



*Research article***A distributional framework for the short-time special affine Fourier transform and quasiasymptotic analysis****Astrit Ferizi^{1,*} and Katerina Hadzi-Velkova Saneva²**¹ Faculty of Mathematics and Natural Sciences, University of Prishtina, George Bush 31, Prishtina, 10000, Kosovo² Faculty of Electrical Engineering and Information Technologies, Ss. Cyril and Methodius University, Rugjer Boshkovikj 18, Skopje, 1000, North Macedonia*** Correspondence:** Email: astrit.ferizi@uni-pr.edu.

Abstract: Existing theory of the short-time special affine Fourier transform (STSFFT) is primarily formulated in the classical L^2 -setting and does not provide a framework for tempered distributions and their quasiasymptotic analysis. To address this gap, we develop a distributional framework for the STSFFT. We establish the continuity of the STSFFT and its synthesis operator on the corresponding Schwartz spaces of test functions and extend both operators to the associated spaces of tempered distributions by duality. Within this framework, we derive a generalized inversion formula, a desingularization formula, and a characterization of bounded subsets of $\mathcal{S}'(\mathbb{R})$ in terms of polynomial growth estimates of the STSFFT. Furthermore, we establish several Abelian- and Tauberian-type results that relate the quasiasymptotic behavior and quasiasymptotic boundedness of tempered distributions at the origin to the scaled asymptotic behavior of their STSFFT. These results provide a rigorous basis for studying singular signals and local asymptotic phenomena, with potential relevance to distribution theory, time-frequency analysis, optics, and communications.

Keywords: fractional Fourier transform; special affine Fourier transform; short-time special affine Fourier transform; distributions; desingularization formula; Abelian- and Tauberian-type results; quasiasymptotic behavior of distributions

Mathematics Subject Classification: 26A12, 40E05, 42A38, 46F05, 46F12

1. Introduction

The fractional Fourier transform (FRFT) was popularized by Namias in [1]. Interpreting the FRFT as a rotation of the Wigner distribution function in phase space [2], Abe and Sheridan further generalized it to the special affine Fourier transform (SAFT). They also showed that the SAFT can

be realized through the combination of only two independent transformations: the lens transformation and free-space propagation [3]. Moreover, the SAFT has been employed to define the almost-Fourier and almost-Fresnel transforms, providing a unified framework for handling small errors in optical operations on wave functions [4]. As a generalization of the classical Fourier and fractional Fourier transforms, the SAFT, also known as the offset linear canonical transform, has become a powerful tool in signal processing, optics, and related applications [5–8]. Compared with the linear canonical transform, the SAFT incorporates two additional offset parameters associated with space shifts and frequency modulations, thereby providing greater flexibility in the representation of signal-processing and optical transformations [7]. Alongside its practical applications, the mathematical theory of the SAFT has been thoroughly explored, addressing topics such as uncertainty principles, convolution and correlation theorems, multichannel sampling, the Poisson summation formula, and convolution and product theorems [9–14].

More broadly, an important aspect of the mathematical theory of integral transforms is their extension to distribution spaces. Such extensions allow integral transforms to be applied to singular distributions such as the Dirac delta and its derivatives, which lie outside the classical L^2 -setting. They also provide a rigorous framework for analyzing weak limits and scaling behavior that may not be well-defined in classical function spaces. Several studies in the literature have already pursued such extensions for different classes of integral transforms [15–18]. In this direction, a distributional framework for the SAFT was developed in [19–21] using a duality approach, thereby enabling a deeper analysis of its properties in the distributional setting.

Because the SAFT fails to support localized time-frequency analysis of signals, Shah et al. [22] introduced its windowed version. They derived Parseval's identity, established the inversion formula, and proposed a discrete version of this transform. In addition, they provided the series expansion, the Poisson summation formula, and several uncertainty principles related to the windowed SAFT.

Recently, Lakshmanan et al. [23] introduced a new convolution operation for the SAFT and used it to define the short-time special affine Fourier transform (STSAFT). In this construction, the signal is localized by a shifted window and then analyzed in the SAFT domain. Thus, the STSAFT describes the local behavior of the signal while remaining explicitly connected to the newly introduced SAFT convolution. The authors of [23] established a Parseval-type identity and an inversion formula, characterized the range of the transform, derived a Heisenberg-type uncertainty principle, and extended the STSAFT to square-integrable quaternion-valued functions. Compared with the windowed SAFT, the additional chirp-phase factors in the STSAFT explicitly incorporate the quadratic and affine phase structure associated with the SAFT convolution. Consequently, the STSAFT provides a natural framework for the local characterization of signals exhibiting chirp-type or affine phase variations while retaining the localization provided by the shifted window.

Beyond the SAFT and STSAFT, other generalized fractional-transform variants have recently been explored in communication systems. In particular, the weighted fractional Fourier transform (WFRFT) has been integrated with rate-splitting multiple access to enhance physical-layer security [24], and the general multi-fractional Fourier transform (GMFRFT) has been employed for signal-domain, multicomponent, secure hybrid precoding in mmWave systems [25]. These developments illustrate the broader potential of generalized fractional-transform techniques in signal-domain processing and communications. In this context, the development of a distributional theory for the STSAFT is a natural step toward extending its applicability beyond the classical L^2 -setting, particularly to singular signals,

distributional phenomena, and localized time-frequency analysis. Moreover, owing to the affine and chirp-dependent structure of the STSAFT, such a distributional framework may also be relevant to optical and communication-system models involving phase, chirp, and affine transformations.

Another line of research directly relevant to the present work concerns the quasiasymptotic behavior of distributions. This concept was originally introduced by Zivialov [26] and subsequently developed with Vladimirov and Drozzinov. It generalizes the notion of the point value of a distribution, initially introduced by Łojasiewicz [27], and represents a fundamental component of distribution theory (see [28–30] and the references therein). Over the years, quasiasymptotic behavior has been extensively studied and characterized via various integral transforms (see, e.g., [15, 16, 31–34]). Related characterizations have also been obtained using other analytical frameworks, such as Gabor frame expansions [35], shearlet coefficients [36], and wavelet-based techniques [37].

Motivated by these developments, the main objectives of the present work are to develop a rigorous distributional framework for the STSAFT and to use it to characterize the quasiasymptotic behavior of tempered distributions at the origin through the asymptotic properties of their STSAFT. However, the existing distributional results for the SAFT do not directly extend to the STSAFT. Owing to its window-dependent structure and two-dimensional parameter space, such an extension requires separate continuity results for the corresponding analysis and synthesis operators. Moreover, the existing L^2 -theory of the STSAFT does not address distributional reconstruction, bounded-set characterizations, or quasiasymptotic analysis.

Given the close structural relationship between the STSAFT and the classical STFT, several results considered in this work, including continuity on Schwartz spaces, reconstruction, and extension to tempered distributions by duality, have natural counterparts in STFT theory. Abelian- and Tauberian-type characterizations of quasiasymptotic behavior based on the STFT are likewise available in the literature [38]. In the present work, the corresponding results are formulated within the STSAFT setting, where the SAFT-dependent phase factors and parameter scalings enter directly into the quasiasymptotic characterizations.

The paper is organized as follows. Section 2 introduces the notation and function spaces used throughout the paper and provides a brief overview of the SAFT and STSAFT. Section 3 derives a representation formula relating the STSAFT to the SAFT and uses it to establish the continuity of the STSAFT on Schwartz spaces of test functions. In Section 4, we introduce the synthesis operator, prove its continuity, and derive the inversion formula (4.4) in the Schwartz space $\mathcal{S}(\mathbb{R})$. Section 5 extends the STSAFT and its synthesis operator to tempered distributions by duality and establishes a generalized inversion formula in this setting. These results further yield a distributional version of Parseval's identity, referred to as the desingularization formula, as well as a characterization of bounded subsets of tempered distributions in terms of polynomial growth estimates of the STSAFT. In Section 6, we investigate the quasiasymptotic behavior and quasiasymptotic boundedness of tempered distributions at the origin and derive the corresponding Abelian- and Tauberian-type results. Finally, Section 7 summarizes the main findings and conclusions of the paper.

2. Preliminaries

2.1. Notation and spaces

The notation $k_1 \lesssim k_2$ is used to indicate that $k_1 \leq k \cdot k_2$ for some positive constant k . The Fourier transform (FT) $\mathcal{F}$ is defined by

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x} \cdot \xi} d\mathbf{x}, \quad f \in L^1(\mathbb{R}^n).$$

The notation $\langle f, \varphi \rangle$ means the dual pairing between the distribution f and φ , whereas $(f, \varphi)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{\varphi(\mathbf{x})} d\mathbf{x}$ stands for the $L^2(\mathbb{R}^n)$ -inner product.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of test functions consists of those $\varphi \in C^\infty(\mathbb{R}^n)$ whose derivatives decay rapidly. Specifically, for all $\nu \in \mathbb{N}_0$,

$$\rho_\nu(\varphi) := \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{m}| \leq \nu}} (1 + |\mathbf{x}|^2)^{\nu/2} |\partial_{\mathbf{x}}^{\mathbf{m}} \varphi(\mathbf{x})| < \infty. \quad (2.1)$$

Endowed with the topology generated by the seminorms (2.1), the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ becomes a nuclear Fréchet space. Its dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. Every tempered distribution is the distributional derivative of some continuous function of at most polynomial growth; see [39, 40]. If a measurable function f on $\mathbb{R}^n$ is of at most polynomial growth on $\mathbb{R}^n$, that is, if there exist $C > 0$ and $\nu \in \mathbb{N}_0$ such that

$$|f(\mathbf{x})| \leq C (1 + |\mathbf{x}|^2)^{\nu/2}, \quad \mathbf{x} \in \mathbb{R}^n,$$

then f defines a regular tempered distribution in $\mathcal{S}'(\mathbb{R}^n)$ by duality:

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (2.2)$$

In this article, all dual spaces will be equipped with the strong dual topology, that is, the topology of bounded convergence [41].

2.2. The SAFT and the STSAFT

This subsection provides a brief summary of well-known results on the SAFT [2, 3, 19] and the STSAFT [22, 23].

For a given matrix $\mathbf{A} = (\alpha, \beta, p; \gamma, \delta, q) = \begin{pmatrix} \alpha & \beta & p \\ \gamma & \delta & q \end{pmatrix} \in \mathbb{R}^{2 \times 3}$ with $\beta \neq 0$ and $\alpha\delta - \beta\gamma = 1$, the SAFT of a function $f \in L^1(\mathbb{R})$, with respect to the matrix $\mathbf{A}$, denoted by $S_{\mathbf{A}}f$, is defined as

$$S_{\mathbf{A}}f(\xi) = \int_{\mathbb{R}} f(x) K_{\mathbf{A}}(x, \xi) dx, \quad (2.3)$$

where the kernel $K_{\mathbf{A}}(x, \xi)$ is given by

$$K_{\mathbf{A}}(x, \xi) = (2\pi|\beta|)^{-1/2} e^{\frac{i}{2\beta}(\alpha x^2 - 2x(\xi - p) + 2\xi(\beta q - \delta p) + \delta \xi^2)} \quad (2.4)$$

for $x, \xi \in \mathbb{R}$ [2, 3]. For a parameter matrix $\mathbf{A} = (\alpha, \beta, 0; \gamma, \delta, 0)$ in (2.3), the SAFT reduces to the linear canonical transform (LCT) [42, 43]. In particular, when $\mathbf{A} = (\cos \theta, \sin \theta, 0; -\sin \theta, \cos \theta, 0)$, $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$, the LCT reduces to the FRFT [44, 45]. Setting $\theta = \pi/2$ further recovers the classical FT. Throughout this paper, we assume that $\mathbf{A} = (\alpha, \beta, p; \gamma, \delta, q)$ is a fixed matrix satisfying $\beta \neq 0$ and $\alpha\delta - \beta\gamma = 1$.

If $f \in L^1(\mathbb{R})$ such that $S_{\mathbf{A}}f \in L^1(\mathbb{R})$, then the following inversion formula holds:

$$f(x) = \int_{\mathbb{R}} S_{\mathbf{A}}f(\xi) K_{\mathbf{A}^*}(\xi, x) d\xi \quad (2.5)$$

for almost everywhere (a.e.) $x \in \mathbb{R}$ ([23], Thm. 3.11). $K_{\mathbf{A}^*}$ is the kernel corresponding to the matrix $\mathbf{A}^* = \begin{pmatrix} \delta & -\beta & \beta q - \delta p \\ -\gamma & \alpha & \gamma p - \alpha q \end{pmatrix}$, as defined in (2.4). One can notice that $K_{\mathbf{A}^*}(\xi, x) = \overline{K_{\mathbf{A}}(x, \xi)}$, $x, \xi \in \mathbb{R}$ [23].

The SAFT operator $S_{\mathbf{A}} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ is continuous ([19], Thm. 2.1; [23], Thm. 3.5), and it extends to the space of tempered distributions ([21], Def. 3.6) by duality:

$$\langle S_{\mathbf{A}}f, \varphi \rangle = \langle f, \overline{S_{\mathbf{A}^*}\varphi} \rangle, \quad f \in \mathcal{S}'(\mathbb{R}), \varphi \in \mathcal{S}(\mathbb{R}).$$

Naturally, Shah et al. in [22] defined the windowed special affine Fourier transform $O_{\psi, \mathbf{A}}f$ of a function $f \in L^2(\mathbb{R})$ with respect to a window function $\psi \in L^2(\mathbb{R})$ as

$$O_{\psi, \mathbf{A}}f(b, a) = \int_{\mathbb{R}} f(x) \overline{\psi(x-b)} K_{\mathbf{A}}(x, a) dx,$$

where $(b, a) \in \mathbb{R} \times \mathbb{R}$, and $K_{\mathbf{A}}$ is given by (2.4) ([22], Def. 2.1).

Recently, Lakshmanan et al. in [23] introduced a novel STSAFT of $f \in L^2(\mathbb{R})$ with respect to a window function $\psi \in L^2(\mathbb{R})$ and a given matrix $\mathbf{A}$ as the function $V_{\psi, \mathbf{A}}f$ defined by

$$V_{\psi, \mathbf{A}}f(b, a) = \int_{\mathbb{R}} f(x) \overline{\psi(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha, \beta}(b)} \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a) dx, \quad (2.6)$$

where $(b, a) \in \mathbb{R} \times \mathbb{R}$, $K_{\mathbf{A}}$ is given by (2.4), and

$$\kappa_{\mathbf{A}}(x) = e^{\frac{i}{2\beta}(\delta x^2 - 2px + 2x(\beta q - \delta p))}, \quad \chi_{\alpha, \beta}(x) = e^{i\frac{\alpha}{2\beta}x^2} \quad (2.7)$$

for $x \in \mathbb{R}$ ([23], Def. 4.1). For the matrix $\mathbf{A} = (0, 1, 0; -1, 0, 0)$, the STSAFT reduces to the short-time Fourier transform (STFT), denoted by V_{ψ} [46].

Unlike the windowed SAFT $O_{\psi, \mathbf{A}}$, which is obtained by applying the SAFT directly to the windowed signal, the STSAFT $V_{\psi, \mathbf{A}}$ incorporates additional chirp-phase factors associated with the newly introduced SAFT convolution structure.

Motivated by (2.6), we define the STSAFT of $f \in \mathcal{S}'(\mathbb{R})$ with respect to $\psi \in \mathcal{S}(\mathbb{R})$ and a given matrix $\mathbf{A}$ as the direct evaluation

$$V_{\psi, \mathbf{A}}f(b, a) = \left\langle f(x), \overline{\psi(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha, \beta}(b)} \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a) \right\rangle_x, \quad (b, a) \in \mathbb{R} \times \mathbb{R}. \quad (2.8)$$

One may observe that all partial derivatives of $\overline{\psi(x-b)}\kappa_{\mathbf{A}}(b-x)\overline{\chi_{\alpha,\beta}(b)}\chi_{\alpha,\beta}(x)K_{\mathbf{A}}(x,a)$ with respect to the variables $b, a \in \mathbb{R}$ belong to $\mathcal{S}(\mathbb{R})$. Consequently, by ([41], Thm. 27.1), the STSAFT $V_{\psi,\mathbf{A}}f(b,a)$ is a smooth function with respect to the variables $(b,a) \in \mathbb{R} \times \mathbb{R}$.

We now proceed to show that $V_{\psi,\mathbf{A}}f(b,a)$ exhibits at most polynomial growth in the variables (b,a) over $\mathbb{R} \times \mathbb{R}$. Because $f \in \mathcal{S}'(\mathbb{R})$, there exist $\nu \in \mathbb{N}_0$ and a constant $C' > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R})$,

$$|\langle f, \varphi \rangle| \leq C' \rho_{\nu}(\varphi). \quad (2.9)$$

Then, the relations (2.8) and (2.9) yield

$$\begin{aligned} |V_{\psi,\mathbf{A}}f(b,a)| &= \left| \left\langle f(x), \overline{\psi(x-b)}\kappa_{\mathbf{A}}(b-x)\overline{\chi_{\alpha,\beta}(b)}\chi_{\alpha,\beta}(x)K_{\mathbf{A}}(x,a) \right\rangle_x \right| \\ &\leq C' \rho_{\nu} \left(\overline{\psi(x-b)}\kappa_{\mathbf{A}}(b-x)\overline{\chi_{\alpha,\beta}(b)}\chi_{\alpha,\beta}(x)K_{\mathbf{A}}(x,a) \right) \\ &\leq C(1 + |b|^2 + |a|^2)^{l/2}, \end{aligned}$$

for some $C > 0$ and $l \in \mathbb{N}_0$. Then, by means of the standard identification (2.2), $V_{\psi,\mathbf{A}}f$ can be viewed as an element of $\mathcal{S}'(\mathbb{R}^2)$ via the action

$$\langle V_{\psi,\mathbf{A}}f(b,a), \Phi(b,a) \rangle_{b,a} = \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi,\mathbf{A}}f(b,a) \Phi(b,a) db da, \quad \Phi \in \mathcal{S}(\mathbb{R}^2).$$

3. The continuity of the STSAFT on $\mathcal{S}(\mathbb{R})$

We first derive a representation formula relating the STSAFT to the SAFT, which will be used in the proof of our first main result. Using Parseval's identity for the FT and the Fourier inversion formula, for every $\psi \in \mathcal{S}(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, we obtain

$$\begin{aligned} V_{\psi,\mathbf{A}}f(b,a) &= \overline{\chi_{\alpha,\beta}(b)} \int_{\mathbb{R}} \mathcal{F}(f(x)\chi_{\alpha,\beta}(x)K_{\mathbf{A}}(x,a))(\xi) e^{i\xi b} \mathcal{F}(\overline{\psi}\kappa_{\mathbf{A}})(\xi) d\xi \\ &= \frac{\overline{\chi_{\alpha,\beta}(b)}}{\sqrt{2\pi}} \int_{\mathbb{R}} S_{\mathbf{A}}(f(x)\chi_{\alpha,\beta}(x)e^{-i\xi x})(a) e^{i\xi b} \mathcal{F}(\overline{\psi}\kappa_{\mathbf{A}})(\xi) d\xi, \end{aligned} \quad (3.1)$$

where $\check{\psi}(x) = \psi(-x)$.

We notice that the STSAFT may be considered as a sesquilinear mapping $V_{\mathbf{A}} : (f, \psi) \rightarrow V_{\psi,\mathbf{A}}f$.

Theorem 3.1. *The mapping $V_{\mathbf{A}} : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}^2)$ is continuous.*

Proof. For the seminorms on the space $\mathcal{S}(\mathbb{R}^2)$, we make the equivalent choice

$$\rho_{s,r}^{l,k}(\Phi) = \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}} (1 + |b|^2)^{r/2} (1 + |a|^2)^{s/2} \left| \frac{d^l}{da^l} \frac{d^k}{db^k} \Phi(b,a) \right| \quad (3.2)$$

for all $s, r, l, k \in \mathbb{N}_0$.

To show the continuity of the sesquilinear map $V_{\mathbf{A}}$, we must prove that for given $s, r, l, k \in \mathbb{N}_0$, there exist $\nu, \tau \in \mathbb{N}_0$ and $C > 0$ such that for all $f \in \mathcal{S}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$,

$$\rho_{s,r}^{l,k}(V_{\psi,\mathbf{A}}f) \leq C\rho_{\nu}(f)\rho_{\tau}(\psi).$$

Step 1. A direct application of Leibniz's formula yields

$$\begin{aligned} & \left| \frac{d^l}{da^l} \frac{d^k}{db^k} (V_{\psi,\mathbf{A}}f(b,a)) \right| \\ &= \left| \frac{d^l}{da^l} \frac{d^k}{db^k} \left(\int_{\mathbb{R}} f(x) \overline{\psi(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha,\beta}(b)} \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a) dx \right) \right| \\ &= \left| \sum_{q_1, q_2, t \leq k} c_{q_1, q_2, t} b^{q_1} \frac{d^l}{da^l} \left(\int_{\mathbb{R}} x^{q_2} f(x) \overline{\psi^{(t)}(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha,\beta}(b)} \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a) dx \right) \right| \\ &= \left| \sum_{\substack{q_1, q_2, t \leq k \\ j, m \leq l}} c_{q_1, q_2, t, j, m} b^{q_1} a^j \int_{\mathbb{R}} x^{m+q_2} f(x) \overline{\psi^{(t)}(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha,\beta}(b)} \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a) dx \right| \\ &\leq (1 + |a|^2)^{l/2} (1 + |b|^2)^{k/2} \sum_{\substack{q_1, q_2, t \leq k \\ j, m \leq l}} |c_{q_1, q_2, t, j, m}| |V_{\psi^{(t)}, \mathbf{A}} f_{m+q_2}(b, a)|, \end{aligned}$$

where $f_{m+q_2}(x) = x^{m+q_2} f(x)$ and $c_{q_1, q_2, t, j, m}$ are constants that depend only on k, l . Because differentiation and multiplication by polynomials are continuous linear operators on $\mathcal{S}(\mathbb{R})$, it suffices to consider the case $l = k = 0$.

Step 2. Because the seminorms (3.2) with even r form a defining family for the topology of $\mathcal{S}(\mathbb{R}^2)$, we may assume that r is even. Using the relation (3.1), we obtain

$$\begin{aligned} & (1 + |b|^2)^{r/2} |V_{\psi,\mathbf{A}}f(b,a)| \\ &= (1 + |b|^2)^{r/2} \left| \overline{\chi_{\alpha,\beta}(b)} \int_{\mathbb{R}} \mathcal{F}(f(x) \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a))(\xi) \mathcal{F}(\overline{\psi} \kappa_{\mathbf{A}})(\xi) e^{i\xi b} d\xi \right| \\ &= \left| \int_{\mathbb{R}} \mathcal{F}(f(x) \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a))(\xi) \mathcal{F}(\overline{\psi} \kappa_{\mathbf{A}})(\xi) \left(1 - \frac{d^2}{d\xi^2}\right)^{r/2} (e^{i\xi b}) d\xi \right| \\ &= \left| \int_{\mathbb{R}} \left(1 - \frac{d^2}{d\xi^2}\right)^{r/2} \left(\mathcal{F}(f(x) \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a))(\xi) \mathcal{F}(\overline{\psi} \kappa_{\mathbf{A}})(\xi) \right) e^{i\xi b} d\xi \right|. \end{aligned}$$

The last equality is obtained by repeated integration by parts, with all boundary terms vanishing because the functions involved are Schwartz functions. Hence, the relation (3.1) yields

$$\begin{aligned} & (1 + |b|^2)^{r/2} |V_{\psi,\mathbf{A}}f(b,a)| \\ &= \left| \sum_{j, m \leq r} c_{j, m} \int_{\mathbb{R}} \mathcal{F}(x^m f(x) \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a))(\xi) \mathcal{F}(x^j \overline{\psi} \kappa_{\mathbf{A}})(\xi) e^{i\xi b} d\xi \right| \\ &\lesssim \sum_{j, m \leq r} \left| \overline{\chi_{\alpha,\beta}(b)} \int_{\mathbb{R}} \mathcal{F}(f_m(x) \chi_{\alpha,\beta}(x) K_{\mathbf{A}}(x,a))(\xi) e^{i\xi b} \mathcal{F}(\overline{\psi}_{[j]} \kappa_{\mathbf{A}})(\xi) d\xi \right| \end{aligned}$$

$$= \sum_{j,m \leq r} |V_{\psi_{[j]}, \mathbf{A}} f_m(b, a)|,$$

where $f_m(x) = x^m f(x)$ and $\psi_{[j]}(x) = (-x)^j \psi(x)$ belong to $\mathcal{S}(\mathbb{R})$, and the constants $c_{j,m}$ depend only on r . Thus, we may reduce the proof to the case $r = 0$.

Step 3. Because the SAFT $S_{\mathbf{A}} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ is a continuous map ([19], Thm. 2.1), by relation (3.1), we obtain

$$\begin{aligned} & (1 + |a|^2)^{s/2} |V_{\psi, \mathbf{A}} f(b, a)| \\ & \lesssim (1 + |a|^2)^{s/2} \int_{\mathbb{R}} |S_{\mathbf{A}}(f(x) \chi_{\alpha, \beta}(x) e^{-i\xi x})(a)| \left| \mathcal{F}(\bar{\psi}_{\kappa_{\mathbf{A}}})(\xi) \right| d\xi \\ & = (1 + |a|^2)^{s/2} \int_{\mathbb{R}} |S_{\mathbf{A}}(f \chi_{\alpha, \beta})(a + \beta \xi)| \left| \mathcal{F}(\bar{\psi}_{\kappa_{\mathbf{A}}})(\xi) \right| d\xi \\ & \lesssim \int_{\mathbb{R}} (1 + |a + \beta \xi|^2)^{s/2} (1 + |\beta \xi|^2)^{s/2} |S_{\mathbf{A}}(f \chi_{\alpha, \beta})(a + \beta \xi)| \left| \mathcal{F}(\bar{\psi}_{\kappa_{\mathbf{A}}})(\xi) \right| d\xi \\ & \lesssim \rho_{q_1}(f) \int_{\mathbb{R}} (1 + |\xi|^2)^{s/2} \left| \mathcal{F}(\bar{\psi}_{\kappa_{\mathbf{A}}})(\xi) \right| d\xi \lesssim \rho_{q_1}(f) \rho_{q_2}(\psi), \end{aligned}$$

for some $q_1, q_2 \in \mathbb{N}_0$. This completes the proof. $\square$

4. The short-time special affine Fourier synthesis operator

Motivated by the reconstruction formula ([23], Thm. 4.5), we introduce the short-time special affine Fourier synthesis operator, which maps functions on $\mathbb{R}^2$ to functions on $\mathbb{R}$.

Definition 4.1. Let $\psi \in \mathcal{S}(\mathbb{R})$. The short-time special affine Fourier synthesis operator of $\Phi \in \mathcal{S}(\mathbb{R}^2)$ with respect to ψ is defined by

$$V_{\psi, \mathbf{A}}^* \Phi(x) := \overline{\chi_{\alpha, \beta}(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(b, a) \psi(x - b) \chi_{\alpha, \beta}(b) \overline{\kappa_{\mathbf{A}}(b - x)} K_{\mathbf{A}^*}(a, x) db da, \quad x \in \mathbb{R}, \quad (4.1)$$

where $K_{\mathbf{A}^*}$, $\kappa_{\mathbf{A}}$ and $\chi_{\alpha, \beta}$ are given by (2.4) and (2.7).

It is immediate to verify that for every $\Phi \in \mathcal{S}(\mathbb{R}^2)$ the integral (4.1) is absolutely convergent. The short-time special affine Fourier synthesis operator extends to a bilinear mapping $V_{\mathbf{A}}^* : (\Phi, \psi) \rightarrow V_{\psi, \mathbf{A}}^* \Phi$. We establish the continuity of this bilinear map.

Theorem 4.1. The mapping $V_{\mathbf{A}}^* : \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ is continuous.

Proof. To prove the theorem, it is sufficient to show that for given $\nu \in \mathbb{N}_0$, there exist $s, \tau \in \mathbb{N}_0$ and $C > 0$ such that for all $\Phi \in \mathcal{S}(\mathbb{R}^2)$ and $\psi \in \mathcal{S}(\mathbb{R})$,

$$\rho_{\nu}(V_{\psi, \mathbf{A}}^* \Phi) \leq C \rho_s(\Phi) \rho_{\tau}(\psi).$$

The rapid decay of Φ and ψ justifies differentiation under the integral sign and implies that $V_{\psi, \mathbf{A}}^* \Phi \in \mathcal{C}^{\infty}(\mathbb{R})$ for every $\Phi \in \mathcal{S}(\mathbb{R}^2)$ and $\psi \in \mathcal{S}(\mathbb{R})$.

We may assume that ν is even. By Leibniz's formula and the definition of the synthesis operator (4.1), for every $m \leq \nu$, we obtain

$$\begin{aligned}
& (1 + |x|^2)^{\nu/2} \left| \frac{d^m}{dx^m} (V_{\psi, \mathbf{A}}^* \Phi(x)) \right| \\
&= (1 + |x|^2)^{\nu/2} \left| \frac{d^m}{dx^m} \left(\overline{\chi_{\alpha, \beta}(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(b, a) \psi(x - b) \chi_{\alpha, \beta}(b) \overline{\kappa_{\mathbf{A}}(b - x)} K_{\mathbf{A}^*}(a, x) db da \right) \right| \\
&= (1 + |x|^2)^{\nu/2} \left| \sum_{k, r, d, t \leq m} c_{r, d, k, t} x^r \overline{\chi_{\alpha, \beta}(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} a^d b^t \Phi(b, a) \psi^{(k)}(x - b) \chi_{\alpha, \beta}(b) \overline{\kappa_{\mathbf{A}}(b - x)} K_{\mathbf{A}^*}(a, x) db da \right| \\
&\lesssim \sum_{k, r, d, t \leq m} \sum_{l \leq \nu/2} |x|^{r+2l} \int_{\mathbb{R}} \int_{\mathbb{R}} |a|^d |b|^t |\Phi(b, a)| |\psi^{(k)}(x - b)| db da \\
&\lesssim \sum_{k, r, d, t \leq m} \sum_{l \leq \nu/2} \sum_{j \leq r+2l} \int_{\mathbb{R}} \int_{\mathbb{R}} |x - b|^j |a|^d |b|^{t+r+2l-j} |\Phi(b, a)| |\psi^{(k)}(x - b)| db da \\
&\lesssim \rho_{2\nu}(\psi) \sum_{k, r, d, t \leq m} \sum_{l \leq \nu/2} \sum_{j \leq r+2l} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |a|^2)^{\nu/2} (1 + |b|^2)^{3\nu/2} |\Phi(b, a)| db da \\
&= \rho_{2\nu}(\psi) \sum_{k, r, d, t \leq m} \sum_{l \leq \nu/2} \sum_{j \leq r+2l} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |a|^2)^{\frac{\nu+2}{2}} (1 + |b|^2)^{\frac{3\nu+2}{2}} |\Phi(b, a)| \frac{db da}{(1 + |a|^2)(1 + |b|^2)} \\
&\lesssim \rho_{2\nu}(\psi) \rho_{4\nu+4}(\Phi) \sum_{k, r, d, t \leq m} \sum_{l \leq \nu/2} \sum_{j \leq r+2l} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |a|^2)^{-1} (1 + |b|^2)^{-1} db da,
\end{aligned}$$

where $c_{r, d, k, t}$ are constants depending only on ν and the given matrix $\mathbf{A}$. Then,

$$\rho_{\nu}(V_{\psi, \mathbf{A}}^* \Phi) \leq C \rho_{2\nu}(\psi) \rho_{4\nu+4}(\Phi)$$

for some constant $C = C(\nu, \mathbf{A}) > 0$. This completes the proof. $\square$

One may show that the short-time special affine Fourier synthesis operator is the adjoint of the STSAFT. In fact, for $f \in L^1(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$, by Fubini's theorem, we obtain

$$\int_{\mathbb{R}} f(x) \overline{V_{\psi, \mathbf{A}}^* \Phi(x)} dx = \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \Phi(b, a) db da, \quad \Phi \in \mathcal{S}(\mathbb{R}^2). \quad (4.2)$$

Because $L^1(\mathbb{R})$ naturally embeds into $\mathcal{S}'(\mathbb{R})$, using Theorem 4.1 and the standard identification for regular distributions on $\mathbb{R}^n$ (2.2), we may write (4.2) as

$$\left\langle f, \overline{V_{\psi, \mathbf{A}}^* \Phi} \right\rangle = \langle V_{\psi, \mathbf{A}} f, \Phi \rangle, \quad \Phi \in \mathcal{S}(\mathbb{R}^2). \quad (4.3)$$

This dual relation serves as the model for defining the distributional STSAFT in Section 5.

Proposition 4.1. (*Inversion formula*) If $f \in \mathcal{S}(\mathbb{R})$, then

$$\frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} (V_{\eta, \mathbf{A}}^* \circ V_{\psi, \mathbf{A}}) f(x) = f(x), \quad x \in \mathbb{R}, \quad (4.4)$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$, namely $(\eta, \psi)_{L^2(\mathbb{R})} \neq 0$.

Proof. Using the relation (3.1), the inversion formula for the SAFT (2.5), Parseval's identity for the FT, and Fubini's theorem, we obtain

$$\begin{aligned}
 (V_{\eta, \mathbf{A}}^* \circ V_{\psi, \mathbf{A}}) f(x) &= \overline{\chi_{\alpha, \beta}(x)} \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \eta(x-b) \chi_{\alpha, \beta}(b) \overline{\kappa_{\mathbf{A}}(b-x)} K_{\mathbf{A}^*}(a, x) db da \\
 &= \frac{\overline{\chi_{\alpha, \beta}(x)}}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \eta(x-b) \overline{\kappa_{\mathbf{A}}(b-x)} K_{\mathbf{A}^*}(a, x) db da \int_{\mathbb{R}} S_{\mathbf{A}}(f(t) \chi_{\alpha, \beta}(t) e^{-i\xi t}) (a) e^{i\xi b} \mathcal{F}(\tilde{\psi} \kappa_{\mathbf{A}})(\xi) d\xi \\
 &= \frac{\overline{\chi_{\alpha, \beta}(x)}}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}(\tilde{\psi} \kappa_{\mathbf{A}})(\xi) d\xi \int_{\mathbb{R}} S_{\mathbf{A}}(f(t) \chi_{\alpha, \beta}(t) e^{-i\xi t}) (a) K_{\mathbf{A}^*}(a, x) da \int_{\mathbb{R}} \eta(x-b) \overline{\kappa_{\mathbf{A}}(b-x)} e^{i\xi b} db \\
 &= \overline{\chi_{\alpha, \beta}(x)} \int_{\mathbb{R}} e^{i\xi x} \mathcal{F}(\tilde{\psi} \kappa_{\mathbf{A}})(\xi) \overline{\mathcal{F}(\tilde{\eta} \kappa_{\mathbf{A}})(\xi)} d\xi \int_{\mathbb{R}} S_{\mathbf{A}}(f(t) \chi_{\alpha, \beta}(t) e^{-i\xi t}) (a) K_{\mathbf{A}^*}(a, x) da \\
 &= f(x) \int_{\mathbb{R}} \mathcal{F}(\tilde{\psi} \kappa_{\mathbf{A}})(\xi) \overline{\mathcal{F}(\tilde{\eta} \kappa_{\mathbf{A}})(\xi)} d\xi = (\eta, \psi)_{L^2(\mathbb{R})} f(x),
 \end{aligned}$$

for all $x \in \mathbb{R}$. □

5. The STSAFT on $\mathcal{S}'(\mathbb{R})$

In this section, we rigorously extend the definition of the STSAFT and its corresponding synthesis operator to the space of tempered distributions. Furthermore, we establish a generalized form of the reconstruction formula (4.4) within the framework of tempered distributions.

Motivated by the relation (4.3), we introduce the STSAFT of a tempered distribution. The STSAFT of $f \in \mathcal{S}'(\mathbb{R})$, with respect to $\psi \in \mathcal{S}(\mathbb{R})$, denoted by $V_{\psi, \mathbf{A}} f$, is defined as a tempered distribution by duality:

$$\langle V_{\psi, \mathbf{A}} f, \Phi \rangle := \left\langle f, \overline{V_{\psi, \mathbf{A}}^* \Phi} \right\rangle, \quad \Phi \in \mathcal{S}(\mathbb{R}^2). \quad (5.1)$$

Theorem 4.1 ensures that $V_{\psi, \mathbf{A}} f$ is well-defined by (5.1). Moreover, by ([41], Corollary of Prop. 19.5), $V_{\psi, \mathbf{A}} : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R}^2)$ is continuous.

The following proposition shows that the duality definition (5.1) agrees with the direct definition (2.8).

Proposition 5.1. *Let $\psi \in \mathcal{S}(\mathbb{R})$ and $f \in \mathcal{S}'(\mathbb{R})$. Then,*

$$\left\langle f, \overline{V_{\psi, \mathbf{A}}^* \Phi} \right\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \Phi(b, a) db da, \quad \Phi \in \mathcal{S}(\mathbb{R}^2). \quad (5.2)$$

Proof. Because $f \in \mathcal{S}'(\mathbb{R})$, by the Schwartz structural theorem (see [40], Thm. VI, p. 239), there exists a continuous function h on $\mathbb{R}$ with at most polynomial growth and an integer $m \in \mathbb{N}_0$ such that $f = h^{(m)}$ in the distributional sense. Then, using the definitions (5.1) and (4.1), Theorem 4.1, Fubini's theorem, and the standard identification of distributions in $\mathcal{S}'(\mathbb{R})$, we obtain

$$\begin{aligned}
 \left\langle f, \overline{V_{\psi, \mathbf{A}}^* \Phi} \right\rangle &= (-1)^m \int_{\mathbb{R}} h(x) \frac{d^m}{dx^m} \left(\overline{V_{\psi, \mathbf{A}}^* \Phi(x)} \right) dx \\
 &= (-1)^m \int_{\mathbb{R}} h(x) \frac{d^m}{dx^m} \left(\chi_{\alpha, \beta}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(b, a) \overline{\psi(x-b)} \overline{\kappa_{\mathbf{A}}(b-x)} \overline{\chi_{\alpha, \beta}(b)} K_{\mathbf{A}}(x, a) db da \right) dx
 \end{aligned}$$

$$\begin{aligned}
&= (-1)^m \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(b, a) db da \int_{\mathbb{R}} h(x) \frac{d^m}{dx^m} \left(\overline{\psi(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha, \beta}(b)} \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a) \right) dx \\
&= (-1)^m \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(b, a) \left\langle h(x), \frac{d^m}{dx^m} \left(\overline{\psi(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha, \beta}(b)} \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a) \right) \right\rangle_x db da \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left\langle f(x), \overline{\psi(x-b)} \kappa_{\mathbf{A}}(b-x) \overline{\chi_{\alpha, \beta}(b)} \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a) \right\rangle_x \Phi(b, a) db da \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \Phi(b, a) db da = \langle V_{\psi, \mathbf{A}} f(b, a), \Phi(b, a) \rangle_{b, a},
\end{aligned}$$

which concludes the proof. $\square$

Analogously, we define the short-time special affine Fourier synthesis operator of $F \in \mathcal{S}'(\mathbb{R}^2)$ with respect to $\psi \in \mathcal{S}(\mathbb{R})$, denoted by $V_{\psi, \mathbf{A}}^* F$, as the distribution in $\mathcal{S}'(\mathbb{R})$ whose action on test functions is given by

$$\langle V_{\psi, \mathbf{A}}^* F, \varphi \rangle := \left\langle F, \overline{V_{\psi, \mathbf{A}} \varphi} \right\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}). \quad (5.3)$$

Theorem 4.1 ensures that $V_{\psi, \mathbf{A}}^* F$ is well-defined by (5.3). Moreover, by ([41], Corollary of Prop. 19.5), $V_{\psi, \mathbf{A}}^* : \mathcal{S}'(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R})$ is continuous.

The following proposition establishes an extension of the reconstruction formula (4.4) to the space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Proposition 5.2. (Inversion formula) *Let $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. Then,*

$$\frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} (V_{\eta, \mathbf{A}}^* \circ V_{\psi, \mathbf{A}}) = Id_{\mathcal{S}'(\mathbb{R})},$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for ψ .

Proof. Let $f \in \mathcal{S}'(\mathbb{R})$. For every $\varphi \in \mathcal{S}(\mathbb{R})$, Definitions (5.1) and (5.3) together with the inversion formula (4.4) imply that

$$\begin{aligned}
\langle (V_{\eta, \mathbf{A}}^* \circ V_{\psi, \mathbf{A}}) f, \varphi \rangle &= \left\langle V_{\psi, \mathbf{A}} f, \overline{V_{\eta, \mathbf{A}} \varphi} \right\rangle = \left\langle f, \overline{V_{\psi, \mathbf{A}}^* (V_{\eta, \mathbf{A}} \varphi)} \right\rangle \\
&= \left\langle f, \overline{(\psi, \eta)_{L^2(\mathbb{R})} \varphi} \right\rangle = (\eta, \psi)_{L^2(\mathbb{R})} \langle f, \varphi \rangle.
\end{aligned}$$

$\square$

An immediate and significant consequence of Propositions 5.1 and 5.2 is the following desingularization formula, which serves as a generalized version of Parseval's identity ([23], Thm. 4.4).

Corollary 5.1. (Desingularization formula) *Let $f \in \mathcal{S}'(\mathbb{R})$. Then,*

$$\langle f, \varphi \rangle = \frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \overline{V_{\eta, \mathbf{A}} \varphi(b, a)} db da, \quad \varphi \in \mathcal{S}(\mathbb{R}), \quad (5.4)$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a synthesis window for $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$.

Proof. By proposition 5.2, Def. (5.3), and the identification (5.2), we obtain

$$\begin{aligned}\langle f, \varphi \rangle &= \frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} \langle V_{\eta, \mathbf{A}}^* (V_{\psi, \mathbf{A}} f), \varphi \rangle = \frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} \left\langle V_{\psi, \mathbf{A}} f, \overline{V_{\eta, \mathbf{A}} \varphi} \right\rangle \\ &= \frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \overline{V_{\eta, \mathbf{A}} \varphi(b, a)} db da.\end{aligned}$$

□

Remark 5.1. For the matrix $\mathbf{A} = (0, 1, 0; -1, 0, 0)$, we obtain the desingularization formula for the STFT that will be used in the proof of Proposition 6.2.

The following result characterizes the bounded subsets of $\mathcal{S}'(\mathbb{R})$ in terms of the STSAFT.

Proposition 5.3. Let $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. The set $\mathcal{B} \subset \mathcal{S}'(\mathbb{R})$ is weakly (equivalently, strongly) bounded in $\mathcal{S}'(\mathbb{R})$ if and only if there exists $l = l_{\mathcal{B}} \in \mathbb{N}_0$ such that

$$\sup_{\substack{f \in \mathcal{B} \\ (b, a) \in \mathbb{R} \times \mathbb{R}}} (1 + |b|^2 + |a|^2)^{-l/2} |V_{\psi, \mathbf{A}} f(b, a)| < \infty. \quad (5.5)$$

Proof. Suppose $\mathcal{B}$ is weakly bounded in $\mathcal{S}'(\mathbb{R})$. By the Banach–Steinhaus theorem, $\mathcal{B}$ is equicontinuous, so there exist $C' = C'_{\mathcal{B}} > 0$ and $\nu = \nu_{\mathcal{B}} \in \mathbb{N}_0$ such that

$$|\langle f, \varphi \rangle| \leq C' \rho_{\nu}(\varphi), \quad \forall f \in \mathcal{B}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

Then, there exist $l = l_{\mathcal{B}} \in \mathbb{N}_0$ and $C = C_{\mathcal{B}} > 0$ such that

$$\begin{aligned}|V_{\psi, \mathbf{A}} f(b, a)| &= \left| \left\langle f(x), \overline{\psi(x-b) \kappa_{\mathbf{A}}(b-x) \chi_{\alpha, \beta}(b) \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a)} \right\rangle_x \right| \\ &\leq C' \rho_{\nu} \left(\overline{\psi(x-b) \kappa_{\mathbf{A}}(b-x) \chi_{\alpha, \beta}(b) \chi_{\alpha, \beta}(x) K_{\mathbf{A}}(x, a)} \right) \\ &\leq C (1 + |b|^2 + |a|^2)^{l/2}\end{aligned}$$

for all $f \in \mathcal{B}$ and $(b, a) \in \mathbb{R} \times \mathbb{R}$, which proves the relation (5.5).

Conversely, assume that (5.5) holds. It suffices to show that $\mathcal{B}$ is a weakly bounded set in $\mathcal{S}'(\mathbb{R})$. Let $\eta \in \mathcal{S}(\mathbb{R})$ be a synthesis window for ψ , and let $\varphi \in \mathcal{S}(\mathbb{R})$. Then, the desingularization formula (5.4) yields

$$\begin{aligned}\sup_{f \in \mathcal{B}} |\langle f, \varphi \rangle| &= \sup_{f \in \mathcal{B}} \left| \frac{1}{(\eta, \psi)_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\psi, \mathbf{A}} f(b, a) \overline{V_{\eta, \mathbf{A}} \varphi(b, a)} db da \right| \\ &\leq \frac{C_{\mathcal{B}}}{|(\eta, \psi)_{L^2(\mathbb{R})}|} \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |b|^2 + |a|^2)^{l/2} |\overline{V_{\eta, \mathbf{A}} \varphi(b, a)}| db da.\end{aligned}$$

Because $V_{\eta, \mathbf{A}}(\varphi) \in \mathcal{S}(\mathbb{R}^2)$, we may conclude that $\mathcal{B}$ is weakly bounded in $\mathcal{S}'(\mathbb{R})$. Because $\mathcal{S}(\mathbb{R})$ is barreled, weak and strong boundedness coincide in $\mathcal{S}'(\mathbb{R})$ by the Banach–Steinhaus theorem ([41], Thm. 33.1). □

6. Abelian- and Tauberian-type results

In this section, we briefly review the asymptotic concepts for tempered distributions that serve as the foundation of our study. The asymptotic behavior of a distribution can be described by comparison with Karamata regularly varying functions [47], leading to the notion of quasiasymptotic behavior of distributions [28–30].

Next, we present several Abelian and Tauberian theorems that describe the quasiasymptotic behavior of tempered distributions at the origin, based on the asymptotic properties of their STSAFT.

6.1. Asymptotic behavior of distributions

Let us recall that a measurable real-valued function L , defined and positive on the interval $(0, M]$ (resp. $[M, \infty)$) for some $M > 0$, is called a *slowly varying function* at the origin (resp. at infinity) if it satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \quad \left(\text{resp.} \quad \lim_{\lambda \rightarrow \infty} \frac{L(a\lambda)}{L(\lambda)} = 1 \right)$$

for all $a > 0$.

We refer to [47] for the properties of slowly varying functions. Throughout this paper, the letter L denotes a slowly varying function at the origin.

It is said that the distribution $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior of degree $\sigma \in \mathbb{R}$ at the origin with respect to a slowly varying function L if the limit

$$\lim_{\varepsilon \rightarrow 0^+} \left\langle \frac{f(\varepsilon x)}{\varepsilon^\sigma L(\varepsilon)}, \varphi(x) \right\rangle_x$$

exists and is finite for every $\varphi \in \mathcal{S}(\mathbb{R})$. One can observe that when studying quasiasymptotic behavior at the origin, the representation theorem ([47], Thm. 1.2) allows us, without loss of generality, to assume that L is defined on the whole of $(0, \infty)$ and has any convenient properties away from the origin.

Suppose $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior of degree $\sigma \in \mathbb{R}$ at the origin with respect to a slowly varying function L . Then, the net $\mathcal{B} = \{f(\varepsilon x)/\varepsilon^\sigma L(\varepsilon)\}_{0 < \varepsilon < 1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R})$. By ([41], Thm. 33.2), it is therefore equicontinuous. Consequently, there exist $C = C_{\mathcal{B}} > 0$ and $\nu = \nu_{\mathcal{B}} \in \mathbb{N}_0$ such that

$$\left| \left\langle \frac{f(\varepsilon x)}{\varepsilon^\sigma L(\varepsilon)}, \varphi(x) \right\rangle_x \right| \leq C \rho_\nu(\varphi), \quad \forall \varepsilon \in (0, 1), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}). \quad (6.1)$$

Now, we define the linear functional g acting on a test function $\varphi \in \mathcal{S}(\mathbb{R})$ by $\langle g(x), \varphi(x) \rangle = \lim_{\varepsilon \rightarrow 0^+} \langle f(\varepsilon x)/\varepsilon^\sigma L(\varepsilon), \varphi(x) \rangle_x$. The relation (6.1) implies that

$$|\langle g, \varphi \rangle| \leq C \rho_\nu(\varphi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}),$$

which establishes the continuity of g , and hence, g is a tempered distribution. If in addition, g is a nonzero distribution, then for every $a > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle g(ax), \varphi(x) \rangle_x = \lim_{\varepsilon \rightarrow 0^+} \left\langle \frac{f(\varepsilon ax)}{\varepsilon^\sigma L(\varepsilon)}, \varphi(x) \right\rangle_x = a^\sigma \lim_{\varepsilon \rightarrow 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} \left\langle \frac{f(\varepsilon ax)}{(a\varepsilon)^\sigma L(a\varepsilon)}, \varphi(x) \right\rangle_x = a^\sigma \langle g(x), \varphi(x) \rangle_x,$$

which proves that g must be a homogeneous distribution of degree σ [28–30]. We use the following notation to express the quasiasymptotic behavior at the origin $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon)g(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$, which should always be interpreted in the weak topology of $\mathcal{S}'(\mathbb{R})$.

We say that the distribution $f \in \mathcal{S}'(\mathbb{R})$ is quasiasymptotically bounded of degree $\sigma \in \mathbb{R}$ at the origin, with respect to L , if

$$\left\langle \frac{f(\varepsilon x)}{\varepsilon^\sigma L(\varepsilon)}, \varphi(x) \right\rangle_x = O(1) \text{ as } \varepsilon \rightarrow 0^+$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. We use the notation $f(\varepsilon x) = O(\varepsilon^\sigma L(\varepsilon))$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$, which should always be interpreted in the weak topology of $\mathcal{S}'(\mathbb{R})$.

The quasiasymptotic analysis at infinity can be carried out in a similar manner.

6.2. Characterization of quasiasymptotics in $\mathcal{S}'(\mathbb{R})$ through the STSAFT

For the purposes of our quasiasymptotic analysis, we use the definition of the STSAFT given in (2.8). Given $\psi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$, we define the scaled window function by $\psi_{1/\varepsilon}(x) = \psi(x/\varepsilon)$.

The first result of this section is an Abelian-type theorem, which describes the scaled asymptotic behavior of the STSAFT for distributions exhibiting quasiasymptotic behavior at the origin.

Proposition 6.1. *Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon)g(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$. Then, the STSAFT of f with respect to the window $\psi \in \mathcal{S}(\mathbb{R})$ satisfies*

$$\frac{1}{\varepsilon^{\sigma+1}L(\varepsilon)} e^{-\frac{i}{\beta}\left(\frac{a}{\varepsilon}(\beta q - \delta p) + \frac{\delta}{2}\left(\frac{a}{\varepsilon}\right)^2\right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f\left(\varepsilon b, \frac{a}{\varepsilon}\right) \rightarrow |\beta|^{-1/2} V_\psi g\left(b, \frac{a}{\beta}\right)$$

as $\varepsilon \rightarrow 0^+$ uniformly for (b, a) in bounded subsets of $\mathbb{R} \times \mathbb{R}$.

Proof. Let K be a bounded subset of $\mathbb{R} \times \mathbb{R}$. By Leibniz's rule, the boundedness of K , and the rapid decay of ψ , for every $\nu \in \mathbb{N}_0$, we have $\sup_{(b,a) \in K} \rho_\nu \left(\overline{\psi(x-b)} e^{-ix\frac{a}{\beta}} \right) < \infty$. Hence, the set $\left\{ \overline{\psi(x-b)} e^{-ix\frac{a}{\beta}} : (b, a) \in K \right\}$ is bounded in $\mathcal{S}(\mathbb{R})$. Then, by ([41], Corollary 2, p. 358),

$$\frac{1}{\varepsilon^\sigma L(\varepsilon)} \left\langle f(\varepsilon x), \overline{\psi(x-b)} e^{-ix\frac{a}{\beta}} \right\rangle_x \rightarrow \left\langle g(x), \overline{\psi(x-b)} e^{-ix\frac{a}{\beta}} \right\rangle_x \quad (6.2)$$

as $\varepsilon \rightarrow 0^+$ uniformly for $(b, a) \in K$. Moreover, using again Leibniz's rule together with the boundedness of K , the rapid decay of ψ , and the smoothness of the phase factors, for every $\nu \in \mathbb{N}_0$,

$$\sup_{(b,a) \in K} \rho_\nu \left(\left(\kappa_{\mathbf{A}}(\varepsilon b - \varepsilon x) \overline{\chi_{\alpha,\beta}(\varepsilon b)} \chi_{\alpha,\beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} - 1 \right) \overline{\psi(x-b)} e^{-ix\frac{a}{\beta}} \right) \rightarrow 0, \quad \varepsilon \rightarrow 0^+.$$

Because $\{f(\varepsilon x)/\varepsilon^\sigma L(\varepsilon)\}_{0 < \varepsilon < 1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R})$, the estimate (6.1) together with the preceding uniform convergence, yields

$$\sup_{(b,a) \in K} \left| \left\langle \frac{f(\varepsilon x)}{\varepsilon^\sigma L(\varepsilon)}, \left(\kappa_{\mathbf{A}}(\varepsilon b - \varepsilon x) \overline{\chi_{\alpha,\beta}(\varepsilon b)} \chi_{\alpha,\beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} - 1 \right) \overline{\psi(x-b)} e^{-ix\frac{a}{\beta}} \right\rangle_x \right| \rightarrow 0 \quad (6.3)$$

as $\varepsilon \rightarrow 0^+$. Combining the preceding convergences (6.2) and (6.3), we obtain

$$\begin{aligned}
 & \frac{1}{\varepsilon^{\sigma+1}L(\varepsilon)} e^{-\frac{i}{\beta}\left(\frac{a}{\varepsilon}(\beta q - \delta p) + \frac{\delta}{2}\left(\frac{a}{\varepsilon}\right)^2\right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f\left(\varepsilon b, \frac{a}{\varepsilon}\right) \\
 &= \frac{1}{\varepsilon^{\sigma+1}L(\varepsilon)} e^{-\frac{i}{\beta}\left(\frac{a}{\varepsilon}(\beta q - \delta p) + \frac{\delta}{2}\left(\frac{a}{\varepsilon}\right)^2\right)} \left\langle f(x), \overline{\psi_{1/\varepsilon}(x - \varepsilon b)} \kappa_{\mathbf{A}}(\varepsilon b - x) \overline{\chi_{\alpha, \beta}(\varepsilon b)} \chi_{\alpha, \beta}(x) K_{\mathbf{A}}\left(x, \frac{a}{\varepsilon}\right) \right\rangle_x \\
 &= \frac{1}{\varepsilon^{\sigma}L(\varepsilon)} e^{-\frac{i}{\beta}\left(\frac{a}{\varepsilon}(\beta q - \delta p) + \frac{\delta}{2}\left(\frac{a}{\varepsilon}\right)^2\right)} \left\langle f(\varepsilon x), \overline{\psi(x - b)} \kappa_{\mathbf{A}}(\varepsilon b - \varepsilon x) \overline{\chi_{\alpha, \beta}(\varepsilon b)} \chi_{\alpha, \beta}(\varepsilon x) K_{\mathbf{A}}\left(\varepsilon x, \frac{a}{\varepsilon}\right) \right\rangle_x \\
 &= \frac{1}{\varepsilon^{\sigma}L(\varepsilon) \sqrt{2\pi|\beta|}} \left\langle f(\varepsilon x), \overline{\psi(x - b)} \kappa_{\mathbf{A}}(\varepsilon b - \varepsilon x) \overline{\chi_{\alpha, \beta}(\varepsilon b)} \chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} e^{-ix\frac{a}{\beta}} \right\rangle_x \\
 &= \frac{1}{\varepsilon^{\sigma}L(\varepsilon) \sqrt{2\pi|\beta|}} \left\langle f(\varepsilon x), \left(\kappa_{\mathbf{A}}(\varepsilon b - \varepsilon x) \overline{\chi_{\alpha, \beta}(\varepsilon b)} \chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} - 1 \right) \overline{\psi(x - b)} e^{-ix\frac{a}{\beta}} \right\rangle_x \\
 &\quad + \frac{1}{\varepsilon^{\sigma}L(\varepsilon) \sqrt{2\pi|\beta|}} \left\langle f(\varepsilon x), \overline{\psi(x - b)} e^{-ix\frac{a}{\beta}} \right\rangle_x \\
 &\rightarrow \frac{1}{\sqrt{2\pi|\beta|}} \left\langle g(x), \overline{\psi(x - b)} e^{-ix\frac{a}{\beta}} \right\rangle_x = |\beta|^{-1/2} V_{\psi} g\left(b, \frac{a}{\beta}\right)
 \end{aligned}$$

as $\varepsilon \rightarrow 0^+$ uniformly for $(b, a) \in K$. $\square$

Example 6.1. Let $f(x) = \delta_0^{(m)}(x)$, where $m \in \mathbb{N}_0$, and δ_0 is the Dirac distribution at the origin. We note that $f(\varepsilon x) \sim \varepsilon^{-m-1}f(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$. Thus, f has quasiasymptotic behavior of degree $\sigma = -m - 1$ at the origin, with $L(\varepsilon) \equiv 1$ and limit distribution $g = f$.

Consequently, the preceding proposition yields, for all $\psi \in \mathcal{S}(\mathbb{R})$ and $(b, a) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0^+} \varepsilon^m e^{-\frac{i}{\beta}\left(\frac{a}{\varepsilon}(\beta q - \delta p) + \frac{\delta}{2}\left(\frac{a}{\varepsilon}\right)^2\right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f\left(\varepsilon b, \frac{a}{\varepsilon}\right) &= |\beta|^{-1/2} V_{\psi} \delta_0^{(m)}\left(b, \frac{a}{\beta}\right) \\
 &= (2\pi|\beta|)^{-1/2} \left\langle \delta_0^{(m)}(x), \overline{\psi(x - b)} e^{-ix\frac{a}{\beta}} \right\rangle_x = (2\pi|\beta|)^{-1/2} \sum_{m_1+m_2=m} \binom{m}{m_1, m_2} (-1)^{m_1} \overline{\psi^{(m_1)}(-b)} (ia/\beta)^{m_2}.
 \end{aligned}$$

The convergence is uniform for (b, a) in bounded subsets of $\mathbb{R} \times \mathbb{R}$. Similarly, for the locally integrable power-law function $h(x) = |x|^\sigma$, $\sigma > -1$, one has $h(\varepsilon x) = \varepsilon^\sigma h(x)$; hence, the same quasiasymptotic argument applies with $L(\varepsilon) \equiv 1$ and limit distribution $g = h$.

Remark 6.1. The proof of Proposition 6.1 does not directly extend to quasiasymptotics at infinity. Indeed, for $\alpha \neq 0$, the set $\left\{ \overline{\psi(x - b)} \kappa_{\mathbf{A}}(\lambda b - \lambda x) \overline{\chi_{\alpha, \beta}(\lambda b)} \chi_{\alpha, \beta}(\lambda x) e^{\frac{i}{2\beta}(\alpha(\lambda x)^2 + 2\lambda x p - 2\lambda a)} \right\}_{\lambda > 1}$ is not bounded in $\mathcal{S}(\mathbb{R})$. Hence, the argument used at the origin cannot be applied directly at infinity; extending the present quasiasymptotic analysis to infinity, possibly under additional assumptions on the SAFT parameters or the window function, is left for future investigation.

The following proposition is a Tauberian converse of Proposition 6.1.

Proposition 6.2. Let $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. The conditions

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^{\sigma+1}L(\varepsilon)} e^{-\frac{i}{\beta}\left(\frac{a}{\varepsilon}(\beta q - \delta p) + \frac{\delta}{2}\left(\frac{a}{\varepsilon}\right)^2\right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f\left(\varepsilon b, \frac{a}{\varepsilon}\right) = M_{b,a} \in \mathbb{C}, \quad (b, a) \in \mathbb{R} \times \mathbb{R} \quad (6.4)$$

and the existence of a constant $l \geq 0$ such that

$$\sup_{\substack{0 < \varepsilon < 1 \\ (b,a) \in \mathbb{R} \times \mathbb{R}}} \frac{1}{\varepsilon^{\sigma+1} L(\varepsilon)} (1 + |b|^2 + |a|^2)^{-l/2} \left| V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon b, \frac{a}{\varepsilon} \right) \right| < \infty \quad (6.5)$$

are necessary and sufficient conditions for the existence of a homogeneous distribution $g \in \mathcal{S}'(\mathbb{R})$ such that $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon) g(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$. In this case, the limit in (6.4) is given by $M_{b,a} = |\beta|^{-1/2} V_\psi g(b, a/\beta)$ for all $(b, a) \in \mathbb{R} \times \mathbb{R}$.

Proof. Assume (6.4) and (6.5). We use the notation $\kappa_{\mathbf{A}, \varepsilon}(x) = \kappa_{\mathbf{A}}(\varepsilon x)$. By using the desingularization formula (5.4) for the STFT and setting $\eta = \overline{\psi \kappa_{\mathbf{A}, \varepsilon}} \in \mathcal{S}(\mathbb{R})$, which serves as a synthesis window for $\psi \kappa_{\mathbf{A}, \varepsilon}$, we obtain

$$\begin{aligned} \frac{1}{\varepsilon^\sigma L(\varepsilon)} \langle f(\varepsilon x), \varphi(x) \rangle_x &= \frac{1}{\varepsilon^\sigma L(\varepsilon)} \left\langle \chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} f(\varepsilon x), \overline{\chi_{\alpha, \beta}(\varepsilon x) e^{-\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} \varphi(x)} \right\rangle_x \\ &= \frac{1}{\varepsilon^\sigma L(\varepsilon) \|\psi\|_2^2} \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} f(\varepsilon x) \right) (b, a) \\ &\quad \overline{V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} \overline{\varphi(x)} \right) (b, a)} db da. \end{aligned}$$

By the changes of variables $a \rightarrow a/\beta$ and $x \rightarrow x/\varepsilon$, we have

$$\begin{aligned} &\frac{1}{\varepsilon^\sigma L(\varepsilon)} \langle f(\varepsilon x), \varphi(x) \rangle_x \\ &= \frac{1}{\varepsilon^\sigma L(\varepsilon) \|\psi\|_2^2 |\beta|} \int_{\mathbb{R}} \int_{\mathbb{R}} V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} f(\varepsilon x) \right) \left(b, \frac{a}{\beta} \right) \\ &\quad \overline{V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} \overline{\varphi(x)} \right) \left(b, \frac{a}{\beta} \right)} db da \\ &= \frac{1}{\varepsilon^\sigma L(\varepsilon) \|\psi\|_2^2 |\beta| \sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left\langle \chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} f(\varepsilon x), \overline{\psi(x-b) \kappa_{\mathbf{A}}(\varepsilon b - \varepsilon x) e^{-ix \frac{a}{\beta}}} \right\rangle_x \\ &\quad \overline{V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} \overline{\varphi(x)} \right) \left(b, \frac{a}{\beta} \right)} db da \\ &= \frac{1}{\varepsilon^{\sigma+1} L(\varepsilon) \|\psi\|_2^2 |\beta| \sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left\langle f(x), \overline{\psi_{1/\varepsilon}(x - \varepsilon b) \kappa_{\mathbf{A}}(\varepsilon b - x) \chi_{\alpha, \beta}(x) e^{\frac{i}{2\beta}(\alpha x^2 + 2xp)} e^{-ix \frac{a}{\varepsilon \beta}}} \right\rangle_x \\ &\quad \overline{V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} \overline{\varphi(x)} \right) \left(b, \frac{a}{\beta} \right)} db da \\ &= \frac{1}{\varepsilon^{\sigma+1} L(\varepsilon) \|\psi\|_2^2 \sqrt{|\beta|}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{i}{\beta} \left(\frac{a}{\varepsilon} (\beta q - \delta p) + \frac{\delta}{2} \left(\frac{a}{\varepsilon} \right)^2 \right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon b, \frac{a}{\varepsilon} \right) \chi_{\alpha, \beta}(\varepsilon b) \\ &\quad \overline{V_{\overline{\psi \kappa_{\mathbf{A}, \varepsilon}}} \left(\chi_{\alpha, \beta}(\varepsilon x) e^{\frac{i}{2\beta}(\alpha(\varepsilon x)^2 + 2\varepsilon x p)} \overline{\varphi(x)} \right) \left(b, \frac{a}{\beta} \right)} db da. \end{aligned}$$

By Assumptions (6.4) and (6.5), the function $(b, a) \mapsto M_{b,a}$, $(b, a) \in \mathbb{R} \times \mathbb{R}$ is measurable and satisfies the estimate $|M_{b,a}| \lesssim (1 + |b|^2 + |a|^2)^{r/2}$ for some $r \in \mathbb{N}_0$. Therefore, Theorem 3.1 together

with the Lebesgue dominated convergence theorem implies that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^\sigma L(\varepsilon)} \langle f(\varepsilon x), \varphi(x) \rangle_x = \frac{1}{\|\psi\|_2^2 \sqrt{|\beta|}} \int_{\mathbb{R}} \int_{\mathbb{R}} M_{b,a} V_\psi \overline{\varphi} \left(b, \frac{a}{\beta} \right) db da.$$

Because the last limit exists and is finite for all $\varphi \in \mathcal{S}(\mathbb{R})$, the family $\{f(\varepsilon x)/\varepsilon^\sigma L(\varepsilon)\}_{0 < \varepsilon < 1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R})$. Hence, by the Banach–Steinhaus theorem ([41], Corollary, p. 348), it is equicontinuous, and therefore the limit defines a continuous linear functional $g \in \mathcal{S}'(\mathbb{R})$. The scaling argument given in Section 6.1 together with the slow variation of L shows that g is homogeneous of degree σ whenever $g \neq 0$; for $g = 0$, this property is trivial. Thus, $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon)g(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$, and by (6.4), we obtain $M_{b,a} = |\beta|^{-1/2} V_\psi g(b, a/\beta)$ for all $(b, a) \in \mathbb{R} \times \mathbb{R}$.

Conversely, suppose that $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon)g(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$. The limit (6.4) is a consequence of Proposition 6.1. Because $\{f(\varepsilon x)/\varepsilon^\sigma L(\varepsilon)\}_{0 < \varepsilon < 1}$ is weakly bounded in $\mathcal{S}'(\mathbb{R})$, (6.5) follows by taking similar steps as in Proposition 5.3 for the net $\mathcal{B} = \{f(\varepsilon x)/\varepsilon^\sigma L(\varepsilon)\}_{0 < \varepsilon < 1}$. $\square$

We have a direct consequence from the proof of Proposition 6.2.

Corollary 6.1. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. Then, $f(\varepsilon x) = O(\varepsilon^\sigma L(\varepsilon))$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$ if and only if there exists $l \geq 0$ such that*

$$\sup_{\substack{0 < \varepsilon < 1 \\ (b,a) \in \mathbb{R} \times \mathbb{R}}} \frac{1}{\varepsilon^{\sigma+1} L(\varepsilon)} (1 + |b|^2 + |a|^2)^{-1/2} \left| V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon b, \frac{a}{\varepsilon} \right) \right| < \infty.$$

The following result follows directly from Proposition 6.1 by replacing b with εb , which yields the finer translation scaling $\varepsilon^2 b$ and a limiting STFT evaluated at $b = 0$.

Corollary 6.2. *Suppose that $f \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon)g(x)$ as $\varepsilon \rightarrow 0^+$ in $\mathcal{S}'(\mathbb{R})$. Then, the STSAFT of f with respect to the window $\psi \in \mathcal{S}(\mathbb{R})$ satisfies*

$$\frac{1}{\varepsilon^{\sigma+1} L(\varepsilon)} e^{-\frac{i}{\beta} \left(\frac{a}{\varepsilon} (\beta q - \delta p) + \frac{\delta}{2} \left(\frac{a}{\varepsilon} \right)^2 \right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon^2 b, \frac{a}{\varepsilon} \right) \rightarrow |\beta|^{-1/2} V_\psi g \left(0, \frac{a}{\beta} \right)$$

as $\varepsilon \rightarrow 0^+$ uniformly for (b, a) in bounded subsets of $\mathbb{R} \times \mathbb{R}$.

Proof. Let $K \subset \mathbb{R} \times \mathbb{R}$ be bounded. Choose $M, N > 0$ such that $K \subset [-M, M] \times [-N, N]$. For convenience, define

$$F_\varepsilon(u, a) = \frac{1}{\varepsilon^{\sigma+1} L(\varepsilon)} e^{-\frac{i}{\beta} \left(\frac{a}{\varepsilon} (\beta q - \delta p) + \frac{\delta}{2} \left(\frac{a}{\varepsilon} \right)^2 \right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon u, \frac{a}{\varepsilon} \right).$$

By Proposition 6.1, $F_\varepsilon(u, a) \rightarrow |\beta|^{-1/2} V_\psi g(u, a/\beta)$ uniformly for (u, a) in bounded subsets of $\mathbb{R} \times \mathbb{R}$. For $(b, a) \in K$ and $0 < \varepsilon \leq 1$, we have $(\varepsilon b, a) \in [-M, M] \times [-N, N]$. Hence,

$$\sup_{(b,a) \in K} \left| F_\varepsilon(\varepsilon b, a) - |\beta|^{-1/2} V_\psi g \left(\varepsilon b, \frac{a}{\beta} \right) \right| \leq \sup_{(u,a) \in [-M,M] \times [-N,N]} \left| F_\varepsilon(u, a) - |\beta|^{-1/2} V_\psi g \left(u, \frac{a}{\beta} \right) \right| \rightarrow 0$$

as $\varepsilon \rightarrow 0^+$. Because

$$F_\varepsilon(\varepsilon b, a) = \frac{1}{\varepsilon^{\sigma+1} L(\varepsilon)} e^{-\frac{i}{\beta} \left(\frac{a}{\varepsilon} (\beta q - \delta p) + \frac{\delta}{2} \left(\frac{a}{\varepsilon} \right)^2 \right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon^2 b, \frac{a}{\varepsilon} \right),$$

the finer translation scaling $\varepsilon^2 b$ follows directly.

Furthermore, because $V_\psi g$ is continuous on $\mathbb{R} \times \mathbb{R}$, its restriction to the compact set $[-M, M] \times [-N/|\beta|, N/|\beta|]$ is uniformly continuous. Moreover, $\sup_{(b,a) \in K} |\varepsilon b| \leq \varepsilon M \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Therefore,

$$\sup_{(b,a) \in K} \left| V_\psi g \left(\varepsilon b, \frac{a}{\beta} \right) - V_\psi g \left(0, \frac{a}{\beta} \right) \right| \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0^+.$$

Combining the two convergences, we obtain

$$\frac{1}{\varepsilon^{\sigma+1} L(\varepsilon)} e^{-\frac{i}{\beta} \left(\frac{a}{\varepsilon} (\beta q - \delta p) + \frac{\delta}{2} \left(\frac{a}{\varepsilon} \right)^2 \right)} V_{\psi_{1/\varepsilon}, \mathbf{A}} f \left(\varepsilon^2 b, \frac{a}{\varepsilon} \right) \rightarrow |\beta|^{-1/2} V_\psi g \left(0, \frac{a}{\beta} \right)$$

as $\varepsilon \rightarrow 0^+$ uniformly for (b, a) in bounded subsets of $\mathbb{R} \times \mathbb{R}$. $\square$

7. Conclusions

In this paper, we developed a distributional framework for the short-time special affine Fourier transform. We proved the continuity of the STSAFT and its synthesis operator on the Schwartz spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^2)$, respectively, and extended both operators to tempered distributions by duality. We also established reconstruction and desingularization formulas and characterized bounded subsets of $\mathcal{S}'(\mathbb{R})$ in terms of polynomial growth estimates of the STSAFT. These results extend the STSAFT from the classical L^2 -setting to a setting suitable for the analysis of tempered distributions.

We further studied the quasiasymptotic behavior of tempered distributions at the origin through the scaled behavior of their STSAFT. The obtained Abelian- and Tauberian-type results relate the quasiasymptotic behavior and quasiasymptotic boundedness of a distribution to corresponding limits and polynomial growth estimates of its transform. Thus, local quasiasymptotic properties of tempered distributions can be characterized through the behavior of their STSAFT. Because the classical STFT is recovered as a special case of the STSAFT, the developed framework reduces to the corresponding results in the classical short-time Fourier setting.

Author contributions

Astrit Ferizi and Katerina Hadzi-Velkova Saneva: Conceptualization, methodology, formal analysis, validation, writing—original draft preparation, and writing—review and editing. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors report that there are no competing interests to declare.

References

1. V. Namias, The fractional order Fourier transform and its application to quantum mechanics, *IMA J. Appl. Math.*, **25** (1980), 241–265. <https://doi.org/10.1093/imamat/25.3.241>
2. S. Abe, J. T. Sheridan, Generalization of the fractional Fourier transformation to an arbitrary linear lossless transformation an operator approach, *J. Phys. A: Math Gen.*, **27** (1994), 4179. <https://doi.org/10.1088/0305-4470/27/12/023>
3. S. Abe, J. T. Sheridan, Optical operations on wave functions as the abelian subgroups of the special affine Fourier transformation, *Opt. Lett.*, **19** (1994), 1801–1803. <https://doi.org/10.1364/OL.19.001801>
4. S. Abe, J. T. Sheridan, Almost-Fourier and almost-Fresnel transformations, *Opt. Commun.*, **113** (1995), 385–388. [https://doi.org/10.1016/0030-4018\(94\)00521-U](https://doi.org/10.1016/0030-4018(94)00521-U)
5. L. Z. Cai, Special affine Fourier transformation in frequency-domain, *Opt. Commun.*, **185** (2000), 271–276. [https://doi.org/10.1016/S0030-4018\(00\)01005-1](https://doi.org/10.1016/S0030-4018(00)01005-1)
6. N. Goel, S. Gabarda, K. Singh, Chirp multiplexing and filtering in the offset linear canonical transform domain, *Optik*, **227** (2021), 165482. <https://doi.org/10.1016/j.ijleo.2020.165482>
7. X. Zhi, D. Wei, W. Zhang, A generalized convolution theorem for the special affine Fourier transform and its application to filtering, *Optik*, **127** (2016), 2613–2616. <https://doi.org/10.1016/j.ijleo.2015.11.211>
8. S. Xu, Y. Chai, Y. Hu, Spectral analysis of sampled band-limited signals in the offset linear canonical transform domain, *Circuits Syst. Signal Process.*, **34** (2015), 3979–3997. <https://doi.org/10.1007/s00034-015-0053-1>
9. H. Huo, Uncertainty principles for the offset linear canonical transform, *Circuits Syst. Signal Process.*, **38** (2019), 395–406. <https://doi.org/10.1007/s00034-018-0863-z>
10. D. Wei, Y. M. Li, Convolution and multichannel sampling for the offset linear canonical transform and their applications, *IEEE Trans. Signal Process.*, **67** (2019), 6009–6024. <https://doi.org/10.1109/TSP.2019.2951191>
11. Q. Xiang, K. Y. Qin, Convolution, correlation, and sampling theorems for the offset linear canonical transform, *Signal Image Video Process.*, **8** (2014), 433–442. <https://doi.org/10.1007/s11760-012-0342-0>
12. A. Bhandari, A. I. Zayed, Convolution and product theorems for the special affine Fourier transform, In: *Frontiers in orthogonal polynomials and q-series*, Singapore: World Scientific, 2018, 119–137. https://doi.org/10.1142/9789813228887_0006
13. Z. H. Zhuo, Poisson summation formulae associated with the special affine Fourier transform and offset Hilbert transform, *Math. Probl. Eng.*, **2017** (2017), 1354129. <https://doi.org/10.1155/2017/1354129>
14. B. Liu, Z. Wu, R. Li, Dynamical sampling associated with the special affine Fourier transform, *Optik*, **238** (2021), 166767. <https://doi.org/10.1016/j.ijleo.2021.166767>

15. A. Ferizi, K. H. V. Saneva, Tauberian theorems for the fractional Stockwell transform of Lizorkin distributions, *J. Pseudo-Differ. Oper. Appl.*, **17** (2026), 17. <https://doi.org/10.1007/s11868-026-00766-2>
16. S. Maksimović, S. Atanasova, Z. D. Mitrović, S. Haque, N. Mlaiki, Abelian and Tauberian results for the fractional Fourier cosine (sine) transform, *AIMS Mathematics*, **9** (2024), 12225–12238. <https://doi.org/10.3934/math.2024597>
17. F. Bartolucci, S. Pilipović, N. Teofanov, Continuity properties of the shearlet transform and the shearlet synthesis operator on the Lizorkin type spaces, *Math. Nachr.*, **295** (2022), 2318–2337. <https://doi.org/10.1002/mana.202000223>
18. S. Kostadinova, S. Pilipović, K. Saneva, J. Vindas, The Ridgelet transform of distributions, *Integral Transforms Spec. Funct.*, **25** (2014), 344–358. <https://doi.org/10.1080/10652469.2013.853057>
19. Bhawna, M. Kumar, Special affine Fourier transform of tempered distributions and pseudo-differential operators, *Integral Transforms Spec. Funct.*, **35** (2024), 561–576. <https://doi.org/10.1080/10652469.2024.2356130>
20. M. Kumar, Bhawna, Special affine Fourier transform on tempered distribution and its application, *Math. Methods Appl. Sci.*, **48** (2025), 6092–6102. <https://doi.org/10.1002/mma.10657>
21. S. Lakshmanan, R. Roopkumar, A. I. Zayed, Special affine Fourier transform on generalized functions, *Integral Transforms Spec. Funct.*, 2025. <https://doi.org/10.1080/10652469.2025.2585492>
22. F. A. Shah, A. A. Teali, A. Y. Tantary, Windowed special affine Fourier transform, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 1389–1420. <https://doi.org/10.1007/s11868-019-00319-w>
23. S. Lakshmanan, R. Roopkumar, A. I. Zayed, A novel special affine convolution and short time special affine Fourier transform, *Integral Transforms Spec. Funct.*, **37** (2026), 169–192. <https://doi.org/10.1080/10652469.2025.2502988>
24. H. Dong, X. Fang, J. Zhao, X. Lin, X. Sha, Z. Li, Secure communication for RSMA systems: Integrating spatial and WFRFT signal domains, *IEEE Commun. Lett.*, **28** (2024), 2498–2502. <https://doi.org/10.1109/LCOMM.2024.3457617>
25. H. Dong, X. Fang, J. Zhao, X. Sha, Z. Li, Signal domain multicomponent-based secure hybrid precoding for mmWave systems, *IEEE Trans. Wireless Commun.*, **24** (2025), 51–67. <https://doi.org/10.1109/TWC.2024.3438265>
26. B. I. Zavialov, Automodel asymptotic of electromagnetic form-factors and the behavior of their Fourier transforms near the light cone, (in Russian), *Teor. Mat. Fiz.*, **17** (1973), 178–188.
27. S. Łojasiewicz, Sur la valeur et la limite d’une distribution en un point, *Stud. Math.*, **16** (1957), 1–36. <https://doi.org/10.4064/sm-16-1-1-36>
28. S. Pilipović, B. Stanković, J. V. Diaz, *Asymptotic behavior of generalized functions*, World Scientific Publishing, 2011. <https://doi.org/10.1142/8266>
29. V. S. Vladimirov, Yu. N. Drozzinov, B. I. Zavialov, *Tauberian theorems for generalized functions*, Dordrecht: Springer, 1988. <https://doi.org/10.1007/978-94-009-2831-2>
30. R. Estrada, R. P. Kanwal, *A distributional approach to asymptotics: Theory and applications*, Boston: Birkhäuser, 2002. <https://doi.org/10.1007/978-0-8176-8130-2>

31. J. V. Buralieva, K. Saneva, S. Atanasova, Directional short-time Fourier transform and quasiasymptotics of distributions, *Funct. Anal. Its Appl.*, **53** (2019), 3–10. <https://doi.org/10.1007/s10688-019-0244-9>
32. A. Ferizi, K. H. Saneva, Directional short-time fractional Fourier analysis of quasiasymptotic properties of distributions, *Anal. Math.*, **52** (2026), 475–492. <https://doi.org/10.1007/s10476-026-00168-y>
33. K. H. Saneva, A. Ferizi, Short-time coupled fractional Fourier transform and asymptotic behaviour of distributions, *Integral Transforms Spec. Funct.*, 2026. <https://doi.org/10.1080/10652469.2026.2621202>
34. J. Vindas, S. Pilipović, D. Rakić, Tauberian theorems for the wavelet transform, *J. Fourier Anal. Appl.*, **17** (2011), 65–95. <https://doi.org/10.1007/s00041-010-9146-1>
35. S. Kostadinova, K. Saneva, J. V. Diaz, Gabor frames and asymptotic behavior of Schwartz distributions, *Appl. Anal. Discrete Math.*, **10** (2016), 292–307. <https://doi.org/10.2298/AADM160511011K>
36. A. R. Ferizi, K. H. Saneva, Shearlet expansion theory on Lizorkin-type spaces, *Funct. Anal. Its Appl.*, **60** (2026), 82–96. <https://doi.org/10.1134/S1234567826010064>
37. K. Saneva, Asymptotic behaviour of wavelet coefficients, *Integral Transforms Spec. Funct.*, **20** (2009), 333–339. <https://doi.org/10.1080/10652460802568200>
38. K. Saneva, R. Aceska, S. Kostadinova, Some Abelian and Tauberian results for the short-time Fourier transform, *Novi Sad J. Math.*, **43** (2013), 81–89.
39. L. Hörmander, *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*, Berlin: Springer, 2003. <https://doi.org/10.1007/978-3-642-61497-2>
40. L. Schwartz, *Théorie des distributions*, Editions Hermann, 1997.
41. F. Trèves, *Topological vector spaces, distributions and kernels*, New York-London: Academic Press, 1967.
42. M. Moshinsky, C. Quesne, Linear canonical transformations and their unitary representations, *J. Math. Phys.*, **12** (1971), 1772–1780. <https://doi.org/10.1063/1.1665805>
43. J. J. Healy, M. A. Kutay, H. M. Ozaktas, J. T. Sheridan, *Linear canonical transforms: Theory and applications*, Springer, 2016.
44. L. B. Almeida, The fractional Fourier transform and time-frequency representations, *IEEE Trans. Signal Process.*, **42** (1994), 3084–3091. <https://doi.org/10.1109/78.330368>
45. A. C. McBride, F. H. Kerr, On Namias’ fractional Fourier transforms, *IMA J. Appl. Math.*, **39** (1987), 159–175. <https://doi.org/10.1093/imamat/39.2.159>
46. K. Gröchenig, *Foundations of time-frequency analysis*, Boston: Birkhäuser, 2001. <https://doi.org/10.1007/978-1-4612-0003-1>
47. E. Seneta, *Regularly varying functions*, Heidelberg: Springer, 1976. <https://doi.org/10.1007/BFb0079658>

