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*Research article***Polyhedral-type reachable set estimation with generalized initial conditions: From positive systems to general linear delay systems****Hyung Tae Choi<sup>1</sup> and Jung Hoon Kim<sup>2,\*</sup>**<sup>1</sup> School of Electrical and Electronics Engineering, Chung-Ang University, Seoul 06974, Republic of Korea<sup>2</sup> Department of Electrical Engineering, Pohang University of Science and Technology (POSTECH), Pohang 37673, Republic of Korea**\* Correspondence:** Email: [junghoonkim@postech.ac.kr](mailto:junghoonkim@postech.ac.kr); Tel: +82542792230; Fax: +82542792903.

**Abstract:** This work developed a polyhedral set-based approach for bounding reachable sets of linear systems with nonzero polyhedral initial sets. We first considered linear positive systems and established sufficient conditions for constructing polyhedral bounds of reachable sets in both delay-free and time-varying-delay cases. For systems without delays, the conventional polyhedral set-bounding approach was extended to accommodate nontrivial initial sets given in polyhedral form. For systems with time-varying delays, the conventional auxiliary functional used for reachable set bounding was generalized to account for the effect of time-varying delays. The proposed framework was then extended to general linear systems that are not necessarily positive. In particular, for the case with delays, the auxiliary functional established for positive systems was further generalized by employing absolute values of the state. The resulting sufficient conditions provided polyhedral reachable set bounds for both delay-free and time-varying-delay systems, and all the proposed bounding sets were shown to be computed via linear programming. Several numerical examples are provided to illustrate the effects of nonzero initial sets and time-varying delay parameters on the resulting bounds, and to demonstrate the applicability of the proposed approach to general linear systems.

**Keywords:** positive systems; time delay; reachable set; auxiliary functional; polyhedral set**Mathematics Subject Classification:** 34H05, 93C23

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**1. Introduction**

The problem of reachable set estimation [1, 2] aims at approximating the reachable set of a dynamical system, in which the end points of all state trajectories are gathered provided that the bounded disturbances are given as inputs. Motivated by the difficulty of exactly computing the shape of

the reachable sets, some conventional results [3–5] have been established based on ellipsoidal sets and they contributed to the development of relevant problems such as linear  $L_1$  control [6–10], nonlinear  $L_1$  control [11–14], and safety verification [15, 16]. Especially, the ellipsoidal sets used in [3–5] were determined from a Lyapunov function (or a Lyapunov-Krasovski functional [17–19]) associated with the given system, and computed by solving relevant linear matrix inequalities (LMIs) [20, 21]. This idea has also been extended to linear delay systems [22] and to a switched linear system [23].

A dynamical system is said to be positive [24] if the state variable remains nonnegative whenever the initial condition and input are nonnegative. For linear positive systems, since copositive Lyapunov functions are shown to be less conservative for their stability and performance analysis [25] rather than the Lyapunov functions used for general systems, their associated level sets, which have a form of hyper-pyramids [26, 27], are used to estimate their reachable sets. However, the resulting hyper-pyramids in [26, 27] were generated only with half-spaces with nonnegative normal vectors, leaving a possibility for a generalization. Motivated by this, it was proved in [28] that a functional, called the auxiliary functional, which is defined by linear functionals with normal vectors not being necessarily nonnegative, provides a tighter polyhedral set bounding for positive delay systems.

However, the framework established in [28] is subject to several restrictions that narrow its applicability to more general reachable set-estimation problems. First, the reachable set in [28] is characterized only under a zero initial condition, i.e.,  $X_0 = 0$ , leaving the case of nonzero or general polyhedral initial sets unaddressed. The consideration of nonzero initial sets is particularly important when the initial state cannot be exactly specified due to uncertainties in measurement or initialization. Second, the arguments developed in [28] for bounding reachable sets are restricted to positive systems, and therefore do not directly apply to general linear systems whose states and system matrices may involve negative entries. Finally, the delay in [28] was assumed to be constant, and thus the effects of time-varying delays were not taken into account.

To address the restrictions associated with [28], this paper develops an extended framework for bounding reachable sets by polyhedral sets. We first allow the initial conditions to be specified by a nontrivial polyhedral set and formulate the corresponding estimating problem of reachable sets for positive systems. Sufficient conditions are then established for constructing polyhedral set bounds in both the delay-free and time-varying-delay cases. For systems with time-varying delays, a delay-dependent result is derived by generalizing the auxiliary functional developed in [28] to deal with time-varying delays. The corresponding bounds for reachable sets are characterized by sublevel sets of the auxiliary functional, which are robustly forward invariant. In each case, it is shown that the polyhedral set bound can be computed by solving linear programming (LP).

The developed framework is next extended in parallel to general linear systems that are not necessarily positive, while maintaining the setting for nonzero polyhedral initial sets. Sufficient conditions for polyhedral set bounding of reachable sets are established for both delay-free and time-varying-delay systems. In the delayed case, to account for state variables possibly with negative components, the auxiliary functional is further generalized by using absolute values of the state variables in the integrand. It is also shown that the sufficient conditions for both the delay-free and time-varying-delay cases can be reduced to LP feasibility conditions. Importantly, since the absolute value is not linear with respect to its argument, we linearize the absolute values taken for the state by using sign enumeration variables and reduce the condition into an LP condition. We highlight the main contributions of this paper in Table 1.

**Table 1.** The contributions of this paper compared to the relevant literature.

|                   | <b>This paper</b>                         | [28]                  | [3, 22]                   |
|-------------------|-------------------------------------------|-----------------------|---------------------------|
| Bounding set      | <b>Polyhedral set</b>                     | <b>Polyhedral set</b> | Ellipsoidal set           |
| System            | <b>Positive &amp; General</b>             | Positive              | <b>General</b>            |
| Initial condition | <b>General polyhedral</b> $\mathcal{X}_0$ | $\{0\}$               | $\{0\}$                   |
| Delay type        | <b>Time-varying delay</b>                 | Constant delay        | <b>Time-varying delay</b> |
| Optimization      | <b>LP</b>                                 | <b>LP</b>             | LMI                       |

The remainder of this paper is organized as follows. Section 2 develops polyhedral reachable-set bounding results for positive systems, considering both delay-free and time-varying-delay cases. Section 3 extends these results to general linear systems that are not necessarily positive. Section 4 presents numerical examples that verify the feasibility and effectiveness of the developed results, and Section 5 concludes the paper. The notation used throughout the paper is summarized in Table 2.

**Table 2.** The notations used in this paper.

| Notation           | Meaning                                                                                          |
|--------------------|--------------------------------------------------------------------------------------------------|
| $\mathbb{N}$       | The set of positive integers                                                                     |
| $\mathbb{R}$       | The set of real numbers                                                                          |
| $\mathbb{R}_+$     | The set of nonnegative real numbers                                                              |
| $\mathbb{R}^\nu$   | The set of $\nu$ -dimensional real vectors                                                       |
| $\mathbb{R}_+^\nu$ | The set of $\nu$ -dimensional real vectors with nonnegative components                           |
| $(\cdot)^\top$     | The matrix transpose                                                                             |
| $\ \cdot\ _\infty$ | The vector $\infty$ norm, i.e., $\ v\ _\infty := \max_i  v_i $                                   |
| $A \geq 0$         | A nonnegative matrix, i.e., $A_{ij} \geq 0, \forall i, j$                                        |
| $\mathcal{P}(p)$   | The $p$ -normal half-plane, i.e., $\mathcal{P}(p) = \{v \in \mathbb{R}^n \mid p^\top v \leq 1\}$ |
| $\mathbb{1}_\nu$   | The $\nu$ -dimensional vector with entries 1, i.e., $\mathbb{1}_i = 1, \forall i$                |

## 2. Reachable set estimation for positive systems with a generalized set of initial conditions

This section develops a polyhedral set-based approach to estimating the reachable set of linear positive systems with respect to a generalized initial condition.

### 2.1. Problem formulation

Consider the continuous-time single-input linear system with a time delay, denoted by  $\Sigma_{\geq 0}$ , which is defined as

$$\Sigma_{\geq 0} : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bw(t), \\ x(t) = \phi(t), \quad \forall t \in (-\infty, 0], \end{cases} \quad (2.1)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $w(t) \in \mathbb{R}$  is the disturbance,  $d(t) > 0$  is the time delay that is possibly varying with respect to  $t \in \mathbb{R}$ , and  $\phi : (-\infty, 0] \rightarrow \mathbb{R}^n$  is the initial condition for  $x$ . To assume the positiveness for  $\Sigma_{\geq 0}$ , the following definition is introduced.

**Definition 2.1.** The system  $\Sigma_{\geq 0}$  is said to be positive if  $x$  is always nonnegative for arbitrary nonnegative  $\phi$  and  $w$ , i.e.,

$$\begin{cases} \phi(t) \geq 0, & \forall t \in (-\infty, 0] \\ w(t) \geq 0, & \forall t \in [0, \infty) \end{cases} \Rightarrow x(t) \geq 0, \quad \forall t \in [0, \infty). \quad (2.2)$$

We assume that  $\Sigma_{\geq 0}$  in (2.1) is positive throughout this section. Since it is known in [24] that  $\Sigma_{\geq 0}$  is positive if and only if  $A$  is Metzler (i.e., every off-diagonal entry of  $A$  is nonnegative) and  $A_d, B$  are nonnegative, we assume those conditions on  $A, A_d, B$  in (2.1) in this section.

Let  $X_0$  be a bounded subset of  $\mathbb{R}_+^n$  and  $W_{\infty,+}$  be a set of nonnegative disturbances with unit magnitude, i.e.,

$$W_{\infty,+} = \{w : \mathbb{R}_+ \rightarrow \mathbb{R} \mid 0 \leq w(t) \leq 1, \forall t \in [0, \infty)\}. \quad (2.3)$$

Then, a reachable set of  $\Sigma_{\geq 0}$  equipped with the initial condition  $X_0$  and the disturbance condition  $W_{\infty,+}$  is denoted by  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+})$ , which is defined as

$$\begin{aligned} \mathcal{R}^{\geq 0}(X_0, W_{\infty,+}) &= \{x_* \in \mathbb{R}_+^n \mid \exists t_* \geq 0, \exists \phi \in L_\infty^n(-\infty, 0] \big|_{X_0}, \\ &\exists w \in W_{\infty,+} \text{ s.t. } \exists \text{ a solution } x : [0, t_*] \rightarrow \mathbb{R}_+^n \text{ to } \Sigma_{\geq 0}, \\ &\text{with initial condition } \phi \text{ and disturbance } w \text{ satisfying } x(t_*) = x_*, \end{aligned} \quad (2.4)$$

where  $L_\infty^n(-\infty, 0] \big|_{X_0}$  in (2.4) is defined by

$$L_\infty^n(-\infty, 0] \big|_{X_0} := \{f \in L_\infty^n(-\infty, 0] \mid f(s) \in X_0 \quad \forall s \in (-\infty, 0]\}. \quad (2.5)$$

Intuitively, the elements in  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+})$  correspond to the endpoints of the solution arcs to  $\Sigma_{\geq 0}$  when the associated initial conditions are given inside  $X_0$  and the disturbances are taken from  $W_{\infty,+}$ .

It is generally known in the literature [28] that it is quite difficult to exactly compute the reachable set. Motivated by this issue, we aim at formulating a problem of bounding the reachable set by taking an arbitrary convex polyhedron as the initial condition. To put it another way, for an arbitrarily given convex polyhedron  $X_0 \subset \mathbb{R}_+^n$ , we tackle the following problem associated with estimating  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+})$ , which is graphically described as shown in Figure 1.

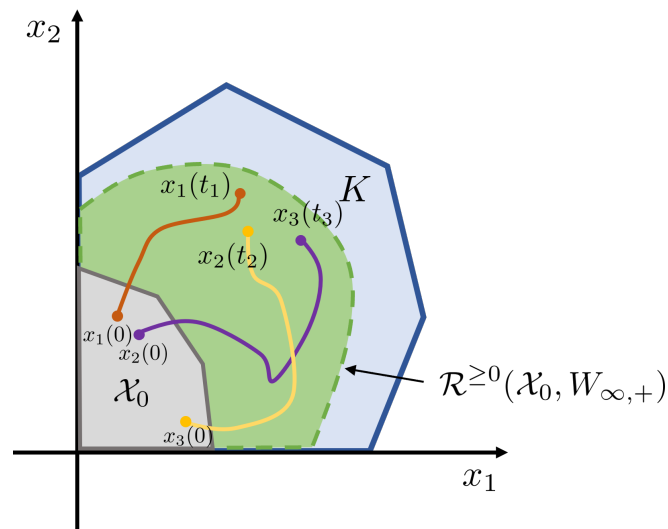


**Figure 1.** The reachable set  $\mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+})$  associated with  $\mathcal{X}_0$  and its bounding polyhedron  $K$ .

**Problem 2.1.** Let  $\Sigma_{\geq 0}$  be a positive system given in (2.1),  $\mathcal{X}_0$  be a convex polyhedron in  $\mathbb{R}_+^n$ , and  $W_{\infty,+}$  be given as in (2.3). Then, find a convex polyhedron  $K$  in  $\mathbb{R}_+^n$  such that

$$\mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+}) \subseteq K. \quad (2.6)$$

Note that  $\mathcal{X}_0$  and  $K$  are assumed in Problem 2.1 to be convex polyhedrons in  $\mathbb{R}_+^n$ . This implies that there exist  $c_i \in \mathbb{R}^n$  ( $i = 1, 2, \dots, N_{\mathcal{X}_0}$ ) and  $p_i \in \mathbb{R}^n$  ( $i = 1, 2, \dots, N_K$ ) such that

$$\mathcal{X}_0 = \left( \bigcap_{i=1}^{N_{\mathcal{X}_0}} \mathcal{P}(c_i) \right) \cap \mathbb{R}_+^n, \quad K = \left( \bigcap_{i=1}^{N_K} \mathcal{P}(p_i) \right) \cap \mathbb{R}_+^n, \quad (2.7)$$

where  $\mathcal{P}(p)$  means the  $p$ -normal half-plane in  $\mathbb{R}^n$  defined in Table 2.

**Remark 2.1.** Problem 2.1 can be interpreted as providing a generalized formulation of that in [28], where the zero initial condition (i.e.,  $\mathcal{X}_0 = \{0\}$ ) and a constant delay (i.e.,  $d(t) \equiv d$ ) are only considered.

## 2.2. The case of positive systems without a time delay

This subsection tackles Problem 2.1 for the case where no delay exists in  $\Sigma_{\geq 0}$  (i.e.,  $A_d = 0$  in (2.1)). To this end, we first introduce the definition of the robust forward invariance of sets [3] as follows.

**Definition 2.2.** Assume that  $A_d = 0$  in (2.1) and the resulting  $\Sigma_{\geq 0}$  is a positive system. Let  $K(\subset \mathbb{R}_+^n)$  be a given closed set and  $W_{\infty,+}$  be the set defined in (2.3). Then,  $K$  is said to be robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$  if every solution  $x$  to  $\Sigma_{\geq 0}$  with any initial condition  $\phi$  and any disturbance  $w$  such that  $\phi(0) \in K$  and  $w \in W_{\infty,+}$  satisfies  $x(t) \in K$  for all  $t \in [0, \infty)$ .

Based on Definition 2.2, we consider the following lemma, which characterizes Problem 2.1 in terms of robust forward invariance of  $K$ .

**Lemma 2.1.** Let  $\Sigma_{\geq 0}$  be a positive system in (2.1) with  $A_d = 0$ . Let  $X_0, K \subset \mathbb{R}_+^n$  be convex polyhedrons given in (2.7), and  $W_{\infty,+}$  be as defined in (2.3). Then, if  $K$  is robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$  such that  $X_0 \subset K$ , then  $K$  satisfies (2.6) in Problem 2.1.

*Proof.* Let  $x_*$  be an arbitrary element taken from  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+})$ . Then, there exist  $t_* \geq 0$ ,  $\phi \in L_{\infty}^n(-\infty, 0] \big|_{X_0}$ ,  $w \in W_{\infty,+}$  such that there exists a corresponding solution  $x$  defined on  $[0, t_*]$  satisfying the assertions in (2.4). Since  $\phi \in L_{\infty}^n(-\infty, 0] \big|_{X_0}$  and  $X_0 \subset K$ , it follows that  $\phi(0) \in X_0 \subset K$ , and thus  $\phi(0) \in K$ . This fact together with  $K$  being robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$  shows that  $x(t) \in K$  for all  $t \in [0, t_*]$ . Since this means that  $x_* = x(t_*)$ , condition (2.6) is established.  $\square$

Since Lemma 2.1 provides a sufficient condition for satisfying (2.6) in Problem 2.1 in terms of the robust forward invariance of  $K$ , we next establish an assertion for ensuring the sufficient condition. The following proposition is effectively exploited to ensure the robust forward invariance of  $K$ , without using explicit solutions  $x$  of  $\Sigma_{\geq 0}$ .

**Proposition 2.1.** Let  $\Sigma_{\geq 0}$  be a positive system in (2.1) with  $A_d = 0$ . Let  $K \subset \mathbb{R}_+^n$  and  $W_{\infty,+}$  be given as (2.7) and (2.3), respectively. Let  $p_i \in \mathbb{R}^n$  ( $i = 1, 2, \dots, N_K$ ) be nonzero vectors used for defining  $K$  in (2.7). If there exist scalars  $\alpha_i \geq 0$ ,  $\tau_{ij} \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ) such that

$$\begin{cases} p_i^{\top} A + \alpha_i p_i^{\top} + \sum_{j \neq i} \tau_{ij} (p_i^{\top} - p_j^{\top}) \leq 0, \\ p_i^{\top} B - \alpha_i \leq 0, \end{cases} \quad \forall i = 1, 2, \dots, N_K, \quad (2.8)$$

then  $K$  is robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$ .

Although Proposition 2.1 is already available in [28], we provide a slightly modified proof tailored to the flow and arguments of this paper, which will also be used in the subsequent developments.

*Proof.* Suppose that  $K$  is not robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$ . Then, there exist  $t_* \geq 0$  and a solution  $x$  to  $\Sigma_{\geq 0}$  with the initial condition  $\phi$  and the disturbance  $w$  such that  $\phi(0) \in K$  and  $w \in W_{\infty,+}$  satisfying  $x(t_*) \in \partial K$  and  $x(t) \notin K$  for all  $t \in [t_*, t_* + \epsilon)$  for some small  $\epsilon > 0$ . This fact together with (2.7) implies that there exists an  $i \in \{1, 2, \dots, N_K\}$  satisfying  $p_i^{\top} x(t_*) = 1$  and  $p_i^{\top} \dot{x}(t_*) > 0$ .

On the other hand, from the fact that  $x(t_*)$  is a point in  $K$  satisfying

$$p_i^{\top} x(t_*) = 1, p_j^{\top} x(t_*) \leq p_i^{\top} x(t_*) = 1,$$

for each  $j = 1, 2, \dots, N_K$ , this fact together with  $w(t_*) \in [0, 1]$  yields

$$(p_i^{\top} - p_j^{\top})x(t_*) \geq 0$$

and

$$p_i^{\top} x(t_*) - w \geq 0.$$

Then, we can rewrite  $p_i^{\top} \dot{x}(t_*)$  as

$$p_i^{\top} \dot{x}(t_*) = p_i^{\top} (Ax(t_*) + Bw(t_*)) \leq (p_i^{\top} A + \alpha_i p_i^{\top} + \sum_{j \neq i} \tau_{ij} (p_i^{\top} - p_j^{\top}))x(t_*) + (p_i^{\top} B - \alpha_i)w(t_*), \quad (2.9)$$

where the right-hand side of (2.9) is nonpositive due to (2.8) and the positivity of  $x(t_*)$ ,  $w(t_*)$ ,  $\alpha_i$ , and  $\tau_{ij}$ . This contradicts the fact that  $p_i^{\top} \dot{x}(t_*) > 0$ , and thus the proof is established.  $\square$

Note that, for a fixed set of  $p_i \in \mathbb{R}^n (i = 1, 2, \dots, N_K)$ , the constraints in (2.8) are linear with respect to  $\alpha_i$  and  $\tau_{ij}$  for  $i, j \in \{1, 2, \dots, N_K\}$ . Therefore, a practical computational procedure corresponds to selecting candidate values of  $p_i$  from a finite set in  $\mathbb{R}^n$ . For example, one might solve the resulting LP feasibility problem with respect to  $\alpha_i$  and  $\tau_{ij}$  for each candidate of  $p_i$ . The feasibility of such an LP problem can be readily verified by using a standard LP solver in MATLAB.

While the robust forward invariance of  $K$  is converted into a computationally tractable condition via Proposition 2.1, it is still required to obtain an  $X_0 \subset K$  in Lemma 2.1. To tackle this issue, we introduce the following lemma.

**Lemma 2.2** (Farkas lemma [29]). *Let  $Q, R$  be matrices and  $v, \phi$  be vectors with appropriate dimensions. Then, the following two statements are equivalent.*

(F1)  $Qv \leq \psi$  holds for all  $v$  such that  $Rv \leq \phi$ .

(F2) There exists a matrix  $U \geq 0$  such that

$$UR = Q, \quad U\phi \leq \psi. \quad (2.10)$$

Note that a direct employment of the assertion (F1) in Lemma 2.2 is not straightforward in general, since there are infinitely many  $v$ 's to be verified. Hence, Lemma 2.2 is useful in the sense that we can exploit the assertion (F2) instead of the assertion (F1). To apply Lemma 2.2, we obtain the following proposition relevant to ensuring  $X_0 \subset K$  for a given  $X_0$ .

**Proposition 2.2.** *Let  $X_0, K \subset \mathbb{R}_+^n$  be given in (2.7) and  $p_i \in \mathbb{R}^n (i = 1, 2, \dots, N_K)$ ,  $c_i \in \mathbb{R}^n (i = 1, 2, \dots, N_{X_0})$  be nonzero vectors with respect to defining such  $X_0$  and  $K$ . Then, the relation  $X_0 \subset K$  is satisfied if and only if there exists  $u_{ij} \geq 0 (i = 1, 2, \dots, N_K, j = 1, 2, \dots, N_{X_0})$  such that*

$$\sum_{j=1}^{N_{X_0}} u_{ij} c_j^\top = p_i^\top, \quad \sum_{j=1}^{N_{X_0}} u_{ij} \leq 1, \quad \forall i = 1, 2, \dots, N_K. \quad (2.11)$$

*Proof.* Suppose that  $X_0 \subset K$  holds. Then, it follows from the definitions of  $X_0$  and  $K$  in (2.7) that

$$X_0 = \{v \in \mathbb{R}_+^n \mid Cv \leq \mathbb{1}_{N_{X_0}}\}, \quad K = \{v \in \mathbb{R}_+^n \mid Pv \leq \mathbb{1}_{N_K}\}, \quad (2.12)$$

where  $\mathbb{1}_k$  is a  $k$ -dimensional real vector consisting of 1's for all components, and  $P \in \mathbb{R}^{N_K \times n}$ ,  $C \in \mathbb{R}^{N_{X_0} \times n}$  are defined by

$$P = \begin{bmatrix} p_1^\top \\ \vdots \\ p_{N_K}^\top \end{bmatrix}, \quad C = \begin{bmatrix} c_1^\top \\ \vdots \\ c_{N_{X_0}}^\top \end{bmatrix}. \quad (2.13)$$

Then, combining  $X_0 \subset K$  with (2.12) implies that the assertion (F1) in Lemma 2.2 holds with

$$Q = P, \quad R = C, \quad \psi = \mathbb{1}_{N_K}, \quad \phi = \mathbb{1}_{N_{X_0}}. \quad (2.14)$$

Applying the assertion (F2) in Lemma 2.2 with the above  $Q, R, \psi, \phi$ , there exists a  $U = [u_{ij}] \geq 0$  such that

$$UC = P, \quad U\mathbb{1}_{N_{X_0}} \leq \mathbb{1}_{N_K}. \quad (2.15)$$

By rewriting (2.15) with the components of  $U, P, C$ , it immediately follows that (2.15) is equivalent to (2.11). Hence, there exists  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K, j = 1, 2, \dots, N_{X_0}$ ) such that (2.11) holds. This completes the proof for the necessary condition. The sufficient condition follows by exactly reversing the above arguments.  $\square$

Note that for given  $p_i$  and  $c_i$ , the feasibility of (2.11) with respect to  $u_{ij}$  can be checked by solving an LP. Combining Propositions 2.1 and 2.2 with Lemma 2.1 leads to the following theorem, which characterizes a solution of Problem 2.1 in terms of LP.

**Theorem 2.1.** *Let  $\Sigma_{\geq 0}$  be a positive system in (2.1) with  $A_d = 0$ . Let  $X_0, K \subset \mathbb{R}_+^n$  be the sets defined in (2.7), and  $W_{\infty,+}$  be as in (2.3). Then, the relation  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+}) \subset K$  (i.e., (2.6)) holds, if there exist  $\alpha_i \geq 0$  ( $i = 1, 2, \dots, N_K$ ),  $\tau_{ij} \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ), and  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K, j = 1, 2, \dots, N_{X_0}$ ) such that (2.8) and (2.11) are simultaneously satisfied.*

Note that Theorem 2.1 implies that the decision variables in Propositions 2.1 and 2.2 can be determined simultaneously once  $p_i$  and  $c_j$  are fixed. Accordingly, for a given set of  $c_j$ , a computational implementation of Theorem 2.1 can be carried out by using a grid search over  $p_i \in \mathbb{R}^n$  and, for each candidate choice of  $p_i$ , solving (2.8) and (2.11) for the other decision variables through some standard LP solver. In contrast to [28], where reachable sets are considered only for  $X_0 = \{0\}$ , Theorem 2.1 considers an extended version of reachable sets to the case where  $X_0$  is an arbitrarily given polyhedral set.

### 2.3. The case of positive systems with a time-varying delay

This subsection aims at solving Problem 2.1 by relaxing the assumption in the preceding subsection that  $\Sigma_{\geq 0}$  is not equipped with a time delay. To this end, we now allow  $A_d$  to be nonzero and assume that the time-varying delay satisfies

$$0 < d(t) \leq h, \quad \forall t \geq 0, \quad (2.16)$$

for a given  $h > 0$ . Then, it is first required to characterize (2.6) in Problem 2.1 with respect to a forward invariance of sets for the state space of  $\Sigma_{\geq 0}$ , in a parallel fashion to Lemma 2.1. However, since  $\Sigma_{\geq 0}$  is equipped with a delay, it is natural to extend the robust forward invariance from the space of state vectors to that of solution arcs.

Regarding this, we define a collection of  $L_\infty$ -bounded signals (or arcs), denoted by  $L_\infty^n(h)$ , as

$$L_\infty^n(h) := \bigcup_{t \geq 0} L_\infty^n[t-h, t], \quad (2.17)$$

where  $L_\infty^n[t-h, t]$  is defined (for each  $t \geq 0$ ) as

$$L_\infty^n[t-h, t] := \{x_t : [t-h, t] \rightarrow \mathbb{R}_+^n \mid \sup_{s \in [t-h, t]} |x_t(s)|_\infty < +\infty\} \quad (2.18)$$

with  $x_t$  in (2.18) being a history segment of  $x$  over  $[t-h, t]$ . On the basis of  $L_\infty^n(h)$ , we define the robust forward invariance to a subset  $K$  of  $L_\infty^n(h)$  as follows.

**Definition 2.3.** *Let  $\Sigma_{\geq 0}$  be a positive system in (2.1) with  $A_d$  not necessarily zero and  $L_\infty^n(h)$  be the set defined in (2.17). Let  $K$  be a closed subset of  $L_\infty^n(h)$  and  $W_{\infty,+}$  be the set defined in (2.3). Then,  $K$  is*

said to be robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$  if every solution arc  $x : [0, \infty) \rightarrow \mathbb{R}_+^n$  for  $\Sigma_{\geq 0}$  with any initial condition  $\phi$  and any disturbance  $w$  such that  $\phi \in K \cap L_\infty^n[-h, 0]$  and  $w \in W_{\infty,+}$  satisfies  $x_t \in K$  for all  $t \in [0, \infty)$ .

To describe a robust forward invariant set that is helpful for characterizing (2.6) in Problem 2.1, we define a linear functional  $V : L_\infty^n(h) \rightarrow \mathbb{R}$  for  $p_i$  ( $i = 1, 2, \dots, N_K$ ) used to define  $K$  in (2.7), an auxiliary vector  $q \in \mathbb{R}_+^n$ , and a scalar  $\alpha \geq 0$  as

$$V(x_t) := \max_{1 \leq i \leq N_K} V_i(x_t), \quad \forall x_t \in L_\infty^n(h), \quad (2.19)$$

where

$$V_i(x_t) := p_i^\top x(t) + \int_{t-d(t)}^t e^{\alpha(s-t)} q^\top x(s) ds, \quad \forall i = 1, 2, \dots, N_K, \quad (2.20)$$

and this  $V(x_t)$  is called an auxiliary functional [28]. This definition allows us to obtain the following lemma associated with characterizing (2.6) in terms of  $V$ .

**Lemma 2.3.** *Let  $\Sigma_{\geq 0}$  be a positive system given in (2.1), equipped with  $d(t)$  satisfying (2.16). Let  $\mathcal{X}_0, K \subset \mathbb{R}_+^n$  be the sets in (2.7), and  $W_{\infty,+}$  be as in (2.3). For  $V$  defined in (2.19), let us define  $V^{-1}((-\infty, 1]) \subset L_\infty^n(h)$  as*

$$V^{-1}((-\infty, 1]) := \{x_t \in L_\infty^n(h) \mid V(x_t) \leq 1\}. \quad (2.21)$$

*Then, the relation (2.6) in Problem 2.1 holds, i.e.,  $\mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+}) \subset K$ , if  $V^{-1}((-\infty, 1])$  is robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$  such that*

$$L_\infty^n[-h, 0] \big|_{\mathcal{X}_0} \subset V^{-1}((-\infty, 1]), \quad (2.22)$$

*where  $L_\infty^n[-h, 0] \big|_{\mathcal{X}_0}$  is defined in (2.5).*

*Proof.* Let  $x_*$  be an arbitrary element taken from  $\mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+})$ . Then, by the definition of  $\mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+})$  in (2.4), there exist  $t_* \geq 0$ ,  $\phi \in L_\infty^n(-\infty, 0] \big|_{\mathcal{X}_0}$ ,  $w \in W_{\infty,+}$  such that there exists a corresponding solution  $x$  defined on  $[0, t_*]$  with initial condition  $\phi$  and disturbance  $w$  satisfying  $x(t_*) = x_*$ . Then,  $\phi \in L_\infty^n(-\infty, 0] \big|_{\mathcal{X}_0}$  together with (2.22) implies that  $\phi \in V^{-1}((-\infty, 1])$ . This fact together with  $V^{-1}((-\infty, 1])$  being robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$  implies that  $x_t \in V^{-1}((-\infty, 1])$  for all  $t \in [0, t_*]$ . Now, (2.21) implies  $V(x_t) \leq 1$  for all  $t \in [0, t_*]$  and the positivity of  $q$  implies that

$$p_i^\top x(t) \leq p_i^\top x(t) + \int_{t-d(t)}^t e^{\alpha(s-t)} q^\top x(s) ds = V_i(x_t) \leq V(x_t) \leq 1 \quad (2.23)$$

holds for each  $i = 1, 2, \dots, N_K$  and  $t \in [0, t_*]$ . Putting  $t = t_*$  in (2.23) leads us to  $p_i^\top x(t_*) = p_i^\top x_* \leq 1$  for each  $i = 1, 2, \dots, N_K$ . Hence,  $x_* \in K$  holds by (2.7). This establishes (2.6), completing the proof.  $\square$

Note that Lemma 2.3 is a generalized version of Lemma 2.1 in the sense that the reachable set is estimated for positive systems that possibly involve delays. More precisely, the robust forward invariance of  $V^{-1}((-\infty, 1])$  is considered instead that of  $K$ , and the associated inclusion is generalized from  $\mathcal{X}_0 \subset K$  to (2.22). Compared with [28], Problem 2.1 allows a general set of initial conditions,

and thus this setting makes Lemma 2.3 additionally require (2.22) together with the robust forward invariance condition. In [28], (2.22) is obviously satisfied since the initial set  $X_0$  is given to be trivial, i.e.,  $X_0 = \{0\}$ .

Our next step is to convert the robust forward invariance of  $V^{-1}((-\infty, 1])$  into more tractable conditions. In this regard, we provide the following proposition on a sufficient condition for a robust forward invariance of  $V^{-1}((-\infty, 1])$ .

**Proposition 2.3.** *Let  $\Sigma_{\geq 0}$  be a positive system in (2.1) and  $W_{\infty,+}$  be as in (2.3). Assume that  $d(t)$  satisfies (2.16) for a given  $h > 0$  and*

$$|\dot{d}(t)| \leq \mu < 1, \quad \forall t \geq 0, \quad (2.24)$$

for some  $\mu \in (0, 1)$ . If there exist  $p_i \in \mathbb{R}^n$ ,  $\tau_{ij} \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ),  $\alpha \geq 0$ ,  $\lambda \in (0, 1)$ , and  $q \in \mathbb{R}_+^n$  such that

$$\begin{cases} p_i^T A + \alpha p_i^T + q^T + \sum_{j \neq i} \tau_{ij}(p_i^T - p_j^T) \leq 0, \\ p_i^T A_d - e^{-\alpha h} q^T (1 - \mu) - \lambda \alpha p_i^T \leq 0, \\ p_i^T B - (1 - \lambda) \alpha \leq 0, \end{cases} \quad \forall i = 1, 2, \dots, N_K, \quad (2.25)$$

then  $V^{-1}((-\infty, 1])$  is robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$ , where  $V$  is generated by using  $p_i$  ( $i = 1, 2, \dots, N_K$ ),  $\alpha$ , and  $q$ .

*Proof.* Suppose that  $V^{-1}((-\infty, 1])$  is not robustly forward invariant with respect to  $(\Sigma_{\geq 0}, W_{\infty,+})$ . Then, by Definition 2.3, there exist  $t_* \geq 0$ ,  $\phi \in K \cap L_\infty^n[-h, 0]$ ,  $w \in W_{\infty,+}$ , and a solution arc  $x$  to  $\Sigma_{\geq 0}$  with initial condition  $\phi$  and disturbance  $w$  such that  $V(x_{t_*}) = 1$  and  $V(x_t) > 1$  for all  $t \in [t_*, t_* + \epsilon)$  for some small  $\epsilon > 0$ . This implies that there exists  $i \in \{1, 2, \dots, N_K\}$  such that  $V_i(x_{t_*}) = 1$  and  $\frac{d}{dt} V_i(x_t)|_{t=t_*} > 0$ .

However, we obtain

$$\begin{aligned} \left. \frac{d}{dt} V_i(x_t) \right|_{t=t_*} &= p_i^T \dot{x}(t_*) + \left. \frac{d}{dt} \left( e^{-\alpha t} \int_{t-d(t)}^t e^{\alpha s} q^T x(s) ds \right) \right|_{t=t_*} \\ &= p_i^T \dot{x}(t_*) - \alpha \int_{t_*-d(t_*)}^{t_*} e^{\alpha(s-t_*)} q^T x(s) ds + q^T x(t_*) - e^{-\alpha d(t_*)} q^T x(t_* - d(t_*))(1 - \dot{d}(t_*)) \\ &= p_i^T \dot{x}(t_*) - \alpha (V_i(x_t)|_{t=t_*} - p_i^T x(t_*)) + q^T x(t_*) - e^{-\alpha d(t_*)} q^T x(t_* - d(t_*))(1 - \dot{d}(t_*)) \\ &= p_i^T (Ax(t_*) + A_d x(t_* - d(t_*)) + Bw(t_*)) - \alpha (1 - p_i^T x(t_*)) + q^T x(t_*) \\ &\quad - e^{-\alpha d(t_*)} q^T x(t_* - d(t_*))(1 - \dot{d}(t_*)). \end{aligned} \quad (2.26)$$

From the choice of  $i$ , we obtain

$$w(t_*) \leq 1, \quad p_i^T x(t_*) \geq p_j^T x(t_*) \quad (j \neq i), \quad p_i^T x(t_* - d(t_*)) \leq 1. \quad (2.27)$$

Then, combining (2.24), (2.26), and (2.27) together with the positivity of  $\alpha$ ,  $\tau_{ij}$ , and  $\lambda$  leads to

$$\begin{aligned} \left. \frac{d}{dt} V_i(x_t) \right|_{t=t_*} &\leq p_i^T (Ax(t_*) + A_d x(t_* - d(t_*)) + Bw(t_*)) - \alpha (1 - p_i^T x(t_*)) + q^T x(t_*) \\ &\quad - e^{-\alpha h} q^T x(t_* - d(t_*))(1 - \mu) + \alpha (1 - \lambda) (1 - w(t_*)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j \neq i} \tau_{ij}(p_i^T - p_j^T)x(t_*) + \lambda\alpha(1 - p_i^T x(t_* - d(t_*))) \\
& = (p_i^T A + \alpha p_i^T + q^T + \sum_{j \neq i} \tau_{ij}(p_i^T - p_j^T))x(t_*) + (p_i^T A_d - e^{-\alpha h} q^T (1 - \mu) - \lambda\alpha p_i^T)x(t_* - d(t_*)) \\
& \quad + (p_i^T B - \alpha(1 - \lambda))w(t_*), \tag{2.28}
\end{aligned}$$

where the last term in (2.28) is nonpositive due to (2.25) and the positivity of  $x(t_*)$ ,  $x(t_* - d(t_*))$ , and  $w(t_*)$ . This contradicts the first assertion and thus the proof is established.  $\square$

Note that Proposition 2.3 is an extended version of Theorem 2 in [28], in the sense that the delays are allowed to be time-varying. The time-varying delay affects the proof of Proposition 2.3 in the last term of (2.26), and such an effect is bounded by using the delay upper bound  $h$  and its velocity bound  $\mu$  as given in (2.16) and (2.24), respectively. For fixed  $p_i$  and  $\alpha$ , the feasibility conditions for (2.25) in Proposition 2.3 are linear in the remaining decision variables  $q$ ,  $\tau_{ij}$ , and  $\lambda$ , for which we can employ an LP solver. The remaining task is to ensure the relation (2.22), and the following proposition provides an associated sufficient condition.

**Proposition 2.4.** *Let  $X_0$  be the set defined in (2.7) and  $c_i \in \mathbb{R}^n$  ( $i = 1, 2, \dots, N_{X_0}$ ) be the vectors used to define  $X_0$  in (2.7). Let  $h > 0$  be a given scalar satisfying (2.16). Then, if there exist  $p_i \in \mathbb{R}^n$ ,  $u_{ij} \geq 0$ ,  $w_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ),  $\alpha \geq 0$ , and  $q \in \mathbb{R}_+^n$  such that*

$$\sum_{j=1}^{N_{X_0}} u_{ij} c_j^\top = p_i^\top, \quad \sum_{j=1}^{N_{X_0}} w_{ij} c_j^\top = q^\top, \quad \sum_{j=1}^{N_{X_0}} u_{ij} + \left( \frac{1 - e^{-\alpha h}}{\alpha} \right) \sum_{j=1}^{N_{X_0}} w_{ij} \leq 1, \quad \forall i = 1, 2, \dots, N_K, \tag{2.29}$$

then relation (2.22) holds for  $X_0$  and  $V$ , where  $V$  is determined by  $p_i$  ( $i = 1, 2, \dots, N_K$ ),  $\alpha$ , and  $q$ .

*Proof.* To show that (2.22) holds, take an arbitrary  $\phi \in L_\infty^n[-h, 0]_{X_0}$ . By its definition,  $\phi \in L_\infty^n[-h, 0]$  satisfying  $\phi(s) \in X_0$  for every  $s \in [-h, 0]$ . Then, it follows from the definition of  $X_0$  in (2.7) that

$$c_j^\top \phi(s) \leq 1, \quad \forall j = 1, 2, \dots, N_{X_0}, \quad \forall s \in [-h, 0]. \tag{2.30}$$

Then, if we multiply  $u_{ij}$  and  $w_{ij}$  on both sides of (2.30) and take the sum from  $j = 1$  to  $N_{X_0}$ , then we obtain

$$\begin{cases} p_i^\top \phi(0) = \sum_{j=1}^{N_{X_0}} u_{ij} c_j^\top \phi(0) \leq \sum_{j=1}^{N_{X_0}} u_{ij}, & \forall i = 1, 2, \dots, N_K, \\ q^\top \phi(s) = \sum_{j=1}^{N_{X_0}} w_{ij} c_j^\top \phi(s) \leq \sum_{j=1}^{N_{X_0}} w_{ij}, & \forall i = 1, 2, \dots, N_K, \quad \forall s \in [-h, 0]. \end{cases} \tag{2.31}$$

Based on (2.31), we obtain

$$\begin{aligned}
V(\phi) &= \max_{1 \leq i \leq N_K} V_i(\phi) \leq \max_{1 \leq i \leq N_K} \left\{ p_i^\top \phi(0) + \int_{-h}^0 e^{\alpha s} q^\top \phi(s) ds \right\} \\
&\leq \max_{1 \leq i \leq N_K} \left\{ \sum_{j=1}^{N_{X_0}} u_{ij} + \int_{-h}^0 e^{\alpha s} \left( \sum_{j=1}^{N_{X_0}} w_{ij} \right) ds \right\} \leq \max_{1 \leq i \leq N_K} \left\{ \sum_{j=1}^{N_{X_0}} u_{ij} + \left( \sum_{j=1}^{N_{X_0}} w_{ij} \right) \int_{-h}^0 e^{\alpha s} ds \right\}
\end{aligned}$$

$$\leq \max_{1 \leq i \leq N_K} \left\{ \sum_{j=1}^{N_{X_0}} u_{ij} + \left( \sum_{j=1}^{N_{X_0}} w_{ij} \right) \left( \frac{1 - e^{-\alpha h}}{\alpha} \right) \right\} \leq 1, \quad (2.32)$$

where the last inequality follows from the last condition in (2.29). Since this means that  $\phi \in V^{-1}((-\infty, 1])$ , the proof is complete.  $\square$

Note that Proposition 2.4 not only gives us a sufficient condition for (2.22) but also provides us with a computationally tractable condition for verifying (2.22). More precisely, for fixed  $p_i$  and  $\alpha$ , the conditions in (2.29) are linear in  $u_{ij}$ ,  $w_{ij}$ , and  $q$ , and their feasibility can therefore be checked via LP. By combining Propositions 2.3 and 2.4 and Lemma 2.3, we obtain the following theorem, which gives a sufficient condition for ensuring (2.6) to positive systems with delays.

**Theorem 2.2.** *Let  $\Sigma_{\geq 0}$  be a positive system given in (2.1) and  $d(t)$  be given to satisfy (2.16) and (2.24). Let  $X_0, K \subset \mathbb{R}_+^n$  be the sets in (2.7), and  $W_{\infty,+}$  be the set in (2.3). Then,  $K$  satisfies (2.6), i.e.,  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+}) \subset K$ , if there exist  $p_i \in \mathbb{R}^n$ ,  $\tau_{ij} \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ),  $\alpha \geq 0$ ,  $\lambda \in (0, 1)$ ,  $q \in \mathbb{R}_+^n$ ,  $u_{ij} \geq 0$ , and  $w_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ) such that (2.25) and (2.29) are simultaneously satisfied.*

By conducting a grid search over  $p_i$  and  $\alpha$ , the remaining decision variables in Theorem 2.2, such as  $\tau_{ij}$ ,  $\lambda$ ,  $q$ ,  $u_{ij}$ , and  $w_{ij}$ , can be determined simultaneously by solving an LP feasibility problem.

### 3. Extensions to general linear systems that are not necessarily positive

This section extends the results in Section 2 to general linear systems that are not necessarily positive, since the arguments in the preceding section are no longer applicable to systems that possibly have negative values for signals.

#### 3.1. Problem formulation

Consider the following continuous-time linear system with a time delay, denoted by  $\Sigma$ , which is not necessarily positive.

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bw(t), \\ x(t) = \phi(t), \quad \forall t \in (-\infty, 0]. \end{cases} \quad (3.1)$$

Here,  $x(t) \in \mathbb{R}^n$ ,  $w(t) \in \mathbb{R}$ ,  $d(t) > 0$ , and  $\phi : (-\infty, 0] \rightarrow \mathbb{R}^n$  have the same meanings as those in (2.1). However,  $\Sigma$  in (3.1) is not assumed to be positive. Hence,  $A$  is not necessarily Metzler, and  $A_d$ ,  $B$ ,  $x(t)$ , and  $w(t)$  are not necessarily nonnegative.

By allowing negative values for the state and disturbance, we define the disturbance set  $W_\infty$  and the reachable set  $\mathcal{R}(X_0, W_\infty)$  associated with an initial set  $X_0 \subset \mathbb{R}^n$ , respectively, by

$$W_\infty := \{w : \mathbb{R}_+ \rightarrow \mathbb{R} \mid -1 \leq w(t) \leq 1, \quad \forall t \in [0, \infty)\}, \quad (3.2)$$

$$\mathcal{R}(X_0, W_\infty) := \{x_* \in \mathbb{R}^n \mid \exists t_* \geq 0, \exists \phi \in L_\infty^n(-\infty, 0] \big|_{X_0}, \exists w \in W_\infty \text{ s.t. } \exists \text{ a solution } x : [0, t_*] \rightarrow \mathbb{R}^n, \\ \text{to } \Sigma \text{ with initial condition } \phi \text{ and disturbance } w \text{ satisfying } x(t_*) = x_*\}. \quad (3.3)$$

Note that  $W_\infty$  and  $\mathcal{R}(X_0, W_\infty)$  are the counterparts for  $W_{\infty,+}$  and  $\mathcal{R}^{\geq 0}(X_0, W_{\infty,+})$  defined in Section 2, respectively, when negative values of the disturbance and state are allowed. The problem of estimating reachable sets considered in this section is graphically illustrated in Figure 2 and formulated as follows.

**Problem 3.1.** Let  $\Sigma$  be the system given in (3.1),  $\mathcal{X}_0$  be a convex polyhedron in  $\mathbb{R}^n$ , and  $W_\infty$  be the set given in (3.2). Then, find a convex polyhedron  $K$  in  $\mathbb{R}^n$  such that

$$\mathcal{R}(\mathcal{X}_0, W_\infty) \subset K. \quad (3.4)$$

Note that Problem 3.1 generalizes Problem 2.1 in the sense that  $\Sigma$  in (3.1) is not required to be positive. This generalization also extends the framework given in [28], in which only positive systems are taken into account. Since  $\mathcal{X}_0$  and  $K$  in Problem 3.1 are convex polyhedrons in  $\mathbb{R}^n$ , they are represented by

$$\mathcal{X}_0 = \bigcap_{i=1}^{N_{\mathcal{X}_0}} \mathcal{P}(c_i), \quad K = \bigcap_{i=1}^{N_K} \mathcal{P}(p_i), \quad (3.5)$$

where, in general,  $\mathcal{X}_0$  and  $K$  are not necessarily contained in  $\mathbb{R}_+^n$ .

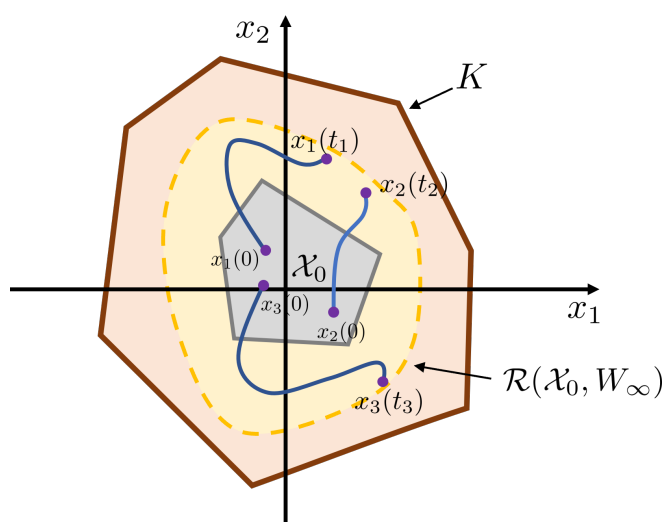


**Figure 2.** The reachable set  $\mathcal{R}(\mathcal{X}_0, W_\infty)$  associated with  $\mathcal{X}_0$  and its bounding polyhedron  $K$ .

### 3.2. The case of general linear systems without a delay

This subsection considers Problem 3.1 for the delay-free case, i.e.,  $A_d = 0$  in (3.1). In a parallel fashion to Lemma 2.1 in Section 2, the following lemma characterizes condition (3.4) in terms of the robust forward invariance of  $K$ .

**Lemma 3.1.** Let  $\Sigma$  be a linear system in (3.1) with  $A_d = 0$ . Let  $\mathcal{X}_0, K \subset \mathbb{R}^n$  be the sets given in (3.5), and  $W_\infty$  be the set defined in (3.2). Then, if  $K$  is robustly forward invariant with respect to  $(\Sigma, W_\infty)$  such that  $\mathcal{X}_0 \subset K$ , then  $K$  satisfies (3.4) in Problem 3.1.

Note that the notion of robust forward invariance used in Lemma 3.1 is defined exactly the same with Definition 2.2, except that the set of disturbances are generalized from  $W_{\infty,+}$  to  $W_\infty$ . The proof of Lemma 3.1 is omitted, since it is parallel to that of Lemma 2.1.

The following proposition characterizes the forward invariance of  $K$  in terms of conditions that are independent to explicit solutions  $x$  to  $\Sigma$ , in a parallel fashion to Proposition 2.1.

**Proposition 3.1.** Let  $\Sigma$  be a system in (3.1) with  $A_d = 0$ . Let  $K \subset \mathbb{R}^n$  and  $W_\infty$  be given as in (3.5) and (3.2), respectively. Let  $p_i$  ( $i = 1, 2, \dots, N_K$ ) be the nonzero vectors used for defining  $K$  in (3.5). If there exist  $\tau_{ij} \geq 0$ ,  $v_i^+ \geq 0$ ,  $v_i^- \geq 0$ , and  $\beta_i \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ) such that

$$\begin{cases} p_i^\top A = \sum_{j=1}^{N_K} \tau_{ij} p_j^\top - \beta_i p_i^\top, \\ p_i^\top B = v_i^+ - v_i^-, \\ \sum_{j=1}^{N_K} \tau_{ij} - \beta_i + v_i^+ + v_i^- \leq 0, \end{cases} \quad \forall i = 1, 2, \dots, N_K, \quad (3.6)$$

then  $K$  is robustly forward invariant with respect to  $(\Sigma, W_\infty)$ .

*Proof.* Suppose that  $K$  is not robustly forward invariant with respect to  $(\Sigma, W_\infty)$ . Then, by essentially the same argument as in Proposition 2.1, there exist  $t_* \geq 0$  and a solution  $x$  to  $\Sigma$  with the initial condition  $\phi$  and the disturbance  $w$  such that  $\phi(0) \in K$  and  $w \in W_\infty$ , and there exists an  $i \in \{1, 2, \dots, N_K\}$  satisfying  $p_i^\top x(t_*) = 1$  and  $p_i^\top \dot{x}(t_*) > 0$ . However, combining (3.6) with  $p_i^\top x(t_*) = 1$ ,  $p_j^\top x(t_*) \leq 1$  for each  $j = 1, 2, \dots, N_K$ , and  $|w(t_*)| \leq 1$  allows us to estimate  $p_i^\top \dot{x}(t_*)$  as

$$\begin{aligned} p_i^\top \dot{x}(t_*) &= p_i^\top (Ax(t_*) + Bw(t_*)) = \left( \sum_{j=1}^{N_K} \tau_{ij} p_j^\top - \beta_i p_i^\top \right) x(t_*) + (v_i^+ - v_i^-) w(t_*) \\ &\leq \sum_{j=1}^{N_K} \tau_{ij} - \beta_i + v_i^+ + v_i^- \leq 0, \end{aligned} \quad (3.7)$$

which contradicts the fact that  $p_i^\top \dot{x}(t_*) > 0$ . Thus, the proof is established.  $\square$

Compared with Proposition 2.1, where the terms  $p_i^\top A$  and  $p_i^\top B$  are handled by the relevant upper bounds as in (2.8), condition (3.6) requires them to be represented explicitly through equalities. This difference arises because  $x(t)$  and  $w(t)$  in the present setting are not necessarily nonnegative, and therefore their contributions in suppressing  $p_i^\top \dot{x}(t)$  as in (3.7) cannot be bounded directly by using their sign information.

Note that, for a fixed set of  $p_i \in \mathbb{R}^n$  ( $i = 1, 2, \dots, N_K$ ), the constraints in (3.6) are linear with respect to  $\tau_{ij}$ ,  $v_i^+$ ,  $v_i^-$ , and  $\beta_i$ . Therefore, one may select candidate values of  $p_i$  from a finite set in  $\mathbb{R}^n$  and solve the resulting LP feasibility problem with respect to  $\tau_{ij}$ ,  $v_i^+$ ,  $v_i^-$ , and  $\beta_i$  for each candidate choice of  $p_i$ .

While Proposition 3.1 provides a computationally tractable condition for the robust forward invariance of  $K$ , the inclusion  $X_0 \subset K$  is still required to be verified to apply Lemma 3.1. This inclusion can be verified by directly employing Proposition 2.2, which is also applicable to polyhedrons  $X_0, K \subset \mathbb{R}^n$ . More precisely,  $X_0 \subset K$  is ensured if there exists  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ) such that (2.11) holds. Combining this fact with Proposition 3.1 and Lemma 3.1 yields the following theorem.

**Theorem 3.1.** Let  $\Sigma$  be a system in (3.1) with  $A_d = 0$ . Let  $X_0, K \subset \mathbb{R}^n$  be the sets defined in (3.5), and  $W_\infty$  be as in (3.2). Then, the relation  $\mathcal{R}(X_0, W_\infty) \subset K$  (i.e., (3.4)) holds, if there exist  $\tau_{ij} \geq 0$ ,  $v_i^+ \geq 0$ ,  $v_i^- \geq 0$ ,  $\beta_i \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ), and  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ) such that (2.11) and (3.6) are simultaneously satisfied.

For a fixed set of  $p_i$ , the remaining decision variables in Theorem 3.1, namely,  $\tau_{ij}$ ,  $v_i^+$ ,  $v_i^-$ ,  $\beta_i$ , and  $u_{ij}$ , can be determined simultaneously by solving an LP problem.

### 3.3. The case of general linear systems with a time-varying delay

This subsection extends the arguments in Section 3.2 to general linear systems equipped with time-varying delays. Throughout this subsection,  $d(t)$  is assumed to satisfy (2.16) and (2.24), for some given  $h > 0$  and  $0 < \mu < 1$ .

In a parallel fashion to the case of a positive system with delays in Section 2, we characterize (3.4) in terms of a robustly forward invariant sublevel set of a linear functional. To this end, we define  $\hat{V} : L_\infty^n(h) \rightarrow \mathbb{R}$  by

$$\hat{V}(x_t) := \max_{1 \leq i \leq N_K} \hat{V}_i(x_t), \quad \forall x_t \in L_\infty^n(h), \quad (3.8)$$

where

$$\hat{V}_i(x_t) := p_i^\top x(t) + \int_{t-d(t)}^t e^{\alpha(s-t)} q^\top |x(s)| ds, \quad \forall i = 1, 2, \dots, N_K, \quad (3.9)$$

where  $p_i \in \mathbb{R}^n$ ,  $q \in \mathbb{R}_+^n$ , and  $\alpha \geq 0$  are nonzero vectors, an auxiliary vector, and a scalar, respectively. Here,  $|x| \in \mathbb{R}^n$  denotes the vector obtained by taking the absolute value of each component of  $x \in \mathbb{R}^n$ , i.e.,

$$|x| := \begin{bmatrix} |x_1| & |x_2| & \cdots & |x_n| \end{bmatrix}^\top, \quad \forall x \in \mathbb{R}^n. \quad (3.10)$$

The difference between  $\hat{V}$  in (3.8) and  $V$  in (2.19) is that the absolute value of  $x(s)$ , taken in a componentwise sense, is used in the integrand of (3.9). Since  $q \geq 0$ , this makes the integral in (3.9) nonnegative, and yields

$$\hat{V}(x_t) \leq 1 \quad \Rightarrow \quad p_i^\top x(t) \leq 1, \quad \forall i = 1, 2, \dots, N_K, \quad (3.11)$$

which equivalently implies that

$$x(t) \in K \left( = \bigcap_{i=1}^{N_K} \mathcal{P}(p_i) \right), \quad \forall x_t \in L_\infty^n(h) \text{ s.t. } \hat{V}(x_t) \leq 1. \quad (3.12)$$

This observation leads us to the following lemma, which characterizes (3.4) in Problem 3.1 in terms of robust forward invariance of a set associated with  $\hat{V}$ .

**Lemma 3.2.** *Let  $\Sigma$  be a system in (3.1) equipped with  $d(t)$  satisfying (2.16). Let  $X_0, K \subset \mathbb{R}^n$  be the sets given in (3.5), and  $W_\infty$  be the set defined in (3.2). For  $\hat{V}$  defined in (3.8), we define  $\hat{V}^{-1}((-\infty, 1])$  as*

$$\hat{V}^{-1}((-\infty, 1]) := \{x_t \in L_\infty^n(h) \mid \hat{V}(x_t) \leq 1\}. \quad (3.13)$$

*Then, (3.4) in Problem 3.1 holds, i.e.,  $\mathcal{R}(X_0, W_\infty) \subset K$ , if  $\hat{V}^{-1}((-\infty, 1])$  is robustly forward invariant with respect to  $(\Sigma, W_\infty)$  such that*

$$L_\infty^n[-h, 0] \Big|_{X_0} \subset \hat{V}^{-1}((-\infty, 1]). \quad (3.14)$$

Note that the notion of robust forward invariance in Lemma 3.2, used for  $\hat{V}^{-1}((-\infty, 1])$ , is defined essentially the same as in Definition 2.3 in Section 2, except that the state and disturbances are allowed to be nonpositive. Lemma 3.2 reduces the estimation problem of reachable sets to ensuring the robust forward invariance of  $\hat{V}^{-1}((-\infty, 1])$  together with (3.14). Our next step is to derive a tractable condition for the robust forward invariance of  $\hat{V}^{-1}((-\infty, 1])$ . However, unlike the case of  $V$  defined for positive systems, differentiation of  $\hat{V}$  involves the terms  $|x(t)|$  and  $|x(t - d(t))|$ , which are nonlinear with respect to  $x(t)$  and  $x(t - d(t))$ .

To handle these absolute-value terms, we introduce the set of sign vectors  $\{-1, 1\}^n \subset \mathbb{R}^n$ , defined by

$$\{-1, 1\}^n := \{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbb{R}^n \mid \sigma_i \in \{-1, 1\}, \quad \forall i \in \{1, 2, \dots, n\}\}. \quad (3.15)$$

Then, it follows that for each  $x \in \mathbb{R}^n$ , there exists  $\sigma \in \{-1, 1\}^n$  such that

$$|x| = D_\sigma x, \quad (3.16)$$

where

$$D_\sigma := \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbb{R}^{n \times n}. \quad (3.17)$$

Note that a sign vector  $\sigma \in \{-1, 1\}^n$  allows  $|x|$  to be linearized as  $D_\sigma x$ . This fact allows us to obtain the following proposition, which provides a sufficient condition for a robust forward invariance of  $\hat{V}^{-1}((-\infty, 1])$ .

**Proposition 3.2.** *Let  $\Sigma$  be a system in (3.1) and  $W_\infty$  be the set defined in (3.2). Assume that  $d(t)$  is given to satisfy (2.16) and (2.24). Let  $D_\sigma, D_\rho$  be defined as in (3.17). If there exist  $p_i \in \mathbb{R}^n$ ,  $\gamma_{ij}^{\sigma,\rho} \geq 0$ ,  $\tau_{ij}^{\sigma,\rho} \geq 0$ ,  $\chi_{ij}^{\sigma,\rho} \geq 0$ ,  $\delta_i^{\sigma,\rho} \in \mathbb{R}_+$ ,  $\omega_i^{\sigma,\rho} \in \mathbb{R}_+$ ,  $\nu_{i,+}^{\sigma,\rho} \geq 0$ , and  $\nu_{i,-}^{\sigma,\rho} \geq 0$  for each  $i, j = 1, 2, \dots, N_K$  and  $\sigma, \rho \in \{-1, 1\}^n$ , and  $\alpha > 0$  such that*

$$\begin{cases} p_i^\top A + \alpha p_i^\top + q^\top D_\sigma = \sum_{j=1}^{N_K} \gamma_{ij}^{\sigma,\rho} p_j^\top + \sum_{j=1}^{N_K} \tau_{ij}^{\sigma,\rho} (p_j^\top - p_i^\top) - (\delta_i^{\sigma,\rho})^\top D_\sigma, \\ p_i^\top A_d - e^{-\alpha h} (1 - \mu) q^\top D_\rho = \sum_{j=1}^{N_K} \chi_{ij}^{\sigma,\rho} p_j^\top - (\omega_i^{\sigma,\rho})^\top D_\rho, \\ p_i^\top B = \nu_{i,+}^{\sigma,\rho} - \nu_{i,-}^{\sigma,\rho}, \\ \sum_{j=1}^{N_K} \gamma_{ij}^{\sigma,\rho} + \sum_{j=1}^{N_K} \chi_{ij}^{\sigma,\rho} + \nu_{i,+}^{\sigma,\rho} + \nu_{i,-}^{\sigma,\rho} \leq \alpha, \end{cases} \quad \begin{array}{l} \forall i = 1, 2, \dots, N_K, \\ \forall \sigma, \rho \in \{-1, 1\}^n, \end{array} \quad (3.18)$$

then  $\hat{V}^{-1}((-\infty, 1])$  is robustly forward invariant with respect to  $(\Sigma, W_\infty)$ , where  $\hat{V}$  is generated by  $p_i$  ( $i = 1, 2, \dots, N_K$ ),  $\alpha$ , and  $q$ .

*Proof.* Suppose that  $\hat{V}^{-1}((-\infty, 1])$  is not robustly forward invariant with respect to  $(\Sigma, W_\infty)$ . Then, it follows from essentially similar arguments with Proposition 2.3 that there exist  $t_* \geq 0$ ,  $\phi \in K \cap L_\infty^n[-h, 0]$ ,  $w \in W_\infty$ , and a solution arc  $x$  to  $\Sigma$  with initial condition  $\phi$  and disturbance  $w$  such that there exists  $i \in \{1, 2, \dots, N_K\}$  such that  $\hat{V}_i(x_{t_*}) = 1$  and  $\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} > 0$ .

On the other hand, we can estimate  $\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*}$  by using (2.16) and (2.24) as

$$\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} \leq p_i^\top (Ax(t_*) + A_d x(t_* - d(t_*)) + Bw(t_*)) - \alpha(1 - p_i^\top x(t_*)) + q^\top |x(t_*)|$$

$$-e^{-\alpha h} q^T |x(t_* - d(t_*))|(1 - \mu). \quad (3.19)$$

For the given fixed  $t_*$ , choose  $\sigma, \rho \in \{-1, 1\}^n$  such that

$$|x(t_*)| = D_\sigma x(t_*), \quad |x(t_* - d(t_*))| = D_\rho x(t_* - d(t_*)). \quad (3.20)$$

Then, by putting (3.20) into (3.19), we obtain that

$$\begin{aligned} \frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} &\leq (p_i^\top A + \alpha p_i^\top + q^\top D_\sigma) x(t_*) \\ &\quad + (p_i^\top A_d - e^{-\alpha h} (1 - \mu) q^\top D_\rho) x(t_* - d(t_*)) + p_i^\top B w(t_*) - \alpha, \end{aligned} \quad (3.21)$$

where substitution of (3.18) into (3.21) allows us to continue the right-hand side of (3.21) by

$$\begin{aligned} &\leq \left( \sum_{j=1}^{N_K} \gamma_{ij}^{\sigma, \rho} p_j^\top + \sum_{j=1}^{N_K} \tau_{ij}^{\sigma, \rho} (p_j^\top - p_i^\top) - (\delta_i^{\sigma, \rho})^\top D_\sigma \right) x(t_*) + \left( \sum_{j=1}^{N_K} \chi_{ij}^{\sigma, \rho} p_j^\top - (\omega_i^{\sigma, \rho})^\top D_\rho \right) x(t_* - d(t_*)) \\ &\quad + (v_{i,+}^{\sigma, \rho} - v_{i,-}^{\sigma, \rho}) w(t_*) - \alpha. \end{aligned} \quad (3.22)$$

Then, from the choice of  $i$ , we use

$$p_j^\top x(t_*) \leq 1, \quad p_j^\top x(t_*) \leq p_i^\top x(t_*),$$

and  $p_j^\top x(t_* - d(t_*)) \leq 1$  for each  $j = 1, 2, \dots, N_K$  to continue (3.22) as

$$\begin{aligned} &\leq \left( \sum_{j=1}^{N_K} \gamma_{ij}^{\sigma, \rho} + \sum_{j=1}^{N_K} \chi_{ij}^{\sigma, \rho} + v_{i,+}^{\sigma, \rho} + v_{i,-}^{\sigma, \rho} \right) - (\delta_i^{\sigma, \rho})^\top |x(t_*)| - (\omega_i^{\sigma, \rho})^\top |x(t_* - d(t_*))| - \alpha \\ &\leq \left( \sum_{j=1}^{N_K} \gamma_{ij}^{\sigma, \rho} + \sum_{j=1}^{N_K} \chi_{ij}^{\sigma, \rho} + v_{i,+}^{\sigma, \rho} + v_{i,-}^{\sigma, \rho} \right) - \alpha, \end{aligned} \quad (3.23)$$

where the last term in (3.23) is nonpositive due to the last inequality in (3.18). This contradicts the fact that  $\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} > 0$ , and hence the proof is established.  $\square$

Note that, for fixed  $p_i$  and  $\alpha$ , the conditions in (3.18) are linear with respect to the remaining decision variables, and can therefore be checked by solving an LP feasibility problem. This LP formulation is enabled by extending the framework in Proposition 3.1 to deal with time-varying delays, while removing the nonlinearity arising from the absolute-value terms through the finite enumeration of the sign variables  $\sigma, \rho \in \{-1, 1\}^n$ . This treatment constitutes the main technical contribution of Proposition 3.2. However, this enumeration increases the computational cost, since the conditions in (3.18) must be imposed for  $N_K 4^n$  combinations of  $i \in \{1, 2, \dots, N_K\}$ ,  $\sigma \in \{1, 2, \dots, 2^n\}$ , and  $\rho \in \{1, 2, \dots, 2^n\}$ , and thus the number of associated constraints (and equations) grows exponentially with respect to the state dimension  $n$ .

To alleviate the computational burden associated with the sign enumeration in Proposition 3.2, we next derive an alternative sufficient condition that does not require enumeration over  $\sigma$  and  $\rho$ . For this purpose, we assume that there exists a known scalar  $M > 0$  such that

$$\begin{cases} \phi \in L_\infty^n[-h, 0] \Big|_{X_0}, \\ w \in W_\infty \end{cases} \quad \Rightarrow \quad |x(t)|_\infty \leq M, \quad \forall t \in [0, \infty). \quad (3.24)$$

Note that (3.24) imposes a uniform bound on the state trajectories generated by the admissible initial conditions and disturbances. Based on this assumption, we obtain the following proposition, which is an alternative result of Proposition 3.2.

**Proposition 3.3.** *Let  $\Sigma$  be a system in (3.1) and  $W_\infty$  be the set defined in (3.2). Assume that  $d(t)$  is given to satisfy (2.16) and (2.24). Let  $M > 0$  be a constant satisfying (3.24) for  $\Sigma$ . If there exist  $p_i \in \mathbb{R}^n$ ,  $\gamma_{ij} \geq 0$ ,  $\tau_{ij} \geq 0$ ,  $r_i \in \mathbb{R}_+^n$ ,  $s_i \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ),  $q \in \mathbb{R}_+^n$ , and  $\alpha > 0$  such that*

$$\begin{cases} p_i^\top A + \alpha p_i^\top = \sum_{j=1}^{N_K} \gamma_{ij} p_j^\top + \sum_{j=1}^{N_K} \tau_{ij} (p_j^\top - p_i^\top), \\ -r_i^\top \leq p_i^\top A_d \leq r_i^\top, \\ -s_i \leq p_i^\top B \leq s_i, \\ \sum_{j=1}^{N_K} \gamma_{ij} + M r_i^\top \mathbb{1}_n + s_i + M q^\top \mathbb{1}_n \leq \alpha, \end{cases} \quad \forall i = 1, 2, \dots, N_K, \quad (3.25)$$

then  $\hat{V}^{-1}((-\infty, 1])$  is robustly forward invariant with respect to  $(\Sigma, W_\infty)$ , where  $\hat{V}$  is generated by using  $p_i$  ( $i = 1, 2, \dots, N_K$ ),  $\alpha$ , and  $q$ .

*Proof.* Suppose that  $\hat{V}^{-1}((-\infty, 1])$  is not robustly forward invariant with respect to  $(\Sigma, W_\infty)$ . Then, similarly to Proposition 3.2, there exist  $t_* \geq 0$ ,  $\phi \in K \cap L_\infty^n[-h, 0]$ ,  $w \in W_\infty$ , and a solution arc  $x$  to  $\Sigma$  with initial condition  $\phi$  and disturbance  $w$  such that there exists  $i \in \{1, 2, \dots, N_K\}$  such that  $\hat{V}_i(x_{t_*}) = 1$  and  $\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} > 0$ .

Then, by discarding the last term depending on  $|x(t_*) - d(t_*)|$  in (3.19), we obtain that

$$\begin{aligned} \frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} &\leq p_i^\top (Ax(t_*) + A_d x(t_* - d(t_*)) + Bw(t_*)) - \alpha(1 - p_i^\top x(t_*)) + q^\top |x(t_*)| \\ &\leq (p_i^\top A + \alpha p_i^\top)x(t_*) + p_i^\top A_d x(t_* - d(t_*)) + p_i^\top Bw(t_*) + q^\top |x(t_*)| - \alpha. \end{aligned} \quad (3.26)$$

Substituting (3.25) and (3.24) into the right-hand side of (3.26) yields

$$\begin{aligned} &\leq \left( \sum_{j=1}^{N_K} \gamma_{ij} p_j^\top + \sum_{j=1}^{N_K} \tau_{ij} (p_j^\top - p_i^\top) \right) x(t_*) + r_i^\top |x(t_*)|_\infty + s_i |w(t_*)| + q^\top |x(t_*)|_\infty - \alpha \\ &\leq \left( \sum_{j=1}^{N_K} \gamma_{ij} p_j^\top + \sum_{j=1}^{N_K} \tau_{ij} (p_j^\top - p_i^\top) \right) x(t_*) + M r_i^\top \mathbb{1}_n + s_i + M q^\top \mathbb{1}_n - \alpha. \end{aligned} \quad (3.27)$$

Combining (3.26) and (3.27) together with  $p_j^\top x(t_*) \leq 1$  and  $p_j^\top x(t_*) \leq p_i^\top x(t_*)$  for each  $j = 1, 2, \dots, N_K$  allows us to obtain

$$\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} \leq \left( \sum_{j=1}^{N_K} \gamma_{ij} + M r_i^\top \mathbb{1}_n + s_i + M q^\top \mathbb{1}_n \right) - \alpha, \quad (3.28)$$

where the last inequality in (3.25) shows that the right-hand side of (3.28) is nonpositive. This contradicts to the fact that  $\frac{d}{dt} \hat{V}_i(x_t) \Big|_{t=t_*} > 0$ , showing that the first assertion is a contradiction. The proof is established.  $\square$

For fixed  $M$  and  $p_i$ , the conditions in (3.25) are linear with respect to  $\gamma_{ij}$ ,  $\tau_{ij}$ ,  $r_i$ ,  $s_i$ ,  $q$ , and  $\alpha$ , and can therefore be checked via LP. Compared with Proposition 3.2, Proposition 3.3 involves fewer constraints and decision variables because no sign enumeration is required. This reduction in computational burden is obtained by employing an additional conservatism from discarding the negative delayed-state term in (3.19).

Since Propositions 3.2 and 3.3 provide sufficient conditions for the robust forward invariance of  $\hat{V}^{-1}((-\infty, 1])$ , the remaining condition in Lemma 3.2 is the inclusion in (3.14). The following proposition provides a sufficient condition for ensuring (3.14).

**Proposition 3.4.** *Let  $X_0$  be the set defined in (3.5) and  $c_i$  ( $i = 1, 2, \dots, N_{X_0}$ ) be the vectors used to define  $X_0$  in (3.5). Let  $h > 0$  be given to satisfy (2.16). Let  $D_\sigma$  be defined as (3.17). Then, if there exist  $p_i \in \mathbb{R}^n$  ( $i = 1, 2, \dots, N_K$ ),  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ),  $q \in \mathbb{R}_+^n$ ,  $\zeta \geq 0$ ,  $r_i^\sigma \geq 0$  ( $i = 1, 2, \dots, N_{X_0}$ ) for each  $\sigma \in \{-1, 1\}^n$  such that*

$$\begin{aligned} \sum_{j=1}^{N_{X_0}} u_{ij} c_j^\top &= p_i^\top, \quad \sum_{j=1}^{N_{X_0}} r_j^\sigma c_j^\top = q^\top D_\sigma, \\ \sum_{j=1}^{N_{X_0}} r_j^\sigma &\leq \zeta, \quad \sum_{j=1}^{N_{X_0}} u_{ij} + \left(\frac{1 - e^{-\alpha h}}{\alpha}\right) \zeta \leq 1, \quad \forall i = 1, 2, \dots, N_K, \\ &\quad \forall \sigma \in \{-1, 1\}^n, \end{aligned} \quad (3.29)$$

then relation (3.14) holds for  $X_0$  and  $\hat{V}$ , where  $\hat{V}$  is determined by  $p_i$  ( $i = 1, 2, \dots, N_K$ ),  $\alpha$ , and  $q$ .

*Proof.* To show that (3.14) holds, take an arbitrary  $\phi \in L_\infty^n[-h, 0]|_{X_0}$ . Then, it follows from the first term in (3.29) together with similar arguments to the proof of Proposition 2.4 that

$$\begin{cases} c_j^\top \phi(s) \leq 1, & \forall j = 1, 2, \dots, N_{X_0}, \quad \forall s \in [-h, 0], \\ p_i^\top \phi(0) = \sum_{j=1}^{N_{X_0}} u_{ij} c_j^\top \phi(0) \leq \sum_{j=1}^{N_{X_0}} u_{ij}, & \forall i = 1, 2, \dots, N_K. \end{cases} \quad (3.30)$$

By using the fact that  $|\phi(s)| = \max_{\sigma \in \{-1, 1\}^n} D_\sigma \phi(s)$ , it follows from (3.29) that

$$q^\top |\phi(s)| = \max_{\sigma \in \{-1, 1\}^n} (q^\top D_\sigma \phi(s)) = \max_{\sigma \in \{-1, 1\}^n} \left( \sum_{j=1}^{N_{X_0}} r_j^\sigma c_j^\top \phi(s) \right) \leq \max_{\sigma \in \{-1, 1\}^n} \sum_{j=1}^{N_{X_0}} r_j^\sigma \leq \zeta, \quad \forall s \in [-h, 0]. \quad (3.31)$$

Then, combining (3.30) and (3.31) with the last inequality in (3.29) leads us to obtain

$$\begin{aligned} \hat{V}(\phi) &= \max_{1 \leq i \leq N_K} \hat{V}_i(\phi) \leq \max_{1 \leq i \leq N_K} \left\{ p_i^\top \phi(0) + \int_{-h}^0 e^{\alpha s} q^\top |\phi(s)| ds \right\} \\ &\leq \max_{1 \leq i \leq N_K} \left\{ \sum_{j=1}^{N_{X_0}} u_{ij} + \zeta \int_{-h}^0 e^{\alpha s} ds \right\} \leq \max_{1 \leq i \leq N_K} \left\{ \sum_{j=1}^{N_{X_0}} u_{ij} + \left(\frac{1 - e^{-\alpha h}}{\alpha}\right) \zeta \right\} \leq 1. \end{aligned} \quad (3.32)$$

Since this means that  $\phi \in \hat{V}^{-1}((-\infty, 1])$ , the proof is complete.  $\square$

Note that, for fixed  $p_i$  and  $\alpha$ , the conditions in (3.29) are linear with respect to  $u_{ij}$ ,  $q$ ,  $\zeta$ , and  $r_j^\sigma$ , and their feasibility can therefore be checked via LP. By combining Propositions 3.2–3.4 with Lemma 3.2, we obtain the following theorem, which provides sufficient conditions for ensuring (3.4) for general linear systems with time-varying delays.

**Theorem 3.2.** *Let  $\Sigma$  be a system in (3.1) and  $d(t)$  be given to satisfy (2.16) and (2.24). Let  $X_0, K \subset \mathbb{R}^n$  be the sets given in (3.5), and  $W_\infty$  be the set in (3.2). Then, the relation  $\mathcal{R}(X_0, W_\infty) \subset K$  (i.e., (3.4)) holds if either one of the following statements holds.*

- *There exist  $p_i \in \mathbb{R}^n$ ,  $\gamma_{ij}^{\sigma,\rho} \geq 0$ ,  $\tau_{ij}^{\sigma,\rho} \geq 0$ ,  $\chi_{ij}^{\sigma,\rho} \geq 0$ ,  $\delta_i^{\sigma,\rho} \in \mathbb{R}_+^n$ ,  $\omega_i^{\sigma,\rho} \in \mathbb{R}_+^n$ ,  $v_{i,+}^{\sigma,\rho} \geq 0$ , and  $v_{i,-}^{\sigma,\rho} \geq 0$ , for each  $i, j = 1, 2, \dots, N_K$  and  $\sigma, \rho \in \{-1, 1\}^n$ , and  $\alpha > 0$ , and  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ),  $\zeta \geq 0$ , and  $r_i^\sigma \geq 0$ , for each  $\sigma \in \{-1, 1\}^n$ , such that (3.18) and (3.29) are simultaneously satisfied.*
- *Assume that there exists  $M > 0$  such that (3.24) holds for  $\Sigma$ , which provides a uniform bound on the state trajectories generated by all initial conditions in  $X_0$  and disturbances in  $W_\infty$ . There exist  $p_i \in \mathbb{R}^n$ ,  $\gamma_{ij} \geq 0$ ,  $\tau_{ij} \geq 0$ ,  $r_i \in \mathbb{R}_+^n$ ,  $s_i \geq 0$  ( $i, j = 1, 2, \dots, N_K$ ),  $q \in \mathbb{R}_+^n$ ,  $\alpha > 0$ ,  $u_{ij} \geq 0$  ( $i = 1, 2, \dots, N_K$ ,  $j = 1, 2, \dots, N_{X_0}$ ),  $\zeta \geq 0$ , and  $r_i^\sigma \geq 0$ , for each  $\sigma \in \{-1, 1\}^n$ , such that (3.25) and (3.29) are simultaneously satisfied.*

Note that the first statement in Theorem 3.2 is obtained by combining Propositions 3.2 and 3.4 with Lemma 3.2, whereas the second statement follows by replacing Proposition 3.2 with Proposition 3.3 in the derivation of the first statement. Since Proposition 3.3 is established under the additional assumption that a uniform state bound  $M$  satisfying (3.24) is available, this assumption is inherited by the second statement. Hence, the second statement provides an alternative sufficient condition to the first statement. Moreover, it reduces the computational burden relative to the first statement, since the number of sign-enumeration indices associated with  $\sigma$  and  $\rho$  is significantly reduced by employing Proposition 3.3 instead of Proposition 3.2.

For the first statement, a grid search over  $p_i$  and  $\alpha$  can be combined with an LP feasibility test for the remaining decision variables in (3.18) and (3.29). For the second statement, once  $M$  is given, a grid search over  $p_i$  can be combined with an LP feasibility test for the remaining decision variables in (3.25) and (3.29). Thus, both sufficient conditions in Theorem 3.2 can be implemented by using finite parameter searches together with standard LP solvers.

## 4. Numerical examples

This section provides numerical examples that verify feasibility and effectiveness of the proposed results for bounding reachable sets.

### 4.1. Examples of positive systems

Consider the system  $\Sigma_{\geq 0}$  in (2.1) with

$$A = \begin{bmatrix} -3 & 0 \\ 0.1 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (4.1)$$

To apply Theorem 2.1, we consider three different initial sets, i.e.,  $\mathcal{X}_0 \in \{\mathcal{X}_0^{(a)}, \mathcal{X}_0^{(b)}, \mathcal{X}_0^{(c)}\}$ , where

$$\mathcal{X}_0^{(a)} = \{0\}, \quad \mathcal{X}_0^{(b)} = \left( \bigcap_{i=1}^5 \mathcal{P}(c_i^{(b)}) \right) \cap \mathbb{R}_+^2, \quad \mathcal{X}_0^{(c)} = \left( \bigcap_{i=1}^5 \mathcal{P}(c_i^{(c)}) \right) \cap \mathbb{R}_+^2, \quad (4.2)$$

with  $c_i^{(l)} \in \mathbb{R}^2$  ( $i = 1, 2, \dots, 5, l \in \{b, c\}$ ) given in matrix form by

$$C^{(b)} = \begin{bmatrix} (c_1^{(b)})^\top \\ (c_2^{(b)})^\top \\ (c_3^{(b)})^\top \\ (c_4^{(b)})^\top \\ (c_5^{(b)})^\top \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 10 & 0 \\ 10 & -10 \\ 0 & 20 \\ -20 & 20 \end{bmatrix}, \quad C^{(c)} = \begin{bmatrix} (c_1^{(c)})^\top \\ (c_2^{(c)})^\top \\ (c_3^{(c)})^\top \\ (c_4^{(c)})^\top \\ (c_5^{(c)})^\top \end{bmatrix} = \begin{bmatrix} 20 & 7.692 \\ 20 & 0 \\ 20 & -20 \\ 0 & 7.692 \\ -7.692 & 7.692 \end{bmatrix}. \quad (4.3)$$

Then, by applying Theorem 2.1 for each  $l \in \{a, b, c\}$ , we find bounding polyhedrons  $K^{(l)}$  such that  $\mathcal{R}^{\geq 0}(\mathcal{X}_0^{(l)}, W_{\infty,+}) \subset K^{(l)}$  is satisfied. More precisely, for each  $l \in \{a, b, c\}$ , the set of vectors  $p_i^{(l)}$  ( $i = 1, 2, 3, 4$ ), which parameterize  $K^{(l)}$  as  $K^{(l)} = \left( \bigcap_{i=1}^4 \mathcal{P}(p_i^{(l)}) \right) \cap \mathbb{R}_+^2$ , is found such that (2.8) and (2.11) are feasible, and the resulting vectors (where the remaining parameters used for verifying the feasibility are provided in [30]) are summarized in matrix form as

$$P^{(a)} = \begin{bmatrix} (p_1^{(a)})^\top \\ (p_2^{(a)})^\top \\ (p_3^{(a)})^\top \\ (p_4^{(a)})^\top \end{bmatrix} = \begin{bmatrix} 2.994 & 0 \\ 50.00 & -50.00 \\ 0 & 2.414 \\ -12.47 & 12.47 \end{bmatrix}, \quad P^{(b)} = \begin{bmatrix} (p_1^{(b)})^\top \\ (p_2^{(b)})^\top \\ (p_3^{(b)})^\top \\ (p_4^{(b)})^\top \end{bmatrix} = \begin{bmatrix} 3.000 & 0 \\ 10.00 & -10.00 \\ 0 & 0.4927 \\ -6.217 & 6.217 \end{bmatrix},$$

$$P^{(c)} = \begin{bmatrix} (p_1^{(c)})^\top \\ (p_2^{(c)})^\top \\ (p_3^{(c)})^\top \\ (p_4^{(c)})^\top \end{bmatrix} = \begin{bmatrix} 3.000 & 0 \\ 20.00 & -20.00 \\ 0 & 0.9854 \\ -6.721 & 6.721 \end{bmatrix}. \quad (4.4)$$

To demonstrate the validity of the sets  $K^{(l)}$  obtained in (4.4), the sets  $K^{(l)}$  and state trajectories initialized from  $\mathcal{X}_0^{(l)}$  are plotted in Figure 3, for each  $l \in \{a, b, c\}$ . As shown in Figure 3, all such state trajectories are contained in  $K^{(l)}$  for each  $l \in \{a, b, c\}$ , numerically confirming that  $\mathcal{R}^{\geq 0}(\mathcal{X}_0^{(l)}, W_{\infty,+}) \subset K^{(l)}$  holds. In particular, since  $\mathcal{X}_0^{(b)}$  and  $\mathcal{X}_0^{(c)}$  are sets other than  $\{0\}$ , Theorem 2.1 is regarded to extend the framework of [28], in which bounding polyhedrons are computed only for zero initial conditions.

To further demonstrate the necessity of developing Theorem 2.1 beyond [28], we compare the resulting polyhedrons in Figure 3. When nonzero initial sets are considered (i.e.,  $\mathcal{X}_0^{(b)}$  and  $\mathcal{X}_0^{(c)}$ ), the corresponding bounding polyhedrons  $K^{(b)}$  and  $K^{(c)}$  become larger than  $K^{(a)}$ , which is equipped with  $\mathcal{X}_0^{(a)} = \{0\}$ . This enlargement is necessary because there are some trajectories initialized from the nonzero initial sets lying outside  $K^{(a)}$ . Theorem 2.1 explicitly accounts for such effects from initial sets and determines bounding polyhedrons that contain the corresponding trajectories, thereby generalizing the result of [28].

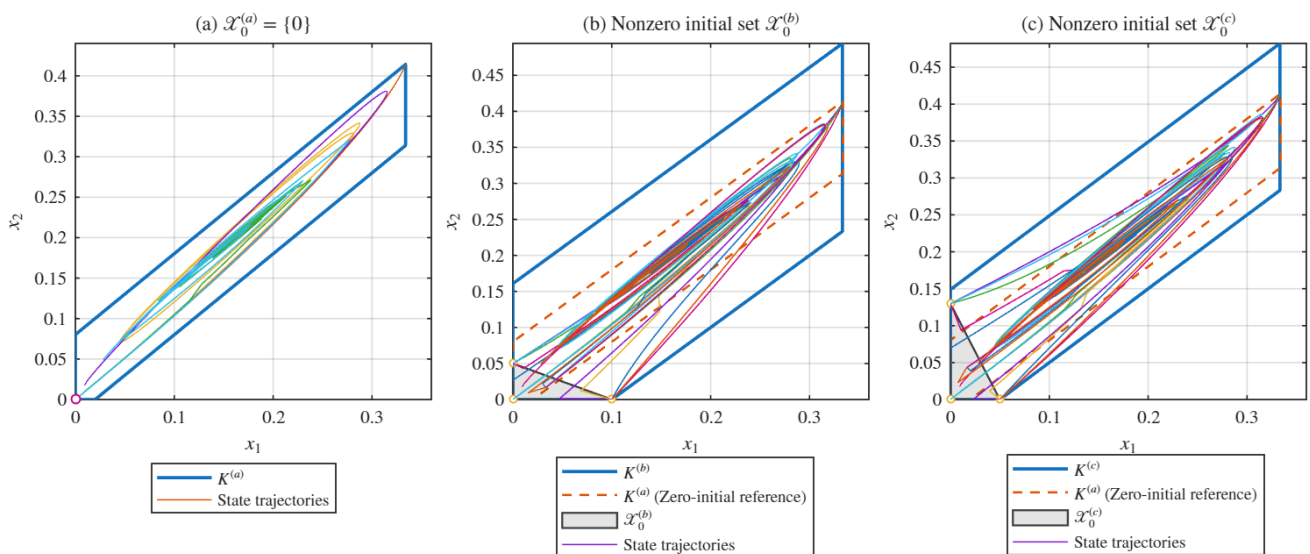


**Figure 3.** Polyhedral sets obtained from Theorem 2.1 bounding reachable sets for three different initial sets  $\mathcal{X}_0^l$  ( $l \in \{a, b, c\}$ ).

We next provide a numerical example to verify the effectiveness of Theorem 2.2, with particular emphasis on the dependence of the resulting bounding polyhedral sets on the delay parameters. Consider the system  $\Sigma_{\geq 0}$  in (2.1) with

$$A = \begin{bmatrix} -3 & 0 \\ 0.1 & -2.5 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (4.5)$$

In contrast to the preceding example for Theorem 2.1, where different initial sets are considered, we fix a nonzero initial set and investigate how the bounding polyhedron varies with the delay parameters  $h$  and  $\mu$ . More precisely, let  $\mathcal{X}_0 = \left(\bigcap_{i=1}^6 \mathcal{P}(c_i)\right) \cap \mathbb{R}_+^2$  be given to be fixed, where  $\hat{c}_i \in \mathbb{R}^2$  ( $i = 1, 2, \dots, 6$ ) are given in matrix form by

$$\hat{C} = \begin{bmatrix} \hat{c}_1^\top \\ \hat{c}_2^\top \\ \hat{c}_3^\top \\ \hat{c}_4^\top \\ \hat{c}_5^\top \\ \hat{c}_6^\top \end{bmatrix} = \begin{bmatrix} 2.2406 & 3.6664 \\ 0.9525 & 13.9706 \\ 2.2406 & 0 \\ 2.2406 & -2.2406 \\ 0 & 13.9706 \\ -13.9706 & 13.9706 \end{bmatrix}. \quad (4.6)$$

For  $\mathcal{X}_0$  given by (4.6), we consider the following four cases of the delay parameters  $(h, \mu)$ :

$$(h, \mu) = (0.1, 0), (0.1, 0.1), (0.5, 0.1), (0.5, 0.5). \quad (4.7)$$

These cases are denoted by (a), (b), (c), and (d), respectively. For each case, the time-varying delay  $d(t)$  is allowed to be any function satisfying (2.16) and (2.24), i.e.,  $0 < d(t) \leq h$  and  $|\dot{d}(t)| \leq \mu$ , with the corresponding pair  $(h, \mu)$  given in (4.7). Then, by solving the constraints in (2.25) and (2.29) in Theorem 2.2 for each case (a)–(d), the values for  $\hat{p}_i^{(l)}$  ( $i = 1, 2, 3, 4$ ,  $l \in \{a, b, c, d\}$ ), ensuring feasibility

of (2.25) and (2.29) in Theorem 2.2, are obtained in matrix form by

$$\begin{aligned}\hat{P}^{(a)} &= \begin{bmatrix} (\hat{p}_1^{(a)})^\top \\ (\hat{p}_2^{(a)})^\top \\ (\hat{p}_3^{(a)})^\top \\ (\hat{p}_4^{(a)})^\top \end{bmatrix} = \begin{bmatrix} 1.7728 & 0 \\ 1.7728 & -1.7728 \\ 0 & 1.9000 \\ -3.3285 & 3.3285 \end{bmatrix}, & \hat{P}^{(b)} &= \begin{bmatrix} (\hat{p}_1^{(b)})^\top \\ (\hat{p}_2^{(b)})^\top \\ (\hat{p}_3^{(b)})^\top \\ (\hat{p}_4^{(b)})^\top \end{bmatrix} = \begin{bmatrix} 1.7150 & 0 \\ 1.7150 & -1.7150 \\ 0 & 1.7150 \\ -3.1282 & 3.1282 \end{bmatrix}, \\ \hat{P}^{(c)} &= \begin{bmatrix} (\hat{p}_1^{(c)})^\top \\ (\hat{p}_2^{(c)})^\top \\ (\hat{p}_3^{(c)})^\top \\ (\hat{p}_4^{(c)})^\top \end{bmatrix} = \begin{bmatrix} 1.6625 & 0 \\ 1.6625 & -1.6625 \\ 0 & 1.6625 \\ -3.1282 & 3.1282 \end{bmatrix}, & \hat{P}^{(d)} &= \begin{bmatrix} (\hat{p}_1^{(d)})^\top \\ (\hat{p}_2^{(d)})^\top \\ (\hat{p}_3^{(d)})^\top \\ (\hat{p}_4^{(d)})^\top \end{bmatrix} = \begin{bmatrix} 1.4250 & 0 \\ 1.4250 & -1.4250 \\ 0 & 1.4250 \\ -2.7212 & 2.7212 \end{bmatrix},\end{aligned}\quad (4.8)$$

where the corresponding auxiliary vectors  $q = q^{(l)} \in \mathbb{R}_+^2$  ( $l \in \{a, b, c, d\}$ ) are determined by

$$q^{(a)} = \begin{bmatrix} 0.2165 \\ 0 \end{bmatrix}, \quad q^{(b)} = \begin{bmatrix} 0.3745 \\ 0.2691 \end{bmatrix}, \quad q^{(c)} = \begin{bmatrix} 0.4985 \\ 0.1042 \end{bmatrix}, \quad q^{(d)} = \begin{bmatrix} 0.7542 \\ 0.2880 \end{bmatrix}. \quad (4.9)$$

The remaining feasible parameters, including  $\tau_{ij}$ ,  $u_{ij}$ , and  $w_{ij}$ , are provided in [30]. It follows from Theorem 2.2 and the above feasibility results that the reachable set for each case  $l \in \{a, b, c, d\}$  is bounded by the polyhedral set  $\widehat{K}^{(l)}$  defined by

$$\widehat{K}^{(l)} = \left( \bigcap_{i=1}^4 \mathcal{P}(\hat{p}_i^{(l)}) \right) \cap \mathbb{R}_+^2, \quad l \in \{a, b, c, d\}, \quad (4.10)$$

where  $\hat{p}_i^{(l)}$  ( $i = 1, 2, 3, 4$ ,  $l \in \{a, b, c, d\}$ ) are given as in (4.8).

The resulting polyhedral sets are compared in Figure 4. The areas of the bounding polyhedra for cases (a)–(d) are computed by 0.2714, 0.3053, 0.3221, and 0.4366, respectively. It is observed that the bounding polyhedron enlarges as the admissible delay bounds become larger. In particular, increasing  $\mu$  from 0 to 0.1 while fixing  $h = 0.1$  enlarges the area of the bound from 0.2714 to 0.3053, and increasing  $h$  from 0.1 to 0.5 with  $\mu = 0.1$  further increases the area to 0.3221. These results demonstrate the delay-dependent nature of Theorem 2.2, since the feasible polyhedral bound varies according to  $h$  and  $\mu$ , even though the system matrices and the initial set  $X_0$  are kept unchanged.

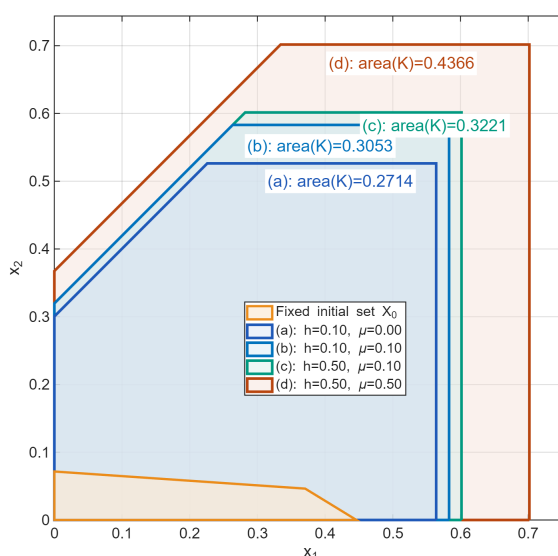


**Figure 4.** Polyhedral sets obtained from Theorem 2.2 bounding reachable sets for four different delay parameters  $(h, \mu)$  given in (4.7).

From a practical point of view, the dynamics in (4.1) and (4.5) can be interpreted as describing a compartmental system [24], which generally models how the amounts or concentrations of substances in interconnected compartments evolve over time. For example, suppose that the states  $x_1(t)$  and  $x_2(t)$  in (4.1) represent the concentrations of a chemical species in two different vessels, referred to as vessels 1 and 2, respectively, and that  $w(t)$  represents a nonnegative external feed supplied to the vessels. Then, the terms  $-3x_1(t)$  and  $-2.5x_2(t)$  can represent first-order removal effects, such as degradation or outflow, in vessels 1 and 2, respectively, whereas the term  $0.1x_1(t)$  represents the transport of the chemical species from vessel 1 to vessel 2. In addition, the delayed terms in (4.5) can represent recirculation effects, which implies a portion of the fluid leaving each vessel may be recirculated back into the vessel through a recycle loop after a time-varying transport delay.

In this specific scenario, the reachable set estimation results in Theorems 2.1 and 2.2 can have practical value in evaluating worst-case operating conditions of the chemical process. For example, suppose that an aggregate hazard measure associated with the concentration of the chemical species over the two vessels is quantified by  $c^T x = c_1 x_1 + c_2 x_2$ , where  $c_i > 0$  ( $i = 1, 2$ ) represents the hazard weight assigned to the concentration in vessel  $i$ . Then, the worst-case hazard level over all admissible initial conditions and external feeds can be characterized by  $\sup_{x \in \mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+})} c^T x$ . Since the exact shape of  $\mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+})$  is generally difficult to compute, the bounding polyhedron  $K (= K^{(l)}, l = a, b, c)$  obtained from the proposed results can instead be used to provide its upper bound

$$\sup_{x \in \mathcal{R}^{\geq 0}(\mathcal{X}_0, W_{\infty,+})} c^T x \leq \sup_{z \in K} c^T z. \quad (4.11)$$

Since  $K$  is a polyhedral set, the right-hand side of (4.11) can be readily computed by solving an LP for any prescribed  $c \in \mathbb{R}_+^2$ . For example, if  $c = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$  is chosen, the right-hand side of (4.11) for the dynamics in (4.1) is computed as 1.1624, 1.3217, and 1.2976, respectively, for each case of  $K = K^{(a)}$ ,  $K^{(b)}$ , and  $K^{(c)}$ , implying that the weighted hazard level is certified not to exceed this value for any

admissible initial condition and external feed.

#### 4.2. Examples of general linear systems

This subsection aims at providing examples to verify the feasibility and effectiveness of the results derived in Section 3, obtained for general linear systems. To this end, consider the system  $\Sigma$  in (3.1) with

$$A = \begin{bmatrix} -1.715 & 0.662 \\ 0.742 & -3.785 \end{bmatrix}, \quad B = \begin{bmatrix} 0.951 \\ 0.309 \end{bmatrix}. \quad (4.12)$$

By taking  $X_0 \subset \mathbb{R}^2$  as  $X_0 = \bigcap_{i=1}^4 \mathcal{P}(c_i)$  with  $c_i \in \mathbb{R}^2$  ( $i = 1, 2, 3, 4$ ) defined by

$$C = \begin{bmatrix} c_1^\top \\ c_2^\top \\ c_3^\top \\ c_4^\top \end{bmatrix} = \begin{bmatrix} -1.268 & -0.4120 \\ 1.717 & -5.284 \\ 1.268 & 0.4120 \\ -1.717 & 5.284 \end{bmatrix}, \quad (4.13)$$

we obtain that the constraints (2.11) and (3.6) in Theorem 3.1 are feasible with  $p_i \in \mathbb{R}^2$  ( $i = 1, 2, 3, 4$ ) taken in matrix form by

$$P = \begin{bmatrix} p_1^\top \\ p_2^\top \\ p_3^\top \\ p_4^\top \end{bmatrix} = \begin{bmatrix} -1.268 & -0.4120 \\ 1.717 & -5.284 \\ 1.268 & 0.4120 \\ -1.717 & 5.284 \end{bmatrix}, \quad (4.14)$$

where the remaining parameters in (2.11) and (3.6) (e.g.,  $\tau_{ij}$ ,  $\beta_i$ ,  $v_i^+$ ,  $v_i^-$ , and  $u_{ij}$ ) used for the feasibility are summarized in [30]. Then, it follows from Theorem 3.1 that the reachable set  $\mathcal{R}(X_0, W_\infty)$  is bounded by the polyhedron  $K$  defined by  $K = \bigcap_{i=1}^4 \mathcal{P}(p_i)$  with  $p_i$  ( $i = 1, 2, 3, 4$ ) in (4.14).

We compare this result with the existing method for bounding reachable sets by an ellipsoidal set [3, 22]. More precisely, if the set of initial conditions  $X_0$  is given as in (4.13), it follows from immediate modification of [3, 22] that an optimal ellipsoid  $K_e := \{x \in \mathbb{R}^2 \mid x^\top Q_e x \leq 1\}$ , which is the smallest possible robust forward invariant ellipsoidal set containing  $X_0$ , is computed by solving

$$\begin{aligned} Q_e = \operatorname{argmax}_{Q \succeq 0} (\det Q), \\ \text{s.t. } \begin{cases} \begin{bmatrix} A^\top Q + QA + \lambda Q & QB \\ B^\top Q & -\lambda \end{bmatrix} \leq 0 \\ v_j^\top Q v_j \leq 1, \quad \forall v_j \in \operatorname{vert}(X_0) \end{cases}, \end{aligned} \quad (4.15)$$

where the symbol  $\succeq$  (and  $\leq$ ) in (4.15) exceptionally represents semidefiniteness of square matrices. As a result, solving (4.15) leads us to the resulting  $Q_e$ , determined by

$$Q_e = \begin{bmatrix} 2.2776 & -4.2741 \\ -4.2741 & 14.0433 \end{bmatrix}. \quad (4.16)$$

For comparison, the polyhedral set  $K$  and the ellipsoidal set  $K_e$ , determined by (4.14) and (4.16), respectively, are plotted in Figure 5. As shown in Figure 5, both  $K$  and  $K_e$  contain the reachable set from  $X_0$ , while  $K$  has a smaller volume than  $K_e$ . In particular,  $K$  closely follows the shape of  $X_0$ ,

whereas  $K_e$  contains a relatively larger region outside  $\mathcal{X}_0$ . This illustrates that a polyhedral bounding set can provide a tighter approximation in general, when the initial set is itself given in a polyhedral form.

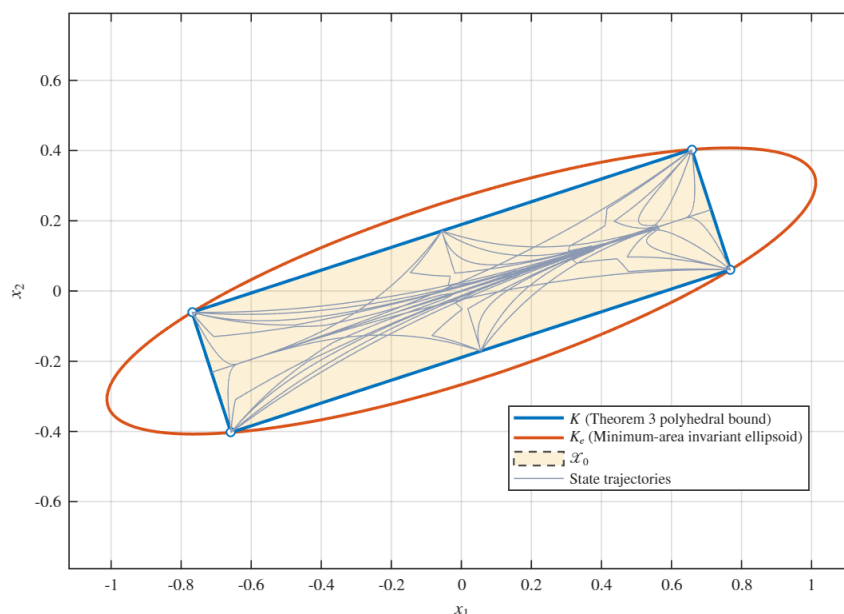


**Figure 5.** Polyhedral and ellipsoidal sets bounding the reachable set for the general linear system without delay given in (4.12).

The next example considers a nonpositive system equipped with a time-varying delay, given by

$$A = \begin{bmatrix} -1.60 & -0.45 \\ -0.35 & -2.20 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.35 & 0.06 \\ 0.05 & 0.28 \end{bmatrix}, \quad B = \begin{bmatrix} 0.80 \\ -0.35 \end{bmatrix}. \quad (4.17)$$

For this system, we take the initial set  $\hat{\mathcal{X}}_0 = \bigcap_{i=1}^4 \mathcal{P}(\hat{c}_i)$ , where  $\hat{c}_i \in \mathbb{R}^2$  ( $i = 1, 2, 3, 4$ ) are given in matrix form by

$$\hat{C} = \begin{bmatrix} \hat{c}_1^\top \\ \hat{c}_2^\top \\ \hat{c}_3^\top \\ \hat{c}_4^\top \end{bmatrix} = \begin{bmatrix} 1.0153 & -0.3695 \\ 2.8319 & 7.7805 \\ -1.0153 & 0.3695 \\ -2.8319 & -7.7805 \end{bmatrix}. \quad (4.18)$$

To illustrate the dependence of the bounding polyhedron on the delay parameters, we consider the four cases of  $h$  and  $\mu$  given in (4.7), i.e.,  $(h, \mu) \in \{(0.1, 0), (0.1, 0.1), (0.5, 0.1), (0.5, 0.5)\}$ , and apply the first statement in Theorem 3.2 to each case. For each pair  $(h, \mu)$ , the admissible time-varying delays  $d(t)$  are allowed to be any function satisfying (2.16) and (2.24). By denoting these four cases by (a), (b), (c), and (d), respectively, it follows that (3.18) and (3.29) in Theorem 3.2 are feasible with  $\hat{p}_i^l$  ( $i = 1, 2, 3, 4$ ,  $l \in \{a, b, c, d\}$ ) given in matrix form by

$$\hat{P}^{(a)} = \begin{bmatrix} (\hat{p}_1^{(a)})^\top \\ (\hat{p}_2^{(a)})^\top \\ (\hat{p}_3^{(a)})^\top \\ (\hat{p}_4^{(a)})^\top \end{bmatrix} = \begin{bmatrix} 0.9725 & -0.3540 \\ 2.7110 & 7.4485 \\ -0.9725 & 0.3540 \\ -2.7110 & -7.4485 \end{bmatrix}, \quad \hat{P}^{(b)} = \begin{bmatrix} (\hat{p}_1^{(b)})^\top \\ (\hat{p}_2^{(b)})^\top \\ (\hat{p}_3^{(b)})^\top \\ (\hat{p}_4^{(b)})^\top \end{bmatrix} = \begin{bmatrix} 0.9617 & -0.3500 \\ 2.6634 & 7.3175 \\ -0.9617 & 0.3500 \\ -2.6634 & -7.3175 \end{bmatrix},$$

$$\hat{P}^{(c)} = \begin{bmatrix} (\hat{p}_1^{(c)})^\top \\ (\hat{p}_2^{(c)})^\top \\ (\hat{p}_3^{(c)})^\top \\ (\hat{p}_4^{(c)})^\top \end{bmatrix} = \begin{bmatrix} 0.9229 & -0.3359 \\ 2.5022 & 6.8749 \\ -0.9229 & 0.3359 \\ -2.5022 & -6.8749 \end{bmatrix}, \quad \hat{P}^{(d)} = \begin{bmatrix} (\hat{p}_1^{(d)})^\top \\ (\hat{p}_2^{(d)})^\top \\ (\hat{p}_3^{(d)})^\top \\ (\hat{p}_4^{(d)})^\top \end{bmatrix} = \begin{bmatrix} 0.8896 & -0.3238 \\ 2.3656 & 6.4995 \\ -0.8896 & 0.3238 \\ -2.3656 & -6.4995 \end{bmatrix}, \quad (4.19)$$

where the scalar  $\alpha$  and the auxiliary vector  $q$  are given to be  $\alpha = 1.1$  and  $q = [0.11 \ 0.13]^\top$ , which are taken to be the same for each  $l \in \{a, b, c, d\}$ . The remaining feasible parameters for Theorem 3.2 are provided in [30].

Theorem 3.2 implies that the reachable sets for each case of  $l \in \{a, b, c, d\}$  are bounded by  $\hat{K}^{(l)}$ , where  $\hat{K}^l$  is defined by

$$\hat{K}^{(l)} = \bigcap_{i=1}^4 \mathcal{P}(\hat{p}_i^{(l)}), \quad l \in \{a, b, c, d\}, \quad (4.20)$$

where  $\hat{p}_i^l$  ( $i = 1, 2, 3, 4, l \in \{a, b, c, d\}$ ) are provided in (4.19).

The resulting polyhedral sets are compared in Figure 6. The areas of the bounding polyhedra for cases (a)–(d) are computed by 0.4876, 0.5019, 0.5567, and 0.6109, respectively, while the area of the fixed initial set  $\hat{\mathcal{X}}_0$  is approximately 0.4471. It is observed that the bounding polyhedron enlarges as the admissible delay bounds become larger. In particular, increasing  $\mu$  from 0 to 0.1 while fixing  $h = 0.1$  enlarges the area of the bound from 0.4876 to 0.5019, and increasing  $h$  from 0.1 to 0.5 with  $\mu = 0.1$  further increases the area to 0.5567. These results demonstrate the delay-dependent nature of Theorem 3.2, since the feasible polyhedral bound varies according to  $h$  and  $\mu$ , even though the system matrices and the initial set  $\hat{\mathcal{X}}_0$  are kept unchanged.

From a practical viewpoint, the general linear systems considered in this subsection can be interpreted, for example, as deviation dynamics of a thermal process [31] around a prescribed operating point. In such a setting, the state variables may represent temperature deviations from their nominal values and can therefore take either positive or negative values. A practical interpretation similar to that discussed in Section 4.1 can then be given to the reachable set bound. For instance, for a prescribed weighting vector  $c \in \mathbb{R}_+^n$ , the quantity  $\sup_{x \in K} c^\top x$  can be interpreted as a certified upper bound on a worst-case weighted temperature rise, or equivalently, a weighted overheating, over all admissible initial conditions and disturbances. Since  $K$  is polyhedral, this quantity can be readily computed by LP.

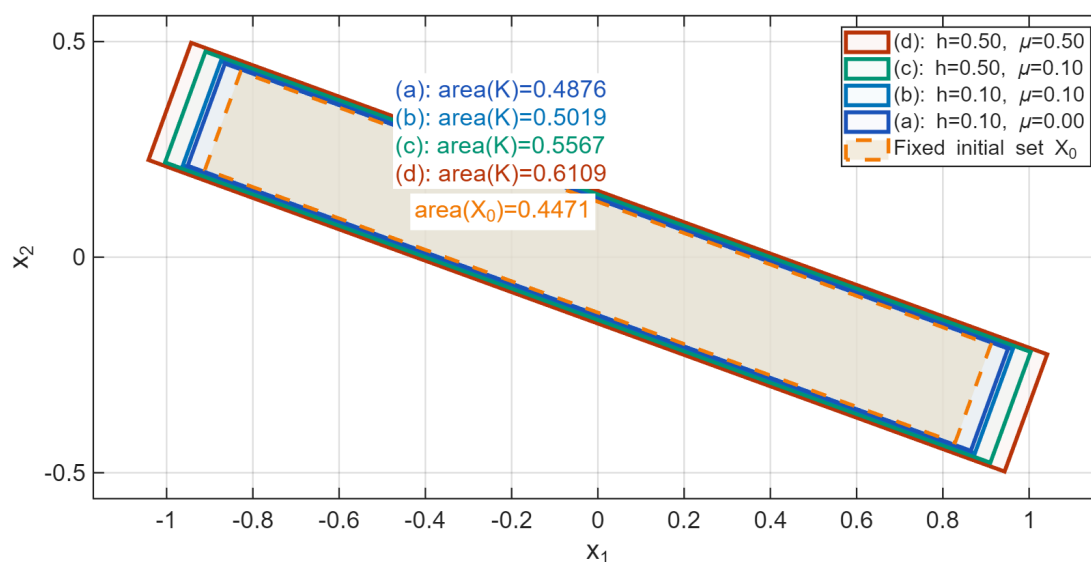


**Figure 6.** Polyhedral sets obtained from Theorem 3.2 bounding reachable sets for four different delay parameters  $(h, \mu)$  given in (4.7).

## 5. Conclusions

This paper developed a polyhedral framework for bounding reachable sets of linear systems under nonzero polyhedral initial conditions. For positive systems, sufficient conditions were established for bounding reachable sets in both delay-free and time-varying-delay cases while accounting for nonzero initial conditions. For systems with time-varying delays, the auxiliary functional was generalized to account for the time-varying delays, leading to delay-dependent conditions involving the parameters for the upper bounds of the delay and its rate. The developed framework was further extended to general linear systems that are not necessarily positive. In particular, an auxiliary functional involving absolute values of the state was introduced for delayed systems, and sufficient conditions were derived for obtaining polyhedral bounding sets for reachable sets in both delay-free and time-varying-delay cases. All the proposed polyhedral bounds were computed using LP. Several numerical examples were provided to demonstrate the effects of nonzero initial sets and delay parameters on the resulting polyhedral bounds, as well as the applicability of the proposed approach to general linear systems.

Regarding future research directions, reducing the computational burden associated with parameter searches in LP would be an important topic. From a control-theoretic perspective, the framework established in this paper also has the potential for several further extensions. Since the system matrices  $A$  and  $B$  are assumed to be exactly known in the present setting, extending the proposed reachable set-estimation framework to systems subject to model uncertainties would be an interesting direction.

Another possible extension is to consider nonlinear systems by representing or approximating their dynamics in a linear parameter-varying form, potentially including models constructed using neural networks [32]. Finally, extending the proposed framework to systems with multiple-delays and multi-input disturbances, by generalizing the auxiliary functional, can also be considered.

### Author contributions

Hyung Tae Choi: Original draft preparation, methodology, investigation, and formal analysis; Jung Hoon Kim: Supervision, validation, manuscript review, project administration, funding acquisition, and revisions. All authors have read and approved the final version of the manuscript for publication.

### Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

Prof. Jung Hoon Kim is a guest editor for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article. The authors declare no conflict of interest.

### References

1. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear matrix inequalities in system and control theory*, Society for Industrial and Applied Mathematics (SIAM), 1994. <http://dx.doi.org/10.1137/1.9781611970777>
2. C. Durieu, E. Walter, B. Polyak, Multi-input multi-output ellipsoidal state bounding, *J. Optim. Theory Appl.*, **111** (2001), 273–303. <http://dx.doi.org/10.1023/A:1011978200643>
3. E. Fridman, U. Shaked, On reachable sets for linear systems with delay and bounded peak inputs, *Automatica*, **39** (2003), 2005–2010. [http://dx.doi.org/10.1016/S0005-1098\(03\)00204-8](http://dx.doi.org/10.1016/S0005-1098(03)00204-8)
4. J. Kim, Improved ellipsoidal bound of reachable sets for time-delayed linear systems with disturbances, *Automatica*, **44** (2008), 2940–2943. <http://dx.doi.org/10.1016/j.automatica.2008.03.015>
5. Z. Zuo, D. W. C. Ho, Y. Wang, Reachable set bounding for delayed systems with polytopic uncertainties: The maximal Lyapunov-Krasovskii functional approach, *Automatica*, **46** (2010), 949–952. <http://dx.doi.org/10.1016/j.automatica.2010.02.022>

6. M. Vidyasagar, Optimal rejection of persistent bounded disturbances, *IEEE Trans. Automat. Control*, **31** (1986), 527–534. <http://dx.doi.org/10.1109/TAC.1986.1104315>
7. D. Kwak, J. H. Kim, T. Hagiwara, Generalized kernel approximation approach to  $\mathcal{L}_1$  optimal control of sampled-data systems, In: *2024 American control conference (ACC)*, Toronto: IEEE, 2024, 5177–5182. <http://dx.doi.org/10.23919/ACC60939.2024.10644972>
8. H. T. Choi, J. H. Kim, An  $L_\infty$  performance control for time-delay systems with time-varying delays: Delay-independent approach via ellipsoidal D-invariance, *AIMS Mathematics*, **9** (2024), 30384–30405. <http://dx.doi.org/10.3934/math.20241466>
9. D. Kwak, J. H. Kim, T. Hagiwara, A new quasi-finite-rank approximation of compression operators on  $L_\infty[0, H)$  with applications to sampled-data and time-delay systems: Piecewise linear kernel approximation approach, *J. Franklin Inst.*, **361** (2024), 107271. <http://dx.doi.org/10.1016/j.jfranklin.2024.107271>
10. J. Kim, J. H. Kim, The  $L_1$ -induced norm analysis for linear multivariable differential equations, *AIMS Mathematics*, **9** (2024), 34205–34223. <http://dx.doi.org/10.3934/math.20241629>
11. H. T. Choi, J. H. Kim, The  $\mathcal{L}_1$  controller synthesis for piecewise continuous nonlinear systems via set invariance principles, *Int. J. Robust Nonlinear Control*, **33** (2023), 8670–8692. <http://dx.doi.org/10.1002/rnc.6843>
12. H. T. Choi, J. H. Kim, T. Hagiwara, Characterizing  $\mathcal{L}_1$  output-feedback controller for nonlinear systems: Existence conditions via output controlled invariance domain, *Int. J. Robust Nonlinear Control*, **34** (2024), 11760–11785. <http://dx.doi.org/10.1002/rnc.7589>
13. H. T. Choi, J. H. Kim, R. G. Sanfelice, A barrier function-based approach for nonlinear  $\mathcal{L}_1$  control, In: *2025 IEEE 64th conference on decision and control (CDC)*, Brazil: IEEE, 2025, 2675–2680. <http://dx.doi.org/10.1109/CDC57313.2025.11312011>
14. O. R. Kang, J. H. Kim, Robust sliding mode control for robot manipulators with analysis on trade-off between reaching time and  $L_\infty$  gain, *Math. Methods Appl. Sci.*, **47** (2024), 7270–7287. <http://dx.doi.org/10.1002/mma.9972>
15. W. Xiang, H. Tran, T. T. Johnson, Output reachable set estimation for switched linear systems and its application in safety verification, *IEEE Trans. Automat. Control*, **62** (2017), 5380–5387. <http://dx.doi.org/10.1109/TAC.2017.2692100>
16. H. Y. Park, J. H. Kim, Robust balancing control of biped robots for external forces, In: *2024 IEEE International conference on robotics and automation (ICRA)*, Japan: IEEE, 2024, 13257–13262. <http://dx.doi.org/10.1109/ICRA57147.2024.10611281>
17. J. P. Richard, Time-delay systems: An overview of some recent advances and open problems, *Automatica*, **39** (2003), 1667–1694. [http://dx.doi.org/10.1016/S0005-1098\(03\)00167-5](http://dx.doi.org/10.1016/S0005-1098(03)00167-5)
18. E. Fridman, Tutorial on Lyapunov-based methods for time-delay systems, *Eur. J. Control*, **20** (2014), 271–283. <http://dx.doi.org/10.1016/j.ejcon.2014.10.001>
19. G. I. Song, J. H. Kim, Time-delay compensation-based robust control of mechanical manipulators: Operator-theoretic analysis and experiment validation, *Math. Methods Appl. Sci.*, **47** (2024), 318–335. <http://dx.doi.org/10.1002/mma.9656>

20. J. Abedor, K. Nagpal, K. Poolla, A linear matrix inequality approach to peak-to-peak gain minimization, *Int. J. Robust Nonlinear Control*, **6** (1996), 899–927. [http://dx.doi.org/10.1002/\(SICI\)1099-1239\(199611\)6:9<10;899::AID-RNC259;3.0.CO;2-G](http://dx.doi.org/10.1002/(SICI)1099-1239(199611)6:9<10;899::AID-RNC259;3.0.CO;2-G)
21. I. Pólik, T. Terlaky, A survey of the S-lemma, *SIAM Rev.*, **49** (2007), 371–418. <http://dx.doi.org/10.1137/S003614450444614X>
22. B. Zhang, J. Lam, S. Xu, Relaxed results on reachable set estimation of time-delay systems with bounded peak inputs, *Int. J. Robust Nonlinear Control*, **26** (2016), 1994–2007. <http://dx.doi.org/10.1002/rnc.3395>
23. Y. Chen, J. Lam, B. Zhang, Estimation and synthesis of reachable set for switched linear systems, *Automatica*, **63** (2016), 122–132. <http://dx.doi.org/10.1016/j.automatica.2015.10.033>
24. L. Farina, S. Rinaldi, *Positive linear systems: Theory and applications*, Chichester: John Wiley & Sons, 2000. <http://dx.doi.org/10.1002/9781118033029>
25. C. Briat, Robust stability and stabilization of uncertain linear positive systems via integral linear constraints:  $L_1$ -gain and  $L_\infty$ -gain characterization, *Int. J. Robust Nonlinear Control*, **23** (2013), 1932–1954. <http://dx.doi.org/10.1002/rnc.2859>
26. B. Du, J. Lam, Z. Shu, Y. Chen, On reachable sets for positive linear systems under constrained exogenous inputs, *Automatica*, **74** (2016), 230–237. <http://dx.doi.org/10.1016/j.automatica.2016.07.048>
27. Y. Chen, J. Lam, Y. Cui, J. Shen, K. W. Kwok, Reachable set estimation and synthesis for periodic positive systems, *IEEE Trans. Cybern.*, **51** (2021), 501–511. <http://dx.doi.org/10.1109/TCYB.2019.2908676>
28. J. Shen, J. Lam, Improved reachable set estimation for positive systems: A polyhedral approach, *Automatica*, **124** (2021), 109167. <http://dx.doi.org/10.1016/j.automatica.2020.109167>
29. J. C. Hennet, Une extension du Lemme de Farkas et son application au probleme de regulation lineaire sous contraintes, *C. R. Acad. Sci. Paris*, **308** (1989), 415–419.
30. H. T. Choi, J. H. Kim, *Polyhedral-type reachable set estimation with generalized initial conditions: From positive systems to general linear delay systems*. Available from: <https://tinyurl.com/fvs2bwss>.
31. B. Bueno, L. Norford, G. Pigeon, R. Britter, A resistance-capacitance network model for the analysis of the interactions between the energy performance of buildings and the urban climate, *Build. Environ.*, **54** (2012), 116–125. <http://dx.doi.org/10.1016/j.buildenv.2012.01.023>
32. G. Zhang, J. Hu, Y. Shen, Exponential lag synchronization for delayed memristive recurrent neural networks, *Neurocomputing*, **154** (2015), 86–93. <http://dx.doi.org/10.1016/j.neucom.2014.12.016>



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