 supplies its coordinate map.

Theorem 3.1. (common-generator AF preservation) Let Φ be a survival generator with $\omega > 0$. Assume that

1. ω is nondecreasing on $(0, \infty)$.
2. η_ω is nonincreasing on $(0, \infty)$.

If

$$X \geq_{\text{st}} Y, \quad X \leq_{\text{AF}} Y,$$

then

$$T_\Phi(X) \leq_{\text{AF}} T_\Phi(Y).$$

Proof. Set

$$x(t) = \Lambda_X(t), \quad q(t) = \frac{\Lambda_Y(t)}{\Lambda_X(t)}.$$

The usual stochastic order gives $q(t) \geq 1$. By Lemma 2.1, Λ_X/Λ_Y is nondecreasing, so q is nonincreasing. The transformed hazard ratio is

$$\frac{h_{T_{\Phi(X)}}(t)}{h_{T_{\Phi(Y)}}(t)} = \frac{\lambda_X(t)}{\lambda_Y(t)} \frac{\omega(x(t))}{\omega(q(t)x(t))}.$$

The first factor is nondecreasing by $X \leq_{\text{AF}} Y$. For the second factor, define $g(t) = \omega(x(t))/\omega(q(t)x(t))$. Direct differentiation yields

$$(\log g)' = \frac{x'}{x} \{\eta_{\omega}(x) - \eta_{\omega}(qx)\} - \frac{q'}{q} \eta_{\omega}(qx).$$

Because $q \geq 1$ and η_{ω} is nonincreasing, the bracketed term is nonnegative. Since ω is nondecreasing, $\eta_{\omega} \geq 0$; because $q' \leq 0$, the second term is also nonnegative. Hence, g is nondecreasing, and the product of the two nonnegative nondecreasing factors is nondecreasing. $\square$

The monotonicity assumption on ω is essential. The next example gives an explicit failure.

Example 3.1. (failure with a decreasing generator) Let $\Omega(u) = 2\sqrt{u}$, so that $\Phi(u) = \exp(-2\sqrt{u})$ and $\omega(u) = u^{-1/2}$. Then $\eta_{\omega}(u) = -1/2$ is constant, but ω is decreasing. Let Y be exponential with rate 1 and let X have

$$\lambda_X(t) = 1 - e^{-t}, \quad \Lambda_X(t) = t - 1 + e^{-t}.$$

Then $X \geq_{\text{st}} Y$ and $X \leq_{\text{AF}} Y$. Nevertheless,

$$\frac{h_{T_{\Phi(X)}}(t)}{h_{T_{\Phi(Y)}}(t)} = (1 - e^{-t}) \sqrt{\frac{t}{t - 1 + e^{-t}}}$$

is not nondecreasing: It is approximately 0.433, 1.042, 1.148, 1.110 at $t = 0.1, 1, 2, 5$, respectively. Thus, nonincreasing η_{ω} alone does not guarantee AF preservation.

The log-elasticity condition is also essential, even when the hazard generator is increasing.

Example 3.2. (failure when η_{ω} increases although ω increases) Let

$$\Omega(u) = u + \frac{u^3}{3}, \quad \omega(u) = 1 + u^2,$$

and set $\Phi(u) = \exp\{-\Omega(u)\}$. Then $\omega'(u) = 2u > 0$, but

$$\eta_{\omega}(u) = \frac{2u^2}{1 + u^2}$$

is strictly increasing. Let $Y \sim \text{Exp}(1)$ and let X have

$$\lambda_X(t) = 1 - e^{-t}, \quad \Lambda_X(t) = t - 1 + e^{-t}.$$

Because $\Lambda_X(t) \leq t = \Lambda_Y(t)$, we have $X \geq_{\text{st}} Y$, while $\lambda_X/\lambda_Y = 1 - e^{-t}$ is increasing, so $X \leq_{\text{AF}} Y$. After transformation,

$$R(t) = (1 - e^{-t}) \frac{1 + (t - 1 + e^{-t})^2}{1 + t^2}.$$

A direct calculation gives

$$R'(1) = \frac{e^{-3}}{2} (-e^3 - 5e + 4 + 4e^2) < 0.$$

Numerically, $R'(1) \approx -0.003005$. Hence, $T_\phi(X) \not\leq_{\text{AF}} T_\phi(Y)$. Thus, monotonicity of ω does not replace the requirement that η_ω will be nonincreasing.

Remark 3.1. (nontrivial baselines and endpoint ratios) If the right limit of λ_X/λ_Y at zero is at least one, its monotonicity implies $\lambda_X \geq \lambda_Y$. Together with $X \geq_{\text{st}} Y$, this forces equality in distribution. The initial-density formulation of this observation is valid when the relevant right-hand limits exist and the right limit of f_Y at zero is positive. It cannot be used by dividing two zero endpoint densities: $\lambda_X(t) = t$ and $\lambda_Y(t) = 2t$ give distinct proper distributions, $X \geq_{\text{st}} Y$ and $X \leq_{\text{AF}} Y$, although both endpoint densities vanish. The examples above also have positive hazards at every interior age.

3.2. Different generators

The common-generator result extends immediately when the two populations are subjected to different cumulative-hazard time changes.

Theorem 3.2. (two-generator AF preservation) Let Φ_1 satisfy the assumptions of Theorem 3.1. Let Φ_2 be another survival generator with positive hazard generator ω_2 . If

$$\frac{\omega_1(u)}{\omega_2(u)} \text{ is nondecreasing in } u,$$

then, for $X \geq_{\text{st}} Y$ and $X \leq_{\text{AF}} Y$,

$$T_{\Phi_1}(X) \leq_{\text{AF}} T_{\Phi_2}(Y).$$

If both generators are differentiable, the cross-generator condition is equivalent to

$$\eta_{\omega_1}(u) \geq \eta_{\omega_2}(u), \quad u > 0.$$

Proof. Theorem 3.1 gives $T_{\Phi_1}(X) \leq_{\text{AF}} T_{\Phi_1}(Y)$. For the same baseline Y ,

$$\frac{h_{T_{\Phi_1}(Y)}(t)}{h_{T_{\Phi_2}(Y)}(t)} = \frac{\omega_1(\Lambda_Y(t))}{\omega_2(\Lambda_Y(t))},$$

which is nondecreasing. Transitivity of $\leq_{\text{AF}}$ gives the desired ordering, and the elasticity equivalence follows from

$$\frac{d}{d \log u} \log \frac{\omega_1(u)}{\omega_2(u)} = \eta_{\omega_1}(u) - \eta_{\omega_2}(u).$$

This completes the proof.

Threshold interpretation. If Z_i has survival function Φ_i , its hazard is ω_i . Therefore, the cross-generator condition in Theorem 3.2 is exactly $Z_1 \leq_{\text{AF}} Z_2$. The theorem separates a baseline comparison from a threshold comparison.

3.3. AFA preservation

For the AFA criterion [14, Theorem 12], let U have distribution function h on $[0,1]$. Then, $Z = -\log U$ has survival function Φ , and its aging intensity is $I_Z(u) = \kappa_\rho(u)$. Thus, the decreasing-aging-intensity assumption in that theorem is exactly the nonincreasing cumulative-elasticity condition below. The identity follows from [14, Lemma 6].

Theorem 3.3. (Common-generator AFA preservation) Let Φ be a survival generator with $\omega > 0$. If κ_Ω is nonincreasing on $(0, \infty)$ and

$$X \geq_{\text{st}} Y, \quad X \leq_{\text{AFA}} Y,$$

then

$$T_\Phi(X) \leq_{\text{AFA}} T_\Phi(Y).$$

Proof. Put $x = \Lambda_X$ and $q = \Lambda_Y/\Lambda_X$. Again $q \geq 1$, while $X \leq_{\text{AFA}} Y$ means Λ_X/Λ_Y is nondecreasing, so q is nonincreasing. The transformed cumulative-hazard ratio is

$$G(t) = \frac{\Omega(x(t))}{\Omega(q(t)x(t))}.$$

Its logarithmic derivative is

$$(\log G)' = \frac{x'}{x} \{\kappa_\Omega(x) - \kappa_\Omega(qx)\} - \frac{q'}{q} \kappa_\Omega(qx).$$

The first term is nonnegative because $q \geq 1$ and κ_Ω is nonincreasing. The second is nonnegative because $q' \leq 0$ and $\kappa_\Omega > 0$. Hence, G is nondecreasing. $\square$

The monotonicity requirement on κ_Ω is not merely technical.

Example 3.3. (failure when cumulative elasticity increases) Take $\Omega(u) = ue^u$, so $\Phi(u) = \exp(-ue^u)$ and $\kappa_\Omega(u) = 1 + u$, which is strictly increasing. Let $X \sim \text{Exp}(1)$ and $Y \sim \text{Exp}(2)$. Then, $X \geq_{\text{st}} Y$ and $X \leq_{\text{AFA}} Y$ because $\Lambda_X/\Lambda_Y = 1/2$. However,

$$\frac{\Lambda_{T_\Phi(X)}(t)}{\Lambda_{T_\Phi(Y)}(t)} = \frac{\Omega(t)}{\Omega(2t)} = \frac{1}{2}e^{-t},$$

which is decreasing. Therefore, AFA preservation fails.

Theorem 3.4. (two-generator AFA preservation) Let Φ_1 satisfy the assumptions of Theorem 3.3, and let Φ_2 be another survival generator satisfying Section 2.2, with positive cumulative generator Ω_2 on $(0, \infty)$. If

$$\frac{\Omega_1(u)}{\Omega_2(u)} \quad \text{is nondecreasing in } u,$$

then, for $X \geq_{\text{st}} Y$ and $X \leq_{\text{AFA}} Y$,

$$T_{\Phi_1}(X) \leq_{\text{AFA}} T_{\Phi_2}(Y).$$

When differentiable, the cross-generator condition is equivalent to

$$\kappa_{\Omega_1}(u) \geq \kappa_{\Omega_2}(u), \quad u > 0.$$

Proof. Theorem 3.3 first gives the common-generator AFA ordering after applying the generator Φ_1 to both baseline lifetimes. The hypothesis that the ratio Ω_1/Ω_2 is nondecreasing then gives the corresponding AFA ordering between the two transformed versions of the same baseline Y . Transitivity of the AFA order completes the proof.

Threshold interpretation. The condition Ω_1/Ω_2 nondecreasing is exactly $Z_1 \leq_{\text{AFA}} Z_2$. Like Theorem 3.2, this extension follows by transitivity after a common-generator comparison.

3.4. A transfer principle linking AF and AFA generator conditions

An elementary integral proof of the transfer implication [14, Remarks 7 and 8] links the two generator conditions.

Proposition 3.1. (elasticity transfer) Suppose $\omega > 0$ is differentiable, η_ω is nonincreasing, and $\eta_\omega(u) > -1$. Then, κ_Ω is nonincreasing. In particular, every generator satisfying the assumptions of Theorem 3.1 also satisfies the generator condition of Theorem 3.3.

Proof. For $0 < s \leq u$, monotonicity of η_ω gives

$$\log \frac{\omega(u)}{\omega(s)} = \int_s^u \frac{\eta_\omega(v)}{v} dv \geq \eta_\omega(u) \log \frac{u}{s}.$$

Hence,

$$\omega(s) \leq \omega(u) \left(\frac{s}{u}\right)^{\eta_\omega(u)}.$$

After integration,

$$\Omega(u) = \int_0^u \omega(s) ds \leq \frac{u\omega(u)}{1 + \eta_\omega(u)},$$

so $\kappa_\Omega(u) \geq 1 + \eta_\omega(u)$. Since

$$\frac{d \log \kappa_\Omega(u)}{d \log u} = 1 + \eta_\omega(u) - \kappa_\Omega(u),$$

we obtain $\kappa_\Omega'(u) \leq 0$.

A similar transfer holds between two generators.

Proposition 3.2. (cross-generator transfer) If $\omega_1(u)/\omega_2(u)$ is nondecreasing, then

$$\frac{\Omega_1(u)}{\Omega_2(u)} \text{ is nondecreasing}$$

and

$$\kappa_{\Omega_1}(u) \geq \kappa_{\Omega_2}(u).$$

Proof. Apply Lemma 2.1 to threshold lifetimes Z_1, Z_2 with hazards ω_1, ω_2 and cumulative hazards Ω_1, Ω_2 . This yields monotonicity of Ω_1/Ω_2 . The second assertion follows from

$$\frac{d}{d \log u} \log \frac{\Omega_1(u)}{\Omega_2(u)} = \kappa_{\Omega_1}(u) - \kappa_{\Omega_2}(u) \geq 0.$$

This also makes it clear why the elasticity inequality is part of the conclusion. $\square$

Propositions 3.1 and 3.2 give a useful hierarchy: A generator pair strong enough for the AF theorem is automatically ordered in the sense needed for the AFA theorem. Thus, within the cumulative-hazard time-change framework, the AF preservation conditions and their cross-generator ordering are naturally stronger than the corresponding AFA conditions. Table 1 summarizes the generator conditions and the corresponding preservation and failure cases.

Table 1. Generator conditions and preservation or failure; cross-reference to Theorem 3.5.

Case/generator	ω nondecreasing	η_ω nonincreasing	κ_Ω nonincreasing	Conclusion
Example 3.1 $\Omega(u) = 2\sqrt{u}$	No (decreasing)	Yes (constant)	Yes (constant)	AF fails; the AFA generator criterion holds.
Example 3.2 $\Omega(u) = u + u^3/3$	Yes	No (increasing)	No (increasing)	AF fails for the exhibited baseline pair.
Example 3.3 $\Omega(u) = ue^u$	Yes	No (increasing)	No (increasing)	AFA fails for the exhibited exponential pair.
Theorem 3.5(a)	Required	Required	Follows	Universal AF; also AFA under its baseline inputs.
Theorem 3.5(b)	Not required	Not required	Required	Universal AFA; AF is not implied.

All rows use the appropriate AF or AFA baseline relation together with $X \geq_{\text{st}} Y$ and the regularity assumptions of Section 2.1. “Fails” refers to the counterexample shown, not to every baseline pair. The universal rows concern a fixed proper generator and all admissible baseline pairs.

3.5. Analytical consequences beyond gamma thresholds

The AF conditions describe a shape-constrained class: $z \mapsto \log \omega(e^z)$ must be nondecreasing and concave. Multiplying Ω and ω by the same positive constant leaves both elasticities unchanged. The following are analytical consequences, not numerical evidence for an unspecified process class. For the power family $\Omega_\alpha(u) = u^\alpha$, $\alpha > 0$, the survival generator is $\Phi_\alpha(u) = e^{-u^\alpha}$ and

$$\eta_{\omega_\alpha}(u) = \alpha - 1, \quad \kappa_{\Omega_\alpha}(u) = \alpha.$$

Thus, the common-generator AFA condition holds for every $\alpha > 0$, whereas the universal AF condition holds for $\alpha \geq 1$. The following two-threshold consequence makes the role of different powers and positive scale constants explicit.

Corollary 3.1. (power thresholds without a magnitude ordering) Let $\Omega_i(u) = c_i u^{\alpha_i}$ with $c_i > 0$. If $\alpha_1 \geq \alpha_2 \geq 1$ and $X \leq_{\text{AF}} Y$, then $T_{\Phi_1} X \leq_{\text{AF}} T_{\Phi_2} Y$. If $\alpha_1 \geq \alpha_2 > 0$ and $X \leq_{\text{AFA}} Y$, then $T_{\Phi_1} X \leq_{\text{AFA}} T_{\Phi_2} Y$. Neither assertion requires $X \geq_{\text{st}} Y$.

Proof. Put $r = \lambda_X/\lambda_Y$ and $\rho_i = \lambda_i/\Lambda_i$. In the first case Lemma 2.1 gives $\rho_X \geq \rho_Y$, and

$$\frac{d}{dt} \log \frac{h_{T_{\Phi_1} X}}{h_{T_{\Phi_2} Y}} = \frac{r'}{r} + (\alpha_2 - 1)(\rho_X - \rho_Y) + (\alpha_1 - \alpha_2)\rho_X \geq 0.$$

In the second case, the cumulative ratio has logarithmic derivative

$$\alpha_2(\rho_X - \rho_Y) + (\alpha_1 - \alpha_2)\rho_X \geq 0.$$

The positive constants affect levels, not these derivatives.

For a concrete non-NHPP use, $\Omega(u) = u^2$ gives $\bar{F}_{T_{\Phi} X}(t) = \exp[-\Lambda_X(t)^2]$: A Weibull

threshold on accumulated risk. The corollary compares such threshold lifetimes analytically, even when only a baseline AF or AFA relation is available. This describes a specified threshold model, not an assertion about arbitrary renewal processes.

A non-power example is

$$\omega_S(u) = 1 - e^{-u}, \quad \Omega_S(u) = u - 1 + e^{-u}, \quad \Phi_S(u) = \exp(-u + 1 - e^{-u}).$$

The induced distortion $h_S(p) = p \exp(1 - p)$ is [14, Example 10]; it also appears in [13, Example 14] under a reversed-hazard comparison. Here, ω_S increases and $\eta_S(u) = u/(e^u - 1)$ decreases: The numerator of its derivative is $e^u(1 - u) - 1$, which vanishes at zero and has derivative $-ue^u < 0$. Theorem 3.1 therefore supplies AF preservation, and Proposition 3.1 supplies AFA preservation. In Section 5.5, we illustrate the resulting margins.

Remark 3.2. (exact identification of the earlier sufficient criteria) Write $d(p) = \Phi(-\log p)$ and $u = -\log p$. In the notation of [13, Theorem 4],

$$\psi(p) = \frac{d(p)}{pd'(p)} = \frac{1}{\omega(u)}, \quad -p \log p \frac{\psi'(p)}{\psi(p)} = \eta_\omega(u).$$

Thus, ψ nondecreasing in p is equivalent to ω nondecreasing in u , and the second expression nondecreasing in p is equivalent to η_ω nonincreasing in u . Together with the same AF and usual stochastic-order assumptions, this identifies the non-parenthetical branch of that theorem with Theorem 3.1. For [14, Lemma 6],

$$K_d(p) = \frac{\log p}{\log d(p)} \frac{pd'(p)}{d(p)}, \quad K_d(e^{-u}) = \kappa_\Omega(u).$$

Theorem 3.3 therefore restates [14, Theorem 12] in these coordinates. In particular, changing variables reverses monotonicity in $p = e^{-u}$.

3.6. Necessary and sufficient conditions for universal preservation

For a fixed proper generator, universal preservation means preservation for every proper baseline pair on $(0, \infty)$ satisfying Section 2.1 and the two indicated order relations. We now characterize this property.

Theorem 3.5. (universal generator characterization) Let Φ satisfy Section 2.2.

- (a) The implication $X \geq_{\text{st}} Y, X \leq_{\text{AF}} Y \Rightarrow T_\Phi X \leq_{\text{AF}} T_\Phi Y$ holds universally if and only if ω is nondecreasing and η_ω is nonincreasing on $(0, \infty)$.
- (b) The implication $X \geq_{\text{st}} Y, X \leq_{\text{AFA}} Y \Rightarrow T_\Phi X \leq_{\text{AFA}} T_\Phi Y$ holds universally if and only if κ_Ω is nonincreasing on $(0, \infty)$.

Proof. Theorems 3.1 and 3.3 prove sufficiency. For necessity in (a), fix $0 < u < v$, take $Y \sim \text{Exp}(1)$ and $X \sim \text{Exp}(c)$, where $c = u/v$. Both required baseline orders hold. At $t = v$, the logarithmic derivative of the transformed hazard ratio is

$$\frac{d}{dt} \log \left[c \frac{\omega(ct)}{\omega(t)} \right]_{t=v} = \frac{\eta_\omega(u) - \eta_\omega(v)}{v}.$$

Universal AF preservation therefore forces $\eta_\omega(u) \geq \eta_\omega(v)$.

To establish the other condition, write $f = \omega'/\omega$. Define $\varepsilon = (u/v)^2$, $\delta = 2uv/(u + v)$, and the continuously differentiable nondecreasing function

$$S(z) = \begin{cases} 0, & z \leq 0, \\ 3z^2 - 2z^3, & 0 < z < 1, \\ 1, & z \geq 1. \end{cases}$$

Set $\lambda_Y(t) = 1$ and

$$r(t) = \lambda_X(t) = \varepsilon + (1 - \varepsilon)S\left(\frac{t - v + \delta}{\delta}\right).$$

Then $0 < r \leq 1$, r is nondecreasing, $r(v) = 1$, $r'(v) = 0$, and $r = 1$ for $t \geq v$. Both lifetimes are proper and satisfy the two baseline orders. Direct integration gives

$$\Lambda_X(v) = \varepsilon v + \frac{(1 - \varepsilon)\delta}{2} = u, \quad \Lambda_Y(v) = v.$$

The transformed log-hazard-ratio derivative at v is $f(u) - f(v)$. Hence, f is nonincreasing. If $f(u_0) < 0$ for some u_0 , then for $u \geq u_0$,

$$\omega(u) \leq \omega(u_0)\exp[f(u_0)(u - u_0)],$$

so $\int_{u_0}^{\infty} \omega(u) du < \infty$. Together with $\Omega(u_0) < \infty$, this contradicts $\Omega(\infty) = \infty$. Therefore, $f \geq 0$, and ω is nondecreasing.

For (b), the same constant-hazard test pair gives transformed cumulative-hazard ratio $\Omega(ct)/\Omega(t)$ and log-derivative $[\kappa_{\Omega}(u) - \kappa_{\Omega}(v)]/v$ at v . Its nonnegativity for every $0 < u < v$ is precisely the required monotonicity. $\square$

The quantifier in Theorem 3.5 is universal; its generator conditions need not be necessary for each individual pair. Corollary 3.1 illustrates this distinction.

4. Nonhomogeneous Poisson epoch times

We now specialize the generator calculations to nonhomogeneous Poisson epoch times.

4.1. Gamma-threshold representation

Let N_X be an NHPP whose intensity $\lambda_X(t)$ is positive and locally continuously differentiable for $t > 0$, and let its mean-value function be

$$\Lambda_X(t) = \int_0^t \lambda_X(s) ds.$$

Assume $\Lambda_X(t) \rightarrow \infty$ so that every epoch time is proper (i.e., finite almost surely). Let $T_{X,n}$ denote the time of the n th event. Standard NHPP theory gives

$$\Pr(T_{X,n} > t) = e^{-\Lambda_X(t)} \sum_{j=0}^{n-1} \frac{\Lambda_X(t)^j}{j!}.$$

Define

$$e_n(u) = \sum_{j=0}^{n-1} \frac{u^j}{j!}, \quad \Phi_n(u) = e^{-u} e_n(u).$$

Then Φ_n is the survival function of a $\text{Gamma}(n, 1)$ random variable, so

$$T_{X,n} \stackrel{d}{=} T_{\Phi_n}(X),$$

where $X = T_{X,1}$ has hazard λ_X and cumulative hazard Λ_X . Equivalently,

$$\Lambda_X(T_{X,n}) \sim \text{Gamma}(n, 1).$$

The corresponding generator functions are

$$\Omega_n(u) = u - \log e_n(u)$$

and

$$\omega_n(u) = \Omega_n'(u) = \frac{u^{n-1}}{(n-1)! e_n(u)}.$$

Therefore, the hazard and cumulative hazard of the n th epoch time are

$$h_{X,n}(t) = \lambda_X(t) \omega_n(\Lambda_X(t))$$

and

$$\Lambda_{X,n}(t) = \Omega_n(\Lambda_X(t)).$$

No independence assumption is required between the two NHPPs compared in any result below. Independent increments are, of course, part of the definition used to obtain the marginal epoch-time distributions, but the comparison theorems depend only on the two marginal intensity structures.

Record-value connection. Let $R_{X,n}$ be the n th upper record from independent and identically distributed observations with lifetime law X , with the first observation counted as record 1. Then $\Lambda_X(R_{X,n}) \sim \text{Gamma}(n, 1)$ [15], so $R_{X,n}$ and $T_{X,n}$ have the same marginal distribution. The identification concerns distributions, not equality of the underlying record and counting processes.

4.2. Generator elasticities for the gamma thresholds

For $n \geq 1$, define

$$\eta_n(u) = \frac{u \omega_n'(u)}{\omega_n(u)}, \quad \kappa_n(u) = \frac{u \omega_n(u)}{\Omega_n(u)}.$$

The following lemma verifies the common-generator AF assumptions in a particularly transparent way.

Lemma 4.1. For every $n \geq 1$, ω_n is nondecreasing and η_n is nonincreasing. Moreover, for $n \geq 2$, ω_n is strictly increasing and η_n is strictly decreasing on $(0, \infty)$.

Proof. For $u > 0$, let $J_{n,u}$ have the truncated-Poisson distribution

$$\Pr(J_{n,u} = j) = \frac{u^j/j!}{e_n(u)}, \quad j = 0, \dots, n-1.$$

A direct logarithmic differentiation gives

$$\eta_n(u) = (n-1) - u \frac{e_{n-1}(u)}{e_n(u)} = (n-1) - \mathbb{E}[J_{n,u}].$$

Since $0 \leq J_{n,u} \leq n-1$, $\eta_n(u) \geq 0$, and hence ω_n is nondecreasing. The family $\{J_{n,u}\}$ is a one-parameter exponential family with natural parameter $\theta = \log u$ and sufficient statistic J . Therefore,

$$\frac{d}{du} \mathbb{E}[J_{n,u}] = \frac{1}{u} \text{Var}(J_{n,u}) \geq 0,$$

with strict inequality for $n \geq 2$. Thus, $\eta_n'(u) \leq 0$ for all $n \geq 1$, with strict inequality for $n \geq 2$. Moreover, for $n \geq 2$ and $u > 0$, the truncated-Poisson distribution assigns positive probability to values below $n-1$, so its expectation is strictly less than $n-1$, and hence $\eta_n(u) > 0$. By the defining identity for η_n , this gives $\omega_n'(u) > 0$. Therefore, ω_n is strictly increasing, and η_n is strictly decreasing on $(0, \infty)$ for $n \geq 2$.

The ordering across epoch indices follows from a power-series ratio argument.

Lemma 4.2. If $m \geq n$, then $\omega_m(u)/\omega_n(u)$ is nondecreasing in $u > 0$. Equivalently,

$$\eta_m(u) \geq \eta_n(u), \quad u > 0.$$

Proof. It is enough to prove the consecutive-index case. From the explicit formula,

$$\frac{\omega_{n+1}(u)}{\omega_n(u)} = \frac{ue_n(u)}{ne_{n+1}(u)}.$$

In the numerator $ue_n(u)$, the coefficient of u^j is 0 for $j = 0$ and $1/(j-1)!$ for $j = 1, \dots, n$. In the denominator $ne_{n+1}(u)$, the coefficient is $n/j!$. The corresponding coefficient ratios are

$$0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n},$$

which form a nondecreasing sequence. The Biernacki-Krzyż monotonicity principle [20,21] therefore implies that ω_{n+1}/ω_n is nondecreasing. Taking the derivative with respect to $\log u$ yields $\eta_{n+1} - \eta_n \geq 0$, and induction gives the result.

The AFA properties now follow without a separate NHPP-specific technical lemma.

Corollary 4.1. For every n , κ_n is nonincreasing. If $m \geq n$, then

$$\kappa_m(u) \geq \kappa_n(u), \quad u > 0,$$

and $\Omega_m(u)/\Omega_n(u)$ is nondecreasing.

Proof. Lemma 4.1 and Proposition 3.1 show that κ_n is nonincreasing. Lemma 4.2 and Proposition 3.2 give the cross-index assertions.

The two elasticities are displayed together in Figure 1. The curves are illustrations of the preceding proofs, not their justification.

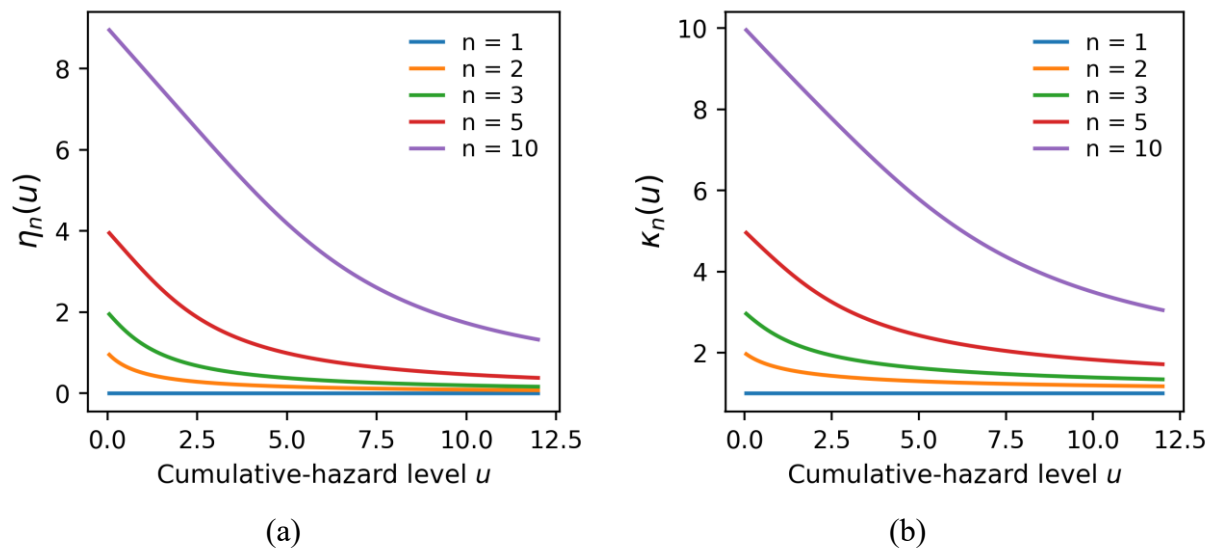


Figure 1. Gamma-threshold elasticities for $n = 1, 2, 3, 5, 10$: (a) Hazard-generator log-elasticity; and (b) cumulative elasticity.

4.3. Cross-epoch preservation

The next result uses the gamma-threshold criteria; its second part requires only the cumulative-order baseline input.

Theorem 4.1. (NHPP cross-epoch relative-aging preservation) Consider two NHPPs with first-event lifetimes X and Y and epoch times $T_{X,m}$ and $T_{Y,n}$. Let $m \geq n$ and assume $X \geq_{\text{st}} Y$.

1. If $X \leq_{\text{AF}} Y$, then

$$T_{X,m} \leq_{\text{AF}} T_{Y,n}.$$

2. If $X \leq_{\text{AFA}} Y$, then

$$T_{X,m} \leq_{\text{AFA}} T_{Y,n}.$$

Proof. For part 1, Lemmas 4.1 and 4.2 verify the assumptions of Theorem 3.2 with $\Phi_1 = \Phi_m$ and $\Phi_2 = \Phi_n$. Part 2 follows from Theorem 3.4 and Corollary 4.1. $\square$

Remark 4.1. The same-index case $m = n$ is a direct preservation result. The cross-index statement $m > n$ is stronger: It combines preservation across populations with the intrinsic relative-aging ordering of the gamma-threshold family.

Remark 4.2. The theorem is a marginal-distribution result. The two NHPPs may be constructed on unrelated probability spaces or coupled in an arbitrary way. Cross-process independence plays no role.

Example 4.1. (an AFA baseline not satisfying AF) Take $\Lambda_Y(t) = t$ and $\Lambda_X(t) = t(0.8 - 0.4e^{-t})$. The cumulative-hazard ratio increases from 0.4 to 0.8, so $X \geq_{\text{st}} Y$ and $X \leq_{\text{AFA}} Y$. However, $\lambda_X(t)/\lambda_Y(t) = 0.8 + 0.4(t-1)e^{-t}$ has derivative $0.4(2-t)e^{-t}$ and is not nondecreasing on the support. Thus, Theorem 4.1(2) applies to a baseline pair to which the AF-input implication in part 1 does not apply. This example distinguishes the hypotheses.

4.4. Limiting behavior

The gamma-threshold representation also clarifies two distinct asymptotic regimes. For fixed n ,

$$\lim_{u \rightarrow \infty} \omega_n(u) = 1, \quad \lim_{u \rightarrow \infty} \eta_n(u) = 0.$$

Moreover,

$$\lim_{u \rightarrow \infty} \kappa_n(u) = 1.$$

Thus, at large cumulative-intensity levels, the epoch-time hazard generator approaches the identity-hazard multiplier. In contrast, for fixed $u > 0$,

$$\lim_{n \rightarrow \infty} \omega_n(u) = 0,$$

because factorial growth dominates u^{n-1} while $e_n(u) \rightarrow e^u$. Hence, for fixed calendar time t and fixed $\Lambda_X(t)$,

$$\lim_{n \rightarrow \infty} h_{X,n}(t) = 0.$$

These two limits should not be conflated: $u \rightarrow \infty$ with fixed n and $n \rightarrow \infty$ with fixed u describe fundamentally different regimes.

5. Numerical investigation and a censored-life-test application

In this section, we complement the analytical results with reproducible computation. We evaluate closed-form preservation margins derived from the preceding theory for explicit baseline and threshold families, record their grid minima and locations, and examine horizon sensitivity inside and outside the certified region (Sections 5.1–5.5). Section 5.6 then illustrates model-based use of the criteria through the public NIST/SEMATECH censored component life test. Throughout, a positive grid minimum is a finite-horizon consistency statement, not a probability of preservation; whenever the analytical hypotheses hold, the preservation guarantee is independent of the horizon and the numerical grid.

Time t and horizons in Sections 5.1–5.5 are dimensionless; no physical time unit is assigned to these synthetic examples. In Section 5.6, we instead use the observed time in hours.

5.1. Baselines, design, and interpretation

We first isolate baseline trend and relative aging through

$$\lambda_Y(t) = a + bt, \quad r(t) = r_\infty - (r_\infty - r_0)e^{-\gamma t}, \quad \lambda_X(t) = r(t)\lambda_Y(t),$$

where $a > 0$, $b \geq 0$, $\gamma > 0$ and $0 < r_0 < r_\infty$. For $r_\infty \leq 1$, r is nondecreasing and $\lambda_X \leq \lambda_Y$, so AF and $X \geq_{\text{st}} Y$ hold. Write

$$y(t) = \Lambda_Y(t) = at + \frac{b}{2}t^2, \quad B_Y(t) = \frac{a(1 - e^{-\gamma t})}{\gamma} + \frac{b[1 - (1 + \gamma t)e^{-\gamma t}]}{\gamma^2},$$

$$x(t) = \Lambda_X(t) = r_\infty y(t) - (r_\infty - r_0)B_Y(t).$$

The linear baseline is selected for transparent integration and control of a constant versus increasing hazard trend, not to represent every engineering lifetime. In Section 5.5, we add decreasing-

and increasing-hazard Weibull baselines. In every case the theorem, not numerical generalization, justifies preservation when its hypotheses hold. Scenarios A, B, and C use $(a, b, r_0, r_\infty, \gamma) = (0.6, 0.1, 0.4, 0.8, 0.5)$, $(1, 0.05, 0.85, 0.98, 0.15)$, and $(0.3, 0.2, 0.1, 0.95, 1.5)$, respectively.

5.2. Margins and their minimum locations

For threshold indices (m, n) , define $\rho_i = \lambda_i/\Lambda_i$ and the log-derivative margins

$$D_{m,n}^{AF}(t) = \frac{d}{dt} \log \frac{h_{X,m}(t)}{h_{Y,n}(t)} = \frac{r'(t)}{r(t)} + \rho_X(t)\eta_m(x(t)) - \rho_Y(t)\eta_n(y(t)),$$

$$D_{m,n}^{AFA}(t) = \frac{d}{dt} \log \frac{\Lambda_{X,m}(t)}{\Lambda_{Y,n}(t)} = \rho_X(t)\kappa_m(x(t)) - \rho_Y(t)\kappa_n(y(t)).$$

These expressions require no numerical differentiation. We evaluate 20,000 equally spaced points in $[0.1, 15]$. Table 2 reports each minimum and the grid point attaining it. All 24 minima occur at the right endpoint. This is a grid statement; it does not assert a positive lower bound on the infinite horizon. The positive margins become smaller near the selected horizon in these examples. Scenario B has $r_\infty = 0.98$ and illustrates proximity to the usual stochastic-order boundary. Appendix A compares 5,000, 10,000, 20,000, and 40,000 points and supports the chosen density for the reported finite-horizon summaries. Figures 2 and 3 display, respectively, the transformed hazard ratios and transformed cumulative-hazard ratios for Scenario A as functions of dimensionless time.

Table 2. Minimum log-derivative margins and their grid locations on $[0.1, 15]$.

Scenario	(m, n)	$\min D^{AF}$	AF min. time	$\min D^{AFA}$	AFA min. time
A	(2,1)	0.006656	15	0.020032	15
A	(3,2)	0.009029	15	0.022514	15
A	(5,3)	0.020915	15	0.042469	15
A	(10,5)	0.076003	15	0.122956	15
B	(2,1)	0.006466	15	0.014140	15
B	(3,2)	0.007288	15	0.014398	15
B	(5,3)	0.013954	15	0.026288	15
B	(10,5)	0.043710	15	0.072949	15
C	(2,1)	0.004674	15	0.014070	15
C	(3,2)	0.005322	15	0.013786	15
C	(5,3)	0.011851	15	0.026246	15
C	(10,5)	0.038595	15	0.071754	15

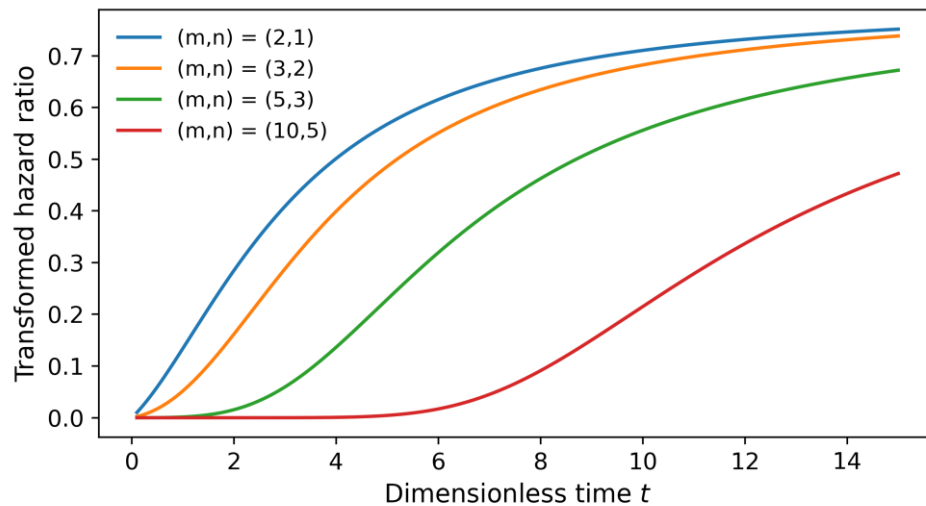


Figure 2. Gamma-threshold hazard ratios for Scenario A. The time axis t is dimensionless.

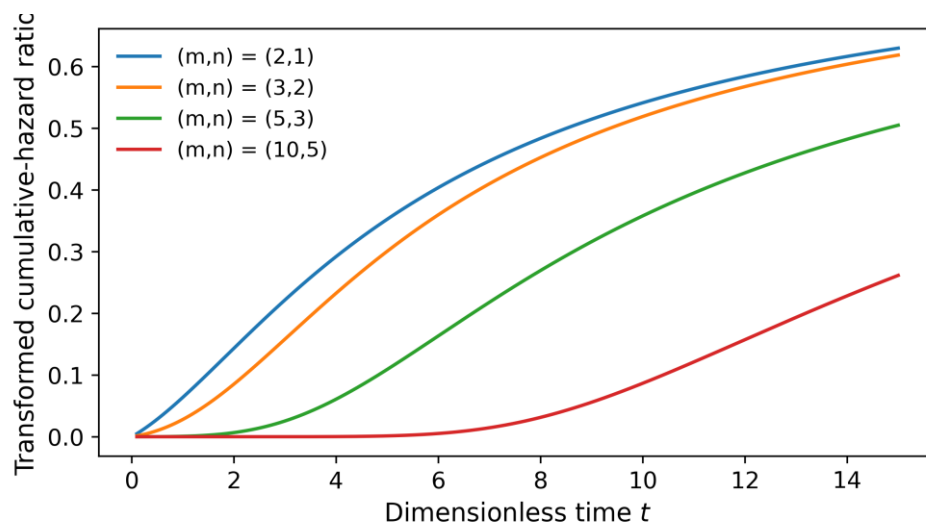


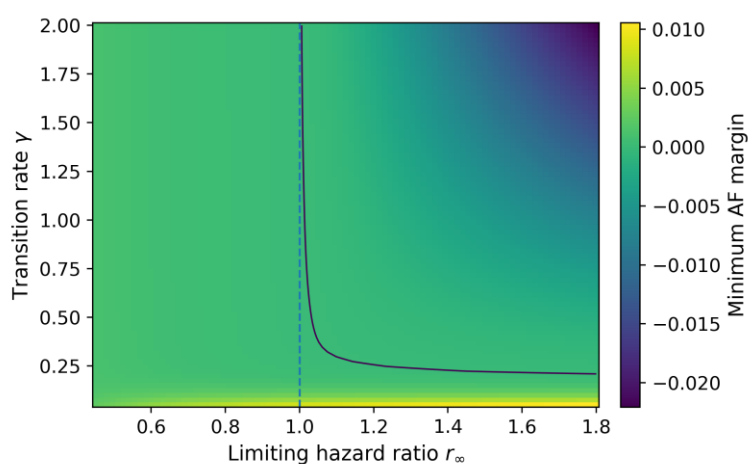
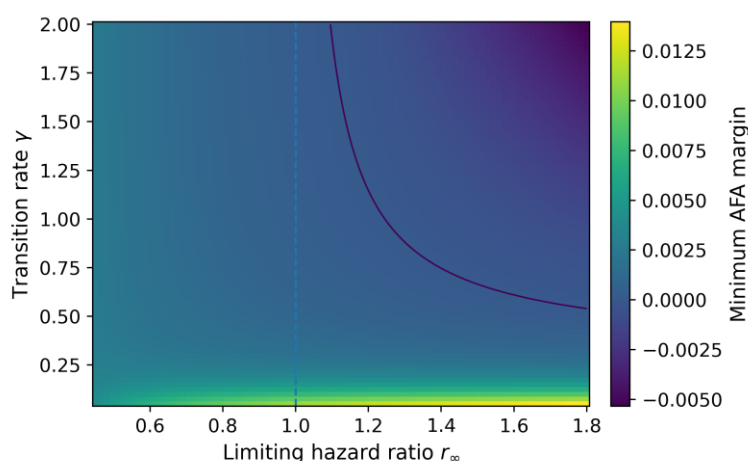
Figure 3. Gamma-threshold cumulative-hazard ratios for Scenario A. The time axis t is dimensionless.

5.3. Finite-horizon sensitivity outside the sufficient region

Set $a = 0.6$, $b = 0.1$, $r_0 = 0.4$, and $(m,n) = (2,2)$. Use 91 equally spaced values of r_∞ in $[0.45, 1.80]$ and 80 values of γ in $[0.05, 2]$, giving 7,280 configurations. Of these, 2,960 have $r_\infty \leq 1$ (“certified”) and 4,320 have $r_\infty > 1$ (“outside”). For each horizon $H = 15, 30, 60, 120$, minimize both margins on 20,000 equally spaced points in $[0.1, H]$. A negative margin below -10^{-10} is counted as a detected violation. Full configuration-level minima and locations are supplied with the code. Grid-density sensitivity is reported in Appendix A. Table 3 summarizes the finite-horizon grid-violation classifications.

Table 3. Percentages with no detected grid violation, not probabilities of global preservation.

Horizon H	Region	Count	AF (%)	AFA (%)	Both (%)
15	certified	2960	100.00	100.00	100.00
15	outside	4320	23.54	64.03	23.54
30	certified	2960	100.00	100.00	100.00
30	outside	4320	12.45	47.11	12.45
60	certified	2960	100.00	100.00	100.00
60	outside	4320	6.06	36.32	6.06
120	certified	2960	100.00	100.00	100.00
120	outside	4320	2.48	29.33	2.48

**Figure 4.** Minimum AF margin on $[0.1, 30]$ over the parameter grid. The minimization time t is dimensionless; the plotted axes are model parameters.**Figure 5.** Minimum AFA margin on $[0.1, 30]$ over the parameter grid. The minimization time t is dimensionless; the plotted axes are model parameters.

The original $H = 30$ percentages are recovered. Extending the horizon reveals additional violations outside the certified region. Within the theorem's domain, the global conclusion is analytical

and does not depend on the chosen horizon. The numerical percentages there are a consistency check, not the source of the guarantee. Figures 4 and 5 display the $H = 30$ margin surfaces; the dashed line is $r_\infty = 1$ and the contour is the zero level.

5.4. Eventual failure and the meaning of the stochastic-order condition

The finite-horizon results in Section 5.3 raise a natural question: Can an “outside” configuration appear preserved on every practically examined time window and yet fail globally? The following result answers this question for the boundary family of Section 5.1. It shows that, once the stochastic-order condition is violated through $r_\infty > 1$, AF and AFA preservation margins under the common second-gamma threshold eventually become negative, even though the violation may occur only at sufficiently large ages. This clarifies the role of the stochastic-order assumption as a condition supporting global preservation rather than merely finite-horizon numerical behavior.

Proposition 5.1. (eventual failure in the specified boundary family) In Section 5.1, suppose $r_\infty > 1$ and apply the common second-gamma threshold. Put $y = \Lambda_Y$. Then, as $t \rightarrow \infty$,

$$D_{2,2}^{AF}(t) \sim \left(\frac{1}{r_\infty} - 1\right) \frac{\lambda_Y(t)}{y(t)^2}, \quad D_{2,2}^{AFA}(t) \sim \left(\frac{1}{r_\infty} - 1\right) \frac{\lambda_Y(t) \log y(t)}{y(t)^2}.$$

Both margins are eventually negative and tend to zero. Thus, none of the “outside” configurations in Table 3 are globally preserved, even when no violation appears in a finite window.

Proof. Put $C = (r_\infty - r_0)(a/\gamma + b/\gamma^2)$. Then $x = r_\infty y - C + O((1+t)e^{-\gamma t})$, $x' = r_\infty y' + O((1+t)e^{-\gamma t})$, and r'/r is exponentially small. Since $\omega_2(u) = u/(1+u)$,

$$D_{2,2}^{AF} = r'/r + \frac{x'}{x(1+x)} - \frac{y'}{y(1+y)},$$

which gives the first equivalent. For $\Omega_2(u) = u - \log(1+u)$,

$$\frac{\omega_2(u)}{\Omega_2(u)} = \frac{1}{u} + \frac{\log u - 1}{u^2} + O\left(\frac{(\log u)^2}{u^3}\right).$$

Substitute this expansion into $D_{2,2}^{AFA} = x'\omega_2(x)/\Omega_2(x) - y'\omega_2(y)/\Omega_2(y)$. The difference $x'/x - y'/y$ is $O(y'/y^2)$, whereas the leading logarithmic term is $(1/r_\infty - 1)y'\log y/y^2$. This proves the second equivalent. Both are negative eventually because $r_\infty > 1$. $\square$

For a simple numerical failure, take $\lambda_Y = 1$ and $r(t) = 1.2 - 0.7e^{-t}$. Then, $\Lambda_X = 1.2t - 0.7(1 - e^{-t})$, the baseline AF relation holds, but the usual stochastic-order condition fails. The transformed AF log-derivative at $t = 12$ is approximately -4.4811×10^{-4} . A relative ordering can therefore reverse at late ages even though it looks preserved earlier. In engineering comparisons, different early-failure and wear-out mechanisms can produce a similar need to check the cumulative-hazard ordering rather than assuming it from a short observation window. The sign failure concerns an ordering assertion; it is not, by itself, a quantified loss in system availability or prediction accuracy. Corollary 3.1 and the broader AF record conditions recalled in [15, Proposition 21] also show why this family-specific result must not be interpreted as a universal necessity of $X \geq_{st} Y$ for every generator.

5.5. Alternative baselines and a non-gamma threshold

Because Proposition 5.1 is family-specific, we next vary the baseline hazard shape and the

threshold family. The calculations below examine whether the positive margins in the certified region persist under alternative admissible specifications, using Weibull baselines with decreasing and increasing hazards and the non-gamma saturating threshold from Section 3.5.

Take $\Lambda_Y(t) = t^p$, $\lambda_Y(t) = pt^{p-1}$, with the same ratio form and $(r_0, r_\infty, \gamma) = (0.35, 0.85, 0.7)$. Then,

$$\Lambda_X(t) = r_\infty t^p - (r_\infty - r_0)p \int_0^t s^{p-1} e^{-\gamma s} ds.$$

The values $p = 0.75$ and $p = 1.75$ give decreasing and increasing baseline hazards, respectively. The singular limit of the first hazard at zero is allowed because regularity is imposed locally on the interior. The integral is evaluated by the incomplete gamma function, with independent quadrature checks in the tests. Each row uses the same 20,000-point grid on $[0.1, 15]$; both minima are attained at $t = 15$ in every row. Grid-density sensitivity is reported in Appendix A. Table 4 reports the resulting minimum margins.

Table 4. Minimum margins for alternative baselines and the saturating threshold.

Baseline	Thresholds	$\min D^{AF}$	$\min D^{AFA}$
Weibull $p=0.75$	Gamma pair (2,1)	0.008023	0.020087
Weibull $p=0.75$	Gamma pair (5,3)	0.033287	0.051201
Weibull $p=1.75$	Gamma pair (2,1)	0.001237	0.006487
Weibull $p=1.75$	Gamma pair (5,3)	0.003027	0.010688
Linear A	Saturating	0.000139	0.006254
Weibull $p=0.75$	Saturating	0.000734	0.008661
Weibull $p=1.75$	Saturating	0.000011	0.002050

These values illustrate already-proved comparisons; they do not establish robustness to misspecified baseline distributions or arbitrary generators.

5.6. Calibration using a public censored component-life-test example

In this section, we reanalyze the component-life-test example reported in the NIST/SEMATECH e-Handbook [24]. The tests involve 100, 50, and 25 components at 85, 105, and 125 °C, respectively, with administrative censoring at 1,000 hours. The numbers of failures are 5, 35, and 24. The public source attributes the failures to one mechanism and discusses lognormal acceleration modeling. Its component type and a separately documented industrial provenance are not specified; we therefore describe this as a public handbook life-test example, not as an independently collected field dataset. The individual times, censor indicators, and source are included in the supplement.

Our purpose is to calibrate a lifetime comparison at the tested temperatures, not to extrapolate an Arrhenius relation to use conditions. Assume independent lognormal lifetimes with group-specific log-locations μ_j and a common log-standard deviation σ . Let $d_i = 1$ for a failure and 0 for censoring, $z_i = (\log t_i - \mu_{j(i)})/\sigma$, and let φ , Q denote the standard normal density and survival function. We maximize

$$\ell(\mu_1, \mu_2, \mu_3, \sigma) = \sum_i \{d_i [\log \varphi(z_i) - \log \sigma - \log t_i] + (1 - d_i) \log Q(z_i)\}.$$

No ordering constraint is imposed on the three locations. The common-scale maximum-likelihood fit gives $\hat{\sigma} = 0.785$. Table 5 reports locations and medians with pointwise 95% parametric-bootstrap percentile intervals. The 85°C fitted median is beyond the 1,000-hour censoring time and is a model-based tail extrapolation.

The bootstrap simulates 4,999 independent life tests from the fitted common-scale model (seed 20260925), retains the original group sizes, and administratively censors each test at 1,000 hours. Each joint refit re-estimates all three locations and the common scale without imposing a location order; the contrasts and transformed ratios are then recomputed from the same joint draw. Of the 4,999 replications, 4,982 have finite estimates and 17 have no failure at 85 °C. Those 17 are retained through the extended likelihood limit $\hat{\mu}_{85}^{(b)} = +\infty$: The remaining locations and scale are fitted to the other cells, and the corresponding 85 °C numerator ratios at 500 hours have limit zero. No bootstrap sample is discarded or replaced. Here, (b) indexes the bootstrap replication. Details and all replicate records are in Supplement S1.

Table 5. Common-scale censored lognormal fit and pointwise 95% bootstrap intervals.

Temperature (°C)	Units / failures	$\hat{\mu}$	95% interval for μ	Median (hours) [95% interval]
85	100 / 5	8.152	[7.807, 8.680]	3471 [2457, 5883]
105	50 / 35	6.487	[6.264, 6.729]	657 [525, 836]
125	25 / 24	5.333	[5.029, 5.632]	207 [153, 279]

A sensitivity fit allowing a separate σ_j gives values 1.2117, 0.7148, and 0.8106. Its likelihood-ratio statistic against the common-scale fit is 1.9571 (two additional parameters; approximate chi-square $p = 0.376$). This limited diagnostic does not prove a common scale, particularly with only five failures at 85 °C. An unrestricted different-scale lognormal pair generally has crossing survival functions on the full positive axis, so the global guarantee below should not be transferred to that alternative fit.

Lemma 5.1. (common-scale lognormal relative aging) If $\log X \sim N(\mu_X, \sigma^2)$ and $\log Y \sim N(\mu_Y, \sigma^2)$ with $\mu_X > \mu_Y$, then $X \geq_{\text{st}} Y$ and $X \leq_{\text{AF}} Y$.

Proof. Put $z_i = (\log t - \mu_i)/\sigma$, $M(z) = \varphi(z)/Q(z)$ and $g(z) = M(z) - z$. Then $z_X < z_Y$, so the survival inequality is immediate. Integration by parts for a standard normal variable N gives

$$E(N \mid N > z) = M(z), \quad \text{Var}(N \mid N > z) = 1 + zM(z) - M(z)^2 > 0.$$

Since $M' = M(M - z)$, we have $g' = -\text{Var}(N \mid N > z) < 0$. The lognormal hazard is $M(z_i)/(\sigma t)$, and hence

$$\frac{d}{dt} \log \frac{\lambda_X(t)}{\lambda_Y(t)} = \frac{g(z_X) - g(z_Y)}{\sigma t} > 0.$$

The hazards are positive at every interior time, and both distributions are proper. $\square$

The fitted locations are ordered $\hat{\mu}_{85} > \hat{\mu}_{105} > \hat{\mu}_{125}$. Thus, the fitted common-scale populations

satisfy the baseline assumptions analytically, and the admissible threshold comparisons follow on all $t > 0$. For illustration, Table 6 gives transformed ratios at 500 hours under common gamma-2, gamma-3/gamma-2, and common saturating thresholds. These are specified threshold scenarios, not threshold parameters estimated from this dataset. No recurrent events are observed, so using gamma thresholds here does not establish an NHPP mechanism for the components.

Table 6. Transformed ratios at 500 hours: Estimate [pointwise 95% bootstrap interval].

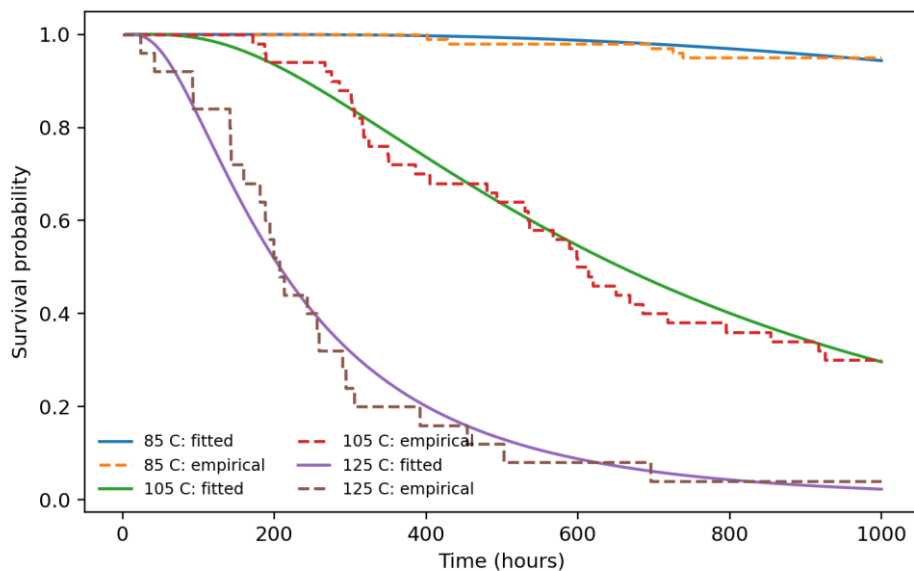
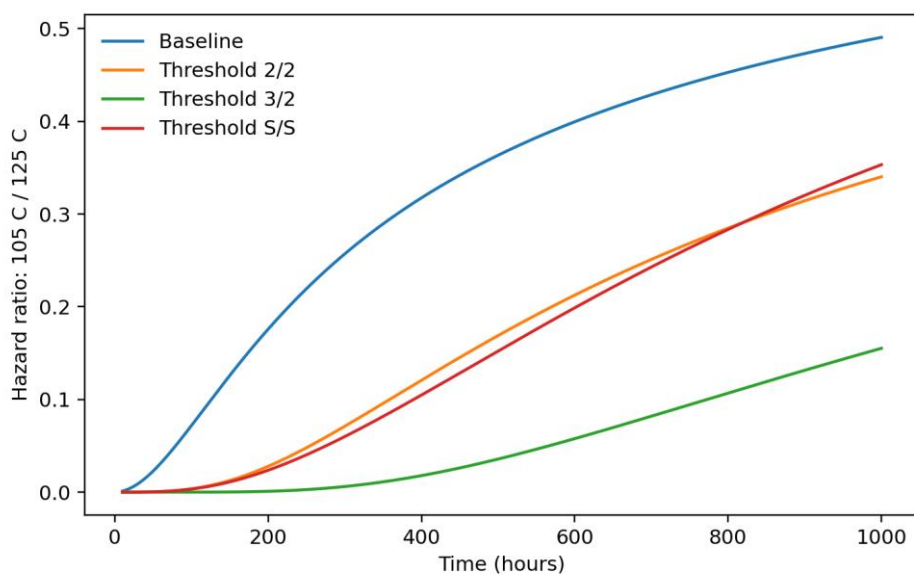
Comparison (°C)	Thresholds	Hazard ratio [95% interval]	Cumulative-hazard ratio [95% interval]
85 vs 105	2/2	0.000701 [1.59e-5, 0.00397]	0.000291 [5.33e-6, 0.00193]
85 vs 105	3/2	2.39e-6 [6.9e-9, 3.2e-5]	6.6e-7 [1.64e-9, 9.92e-6]
85 vs 105	S/S	0.000602 [1.35e-5, 0.00346]	0.000261 [4.82e-6, 0.00176]
105 vs 125	2/2	0.169 [0.0796, 0.301]	0.0857 [0.0314, 0.187]
105 vs 125	3/2	0.0357 [0.0117, 0.0824]	0.012 [0.00315, 0.0341]
105 vs 125	S/S	0.152 [0.0704, 0.28]	0.076 [0.0271, 0.172]
85 vs 125	2/2	0.000119 [2.28e-6, 0.000674]	2.49e-5 [3.64e-7, 0.000176]
85 vs 125	3/2	4.03e-7 [1.03e-9, 5.51e-6]	5.66e-8 [1.12e-10, 9.07e-7]
85 vs 125	S/S	9.17e-5 [1.76e-6, 0.000526]	1.98e-5 [2.87e-7, 0.000141]

Ratios and interval endpoints are rounded to three significant digits; scientific notation is used for small values. The intervals quantify joint location-and-scale estimation uncertainty at the single stated time, not simultaneous bands over time. Thresholds 2/2, 3/2, and S/S retain the definitions used in the preceding analysis.

Table 7 reports the pairwise location contrasts $\Delta_{a,b} = \mu_a - \mu_b$. The interval for the common log-standard deviation is [0.636, 0.920]. Positive contrast intervals are conditional on the common-scale lognormal model; they are not distribution-free evidence of population AF/AFA ordering.

Table 7. Pairwise log-location contrasts and pointwise 95% bootstrap intervals.

Comparison (°C)	$\hat{\Delta}_{a,b}$	95% interval
85 vs 105	1.665	[1.269, 2.212]
105 vs 125	1.154	[0.778, 1.543]
85 vs 125	2.819	[2.360, 3.415]

**Figure 6.** Empirical survival steps and fitted common-scale lognormal survival curves at the tested temperatures.**Figure 7.** Baseline and transformed fitted hazard ratios for the 105°C/125°C comparison.

The 85 °C median interval is approximately 2,457–5,883 hours, wholly beyond the observation endpoint, and the transformed-ratio intervals span substantial relative ranges despite small plug-in values. These approximate, model-conditional intervals do not resolve the weak tail information supplied by five failures, establish a physical threshold or NHPP mechanism, or guarantee nominal coverage under model misspecification. The infinite bootstrap locations are boundary likelihood limits used only for uncertainty bookkeeping; they are not additional proper baselines to which the preservation theorems are applied. The fitted-model grid minima on [10,1000] hours and complete bootstrap diagnostics are supplied in Supplement S1. Figure 6 compares empirical survival steps with the fitted curves, while Figure 7 displays the fitted hazard-ratio comparisons.

6. Discussion and conclusions

Within the stated class of proper thresholds and positive-interior-hazard baselines, Theorem 3.5 establishes an exact distinction between universal and pair-specific preservation. Under $X \geq_{st} Y$, universal AF preservation is equivalent to a nondecreasing hazard generator with nonincreasing log-elasticity; universal AFA preservation is equivalent to nonincreasing cumulative elasticity. The constructive necessity proof uses constant hazard ratios and smoothly increasing ratios that reach one. These statements do not say that every individual preserved pair must satisfy the same sufficient assumptions. The power-threshold consequence makes that limitation concrete.

The numerical study has a correspondingly restricted role. It reproduces analytical margins, records where their grid minima occur, and examines horizon sensitivity. Proposition 5.1 explains why apparently preserved configurations outside the usual stochastic-order region can eventually fail in the specified linear-baseline/gamma-2 family. Its conclusion must not be generalized to all thresholds or all baseline families. Margins have units of inverse time, whereas generator elasticities are dimensionless; neither is a direct measure of operational cost or predictive accuracy.

The public NIST calibration shows how a baseline comparison can be linked to threshold consequences after fitting a censored lifetime model. The common lognormal scale is an assumption with limited diagnostic support, not a fact proved from the data. The model's ordered locations imply AF and usual stochastic ordering analytically, but this does not establish an NHPP recurrence mechanism or distribution-free population order. Heavy censoring is reflected in the reported intervals, and all-censored bootstrap cells are retained as explicit likelihood-boundary cases.

The theory remains marginal and univariate. No independence between the compared populations is required, but no multivariate order for a vector of epoch times is inferred. The process literature [16–19] addresses different dependence questions. The domain does not automatically include defective thresholds, zero-hazard intervals, or arbitrary trend-renewal and random-environment mechanisms. Extensions to those settings require new statements, not a denominator-positive-region convention. Useful further work includes pair-adaptive conditions, statistically calibrated shape tests, and applications in which the accumulated-risk threshold is physically identified.

Appendix A. Grid-density sensitivity

We repeat the experiments in Sections 5.2, 5.3, and 5.5 using $N = 5,000, 10,000, 20,000$, and $40,000$ equally spaced points, including both endpoints of each original time interval. The 40,000-point calculations are the numerical reference, not an exact continuous-time optimum. For Section 5.3, every density is applied to all 7,280 parameter configurations at each horizon $H = 15, 30, 60, 120$, using the original violation cutoff -10^{-10} . This yields 116,480 grid/configuration/horizon records; no configuration is selected after inspecting its behavior.

Appendix A1 summarizes the largest absolute difference between a minimum margin on an N -point grid and on the reference grid, maximized across all 29,120 configuration/horizon pairs. The final column counts changes in the two violation classifications. Grid-specific summaries, minimum locations, and every underlying record are provided in the code-and-data archive.

Table A1. Grid refinement relative to 40,000 points, across all boundary configurations and horizons.

Grid points N	Max. absolute AF difference	Max. absolute AFA difference	Changed AF / AFA classifications
5,000	$7.77\text{e-}7$	$2.57\text{e-}8$	0 / 0
10,000	$2.04\text{e-}7$	$6.89\text{e-}9$	0 / 0
20,000	$4.44\text{e-}8$	$1.7\text{e-}9$	0 / 0
40,000	0	0	0 / 0

At 20,000 points, the largest discrepancies relative to 40,000 points are 4.44×10^{-8} for AF and 1.70×10^{-9} for AFA. No classification changes occur at any tested density, so every percentage in Table 3 is unchanged. In Sections 5.2 and 5.5, all 38 reported minima remain at the common right endpoint $t = 15$ and are unchanged across densities; their full grid records are also supplied. These checks support 20,000 points for the reported finite-horizon summaries. They do not provide a continuous-time error bound or replace the analytical preservation and eventual-failure results.

Author contributions

Ibrahim A. Nafisah: Conceptualization, methodology, formal analysis, investigation, writing—original draft, writing—review & editing; Mohamed Kayid: Conceptualization, methodology, formal analysis, investigation, writing—original draft, writing—review & editing; Tarek M. AlSayed: Conceptualization, methodology, formal analysis, investigation, writing—original draft, writing—review & editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

The empirical analysis reuses the publicly reported NIST/SEMATECH component-life-test example [24]; no new measurements were collected. The transcribed times and censor indicators, analytical computation code, dependency versions, complete numerical outputs and bootstrap diagnostics accompany Supplementary File S1. The scripts regenerate the numerical tables and figures in Section 5, the grid-density appendix and the joint-bootstrap intervals. The source and the distinction between empirical observations, synthetic grids and fitted threshold scenarios are documented in the supplement.

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