



*Research article***Explicit estimates for coupled weakly singular retarded integral inequalities and applications to fractional Volterra-type systems****Liqiang Chen***

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Abstract: This paper establishes a unified comparison framework for coupled weakly singular retarded integral inequalities arising in fractional Volterra-type systems. The inequalities contain powered unknowns, several weakly singular kernels of different orders, delayed arguments, nonconstant outside multipliers, and piecewise continuous inhomogeneous data. In addition to a closed Mittag–Leffler majorant, a variable-coefficient resolvent estimate is derived that retains the coefficient envelopes inside the fractional integral. A further extension treats state-dependent impulse amplitudes at prescribed times and gives a product bound for their cumulative amplification. The proofs use running maxima and Volterra resolvents. Applications yield a priori bounds, continuous dependence, uniqueness, and a classical Caputo formulation in an explicitly stated solution class, and a numerical illustration compares the computed solution with the analytical majorant.

Keywords: Gronwall–Bellman inequality; weakly singular integral inequality; retarded argument; coupled integral system; explicit estimate; Mittag–Leffler function; fractional Volterra-type system

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1. Introduction

Integral inequalities of Gronwall–Bellman type remain one of the standard tools for studying qualitative properties of differential, integral, fractional, and integro-differential equations. The classical Gronwall inequality [1] and Bellman’s stability estimate [2] provide an explicit exponential upper bound for a nonnegative function satisfying a linear Volterra inequality. Pachpatte’s refinements [3] and Bihari’s nonlinear extension [4] show that a bound may still be obtained when the integrand contains nonlinear functions of the unknown. These early estimates are not merely technical lemmas; they are often the mechanism by which existence, uniqueness, continuous dependence, boundedness, and stability are proved for equations whose exact solutions cannot be written down.

The systematic treatment of integral inequalities for differential and integral equations was

developed further in Pachpatte's monographs [5, 6]. The weakly singular case is especially relevant for fractional models, because kernels of the form

$$K_\alpha(t, s) = \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha \leq 1, \quad (1.1)$$

arise naturally from Riemann–Liouville and Caputo fractional integrals. Medveď [7] introduced an influential approach to Henry-type and Bihari-type weakly singular inequalities, and Henry's semigroup work [8] became a standard reference for fractional and parabolic problems. For $0 < \alpha < 1$, the factor $(t - s)^{\alpha-1}$ assigns a slowly decaying weight to the past in a fractional evolution model. It therefore represents memory or hereditary effects: recent states receive a stronger weight, but remote states are not discarded. Different orders naturally occur when two components have different memory scales, while retarded arguments describe transport, reaction, or feedback delays. State-dependent impulses model abrupt interventions whose size depends on the pre-impulse state rather than on prescribed external data. These features motivate treating weak singularity, coupling, delay, variable coefficients, and impulses in one comparison framework. Subsequent studies developed retarded, singular, and nonlinear variants: Agarwal et al. [9] studied retarded Gronwall-like inequalities; Ma and Pečarić [10] obtained explicit bounds for weakly singular integral inequalities; Abdeldaim and Yakout [11] investigated Gronwall–Bellman–Pachpatte type inequalities; Zheng [12] derived explicit estimates for fractional integral equations.

Four motivating papers illustrate several overlapping directions. Luo et al. [13] considered scalar discontinuous inequalities with delayed evaluations, a nondecreasing outside multiplier, and unknown-dependent impulse terms. Ouyang and Wang [14] studied coupled powered inequalities with nonconstant outside multipliers and retarded integration limits. Huang and Wang [15] treated a scalar inequality with three nested weakly singular integrals of a common order and nonlinear functions of the unknown. Zhong and Wang [16] studied coupled powered inequalities containing weakly singular kernels and nonconstant outside multipliers, using Hölder estimates and Gamma and Beta functions. These models share several features but differ in their kernels, nonlinearities, and treatment of delay. Subsection 3.5 records the relevant equations and distinguishes retarded upper limits from delayed evaluations of the unknowns.

Singularity, delay, and impulses have also been combined in scalar inequalities. Tatar [17] considered two singular kernels, variable outside factors, a delayed power, and impulses, and Deng and Prather [18] extended this direction to several nonlinear terms. Li and Wang [19] obtained weakly singular retarded estimates for discontinuous functions. Coupled fractional comparison principles have been developed in other settings as well: Etemad, Akgül, and Conejero [20] established coupled Hadamard–Gronwall–Bellman inequalities with applications to fractional delay systems. The present work considers a coupled convolution-kernel system and derives finite-interval bounds through running envelopes.

A parallel group of papers pushed the retarded and powered directions. Wang et al. [21] considered retarded nonlinear inequalities for differential-integral equations. And Abdeldaim and El-Deeb [22, 23] developed generalized retarded nonlinear inequalities with iterated integrals. Li and Wang [24, 25] studied powered retarded Gronwall–Bellman inequalities and their applications. Shakoor and coauthors [26, 27] developed further nonlinear retarded inequalities; Tian and Fan [28] treated a nonlinear inequality with power for delay integro-differential equations. Gronwall-type

inequalities on time scales and dynamic equations were also investigated (e.g., [29]). Recent literature shows that the topic remains active: Shakoor et al. [30] studied Gronwall–Bellman–Pachpatte type retarded inequalities; Kale [31] obtained new retarded nonlinear inequalities; Samar et al. [32] developed new retarded nonlinear inequalities with applications; Yang et al. [33] proposed a powered Wendroff-type inequality for fractional partial differential equations; Al-Refai et al. [34] formulated a Gronwall-type inequality for general fractional integrals with Sonin kernels; Sousa and Oliveira [35] established a Gronwall-type inequality associated with the ψ -Hilfer fractional operator and applied it to a Cauchy-type problem; impulsive dynamics with variable times have also been studied recently [36]. Related fractional and time-scale tools appear in [37, 38].

Scope and main results. For the coupled fractional Volterra-type inequalities considered in this paper, the following five features occur simultaneously:

- (i) two unknown nonnegative functions u, v entering through their powers u^p, v^q
- (ii) finitely many weakly singular kernels of *different* fractional orders in each equation
- (iii) nonconstant multipliers outside the integral signs
- (iv) retarded arguments in both components
- (v) piecewise continuous inhomogeneous terms representing prescribed jumps

The contribution is the explicit comparison construction for the stated coefficient and delay class, together with its resolvent and impulsive extensions. The main results are summarized as follows:

- (C1) Theorem 3.4 provides an explicit Mittag–Leffler bound for the coupled inequality (3.10) and (3.11) and recovers familiar estimates in special cases (Remark 3.9).
- (C2) Its proof combines a finite-interval reduction of kernels of different orders with a scalar comparison for the envelope $W = u^p + v^q$, using running maxima and Volterra iteration.
- (C3) Theorem 3.5 retains the running coefficient envelopes inside the fractional integral and yields a nonnegative resolvent-series estimate, with the closed Mittag–Leffler bound as a corollary.
- (C4) Theorem 3.7 extends the estimate to state-dependent impulses through an explicit product factor controlling cumulative jump amplification.
- (C5) Section 4 derives a priori bounds and a continuous-dependence/uniqueness criterion for coupled Volterra-type systems, including the Hölder estimates required to obtain the powered integral inequalities.

Role of Lemma 2.2. Lemma 2.2 follows directly from the monotonicity of $(t - s)^{\rho - \alpha}$ on $[0, L]$. Its role is to reduce kernels of different fractional orders to a common kernel before the running-envelope comparison is applied.

Function-space assumptions. The unknown functions are assumed to be continuous so that their running maxima are continuous, whereas the prescribed inhomogeneous terms and outside coefficients may be piecewise continuous and may therefore include known jumps. Theorem 3.4 addresses the particular combination of coupling, different convolution orders, measurable retarded maps, variable outside factors, and discontinuous known data specified below.

Use of the estimates. The results are intended for a priori analysis after existence has been established by an appropriate method, such as a fixed-point argument, compactness, monotone iteration, or successive approximation. Once a solution is available on a local interval, its norm can be estimated through the inequalities developed below.

Organization. Section 2 fixes notation and states the preliminary lemmas. Section 3 contains the scalar estimate, the coupled Mittag–Leffler estimate, a variable-coefficient resolvent refinement, a state-dependent impulsive extension, two corollaries, and an explicit comparison with [13–16]. Section 4 gives an analytic example, proves an a priori estimate for a coupled weakly singular Volterra-type system, writes an explicit Caputo formulation, and derives continuous dependence and uniqueness. Section 5 summarizes the contribution and lists open problems.

2. Preliminaries

Throughout the paper, let

$$J = [t_0, T], \quad -\infty < t_0 < T < \infty, \quad L := T - t_0, \quad \mathbb{R}_+ := [0, \infty).$$

For a finite set of points

$$\Pi = \{\tau_1, \tau_2, \dots, \tau_m\}, \quad t_0 < \tau_1 < \dots < \tau_m < T,$$

denote by $\text{PC}(J, \mathbb{R}_+)$ the class of functions $w: J \rightarrow \mathbb{R}_+$ that are continuous between the points of Π , have finite one-sided limits at every partition point and both endpoints, are right-continuous on $[t_0, T)$, and satisfy $w(T) = w(T^-)$. In particular, these functions are bounded and Borel measurable. If $\Pi = \emptyset$, this class reduces to $C(J, \mathbb{R}_+)$. Partitions for finitely many data functions can be combined by taking their union.

For every $\rho > 0$, write

$$K_\rho(t, s) = (t - s)^{\rho-1} / \Gamma(\rho)$$

for $s < t$ and

$$\mathcal{I}_\rho h(t) = \int_{t_0}^t K_\rho(t, s) h(s) \, ds.$$

The main inequalities use $0 < \rho \leq 1$, as in (1.1); the larger orders also occur in iterated convolutions. Integrals at $t = t_0$ are defined to be zero. The Mittag–Leffler function is

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, \quad z \in \mathbb{C}, \quad 0 < \rho \leq 1. \quad (2.1)$$

A map $\sigma: J \rightarrow J$ is called retarded if it is measurable and satisfies

$$t_0 \leq \sigma(t) \leq t, \quad t \in J. \quad (2.2)$$

The following standard form of the fractional Gronwall inequality is used as a building block.

Lemma 2.1. (*Fractional Gronwall estimate*) Let $0 < \alpha \leq 1$, let $A: J \rightarrow \mathbb{R}_+$ be nondecreasing, and let $b: J \rightarrow \mathbb{R}_+$ be bounded and measurable. If $z: J \rightarrow \mathbb{R}_+$ is bounded and measurable and satisfies

$$z(t) \leq A(t) + \int_{t_0}^t K_\alpha(t, s) b(s) z(s) \, ds, \quad t \in J, \quad (2.3)$$

then

$$z(t) \leq A(t) E_\alpha(B(t) (t - t_0)^\alpha), \quad B(t) := \sup_{t_0 \leq s \leq t} b(s). \quad (2.4)$$

Proof. Fix $t > t_0$ and put $a = A(t)$ and $b_0 = B(t)$. On $[t_0, t]$ the hypothesis gives $z \leq a + b_0 \mathcal{I}_\alpha z$. Iteration and the convolution identity $K_\rho * K_\nu = K_{\rho+\nu}$ yield, for $n \geq 1$,

$$z(t) \leq a \sum_{j=0}^{n-1} \frac{b_0^j (t - t_0)^{j\alpha}}{\Gamma(j\alpha + 1)} + \|z\|_\infty \frac{b_0^n (t - t_0)^{n\alpha}}{\Gamma(n\alpha + 1)}.$$

The final term tends to zero. The series gives the asserted bound, and the case $t = t_0$ follows directly from the hypothesis. $\square$

When $\alpha = 1$, (2.4) reduces to the exponential Gronwall–Bellman estimate. The fractional form is consistent with [37] and with [35, Corollary 2, p. 95], after choosing $\psi(t) = t$ and normalizing the kernel. Because K_α contains $1/\Gamma(\alpha)$, no additional factor $\Gamma(\alpha)$ occurs in the Mittag–Leffler argument.

The next observation is elementary but is the bookkeeping device that lets several weakly singular kernels of different orders be controlled by a single kernel of the smallest order on a finite interval.

Lemma 2.2. (*Kernel reduction*) Let $0 < \alpha \leq \rho \leq 1$ and $L = T - t_0$. Then, for every $t_0 \leq s < t \leq T$,

$$K_\rho(t, s) \leq C_{\rho, \alpha}(L) K_\alpha(t, s), \quad C_{\rho, \alpha}(L) := \frac{\Gamma(\alpha)}{\Gamma(\rho)} L^{\rho-\alpha}. \quad (2.5)$$

Proof. Since $0 < t - s \leq L$ and $\rho - \alpha \geq 0$, one has

$$(t - s)^{\rho-1} = (t - s)^{\alpha-1} (t - s)^{\rho-\alpha} \leq L^{\rho-\alpha} (t - s)^{\alpha-1}.$$

Dividing by $\Gamma(\rho)$ and multiplying by $\Gamma(\alpha)/\Gamma(\alpha)$ gives (2.5). $\square$

Lemma 2.3. (*Monotonicity of fractional integrals*) Let $0 < \alpha \leq 1$ and let $h: J \rightarrow \mathbb{R}_+$ be nondecreasing. Then,

$$(\mathcal{I}_\alpha h)(t) := \int_{t_0}^t K_\alpha(t, s) h(s) \, ds$$

is nondecreasing on J .

Proof. After the change of variables $s = t_0 + (t - t_0)r$, one obtains

$$(\mathcal{I}_\alpha h)(t) = \frac{(t - t_0)^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - r)^{\alpha-1} h(t_0 + (t - t_0)r) \, dr.$$

Both the factor $(t - t_0)^\alpha$ and the integrand are nondecreasing in t for every $r \in [0, 1]$. Hence, $\mathcal{I}_\alpha h$ is nondecreasing. $\square$

Lemma 2.4. (Nonnegative Volterra resolvent) Let $0 < \alpha \leq 1$ and let k be a nonnegative measurable kernel on $\Delta_J := \{(t, s) : t_0 \leq s < t \leq T\}$ satisfying $k(t, s) \leq CK_\alpha(t, s)$ for some $C \geq 0$. Define

$$k_1(t, s) := k(t, s), \quad k_{n+1}(t, s) := \int_s^t k(t, r)k_n(r, s) \, dr,$$

and $R(t, s) := \sum_{n=1}^{\infty} k_n(t, s)$. Then, the series converges for $s < t$, and

$$0 \leq R(t, s) \leq \sum_{n=1}^{\infty} C^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}.$$

If a bounded nonnegative measurable function z satisfies

$$z(t) \leq A(t) + \int_{t_0}^t k(t, s)z(s) \, ds,$$

where A is nonnegative and nondecreasing, then

$$z(t) \leq A(t) + \int_{t_0}^t R(t, s)A(s) \, ds \leq A(t) \left(1 + \int_{t_0}^t R(t, s) \, ds\right). \quad (2.6)$$

Proof. The convolution identity $K_\alpha * K_\beta = K_{\alpha+\beta}$ gives $k_n(t, s) \leq C^n K_{n\alpha}(t, s)$, which proves convergence on compact subtriangles and yields the displayed majorant. Iterating the inequality m times gives

$$z \leq A + \sum_{n=1}^m k_n * A + k_{m+1} * z.$$

The last term tends to zero uniformly on J because z is bounded and

$$\int_{t_0}^t C^{m+1} K_{(m+1)\alpha}(t, s) \, ds = C^{m+1} (t-t_0)^{(m+1)\alpha} / \Gamma((m+1)\alpha + 1).$$

Passing to the limit proves (2.6). The argument applies to bounded measurable functions. $\square$

Finally, for a piecewise continuous nonnegative function h , define its monotone envelope by

$$h^*(t) := \sup_{t_0 \leq r \leq t} h(r), \quad t \in J. \quad (2.7)$$

The envelope is nondecreasing, finite on J , and satisfies $h(t) \leq h^*(t)$. If σ is retarded as in (2.2), then

$$h(\sigma(s)) \leq h^*(s), \quad t_0 \leq s \leq T. \quad (2.8)$$

This is the reason no differentiability of the unknown function is required in the main theorems.

For an essentially bounded nonnegative function $g \in L^\infty(J, \mathbb{R}_+)$, we shall also use

$$g^\#(t) := \operatorname{ess\,sup}_{t_0 \leq r \leq t} g(r), \quad t_0 < t \leq T, \quad g^\#(t_0) := \lim_{t \downarrow t_0} g^\#(t). \quad (2.9)$$

The function $g^\#$ is nondecreasing and finite, so the right limit used at t_0 exists. Moreover, $g(s) \leq g^\#(s)$ for almost every $s \in J$. Values of an L^∞ coefficient on null sets do not affect the integrals. For piecewise continuous outside multipliers the ordinary envelope g^* in (2.7) will be used instead.

3. Main results

Throughout this section $J = [t_0, T]$ and $L = T - t_0$. The unknown functions, inhomogeneous terms, and coefficient functions are nonnegative on their domains. We use $\mathbb{N} = \{1, 2, \dots\}$; hence the integers N and N_i occurring in the main theorems are strictly positive. Cases without an integral term are elementary and are treated separately rather than through an empty-sum convention.

3.1. A scalar powered inequality (preparatory result)

The first theorem treats one unknown function, finitely many weakly singular kernels, retarded arguments, and a piecewise continuous inhomogeneous term. The discontinuous data may represent prescribed jumps or known impulsive inputs. Although Theorem 3.1 is itself the special case $\nu \equiv 0$ of Theorem 3.4 below, it is stated separately because its proof exhibits the supremum envelope argument used in Theorem 3.4 without the additional notation of the coupled setting, and because it suffices for the single-component applications.

Theorem 3.1. (*Scalar inequality*) Let $p > 0$, $N \in \mathbb{N}$, and let $u \in C(J, \mathbb{R}_+)$. For $\ell = 1, 2, \dots, N$, assume

$$0 < \alpha_\ell \leq 1, \quad c_\ell \in L^\infty(J, \mathbb{R}_+), \quad M_\ell \in \text{PC}(J, \mathbb{R}_+),$$

and let $\sigma_\ell: J \rightarrow J$ be retarded. Let $A: J \rightarrow \mathbb{R}_+$ be nondecreasing and piecewise continuous. Suppose that

$$u^p(t) \leq A(t) + \sum_{\ell=1}^N M_\ell(t) \int_{t_0}^t K_{\alpha_\ell}(t, s) c_\ell(s) u^p(\sigma_\ell(s)) \, ds, \quad t \in J. \quad (3.1)$$

Set

$$\alpha_0 := \min_{1 \leq \ell \leq N} \alpha_\ell, \quad C_\ell := C_{\alpha_\ell, \alpha_0}(L) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_\ell)} L^{\alpha_\ell - \alpha_0}, \quad (3.2)$$

$$G(t) := \sum_{\ell=1}^N C_\ell M_\ell^*(t) c_\ell^\#(t), \quad t \in J, \quad (3.3)$$

$$\Phi(t) := A(t) E_{\alpha_0}(G(t)(t - t_0)^{\alpha_0}). \quad (3.4)$$

Then,

$$u(t) \leq \Phi(t)^{1/p}, \quad t \in J. \quad (3.5)$$

Proof. Using the retarded condition and Lemma 2.2, we obtain, for every $t \in J$,

$$\begin{aligned} u^p(t) &\leq A(t) + \sum_{\ell=1}^N M_\ell(t) \int_{t_0}^t K_{\alpha_\ell}(t, s) c_\ell(s) u^p(\sigma_\ell(s)) \, ds \\ &\leq A(t) + G(t) \int_{t_0}^t K_{\alpha_0}(t, s) U(s) \, ds, \end{aligned} \quad (3.6)$$

where

$$U(t) := \sup_{t_0 \leq r \leq t} u^p(r).$$

Since u is continuous, U is continuous and nondecreasing. By Lemma 2.3, the map

$$t \mapsto \int_{t_0}^t K_{\alpha_0}(t, s)U(s) \, ds$$

is nondecreasing. Consequently, taking the supremum of (3.6) over $r \in [t_0, t]$ yields

$$U(t) \leq A(t) + G(t) \int_{t_0}^t K_{\alpha_0}(t, s)U(s) \, ds. \quad (3.7)$$

Fix $t \in J$. For every $r \in [t_0, t]$, the monotonicity of A and G gives

$$U(r) \leq A(t) + G(t) \int_{t_0}^r K_{\alpha_0}(r, s)U(s) \, ds.$$

Applying Lemma 2.1 on the interval $[t_0, t]$ with the constant majorants $A(t)$ and $G(t)$ gives

$$U(t) \leq A(t)E_{\alpha_0}(G(t)(t - t_0)^{\alpha_0}) = \Phi(t).$$

Since $u^p(t) \leq U(t)$, estimate (3.5) follows. The estimate remains valid when the prescribed data have upward jumps. $\square$

Remark 3.2. The proof reveals why the retarded condition is essential. If one allowed $\sigma_\ell(s) > s$, the envelope $U(s)$ would not control the delayed term, and an advance argument or assumption on future values would be required, which is unsuitable for Volterra problems. Similarly, the positivity of the coefficients permits the replacements $u^p(\sigma_\ell(s)) \rightarrow U(s)$ and $K_{\alpha_\ell} \rightarrow C_\ell K_{\alpha_0}$; if signs were allowed to change, a different argument based on absolute values or signed resolvents would be needed.

Remark 3.3. If

$$A(t) = a(t) + \sum_{\tau_i \leq t} \eta_i, \quad a(t) > 0, \quad \eta_i \geq 0, \quad (3.8)$$

then A is piecewise continuous and nondecreasing whenever $a \in \text{PC}(J, \mathbb{R}_+)$ is nondecreasing. Thus, (3.5) covers inequalities with prescribed discontinuous jumps. This is weaker than inequalities with unknown-dependent impulses, but it is sufficient for many a priori estimates in which the jump size is determined by known data.

3.2. The coupled weakly singular system (main result)

The main result of the paper estimates two unknown functions simultaneously. It is designed for systems in which the outside factors are nonconstant and the components are coupled through delayed arguments.

Theorem 3.4. (Coupled inequality) Let $p, q > 0$, let $N_1, N_2 \in \mathbb{N}$, and let $u, v \in C(J, \mathbb{R}_+)$. For $i = 1, 2$ and $\ell = 1, 2, \dots, N_i$, assume

$$0 < \alpha_{i\ell} \leq 1, \quad M_{i\ell} \in \text{PC}(J, \mathbb{R}_+), \quad a_{i\ell}, b_{i\ell} \in L^\infty(J, \mathbb{R}_+), \quad (3.9)$$

and let $\sigma_{i\ell}, \rho_{i\ell}: J \rightarrow J$ be retarded. Let $A_1, A_2: J \rightarrow \mathbb{R}_+$ be nondecreasing and piecewise continuous. Suppose that, for every $t \in J$,

$$u^p(t) \leq A_1(t) + \sum_{\ell=1}^{N_1} M_{1\ell}(t) \int_{t_0}^t K_{\alpha_{1\ell}}(t, s) \{a_{1\ell}(s) u^p(\sigma_{1\ell}(s)) + b_{1\ell}(s) v^q(\rho_{1\ell}(s))\} ds, \quad (3.10)$$

$$v^q(t) \leq A_2(t) + \sum_{\ell=1}^{N_2} M_{2\ell}(t) \int_{t_0}^t K_{\alpha_{2\ell}}(t, s) \{a_{2\ell}(s) u^p(\sigma_{2\ell}(s)) + b_{2\ell}(s) v^q(\rho_{2\ell}(s))\} ds. \quad (3.11)$$

Put

$$\alpha^* := \min\{\alpha_{i\ell}: i = 1, 2, 1 \leq \ell \leq N_i\}, \quad (3.12)$$

$$D_{i\ell} := \frac{\Gamma(\alpha^*)}{\Gamma(\alpha_{i\ell})} L^{\alpha_{i\ell} - \alpha^*}, \quad (3.13)$$

$$H(t) := \sum_{i=1}^2 \sum_{\ell=1}^{N_i} D_{i\ell} M_{i\ell}^*(t) \{a_{i\ell}^\#(t) + b_{i\ell}^\#(t)\}, \quad (3.14)$$

$$A(t) := A_1(t) + A_2(t), \quad (3.15)$$

$$B(t) := A(t) E_{\alpha^*}(H(t)(t - t_0)^{\alpha^*}). \quad (3.16)$$

Then,

$$u(t) \leq B(t)^{1/p}, \quad v(t) \leq B(t)^{1/q}, \quad t \in J. \quad (3.17)$$

Proof. Let

$$W(t) := u^p(t) + v^q(t), \quad t \in J, \quad (3.18)$$

and define its running maximum by

$$Y(t) := \sup_{t_0 \leq r \leq t} W(r).$$

Adding (3.10) and (3.11), using the retarded condition, and reducing all kernels to K_{α^*} gives

$$\begin{aligned} W(t) &\leq A(t) + \sum_{i=1}^2 \sum_{\ell=1}^{N_i} D_{i\ell} M_{i\ell}(t) \int_{t_0}^t K_{\alpha^*}(t, s) \times \{a_{i\ell}(s) W(\sigma_{i\ell}(s)) + b_{i\ell}(s) W(\rho_{i\ell}(s))\} ds \\ &\leq A(t) + H(t) \int_{t_0}^t K_{\alpha^*}(t, s) Y(s) ds. \end{aligned} \quad (3.19)$$

Because u and v are continuous, W and its running maximum Y are continuous, and Y is nondecreasing. Lemma 2.3 therefore shows that the fractional integral of Y in (3.19) is nondecreasing. Taking the supremum over $r \in [t_0, t]$ gives

$$Y(t) \leq A(t) + H(t) \int_{t_0}^t K_{\alpha^*}(t, s) Y(s) ds. \quad (3.20)$$

Fix $t \in J$. For every $r \in [t_0, t]$, the monotonicity of A and H implies

$$Y(r) \leq A(t) + H(t) \int_{t_0}^r K_{\alpha^*}(r, s) Y(s) ds.$$

Lemma 2.1, applied on $[t_0, t]$ with the constant majorants $A(t)$ and $H(t)$, yields

$$Y(t) \leq A(t)E_{\alpha^*}(H(t)(t - t_0)^{\alpha^*}) = B(t).$$

Thus, $W(t) \leq Y(t) \leq B(t)$. Since $u^p(t) \leq W(t)$ and $v^q(t) \leq W(t)$, (3.17) follows. $\square$

The scalar quantity $W = u^p + v^q$ gives one common majorant and then componentwise bounds through $u^p \leq W$ and $v^q \leq W$. This step loses some information about the separate coupling coefficients; matrix comparison methods can retain more of that information. The powers here are changes of nonnegative variables: the comparison inequalities are linear in u^p and v^q .

3.3. A variable-coefficient resolvent refinement

Theorem 3.5. (Resolvent refinement) Assume the hypotheses of Theorem 3.4. Fix $t \in J$ and define, for $t_0 \leq s < r \leq t$,

$$h_t(s) := \sum_{i=1}^2 \sum_{\ell=1}^{N_i} D_{i\ell} M_{i\ell}^*(t)(a_{i\ell}^\#(s) + b_{i\ell}^\#(s)), \quad (3.21)$$

$$k_t(r, s) := K_{\alpha^*}(r, s)h_t(s). \quad (3.22)$$

Let R_t be the resolvent of k_t as in Lemma 2.4. Then,

$$u^p(t) + v^q(t) \leq A(t) \left(1 + \int_{t_0}^t R_t(t, s) \, ds \right). \quad (3.23)$$

Consequently, each component is bounded by the right-hand side of (3.23) raised to the powers $1/p$ and $1/q$, respectively.

Proof. Let

$$Y(r) = \sup_{t_0 \leq \xi \leq r} W(\xi).$$

For each fixed t and every $r \leq t$, the same kernel reduction used in Theorem 3.4, but without replacing the inside coefficients by their terminal suprema, gives

$$Y(r) \leq A(t) + \int_{t_0}^r K_{\alpha^*}(r, s)h_t(s)Y(s) \, ds.$$

Indeed, $a_{i\ell}^\# + b_{i\ell}^\#$ and Y are nondecreasing, so Lemma 2.3 justifies taking the running maximum. Lemma 2.4 now yields (3.23). $\square$

Remark 3.6. (Resolvent and closed-form bounds) The function $h_t(s)$ remains inside the fractional integral and retains the time variation of the running coefficient envelopes. Since $h_t(s) \leq H(t)$, the resolvent estimate implies the Mittag–Leffler majorant in Theorem 3.4. If the coefficient functions are nondecreasing, then $a_{i\ell}^\# = a_{i\ell}$ and $b_{i\ell}^\# = b_{i\ell}$ almost everywhere. For oscillatory coefficients, the running envelopes are required by the supremum reduction in the presence of retarded arguments. Thus, the resolvent majorant is no larger than the closed-form majorant in Theorem 3.4. Equality is possible for some coefficient choices.

3.4. State-dependent impulse amplitudes at prescribed times

For $i = 1, 2$, define the integral operators

$$\mathcal{V}_i[u, v](t) := \sum_{\ell=1}^{N_i} M_{i\ell}(t) \int_{t_0}^t K_{\alpha_{i\ell}}(t, s) \{a_{i\ell}(s)u^p(\sigma_{i\ell}(s)) + b_{i\ell}(s)v^q(\rho_{i\ell}(s))\} ds.$$

Theorem 3.7. (*Impulses at prescribed times*) Let $t_0 < \tau_1 < \cdots < \tau_m < T$, assume the coefficient and delay hypotheses of Theorem 3.4, and let $A_1, A_2: J \rightarrow \mathbb{R}_+$ be nondecreasing and piecewise continuous. Let $u, v \in \text{PC}(J, \mathbb{R}_+)$ and suppose

$$u^p(t) \leq A_1(t) + \mathcal{V}_1[u, v](t) + \sum_{\tau_j \leq t} I_{1j}(u(\tau_j^-), v(\tau_j^-)), \quad (3.24)$$

$$v^q(t) \leq A_2(t) + \mathcal{V}_2[u, v](t) + \sum_{\tau_j \leq t} I_{2j}(u(\tau_j^-), v(\tau_j^-)), \quad (3.25)$$

where $I_{ij}: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and

$$I_{1j}(x, y) + I_{2j}(x, y) \leq \lambda_j(x^p + y^q), \quad \lambda_j \geq 0. \quad (3.26)$$

For fixed t , put

$$\mathcal{E}_t(r) := E_{\alpha^*}(H(t)(r - t_0)^{\alpha^*}), \quad t_0 \leq r \leq t.$$

Then,

$$u^p(t) + v^q(t) \leq A(t)\mathcal{E}_t(t) \prod_{\tau_j \leq t} (1 + \lambda_j \mathcal{E}_t(\tau_j)). \quad (3.27)$$

In particular, for a prescribed amplification budget $P > 1$, the explicit condition

$$\sum_{j=1}^m \lambda_j \mathcal{E}_T(\tau_j) \leq \log P \quad (3.28)$$

implies

$$\prod_{j=1}^m (1 + \lambda_j \mathcal{E}_T(\tau_j)) \leq P$$

and bounds the multiplicative amplification on J by P . This budget condition is sufficient but is not needed for the validity of (3.27) when there are finitely many impulses.

Proof. Fix $t \in J$, add the two inequalities, and set

$$W = u^p + v^q \quad \text{and} \quad Y(r) = \sup_{t_0 \leq \xi \leq r} W(\xi).$$

The same kernel reduction as in Theorem 3.4 gives, for $r \leq t$,

$$W(r) \leq A(t) + \sum_{\tau_j \leq r} \lambda_j W(\tau_j^-) + H(t) \int_{t_0}^r K_{\alpha^*}(r, s) Y(s) ds.$$

The step function

$$C(r) = A(t) + \sum_{\tau_j \leq r} \lambda_j W(\tau_j^-)$$

is nonnegative and nondecreasing. The bounded running supremum Y is nondecreasing and measurable. Lemma 2.3 applies on the entire interval, including across impulse times, so taking the running supremum gives

$$Y(r) \leq C(r) + H(t) \int_{t_0}^r K_{\alpha^*}(r, s) Y(s) \, ds.$$

The bounded-measurable version of Lemma 2.1, applied on $[t_0, r]$ with the nondecreasing forcing C , yields

$$W(r) \leq Y(r) \leq \mathcal{E}_t(r) \left[A(t) + \sum_{\tau_j \leq r} \lambda_j W(\tau_j^-) \right].$$

If $t < \tau_1$, the desired bound follows with an empty product equal to one. Otherwise put $k = \max\{j : \tau_j \leq t\}$. Taking the limit before the first impulse gives $W(\tau_1^-) \leq A(t)\mathcal{E}_t(\tau_1)$. Successive substitution at $\tau_2^-, \dots, \tau_k^-$ and the identity

$$1 + \sum_{j=1}^k c_j \prod_{r < j} (1 + c_r) = \prod_{j=1}^k (1 + c_j)$$

prove (3.27). Finally, $\log(1 + x) \leq x$ for $x \geq 0$ shows that (3.28) implies the stated product bound. $\square$

Remark 3.8. Theorem 3.7 permits jumps of the unknowns, whereas Theorem 3.4 uses continuous unknowns. The times τ_j are prescribed; it is the impulse amplitude that depends on the pre-impulse state. The fractional memory always begins at t_0 and is not restarted at an impulse. Both inequalities and the impulse sums use the right-continuous convention $\tau_j \leq t$.

Remark 3.9. Theorem 3.4 contains several known forms as special cases.

- (1) If $N_1 = N_2 = 1$, $\alpha_{11} = \alpha_{21} = 1$, $M_{i\ell}$ are constants, and no retarded arguments occur, the result reduces to an ordinary coupled Gronwall–Bellman estimate.
- (2) The scalar estimate follows by setting $v \equiv 0$, $A_2 \equiv 0$, $N_2 = 1$, $\alpha_{21} = \min_{\ell} \alpha_{1\ell}$, all coefficients in the second inequality to zero, and $b_{1\ell} \equiv 0$.
- (3) If the functions A_i are continuous and the kernels share a common order, the reduction constants $D_{i\ell}$ disappear (see Corollary 3.11).

Remark 3.10. (Closed-form estimate) Because (3.17) replaces time-dependent coefficients by running envelopes, it is generally conservative. Theorem 3.5 retains these envelopes inside the resolvent kernel and may give a smaller bound. Estimate (3.17) remains useful when a closed expression is required and no differentiability or monotonicity of the unknown functions is available.

Corollary 3.11. (Equal-order continuous case) Assume the hypotheses of Theorem 3.4, and suppose, in addition, that

$$\alpha_{i\ell} = \alpha \in (0, 1], \quad i = 1, 2, 1 \leq \ell \leq N_i, \quad (3.29)$$

and that $A_1, A_2, M_{i\ell}$ are continuous. Then, $D_{i\ell} = 1$, the function H in (3.14) simplifies to

$$H(t) = \sum_{i=1}^2 \sum_{\ell=1}^{N_i} M_{i\ell}^*(t) \{a_{i\ell}^\#(t) + b_{i\ell}^\#(t)\}$$

and

$$u(t) \leq \left[(A_1(t) + A_2(t)) E_\alpha(H(t)(t - t_0)^\alpha) \right]^{1/p}, \quad (3.30)$$

$$v(t) \leq \left[(A_1(t) + A_2(t)) E_\alpha(H(t)(t - t_0)^\alpha) \right]^{1/q}. \quad (3.31)$$

Proof. Under (3.29) the minimum order in (3.12) is α , and the reduction constants in (3.13) are all equal to one. Equations (3.30) and (3.31) are exactly (3.17) under this simplification. $\square$

Corollary 3.12. (*Boundedness criterion on a finite interval*) Under the hypotheses of Theorem 3.4, suppose that

$$A(t) \leq A_0, \quad H(t) \leq H_0, \quad t \in J, \quad (3.32)$$

for finite constants A_0, H_0 . Then,

$$\sup_{t \in J} u(t) \leq [A_0 E_{\alpha^*}(H_0 L^{\alpha^*})]^{1/p}, \quad (3.33)$$

$$\sup_{t \in J} v(t) \leq [A_0 E_{\alpha^*}(H_0 L^{\alpha^*})]^{1/q}. \quad (3.34)$$

This criterion makes the uniform bound explicit on the fixed interval. Under the stated assumptions, $A(T)$ and $H(T)$ are finite and may be used for A_0 and H_0 ; the corollary alone does not assert continuation or global existence of a solution.

3.5. Comparison with the motivating literature

The four motivating inequalities can be located precisely in their original publications: [13, Eq (1.3), pp. 1177], [14, Eqs (6) and (7), pp. 275], [15, Eq (4), pp. 9], and [16, Eq (5), pp. 46]. Table 1 records features of these displayed models. A dash means that the feature is absent from that model, not that it is excluded from every consequence or variant of the cited work.

Table 1. Features of four specified inequalities and the main continuous-unknown estimate. Impulses are treated separately in Theorem 3.7.

Feature of the displayed model	[13]	[14]	[15]	[16]	Thmorem 3.4
Two coupled unknown functions	–	✓	–	✓	✓
Weakly singular kernels	–	–	✓	✓	✓
Nonconstant multiplicative factors outside integrals	✓	✓	–	✓	✓
Delayed evaluations inside integrands	✓	–	–	–	✓
Retarded integration upper limits	–	✓	–	–	–
Discontinuous unknown functions with impulse terms	✓	–	–	–	–

In [13], the factor $q(t)$ multiplies the integral and impulse terms, and the nonlinear transformation $\psi(u)$ is more general than a fixed power. In [15], the term $b(t)$ is additive, and the three nested singular integrals in Eq (4) share the same order. In [16], the factors $b_i(t)$ and $d_i(t)$ multiply two types of singular integrals, including a kernel of the form $(t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1}$. Theorem 3.4 concerns convolution kernels and integrands linear in the powered unknowns.

The present theorem permits a finite family of convolution kernels with independently specified orders $\alpha_{i\ell}$, bounded measurable inside coefficients, piecewise continuous outside factors, and measurable delayed evaluations in both components. Its explicit majorant follows after a scalar envelope reduction. The earlier scalar impulsive estimates [17–19] and the coupled Hadamard estimates [20] provide additional context beyond the four models in the table.

For a given inequality, the quantities in (3.17) are obtained as follows:

- (1) Express the unknown quantities through the powers u^p and v^q .
- (2) Include the known discontinuous terms in A_1 and A_2 .
- (3) Verify the retarded conditions $\sigma(t) \leq t$ and $\rho(t) \leq t$.
- (4) Set $\alpha^* = \min_{i,\ell} \alpha_{i\ell}$ and compute the reduction constants $D_{i\ell}$.
- (5) Form the coefficient envelope $H(t)$ from the separate suprema in (3.14).

4. Applications to coupled fractional integral and differential equations

This section applies Theorem 3.4 to a coupled weakly singular Volterra-type integral system with delayed arguments and to inequalities with prescribed jumps in their known majorants. The principal goal is to derive, from a single application of the inequality, a priori bounds and a continuous-dependence/uniqueness result. We also write the corresponding Caputo fractional differential formulation explicitly and exhibit a concrete class of nonlinearities satisfying the powered growth assumptions.

4.1. An explicit analytic example

Let $J = [0, 1]$, fix $\tau_1 = 1/3$ and $\tau_2 = 2/3$, and define

$$\begin{aligned} A_1(t) &= 1 + t + \eta_{11} \chi_{[\tau_1, 1]}(t) + \eta_{12} \chi_{[\tau_2, 1]}(t), \\ A_2(t) &= 2 + t + \eta_{21} \chi_{[\tau_1, 1]}(t) + \eta_{22} \chi_{[\tau_2, 1]}(t), \end{aligned} \quad (4.1)$$

where $\eta_{ij} \geq 0$. Consider nonnegative continuous functions u, v satisfying

$$\begin{aligned} u^2(t) &\leq A_1(t) + (1+t) \int_0^t K_{1/2}(t, s) s \{u^2(s/2) + v^3(s/2)\} ds + (2+t) \int_0^t K_{3/4}(t, s) (1+s) u^2(s) ds, \\ v^3(t) &\leq A_2(t) + (1+2t) \int_0^t K_{1/2}(t, s) (1+s) \{u^2(s/3) + v^3(s/3)\} ds + t \int_0^t K_1(t, s) v^3(s) ds. \end{aligned} \quad (4.2)$$

The retarded maps are $s \mapsto s/2$, $s \mapsto s/3$, and the identity map; the kernel orders are $1/2$, $3/4$, and 1 ; the outside factors are $1+t$, $2+t$, $1+2t$, t . This example combines the coupled powers, different convolution orders, and discontinuous known majorants in the assumptions of Theorem 3.4.

Here

$$\alpha^* = \frac{1}{2}, \quad L = 1, \quad D_{1/2} = 1, \quad D_{3/4} = \frac{\Gamma(1/2)}{\Gamma(3/4)}, \quad D_1 = \Gamma(1/2). \quad (4.3)$$

Since all coefficient functions in (4.2) are nondecreasing on $[0, 1]$, the envelope in (3.14) gives

$$H(t) = 2(1+t)t + (2+t)D_{3/4}(1+t) + 2(1+2t)(1+t) + tD_1. \quad (4.4)$$

Moreover,

$$A(t) = A_1(t) + A_2(t) = 3 + 2t + \sum_{j=1}^2 (\eta_{1j} + \eta_{2j}) \chi_{[\tau_j, 1]}(t). \quad (4.5)$$

Theorem 3.4 yields

$$B(t) = A(t) E_{1/2}(H(t) t^{1/2}), \quad 0 \leq t \leq 1, \quad (4.6)$$

and therefore

$$u(t) \leq B(t)^{1/2}, \quad v(t) \leq B(t)^{1/3}. \quad (4.7)$$

The first inequality contains two different singular orders and one retarded argument; the second contains both a weakly singular term and a nonsingular Volterra term. The system is coupled through u^2 and v^3 while the inhomogeneous terms have finite jumps. The estimate (4.7) is closed-form in terms of $A(t)$, $H(t)$, $\Gamma(1/2)$, $\Gamma(3/4)$, and the Mittag-Leffler function $E_{1/2}$.

4.2. A numerical illustration

Consider on $[0, 1]$ the linear weakly singular system

$$\begin{aligned} x(t) &= 1 + 0.08 \int_0^t K_{0.6}(t, s)x(s) \, ds + 0.04 \int_0^t K_{0.6}(t, s)y(s) \, ds, \\ y(t) &= 0.5 + 0.03 \int_0^t K_{0.8}(t, s)x(s) \, ds + 0.06 \int_0^t K_{0.8}(t, s)y(s) \, ds. \end{aligned} \quad (4.8)$$

We computed a product-integration approximation on a uniform mesh with 400 subintervals. We used the left-endpoint product-integration rule

$$\int_0^{t_n} K_\alpha(t_n, s)f(s) \, ds \approx \sum_{j=0}^{n-1} \omega_{n-j}^{(\alpha)} f(t_j), \quad \omega_k^{(\alpha)} = \frac{h^\alpha}{\Gamma(\alpha+1)} (k^\alpha - (k-1)^\alpha),$$

with $h = 1/400$ and $t_n = nh$. Here $p = q = 1$, $A = 1.5$, $\alpha^* = 0.6$, and

$$H = 0.12 + \frac{\Gamma(0.6)}{\Gamma(0.8)}(0.09).$$

The analytical envelope is therefore $B(t) = 1.5E_{0.6}(Ht^{0.6})$. Table 2 compares the numerical approximation with this bound.

Table 2. Product-integration approximation of (4.8) and the analytical envelope.

t	$x_h(t)$	$y_h(t)$	$x_h(t) + y_h(t)$	$B(t)$
0.25	1.0504	0.5219	1.5723	1.6871
0.50	1.0779	0.5388	1.6167	1.7968
0.75	1.1010	0.5545	1.6555	1.8932
1.00	1.1217	0.5697	1.6914	1.9833

The computed values lie below the analytical envelope. The theorem bounds the exact solution, while the product-integration values are numerical approximations without a certified discretization-error bound. The gap reflects the reduction of the two-component system to the sum $x + y$ and the use of a common smallest-order kernel. The example illustrates the comparison estimate.

4.3. A coupled fractional Volterra-type system

To apply Theorem 3.4 to a Volterra-type system, we use Hölder's inequality to estimate the powers of the integrals. The resulting constants are given below.

Assume $p, q \geq 1$, $N_1, N_2 \in \mathbb{N}$, and $0 < \alpha_{i\ell} \leq 1$. Consider solutions $x, y \in C(J, \mathbb{R})$ of

$$\begin{aligned} x(t) &= x_0(t) + \sum_{\ell=1}^{N_1} m_{1\ell}(t) \int_{t_0}^t K_{\alpha_{1\ell}}(t, s) F_{1\ell}(s, x(\sigma_{1\ell}(s)), y(\rho_{1\ell}(s))) \, ds, \\ y(t) &= y_0(t) + \sum_{\ell=1}^{N_2} m_{2\ell}(t) \int_{t_0}^t K_{\alpha_{2\ell}}(t, s) F_{2\ell}(s, x(\sigma_{2\ell}(s)), y(\rho_{2\ell}(s))) \, ds, \end{aligned} \quad (4.9)$$

where $x_0, y_0 \in C(J, \mathbb{R})$, $m_{i\ell} \in C(J, \mathbb{R}_+)$, the delay functions are retarded, and

$$F_{i\ell}: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

are Carathéodory functions (measurable in t and continuous in the state variables). Put

$$\Lambda_\alpha := \int_{t_0}^T K_\alpha(T, s) \, ds = \frac{L^\alpha}{\Gamma(\alpha + 1)}.$$

Suppose that the nonlinearities satisfy the powered growth conditions

$$|F_{1\ell}(s, \xi, \zeta)|^p \leq a_{1\ell}(s)|\xi|^p + b_{1\ell}(s)|\zeta|^q, \quad (4.10)$$

$$|F_{2\ell}(s, \xi, \zeta)|^q \leq a_{2\ell}(s)|\xi|^p + b_{2\ell}(s)|\zeta|^q, \quad (4.11)$$

with $a_{i\ell}, b_{i\ell} \in L^\infty(J, \mathbb{R}_+)$. By the elementary inequality

$$|z_0 + z_1 + \cdots + z_N|^r \leq (N + 1)^{r-1} \sum_{j=0}^N |z_j|^r, \quad r \geq 1,$$

and, for $t > t_0$, by Hölder's inequality,

$$\left| \int_{t_0}^t K_\alpha(t, s) f(s) \, ds \right|^r \leq \left(\int_{t_0}^t K_\alpha(t, s) \, ds \right)^{r-1} \int_{t_0}^t K_\alpha(t, s) |f(s)|^r \, ds \leq \Lambda_\alpha^{r-1} \int_{t_0}^t K_\alpha(t, s) |f(s)|^r \, ds.$$

At $t = t_0$, the final inequality is immediate because both integrals vanish. Therefore, for $u(t) = |x(t)|$ and $v(t) = |y(t)|$, the system (4.9) implies inequalities of the form (3.10) and (3.11) with

$$\begin{aligned} A_1(t) &= (N_1 + 1)^{p-1} \sup_{t_0 \leq r \leq t} |x_0(r)|^p, & M_{1\ell}(t) &= (N_1 + 1)^{p-1} m_{1\ell}(t)^p \Lambda_{\alpha_{1\ell}}^{p-1}, \\ A_2(t) &= (N_2 + 1)^{q-1} \sup_{t_0 \leq r \leq t} |y_0(r)|^q, & M_{2\ell}(t) &= (N_2 + 1)^{q-1} m_{2\ell}(t)^q \Lambda_{\alpha_{2\ell}}^{q-1}. \end{aligned}$$

The functions A_1, A_2 are continuous and nondecreasing, so Theorem 3.4 gives explicit a priori bounds for every solution of (4.9) for which the stated hypotheses hold.

Theorem 4.1 (A priori estimate for the Volterra system). *Assume the hypotheses stated above, including (4.10) and (4.11). Define A_i and $M_{i\ell}$ by the preceding formulas and form α^* , H , and B as in Eqs (3.12)–(3.16). Then, every solution $(x, y) \in C(J, \mathbb{R})^2$ of (4.9) satisfies*

$$|x(t)| \leq B(t)^{1/p}, \quad |y(t)| \leq B(t)^{1/q}, \quad t \in J.$$

Proof. The Hölder estimate displayed above converts (4.9) into (3.10) and (3.11) for $u = |x|$ and $v = |y|$. The conclusion is therefore an immediate application of Theorem 3.4. $\square$

4.4. An explicit Caputo fractional differential system

We use the classical Caputo derivative on $AC(J)$, the space of absolutely continuous functions. For $x \in AC(J)$ and $0 < \beta < 1$, set

$${}^C D_{t_0+}^\beta x = \mathcal{I}_{1-\beta} x', \quad {}^C D_{t_0+}^1 x = x',$$

where derivatives and differential equations are understood almost everywhere. Let $0 < \beta_1, \beta_2 \leq 1$ and consider

$$\begin{aligned} {}^C D_{t_0+}^{\beta_1} x(t) &= F_1(t, x(\sigma_1(t)), y(\rho_1(t))), & x(t_0) &= x_*, \\ {}^C D_{t_0+}^{\beta_2} y(t) &= F_2(t, x(\sigma_2(t)), y(\rho_2(t))), & y(t_0) &= y_*. \end{aligned} \quad (4.12)$$

Assume $x, y \in AC(J)$ and that the composed right-hand sides $g_i(t) = F_i(t, x(\sigma_i(t)), y(\rho_i(t)))$ are bounded and measurable. Within this solution class, (4.12) is equivalent to the following identities for every $t \in J$:

$$\begin{aligned} x(t) &= x_* + \int_{t_0}^t K_{\beta_1}(t, s) F_1(s, x(\sigma_1(s)), y(\rho_1(s))) \, ds, \\ y(t) &= y_* + \int_{t_0}^t K_{\beta_2}(t, s) F_2(s, x(\sigma_2(s)), y(\rho_2(s))) \, ds. \end{aligned}$$

Indeed, $\mathcal{I}_\beta {}^C D_{t_0+}^\beta x = x - x(t_0)$ for $x \in AC(J)$ follows from the semigroup identity for fractional integrals. Conversely, applying this identity to an absolutely continuous solution of the integral equation gives $\mathcal{I}_\beta ({}^C D_{t_0+}^\beta x - g_1) = 0$; injectivity of $\mathcal{I}_\beta$ on $L^1(J)$ yields the differential equation almost everywhere. The argument for y is identical, and $\beta = 1$ is the fundamental theorem of calculus. Boundedness of g_i makes the fractional integrals continuous, so the integral identities hold at every point. A merely continuous

Volterra solution need not be absolutely continuous; for such a solution only the integral formulation is asserted.

Thus, Theorem 4.1 applies with $N_1 = N_2 = 1$, $m_{11} = m_{21} = 1$, $\alpha_{11} = \beta_1$, and $\alpha_{21} = \beta_2$. In particular, if

$$|F_1(t, \xi, \zeta)|^p \leq a_1(t)|\xi|^p + b_1(t)|\zeta|^q, \quad |F_2(t, \xi, \zeta)|^q \leq a_2(t)|\xi|^p + b_2(t)|\zeta|^q,$$

with bounded nonnegative coefficients, then every solution admits the explicit componentwise estimate supplied by Theorem 4.1.

Example 4.2. (Concrete nonlinearities) Let $\lambda_i, \mu_i \in C(J, \mathbb{R}_+)$ be bounded and set

$$F_1(t, \xi, \zeta) = \lambda_1(t)\xi + \mu_1(t)|\zeta|^{q/p}, \quad F_2(t, \xi, \zeta) = \lambda_2(t)|\xi|^{p/q} + \mu_2(t)\zeta.$$

For $p, q \geq 1$, the inequality $|r + s|^m \leq 2^{m-1}(|r|^m + |s|^m)$ gives

$$\begin{aligned} |F_1(t, \xi, \zeta)|^p &\leq 2^{p-1}\lambda_1(t)^p|\xi|^p + 2^{p-1}\mu_1(t)^p|\zeta|^q, \\ |F_2(t, \xi, \zeta)|^q &\leq 2^{q-1}\lambda_2(t)^q|\xi|^p + 2^{q-1}\mu_2(t)^q|\zeta|^q. \end{aligned}$$

Hence, these nonlinearities satisfy the powered growth assumptions with explicitly identifiable coefficient functions. Every continuous Volterra solution has bounded measurable composed right-hand sides and satisfies the a priori bound; the classical Caputo interpretation additionally requires the absolute continuity specified above. These growth conditions alone do not imply the difference estimates in the next subsection.

4.5. Continuous dependence and uniqueness

Retain $p, q \geq 1$ and the setting of Subsection 4.3. Let (x_1, y_1) and (x_2, y_2) be two solutions of Eq (4.9) with the same kernels, multipliers, retarded maps, and nonlinearities, but possibly with different prescribed functions x_0, y_0 . Define

$$u(t) := |x_1(t) - x_2(t)|, \quad v(t) := |y_1(t) - y_2(t)|. \quad (4.13)$$

Assume powered Lipschitz estimates

$$|F_{1\ell}(s, \xi_1, \zeta_1) - F_{1\ell}(s, \xi_2, \zeta_2)|^p \leq L_{1\ell}(s)(|\xi_1 - \xi_2|^p + |\zeta_1 - \zeta_2|^q), \quad (4.14)$$

$$|F_{2\ell}(s, \xi_1, \zeta_1) - F_{2\ell}(s, \xi_2, \zeta_2)|^q \leq L_{2\ell}(s)(|\xi_1 - \xi_2|^p + |\zeta_1 - \zeta_2|^q), \quad (4.15)$$

where $L_{i\ell} \in L^\infty(J, \mathbb{R}_+)$. Applying the same Hölder estimate used in Subsection 4.3 to the difference of the two equations, we obtain (3.10) and (3.11) with

$$\begin{aligned} A_1(t) &= (N_1 + 1)^{p-1} \sup_{t_0 \leq r \leq t} |x_0^{(1)}(r) - x_0^{(2)}(r)|^p, \\ A_2(t) &= (N_2 + 1)^{q-1} \sup_{t_0 \leq r \leq t} |y_0^{(1)}(r) - y_0^{(2)}(r)|^q, \end{aligned}$$

where the supremum is essential to obtain nondecreasing known data. Put $r_1 = p$, $r_2 = q$, and take

$$M_{i\ell}(t) = (N_i + 1)^{r_i-1} m_{i\ell}(t)^{r_i} \Lambda_{\alpha_{i\ell}}^{r_i-1}, \quad a_{i\ell} = b_{i\ell} = L_{i\ell}.$$

Define H and B from these coefficients and the displayed envelopes, as in (3.14)–(3.16). Theorem 3.4 therefore yields

$$|x_1(t) - x_2(t)| \leq B(t)^{1/p}, \quad |y_1(t) - y_2(t)| \leq B(t)^{1/q}. \quad (4.16)$$

For fixed coefficients (or a family with a uniform bound on $H(T)$), if the prescribed-data difference envelopes converge uniformly to zero, (4.16) gives continuous dependence. In the special case of identical prescribed data, $A_1 \equiv A_2 \equiv 0$, and the comparison bound gives $B \equiv 0$; hence, $x_1 = x_2$ and $y_1 = y_2$ on J . Thus, uniqueness follows in the class satisfying the powered Lipschitz assumptions.

Remark 4.3. (Scope of the powered difference conditions) Conditions (4.14) and (4.15) are additional assumptions, not consequences of (4.10) and (4.11). If imposed for all real state variables with $q > p$, the first condition makes $F_{1\ell}$ independent of ζ for almost every time. Indeed, with ξ fixed it gives a Hölder bound of exponent $q/p > 1$; subdivision of any interval into n equal pieces bounds the total variation between its endpoints by a constant times $n^{1-q/p}$, which tends to zero. Similarly, if $p > q$, the second condition makes $F_{2\ell}$ independent of ξ . For mutually coupled systems, the choice $p = q \geq 1$ avoids this restriction. Ordinary Lipschitz estimates in both variables imply the powered conditions when $p = q$, and one may in particular apply the comparison theorem with $p = q = 1$ to differences even when different powers are used for a priori growth bounds.

5. Concluding remarks

5.1. Main results

Theorem 3.4 gives an explicit estimate for coupled weakly singular retarded integral inequalities involving powered unknowns, kernels of different fractional orders, nonconstant outside multipliers, retarded arguments, and piecewise continuous inhomogeneous terms. The proof reduces the kernels to a common smallest order and applies a scalar comparison to the envelope $u^p + v^q$. Theorem 3.1 gives the scalar form, Theorem 3.5 retains the variable coefficient envelopes in a resolvent bound, and Theorem 3.7 treats state-dependent impulses. Corollaries 3.11 and 3.12 record the equal-order and boundedness cases.

Section 4.5 applies the estimate to the difference of two solutions of a coupled fractional Volterra-type system. Under the additional powered difference conditions, this gives continuous dependence on the prescribed data and uniqueness. For unequal powers, the restrictions on mutual coupling are stated explicitly in that subsection.

5.2. Limitations

The use of running envelopes and a common smallest-order kernel may enlarge the resulting constants. A quantitative comparison with [13–16] requires a common specialized model and an explicit comparison of the resulting constants. The present bounds are intended for problems in which the coupled, delayed, variable-coefficient, and impulsive features occur together.

5.3. Further directions

Further work may address resolvent estimates without running coefficient envelopes, infinite impulse sequences without finite accumulation points, and matrix-valued comparison principles that retain more componentwise information than the scalar sum envelope.

Use of Generative-AI tools declaration

During the preparation of this manuscript, the author used Writefull within Overleaf for language editing and zbMATH Open for the initial retrieval and screening of relevant literature.

Conflict of interest

The author declares that there is no conflict of interest.

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