



Research article

Numerical investigation of nonlinear elastic wave propagation in a binary mixture of two elastic solids

A. R. El-Dhaba

Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia

* **Correspondence:** Email: aemam@kfu.edu.sa.

Abstract: This manuscript presents a numerical investigation of a coupled system of nonlinear partial differential equations used to model wave propagation in a binary mixture composed of two elastic solids. The proposed model is derived by considering the Green-Lagrange strain tensor in a Lagrangian coordinate system, together with the constitutive relations and governing field equations, subject to appropriate boundary conditions that characterize the deformation of the individual constituents of the binary mixture. A finite difference scheme is employed to obtain numerical solutions for the one-dimensional case under prescribed initial and boundary conditions for the displacements of both constituents. The numerical results are fitted, plotted, and analyzed. Furthermore, it is shown that the induced waves propagate at relatively low velocities; they do not dissipate over time but gradually decay to zero as the distance tends to infinity.

Keywords: binary mixture; elastic solids; nonlinear elastic waves; weak nonlinearity; finite difference method

Mathematics Subject Classification: 74B20, 74F20, 74J30, 74S20

1. Introduction

Wave propagation in binary mixtures of elastic solids is an important area of research, as it enables the characterization of coupling effects and complex behaviors within such media, including wave dispersion arising from material heterogeneity. This line of research not only enhances the theoretical understanding of wave propagation and dynamics in composite and heterogeneous materials but also supports practical applications in material characterization, nondestructive evaluation, and advanced engineering systems. Significant contributions have been made in this area; for example, Biot [1, 2] presented a mathematical model for poro-elasticity to study the propagation of elastic waves in a porous elastic material saturated by a fluid for low-frequency range by formulating a theory of elastic wave

propagation in porous medium saturated by fluid, where this model predicts the speed of compression waves for low-frequency range and higher-frequency range. While in [3], Biot studied the stress-dependent of seismic wave and dispersion in granular medium saturated by a fluid. Also, Mills and Steel [4] studied the nonlinear deformation in mixtures of two isotropic elastic solids within finite elasticity theory. The obtained results gave some first indication on the coupled mechanical response of the interacting solid constituents. Bedford and Drumheller [5] developed a variational formulation in order to deduce the governing equations and boundary conditions through the application of Hamilton's principle within immiscible mixtures. Shi et al. [6] employed the theory of interacting continua to describe the transport of fluid through a nonlinear elastic medium, where great emphasis was placed on the impact of material nonlinearity on transport and deformation processes. Baer and Nunziato [7] utilized a two-phase mixture theory in order to model the process of deflagration-to-detonation transitions in reactive granular materials. Within this framework, the phase interactions, compaction, and reaction kinetics are taken into consideration in a unified way. Ieşan [8] extended the classical mixture theory to the non-simple elastic solids by incorporating higher-order gradients, which is essential for capturing microstructural effects in the elastic mixture's mechanical response. Moreover, Rushchitskii [9] presented a comprehensive review of nonlinear wave propagation in solid mixtures, emphasizing the role of both dispersion and nonlinearity. In addition, Rushchitskii [10] examined the propagation of solitary waves in solid mixtures. Also, an extension for microstructural mixture theory was presented by Rushchitskii [11] to include the composite materials, where the internal length scales was introduced to model heterogeneous solids with complex microstructures. Another generalization was introduced by Eringen [12] to include the micropolar mixture theory for porous media, where the size-dependent and microstructural effects are taken into account to describe the micro-rotational degrees of freedom which complicate the mathematical model. Another direction of this research was introduced by Carcione et al. [13] to investigate the numerical analysis for wave propagation in elastic porous medium saturated by a fluid. Also, the second slow compressional waves existence are confirmed and analyzed. Carcione et al. [13] presented a detailed numerical study of slow-wave propagation in porous materials which are partially saturated with fluid, and the effects of saturation, permeability, and frequency are analyzed. Galeş [14] introduced a potential-based approach for the linear theory of swelling porous elastic medium, where he offered an analytical simplification for coupled deformation-diffusion problems. Moreover, Gales [15] presented a mixture theory for micropolar thermoelastic solids, incorporating thermal effects and micro-rotations into the continuum description. Also, Ieşan [16] developed a strain gradient theory of elasticity for porous materials by introducing a higher-order strain tensor to capture the size-dependent effects and nonlocal behavior. Kumar and Batra [17] derived fundamental solutions for steady oscillations in swelling porous thermoelastic material, by providing the analytical solution for coupled thermo-mechanical problems. In addition, Andrianov et al. [18] presented a comprehensive treatment of linear and nonlinear wave propagation in microstructured materials by combining homogenization techniques with asymptotic methods. Also, Magaña et al. [19] studied the decay properties of solutions in strain-gradient mixture models, emphasizing the stabilizing role of higher-order gradients. Schumacher and Baer [20] introduced a generalized continuum mixture theory for multi-material systems, unifying classical mixture theory with generalized continua to model complex material interfaces. Also, Hilal [21] studied the effect of the magnetic field upon the wave propagation in a binary mixture composed of two thermoelastic materials. As Chen et al. [22] investigated seismic wave propagation

in porous elastic media partially saturated with fluids, the role of wave-induced fluid flow (WIFF) mechanisms was examined, demonstrating their strong influence on wave dispersion and attenuation across different frequency bands. Two multiscale wave-propagation models were introduced, which can describe wave propagation in porous media containing two immiscible fluids over a broadband frequency range. The stability of mathematical models governing wave propagation in elastic and porous mixtures has been investigated by many researchers in order to elucidate the interactions among coupling mechanisms, material parameters, and solution behavior. For instance, Ciarletta [23] analyzed the response of mixtures of nonsimple elastic solids within the framework of higher-gradient elasticity theory, focusing on the uniqueness of solutions in linear elastodynamics without imposing restrictions on the initial stress. Also, Passarella and Zampoli [24] studied the spatial behavior and decay properties for some classes of isotropic simple elastic mixtures, for which the internal energy density is not always a positive definite from quadratic form. In addition, for elastodynamic mixtures of two elastic solids with weak damping and different wave velocities, strong stability was demonstrated by Tebou [25] under weaker parameter constraints using semigroup analysis. Moreover, exponential stability was obtained under additional appropriate conditions imposed on the parameters of the coupled system. He showed that for some sets of parameters, the mathematical model is asymptotically unstable. In a related but distinct setting, Yeremeyev et al. [26] analyzed the loss of stability in elastic solids undergoing phase transitions, showing that non-uniqueness of equilibrium solutions arises from the non-linearity associated with the unknown interface. Finally, Motta et al. [27] examined the stability of linear models describing mixtures of two porous solids, showing that small variations in the characteristic coefficients governing the coupling of mechanical effects lead to correspondingly small variations in the solutions of the associated initial boundary-value problems. Consequently, it was concluded that the mixture model for porous materials is mathematically consistent.

Yang et al. [28] investigated dark rogue-wave solutions of an extended Boussinesq system using Hirota bilinearization and the multiple-rogue-wave method, obtaining first- and second-order rational solutions and analyzing their evolution and parameter dependence. Zhong et al. [29] constructed rogue-wave solutions of a $(2 + 1)$ -dimensional nonlinear Zakharov system through a self-similar reduction to the nonlinear Schrödinger equation. Zhong and Belić [30] derived novel nonlinear excitations for a $(2 + 1)$ -dimensional asymmetric Nizhnik-Novikov-Veselov system using Hirota bilinearization and variable separation. Wang et al. [31] investigated the electronic and transport properties of arsenic-, phosphorus-, and antimony-based monolayers using first-principles calculations, highlighting their stability, strain-dependent band structures, and anisotropic carrier mobility.

In this study, we develop a coupled nonlinear model for a binary mixture of two elastic solids and introduce a one-dimensional formulation to visualize nonlinear wave propagation. The model is derived from the geometrical Green-Lagrange strain tensor and the internal energy functional, initially for anisotropic materials. For isotropic materials, even-order tensors are expressed using the Kronecker delta, while odd-order tensors vanish, simplifying the formulation. Therefore, the paper is organized as follows. Section 2 reviews the governing equations and constitutive relations. Section 3 presents the one-dimensional application, including the coupled field equations, constitutive relations, and initial-boundary conditions. Section 4 details the discretization of the domain, field equations, and constitutive relations using the central finite difference method. In Section 5, material parameters and boundary conditions at the origin are specified, and numerical results are fitted, discussed, and analyzed. Finally, Section 6 concludes the study.

2. Mathematical model

In this section, we consider a mixture composed of two continua, s_1 and s_2 . It is assumed that the mass of each constituent is conserved and that each constituent behaves as a nonlinear elastic solid. The material elements of both constituents are described using a Lagrangian coordinate system X_i ($i = 1, 2, 3$). Accordingly, the Green-Lagrange strain tensor for the two elastic solids is defined as follows:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}), \quad g_{ij} = \frac{1}{2}(w_{i,j} + u_{j,i} + u_{k,i}w_{k,j}), \quad (2.1)$$

where u_i and w_i , $i = 1, 2, 3$, are the components of the displacement vector fields associated with the two constituents s_1 and s_2 , respectively. Details are listed in the Appendix.

By assuming that the initial state of the medium is free from any external actions, therefore, the internal energy takes the following form:

$$U(\varepsilon_{ij}, g_{ij}, d_i) = \frac{1}{2}A_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}B_{ijkl}g_{ij}g_{kl} + C_{ijkl}\varepsilon_{ij}g_{kl} + \frac{1}{2}S_{ij}d_id_j + E_{ijk}\varepsilon_{ij}d_k + F_{ijk}g_{ij}d_k, \quad (2.2)$$

where

$$d_k = u_k - w_k.$$

The constitutive coefficients are assumed to be defined for homogeneous medium, and the tensors A_{ijkl} , B_{ijkl} , C_{ijkl} , and S_{ij} have major symmetry while E_{ijk} and F_{ijk} have minor symmetry. By Eq (2.2) with respect to the internal state variables, one can write the constitutive relations as follows:

$$T_{ij} = A_{ijkl}\varepsilon_{kl} + C_{ijkl}g_{kl} + E_{ijk}d_k, \quad (2.3)$$

$$H_{ij} = B_{ijkl}g_{kl} + C_{ijkl}\varepsilon_{kl} + F_{ijk}d_k, \quad (2.4)$$

$$D_k = S_{jk}d_j + E_{ijk}\varepsilon_{ij} + F_{ijk}g_{ij}. \quad (2.5)$$

In isotropic materials, odd-order tensors vanish, whereas even-order tensors can be constructed from products of the Kronecker delta δ_{ij} . Therefore, for symmetric and nonsymmetric tensors, respectively, these products take the following forms:

$$\begin{aligned} A_{ijkl} &= \lambda_1\delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), & B_{ijkl} &= \lambda_2\delta_{ij}\delta_{kl} + \mu_2\delta_{ik}\delta_{jl} + \mu_3\delta_{il}\delta_{jk}, \\ C_{ijkl} &= \gamma\delta_{ij}\delta_{kl} + \xi\delta_{ik}\delta_{jl} + \zeta\delta_{il}\delta_{jk}, & S_{jk} &= S\delta_{jk}, \end{aligned}$$

where $\lambda_1, \mu_1, \lambda_1, \mu_2, \mu_3, \gamma, \xi, \zeta$, and S are constants related to the components of the mixture.

Therefore, the constitutive relations can be written for isotropic materials in terms of the displacement fields as follows:

$$\begin{aligned} T_{ij} &= \lambda_1\delta_{ij}u_{k,k} + \mu_1u_{i,j} + \mu_1u_{j,i} + \gamma\delta_{ij}w_{k,k} + \frac{1}{2}(\xi + \zeta)u_{j,i} + \frac{1}{2}(\xi + \zeta)w_{i,j} \\ &\quad + \frac{1}{2}\lambda_1\delta_{ij}u_{m,k}u_{m,k} + \mu_1u_{k,i}u_{k,j} + \frac{1}{2}\gamma\delta_{ij}u_{m,k}w_{m,k} + \frac{1}{2}(\xi + \zeta)u_{k,i}w_{k,j}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} H_{ij} &= \frac{1}{2}\lambda_2\delta_{ij}w_{k,k} + \frac{1}{2}\mu_2w_{i,j} + \frac{1}{2}\mu_3w_{j,i} + \frac{1}{2}(\mu_3 + \xi + \zeta)u_{i,j} + \frac{1}{2}(\mu_2 + \xi + \zeta)u_{j,i} + \left(\frac{1}{2}\lambda_2 + \gamma\right)\delta_{ij}u_{k,k} \\ &\quad + \frac{1}{2}\lambda_2\delta_{ij}u_{m,k}w_{m,k} + \frac{1}{2}\mu_2u_{k,i}w_{k,j} + \frac{1}{2}\mu_3u_{k,j}w_{k,i} + \frac{1}{2}\gamma\delta_{ij}u_{m,k}u_{m,k} + \frac{1}{2}(\xi + \zeta)u_{k,i}u_{k,j}, \end{aligned} \quad (2.7)$$

$$D_k = S(u_k - w_k). \quad (2.8)$$

We note that the tensor H_{ij} is generally nonsymmetric and should therefore not be interpreted as a conventional symmetric Cauchy stress tensor of a single classical continuum. Instead, H_{ij} represents the internal mechanical interactions between the two constituents of the mixture [32]. Accordingly, the balance of angular momentum should be considered consistently with the linear momentum balances of the individual constituents and the overall mixture. For more details, see [33, 34].

Following [21], the balances of momentum for binary solid mixtures are defined as follows:

$$T_{jk,j} - D_k + F_k^{(1)} = \rho_1 \ddot{u}_k, \quad (2.9)$$

$$H_{jk,j} + D_k + F_k^{(2)} = \rho_2 \ddot{w}_k, \quad (2.10)$$

where ρ_1 and ρ_2 are the constants densities, T_{ji} and S_{ji} are components of the stress tensors associated with the two constituents s_1 and s_2 , and D_i are the components of the diffusive vectors. Now, Eqs (2.6)–(2.8) into Eqs (2.9) and (2.10) to obtain the field equations without considering body force as follows:

$$(\lambda_1 + \mu_1 + \frac{1}{2}(\xi + \zeta))u_{j,ji} + \mu_1 u_{i,jj} + \gamma w_{k,ki} + \frac{1}{2}(\xi + \zeta)w_{i,jj} + (\lambda_1 + \mu_1)u_{m,j}u_{m,ij} + \mu_1 u_{k,i}u_{k,jj} \\ + \frac{1}{2}(\gamma + \xi + \zeta)u_{m,ij}w_{m,j} + \frac{1}{2}\gamma u_{m,k}w_{m,ik} + \frac{1}{2}(\xi + \zeta)u_{k,i}w_{k,jj} - S(u_i - w_i) = \rho_1 \ddot{u}_i, \quad (2.11)$$

$$\frac{1}{2}(\lambda_2 + \mu_2)w_{j,ji} + \frac{1}{2}\mu_3 w_{i,jj} + \frac{1}{2}(\mu_3 + \xi + \zeta + \lambda_2 + 2\gamma)u_{k,ki} + \frac{1}{2}(\mu_2 + \xi + \zeta)u_{i,jj} \\ + \frac{1}{2}(\lambda_2 + \mu_3)w_{m,j}u_{m,ij} + \frac{1}{2}(\lambda_2 + \mu_2)u_{m,j}w_{m,ij} + \frac{1}{2}\mu_2 w_{k,i}u_{k,jj} + \frac{1}{2}\mu_3 u_{k,i}w_{k,jj} \\ + \frac{1}{2}(2\gamma + \xi + \zeta)u_{m,j}u_{m,ij} + \frac{1}{2}(\xi + \zeta)u_{k,i}u_{k,jj} + S(u_i - w_i) = \rho_2 \ddot{w}_i. \quad (2.12)$$

By neglecting the nonlinear terms, Eqs (2.11) and (2.12) reduce to the equations in [10] and [24].

3. Application for 1D

In this section, the one-dimensional mathematical model is formulated by assuming that the non-vanishing components of the state variables are given by

$$u = u_1(x, t), \quad w = w_1(x, t), \quad T = T_{11}, \quad H = H_{11}, \quad D = D_1.$$

The reduction of the three-dimensional governing equations to the corresponding one-dimensional form is obtained by assuming that all field variables depend only on the axial coordinate $x = x_1$. Accordingly, the derivatives with respect to the transverse coordinates x_2 and x_3 vanish. Since Eqs (2.11) and (2.12) are expressed in tensor form, with the indices ranging over 1, 2, 3, the one-dimensional formulation is obtained by retaining only the tensor components and spatial derivatives consistent with the adopted one-dimensional kinematics.

Substitution of these assumptions into Eqs (2.11) and (2.12), followed by the elimination of the vanishing transverse derivatives and components, directly yields the corresponding one-dimensional field equations. Thus, when all physical quantities are regarded as functions of x and t , Eqs (2.11) and (2.12) take the following form:

$$\alpha_1 \frac{\partial^2 u}{\partial x^2} + \alpha_2 \frac{\partial^2 w}{\partial x^2} + 2\alpha_3 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \alpha_4 \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) - S(u - w) = \rho_1 \frac{\partial^2 u}{\partial t^2}, \quad (3.1)$$

$$\beta_1 \frac{\partial^2 w}{\partial x^2} + \beta_2 \frac{\partial^2 u}{\partial x^2} + 2\beta_3 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \beta_4 \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) + S(u - w) = \rho_2 \frac{\partial^2 w}{\partial t^2}, \quad (3.2)$$

with

$$\begin{aligned} \alpha_1 &= \lambda_1 + 2\mu_1 + \frac{1}{2}(\xi + \zeta), & \alpha_2 &= \gamma + \frac{1}{2}(\xi + \zeta), & \alpha_3 &= \frac{1}{2}(\lambda_1 + 2\mu_1), & \alpha_4 &= \frac{1}{2}\beta_3, \\ \beta_1 &= \frac{1}{2}(\lambda_2 + \mu_2 + \mu_3), & \beta_2 &= \frac{1}{2}(\beta_1 + \beta_3), & \beta_3 &= \gamma + \xi + \zeta, & \beta_4 &= \beta_1. \end{aligned}$$

Equations (3.1) and (3.2) describe the coupled nonlinear longitudinal deformation of a binary mixture consisting of two interacting continua, s_1 and s_2 , with displacement fields $u(x, t)$ and $w(x, t)$, respectively. The equations account for the linear elastic response of the individual constituents and their mutual elastic coupling, as well as the geometric nonlinearities associated with finite deformation and the nonlinear coupling between the deformation gradients of the two constituents. The interaction term $S(u - w)$ represents the internal restoring force between the two continua; it acts in opposite directions on the two constituents and tends to reduce their relative displacement. Finally, the terms on the right-hand sides represent the inertial forces of the two constituents. Thus, the system accounts for linear elasticity, nonlinear deformation, inter-constituent interaction, and inertia in a two-continuum mixture.

The constitutive relations (2.9) and (2.10) are

$$T = \alpha_1 \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial w}{\partial x} + \alpha_3 \left(\frac{\partial u}{\partial x} \right)^2 + \alpha_4 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}, \quad (3.3)$$

$$H = \beta_1 \frac{\partial w}{\partial x} + 2\beta_2 \frac{\partial u}{\partial x} + \frac{1}{2}\beta_3 \left(\frac{\partial u}{\partial x} \right)^2 + \beta_4 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}, \quad (3.4)$$

$$D = S(u - w). \quad (3.5)$$

To facilitate the mathematical analysis of the considered problem and make the mathematical formulation independent of the choice of measurement units, the state variables, together with the spatial and temporal coordinates, are normalized using appropriate characteristic scales of the material. This normalization allows the governing equations to be expressed in terms of dimensionless variables. Accordingly, suitable characteristic length, time, and other physical scales are selected, and the dimensionless variables are introduced as follows:

$$\begin{aligned} t_0 &= \sqrt{\frac{\rho_1 L^2}{\lambda_1 + 2\mu_1}}, & t^* &= \frac{t}{t_0}, & x^* &= \frac{x}{L}, & u^* &= \frac{u}{L}, & w^* &= \frac{w}{L}, & \frac{\partial^2}{\partial t^2} &= \frac{1}{t_0^2} \frac{\partial^2}{\partial t^{*2}}, \\ \frac{\partial^n}{\partial x^n} &= \frac{1}{L^n} \frac{\partial^n}{\partial x^{*n}}, & T^* &= \frac{T}{\lambda_1 + 2\mu_1}, & H^* &= \frac{H}{\lambda_1 + 2\mu_1}, & D^* &= \frac{DL^2}{\lambda_1 + 2\mu_1}. \end{aligned}$$

Therefore, the field equations (3.1) and (3.2) are

$$\alpha_1^* \frac{\partial^2 u}{\partial x^{*2}} + \alpha_2^* \frac{\partial^2 w}{\partial x^{*2}} + 2\alpha_3^* \frac{\partial u}{\partial x^*} \frac{\partial^2 u}{\partial x^{*2}} + \alpha_4^* \left(\frac{\partial^2 u}{\partial x^{*2}} \frac{\partial w}{\partial x^*} + \frac{\partial u}{\partial x^*} \frac{\partial^2 w}{\partial x^{*2}} \right) - S^*(u - w) = \frac{\partial^2 u}{\partial t^{*2}}, \quad (3.6)$$

$$\beta_1^* \frac{\partial^2 w}{\partial x^2} + \beta_2^* \frac{\partial^2 u}{\partial x^2} + 2\beta_3^* \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \beta_4^* \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) + S^*(u - w) = \frac{\rho_2}{\rho_1} \frac{\partial^2 w}{\partial t^{*2}}. \quad (3.7)$$

Also, the constitutive relations (3.3)–(3.5) are

$$T = \alpha_1^* \frac{\partial u}{\partial x} + \alpha_2^* \frac{\partial w}{\partial x} + \alpha_3^* \left(\frac{\partial u}{\partial x} \right)^2 + \alpha_4^* \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}, \quad (3.8)$$

$$H = \beta_1^* \frac{\partial w}{\partial x} + \beta_2^* \frac{\partial u}{\partial x} + \beta_3^* \left(\frac{\partial u}{\partial x} \right)^2 + \beta_4^* \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}, \quad (3.9)$$

$$D = S^*(u - w), \quad S^* = \frac{SL^2}{\lambda_1 + 2\mu_1}, \quad (3.10)$$

where

$$\alpha_1^* = \frac{\alpha_1}{\lambda_1 + 2\mu_1}, \quad \alpha_2^* = \frac{\alpha_2}{\lambda_1 + 2\mu_1}, \quad \alpha_3^* = \frac{\alpha_3}{\lambda_1 + 2\mu_1}, \quad \alpha_4^* = \frac{\alpha_4}{\lambda_1 + 2\mu_1},$$

$$\beta_1^* = \frac{\beta_1}{\lambda_1 + 2\mu_1}, \quad \beta_2^* = \frac{2\beta_2}{\lambda_1 + 2\mu_1}, \quad \beta_3^* = \frac{1}{2} \frac{\beta_3}{\lambda_1 + 2\mu_1}, \quad \beta_4^* = \frac{\beta_4}{\lambda_1 + 2\mu_1}.$$

Let us that consider that the boundary conditions are

$$u(0, t) = u_0, \quad w(0, t) = w_0, \quad (3.11)$$

where u_0 and w_0 are initials values. Moreover, these choices guarantee that the initial conditions are satisfied as follows:

$$\left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0} = 0, \quad \left. \frac{\partial w(x, t)}{\partial t} \right|_{t=0} = 0. \quad (3.12)$$

4. Numerical solution

4.1. Discretization for the domain

Let's discretize the domain $[0, T] \times [0, L]$ into a uniformly spaced grid, which is represented by Δx in the spatial direction and Δt in the temporal direction. We approximate the analytical solution $u(x, t)$ and $w(x, t)$ by finite difference scheme, denoted as $u(i\Delta x, j\Delta t) = u_{i,j}$ and $w(i\Delta x, j\Delta t) = w_{i,j}$, where we have $i = 0, 1, \dots, N$ and $j = 0, 1, \dots, N$, respectively.

Consequently, the time and space derivatives are represented as follows:

$$\left(\frac{\partial u}{\partial t} \right)_{i,k} = \frac{1}{2\Delta t} (u_{i,k+1} - u_{i,k-1}) + O(\Delta t^2), \quad \left(\frac{\partial w}{\partial t} \right)_{i,k} = \frac{1}{2\Delta t} (w_{i,k+1} - w_{i,k-1}) + O(\Delta t^2), \quad (4.1)$$

$$\left(\frac{\partial u}{\partial x} \right)_{i,k} = \frac{1}{2\Delta x} (u_{i+1,k} - u_{i-1,k}) + O(\Delta x^2), \quad \left(\frac{\partial w}{\partial x} \right)_{i,k} = \frac{1}{2\Delta x} (w_{i+1,k} - w_{i-1,k}) + O(\Delta x^2). \quad (4.2)$$

The second derivatives also are represented by the second-order central differences as follows:

$$\left(\frac{\partial^2 u}{\partial t^2} \right)_{i,k} = \frac{1}{\Delta t^2} (u_{i,k+1} - 2u_{i,k} + u_{i,k-1}) + O(\Delta t^2), \quad \left(\frac{\partial^2 w}{\partial t^2} \right)_{i,k} = \frac{1}{\Delta t^2} (w_{i,k+1} - 2w_{i,k} + w_{i,k-1}) + O(\Delta t^2), \quad (4.3)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,k} = \frac{1}{\Delta x^2}(u_{i+1,k} - 2u_{i,k} + u_{i-1,k}) + O(\Delta t^2), \quad \left(\frac{\partial^2 w}{\partial x^2}\right)_{i,k} = \frac{1}{\Delta x^2}(w_{i+1,k} - 2w_{i,k} + w_{i-1,k}) + O(\Delta t^2). \quad (4.4)$$

The average can be defined as follows:

$$(u(x, t))_{i,k} = \frac{1}{2}(u_{i,k+1} + u_{i,k-1}), \quad (w(x, t))_{i,k} = \frac{1}{2}(w_{i,k+1} + w_{i,k-1}), \quad (4.5)$$

where $\Delta t \neq 0$ and $\Delta x \neq 0$ are required in the numerical calculations to avoid division by zero and the resulting singularities.

4.2. Discretization of the field equations

By substituting Eqs (4.1)–(4.5) into Eqs (3.6) and (3.7), we have

$$\begin{aligned} u_{i,j+1} - q_1 w_{i,j+1} &= q_1 w_{i,j-1} + q_2 u_{i,j} - u_{i,j-1} + q_3(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \\ &\quad + q_4(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}) + q_5(u_{i+1,j}^2 - 2u_{i+1,j}u_{i,j} + 2u_{i-1,j}u_{i,j} - u_{i-1,j}^2) \\ &\quad + q_6(u_{i+1,j}w_{i+1,j} - u_{i,j}w_{i+1,j} + u_{i,j}w_{i-1,j} - u_{i+1,j}w_{i,j} + u_{i-1,j}w_{i,j} - u_{i-1,j}w_{i-1,j}), \end{aligned} \quad (4.6)$$

$$\begin{aligned} w_{i,j+1} - p_1 u_{i,j+1} &= p_1 u_{i,j-1} + p_2 w_{i,j} - w_{i,j-1} + p_3(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}) \\ &\quad + p_4(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + p_5(u_{i+1,j}^2 - 2u_{i+1,j}u_{i,j} + 2u_{i-1,j}u_{i,j} - u_{i-1,j}^2) \\ &\quad + p_6(u_{i+1,j}w_{i+1,j} - u_{i,j}w_{i+1,j} + u_{i,j}w_{i-1,j} - u_{i+1,j}w_{i,j} + u_{i-1,j}w_{i,j} - u_{i-1,j}w_{i-1,j}), \end{aligned} \quad (4.7)$$

with

$$\begin{aligned} q_1 &= \frac{\frac{1}{2}S^*\Delta t^2}{1 + \frac{1}{2}S^*\Delta t^2}, & q_2 &= \frac{2}{1 + \frac{1}{2}S^*\Delta t^2}, & q_3 &= \frac{\Delta t^2}{\Delta x^2} \frac{\alpha_1^*}{1 + \frac{1}{2}S^*\Delta t^2}, \\ q_4 &= \frac{\Delta t^2}{\Delta x^2} \frac{\alpha_2^*}{1 + \frac{1}{2}S^*\Delta t^2}, & q_5 &= \frac{\Delta t^2}{\Delta x^3} \frac{\alpha_3^*}{1 + \frac{1}{2}S^*\Delta t^2}, & q_6 &= \frac{\Delta t^2}{\Delta x^3} \frac{\alpha_4^*}{1 + \frac{1}{2}S^*\Delta t^2}, \end{aligned}$$

and

$$\begin{aligned} p_1 &= \frac{\frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}{1 + \frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}, & p_2 &= \frac{2}{1 + \frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}, & p_3 &= \frac{\beta_1^*(\rho_1/\rho_2) \Delta t^2/\Delta x^2}{1 + \frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}, \\ p_4 &= \frac{\beta_2^*(\rho_1/\rho_2) \Delta t^2/\Delta x^2}{1 + \frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}, & p_5 &= \frac{\beta_3^*(\rho_1/\rho_2) \Delta t^2/\Delta x^3}{1 + \frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}, & p_6 &= \frac{\beta_4^*(\rho_1/\rho_2) \Delta t^2/\Delta x^3}{1 + \frac{1}{2}S^*\Delta t^2 \rho_1/\rho_2}. \end{aligned}$$

Solving Eqs (4.6) and (4.7), we obtain

$$u_{i,j+1} = F_{i,j}^L + F_{i,j}^N, \quad (4.8)$$

$$w_{i,j+1} = G_{i,j}^L + G_{i,j}^N, \quad (4.9)$$

where the linear parts are given by

$$\begin{aligned} F_{i,j}^L &= A_3(u_{i+1,j} + u_{i-1,j}) + (A_1 - 2A_3)u_{i,j} - u_{i,j-1} + (A_2 - 2A_4)w_{i,j} + A_4(w_{i+1,j} + w_{i-1,j}), \\ G_{i,j}^L &= B_3(w_{i+1,j} + w_{i-1,j}) + (B_1 - 2B_3)w_{i,j} - w_{i,j-1} + (B_2 - 2B_4)u_{i,j} + B_4(u_{i+1,j} + u_{i-1,j}), \end{aligned}$$

while the nonlinear parts are

$$\begin{aligned} F_{i,j}^N &= A_5(u_{i+1,j}^2 - 2u_{i+1,j}u_{i,j} + 2u_{i-1,j}u_{i,j} - u_{i-1,j}^2) \\ &\quad + A_6(u_{i+1,j}w_{i+1,j} - u_{i,j}w_{i+1,j} + u_{i,j}w_{i-1,j} - u_{i+1,j}w_{i,j} + u_{i-1,j}w_{i,j} - u_{i-1,j}w_{i-1,j}), \\ G_{i,j}^N &= B_5(u_{i+1,j}^2 - 2u_{i+1,j}u_{i,j} + 2u_{i-1,j}u_{i,j} - u_{i-1,j}^2) \\ &\quad + B_6(u_{i+1,j}w_{i+1,j} - u_{i,j}w_{i+1,j} + u_{i,j}w_{i-1,j} - u_{i+1,j}w_{i,j} + u_{i-1,j}w_{i,j} - u_{i-1,j}w_{i-1,j}), \end{aligned}$$

with

$$\begin{aligned} A_1 &= \frac{q_2}{1 - p_1 q_1}, & A_2 &= \frac{p_2 q_1}{1 - p_1 q_1}, & A_3 &= \frac{q_3 + p_4 q_1}{1 - p_1 q_1}, \\ A_4 &= \frac{q_4 + p_3 q_1}{1 - p_1 q_1}, & A_5 &= \frac{q_5 + p_5 q_1}{1 - p_1 q_1}, & A_6 &= \frac{q_6 + p_6 q_1}{1 - p_1 q_1}, \\ B_1 &= \frac{p_2}{1 - p_1 q_1}, & B_2 &= \frac{q_2 p_1}{1 - p_1 q_1}, & B_3 &= \frac{p_3 + p_1 q_4}{1 - p_1 q_1}, \\ B_4 &= \frac{p_4 + p_1 q_3}{1 - p_1 q_1}, & B_5 &= \frac{p_5 + p_1 q_5}{1 - p_1 q_1}, & B_6 &= \frac{p_6 + p_1 q_6}{1 - p_1 q_1}. \end{aligned}$$

4.3. Initial and boundary conditions

(1) At the origin,

$$u_{0,0} = u_0, \quad w_{0,0} = w_0. \quad (4.10)$$

(2) At $t = 0$,

$$u_{i,0} = 0, \quad w_{i,0} = 0, \quad i = 1, 2, \dots, N. \quad (4.11)$$

(3) At $x = 0$,

$$u_{0,j} = 0, \quad w_{0,j} = 0, \quad j = 1, 2, \dots, N-1. \quad (4.12)$$

(4) At $x = L$,

$$u_{N,j} = 0, \quad w_{N,j} = 0, \quad j = 1, 2, \dots, N-1. \quad (4.13)$$

(5) Virtual level: In this level, we use the central difference approximation for the first time derivative, Eq (3.12), as follows:

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \frac{u_{i,1} - u_{i,-1}}{2\Delta t} = 0, \quad u_{i,-1} = u_{i,1}, \quad (4.14)$$

$$\left. \frac{\partial w}{\partial t} \right|_{t=0} = \frac{w_{i,1} - w_{i,-1}}{2\Delta t} = 0, \quad w_{i,-1} = w_{i,1}. \quad (4.15)$$

(6) First level: According to the relations (4.14) and (4.15), the first level where $j = 0$ and $i = 1, 2, \dots, N$ is calculated from the following relations:

$$u_{i,1} = \frac{1}{2} (F_{i,0}^L + F_{i,0}^N), \quad w_{i,1} = \frac{1}{2} (G_{i,0}^L + G_{i,0}^N), \quad (4.16)$$

with

$$F_{i,0}^L = A_3 u_{i+1,0} + A_3 u_{i-1,0} + (A_1 - 2A_3) u_{i,0} + (A_2 - 2A_4) w_{i,0} + A_4 w_{i+1,0} + A_4 w_{i-1,0},$$

$$\begin{aligned}
G_{i,0}^L &= B_3 w_{i+1,0} + B_3 w_{i-1,0} + (B_1 - 2B_3) w_{i,0} + (B_2 - 2B_4) u_{i,0} + B_4 u_{i+1,0} + B_4 u_{i-1,0}, \\
F_{i,0}^N &= A_5 (u_{i+1,0}^2 - 2u_{i+1,0} u_{i,0} + 2u_{i-1,0} u_{i,0} - u_{i-1,0}^2) \\
&\quad + A_6 (u_{i+1,0} w_{i+1,0} - u_{i,0} w_{i+1,0} + u_{i,0} w_{i-1,0} - u_{i+1,0} w_{i,0} + u_{i-1,0} w_{i,0} - u_{i-1,0} w_{i-1,0}), \\
G_{i,0}^N &= B_5 (u_{i+1,0}^2 - 2u_{i+1,0} u_{i,0} + 2u_{i-1,0} u_{i,0} - u_{i-1,0}^2) \\
&\quad + B_6 (u_{i+1,0} w_{i+1,0} - u_{i,0} w_{i+1,0} + u_{i,0} w_{i-1,0} - u_{i+1,0} w_{i,0} + u_{i-1,0} w_{i,0} - u_{i-1,0} w_{i-1,0}).
\end{aligned}$$

Applying the boundary conditions, Eq (4.16) is reduced to the following:

$$u_{1,1} = \frac{1}{2} (A_3 u_{0,0} + A_4 w_{0,0} - A_5 u_{0,0}^2 - A_6 u_{0,0} w_{0,0}), \quad u_{i,1} = 0, \quad i = 2, 3, \dots, N-1, \quad (4.17)$$

$$w_{1,1} = \frac{1}{2} (B_3 w_{0,0} + B_4 u_{0,0} - B_5 u_{0,0}^2 - B_6 u_{0,0} w_{0,0}), \quad w_{i,1} = 0, \quad i = 2, 3, \dots, N-1. \quad (4.18)$$

It is noted that each point in the first level is computed from three points in the previous level as shown in Figure 1.

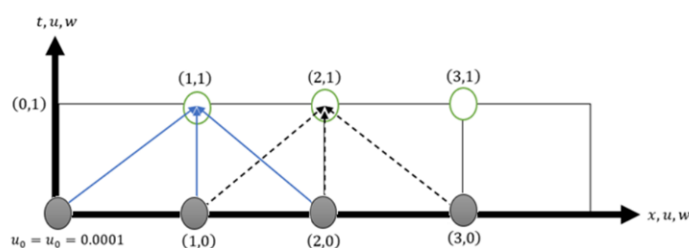


Figure 1. First level computational approach.

- (7) General levels: For the remaining levels, where $j = 1, 2, \dots, M-1$ and $i = 2, \dots, N-1$, Eqs (4.8) and (4.9) are employed to compute the unknown values of the displacement fields at the nodal points. Each value of the displacements at the i -level is evaluated using the value at the previous level $i-1$, together with a single nodal value from the $i-2$ level, as illustrated in Figure 2.

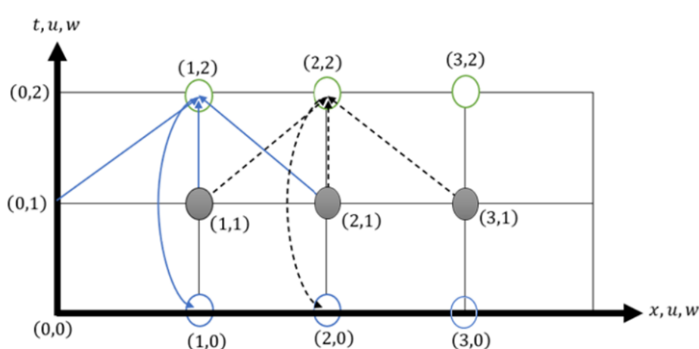


Figure 2. General levels computational approach.

Finally, the displacement fields are fully evaluated at all levels, starting from the first time level $j = 0$. At this initial level, the nodal values of the displacement fields are determined directly from the initial and boundary conditions, where the values $u_{0,0}$ and $w_{0,0}$ are specified. These values serve as the starting point for the time-stepping discretization procedure. For all subsequent time levels, the

displacement at each nodal point is computed sequentially using the previously calculated values from earlier levels. This procedure is repeated over the entire spatial and temporal domain, advancing step by step in time until the displacement fields are obtained at all grid points. This approach ensures that the evolution of the displacement fields is consistently and accurately determined throughout the computational domain, while fully satisfying the imposed initial and boundary conditions.

4.4. Discretization of the constitutive relations

By substituting Eqs (4.1) and (4.2) into Eqs (3.8) and (3.10), we have

$$T_{ij} = \frac{\alpha_1^*}{2\Delta x}(u_{i+1,j} - u_{i-1,j}) + \frac{\alpha_2^*}{2\Delta x}(w_{i+1,j} - w_{i-1,j}) + \frac{\alpha_3^*}{4\Delta x^2}(u_{i+1,j} - u_{i-1,j})^2 + \frac{\alpha_4^*}{4\Delta x^2}(u_{i+1,j} - u_{i-1,j})(w_{i+1,j} - w_{i-1,j}), \quad (4.19)$$

$$H_{ij} = \frac{\beta_1^*}{2\Delta x}(w_{i+1,j} - w_{i-1,j}) + \frac{\beta_2^*}{2\Delta x}(u_{i+1,j} - u_{i-1,j}) + \frac{\beta_3^*}{4\Delta x^2}(u_{i+1,j} - u_{i-1,j})^2 + \frac{\beta_4^*}{4\Delta x^2}(u_{i+1,j} - u_{i-1,j})(w_{i+1,j} - w_{i-1,j}), \quad (4.20)$$

$$D_{ij} = \frac{S^*}{2}(u_{i,j+1} - w_{i,j+1} + u_{i,j-1} - w_{i,j-1}). \quad (4.21)$$

5. Results and discussion

In this section, the physical parameters of the two materials are adopted from Hilal [21] and Kumar and Batra [17] and are listed in Table 1.

Table 1. Material parameters for the binary mixture.

Parameter	Value	Description
ρ_1	$1.74 \times 10^3 \text{ kg/m}^3$	Density of the first material
ρ_2	$1.35 \times 10^3 \text{ kg/m}^3$	Density of the second material
λ_1	$2.17 \times 10^{10} \text{ N/m}^2$	Lame's constant of the first material
μ_1	$3.278 \times 10^{10} \text{ N/m}^2$	Lame's constant of the first material
λ_2	$1.85 \times 10^{10} \text{ N/m}^2$	Lame's constant of the second material
μ_2	$0.5 \times 10^{10} \text{ N/m}^2$	Lame's constant of the second material
ξ	$0.85 \times 10^{10} \text{ N/m}^2$	Elastic constant
γ	$1.95 \times 10^{10} \text{ N/m}^2$	Elastic constant
ζ	$0.75 \times 10^{10} \text{ N/m}^2$	Elastic constant
S	$0.075 \times 10^{12} \text{ N} \cdot \text{s/m}^4$	Material parameter
u_0	0.001	Initial displacement of the first material
w_0	0.001	Initial displacement of the second material

The numerical results are computed with the following restrictions:

$$0 \leq x \leq L, \quad 0 \leq t \leq T, \quad \Delta x = \frac{L}{N}, \quad \Delta t = \frac{T}{M}, \quad N = M = 100, \quad L = 1, \quad T = 0.5.$$

The stability condition of the numerical scheme is satisfied when

$$\frac{\Delta t}{\Delta x} < 1.$$

Under the above restrictions, the following figures are obtained.

Figures 3 and 4 present the numerical simulations of the displacements of the first and second components of the binary mixture, respectively. The two components exhibit similar wave-propagation patterns, indicating strong mechanical interaction between them during the propagation process. The deformation of one component influences the motion of the other through the coupling terms in Eqs (3.6)–(3.7), resulting in a coupled wave response throughout the mixture. As the waves propagate, the initially localized deformation spreads over a wider spatial region, while the peak displacement gradually decreases. This behavior can be attributed to the spatial redistribution of the deformation within the coupled system, together with the dispersive and nonlinear characteristics of the governing equations. The differences in the displacement amplitudes and wave profiles of the two components arise from their distinct elastic and inertial properties, as well as their mutual coupling. Overall, the results demonstrate that elastic-wave propagation in the binary mixture is governed by the coupled motion of its constituents, leading to the progressive evolution of the wave profiles as they propagate through the medium.

In addition, Figures 5 and 6 illustrate the wave velocities associated with the first and second components of the binary elastic mixture, respectively. The wave velocities are governed by the combined effects of inertia, represented by the mass densities ρ_1 and ρ_2 , the elastic coefficients of the two constituents, and the inter-component coupling characterized by the parameter S . Consequently, the wave velocities associated with the two components are intrinsically coupled and may differ because of their distinct elastic properties. Furthermore, because Eqs (3.6)–(3.7) contain nonlinear terms involving the displacement gradients, the propagation characteristics may also depend on the magnitude and spatial distribution of the deformation. Thus, the observed variation in the wave velocity reflects the combined effects of elasticity, inter-component coupling, and nonlinearity.

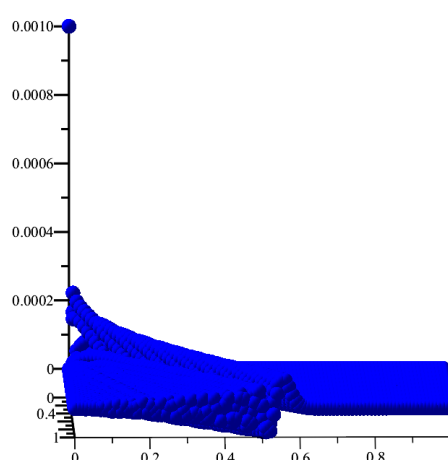


Figure 3. Displacement of the first component of the mixture.

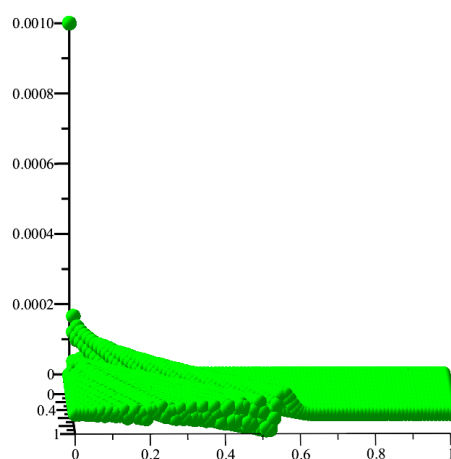


Figure 4. Displacement of the second component of the mixture.

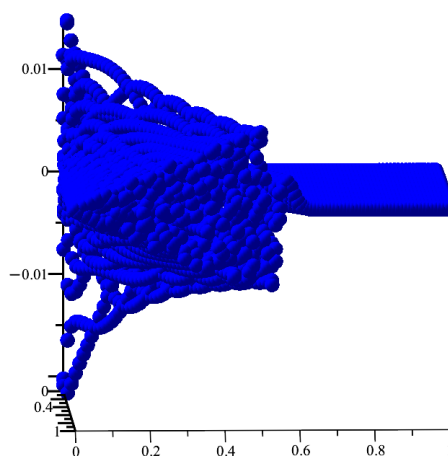


Figure 5. Wave velocity of the first component of the mixture.

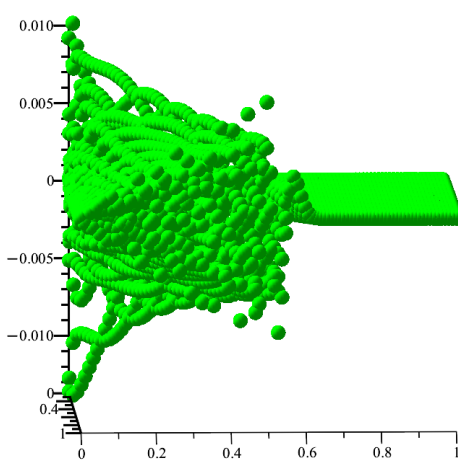


Figure 6. Wave velocity of the second component of the mixture.

Figures 7 and 8 show the temporal evolution of the displacement fields $u(:, t)$ and $w(:, t)$, respectively, at three different locations, $x = 0.1$, $x = 0.5$, and $x = 1$, for the binary mixture. The wave propagation for both components exhibits a strong dependence on spatial position, reflecting the effects of dispersion, propagation characteristics, and inter-constituent coupling within the mixture. Moreover, for both displacement fields, the wave amplitude is largest at the location closest to the excitation region ($x = 0.1$), where the waveforms are relatively sharp and temporally localized, indicating that the energy is initially concentrated near the source. As the observation point moves farther away from the excitation region ($x = 0.5$), the amplitudes decrease and the wave amplitudes become increasingly oscillatory and temporally broadened. This behavior is characteristic of dispersive wave propagation, in which different frequency components travel with distinct velocities, leading to waveform propagation and a reducing in peak amplitude. At $x = 1$, the wave amplitudes are significantly attenuated for both components, and the oscillations appear smoother and more diffuse. The gradual decay of the peak amplitudes without sharp disappearance indicates that the observed attenuation is primarily governed by dispersive propagation and energy redistribution rather than by material dissipation. A comparison between the two figures reveals some similarities in the overall propagation of u and w , while the difference is in amplitude and oscillatory behavior, which indicate the distinct mechanical properties and coupling between the two elastic constituents.

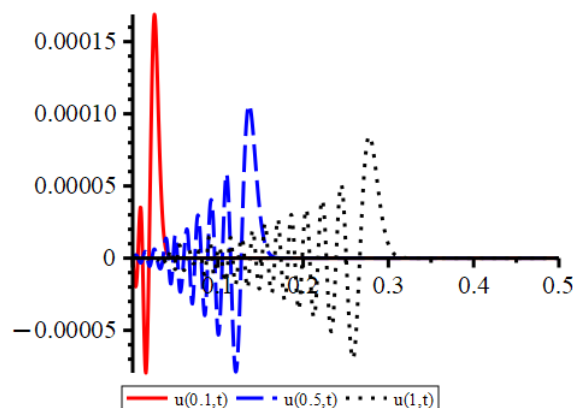


Figure 7. $u(:, t)$ at different positions.

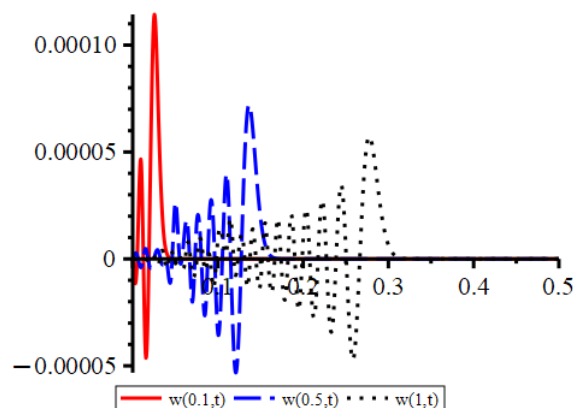


Figure 8. $w(:, t)$ at different positions.

Figures 9 and 10 present the nonlinear dynamic response of the displacement fields $u(x, :)$ and $w(x, :)$ for both components of the mixture across discrete time intervals $t \in \{0.25, 0.50, 0.75\}$. One can note that, the two figures exhibit a well-defined transnational propagation along the spatial coordinate x , accompanied by noticeable dispersion effects and attenuation. As time progresses, the primary pulse exhibits a systematic reduction in amplitude in the first phase. This attenuation can be attributed to the coupling between the elastic components of the binary mixture, which facilitates the redistribution of wave energy between the two constituents.

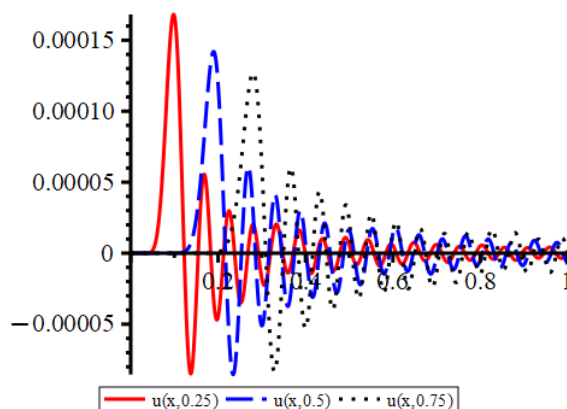


Figure 9. $u(x, :)$ at different times.

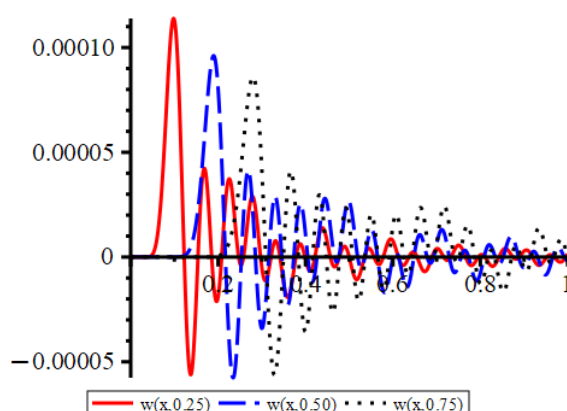


Figure 10. $w(x, :)$ at different times.

Figures 11 and 12 present the temporal evolution of the stress for the first and second constituents at various positions. The stress responses exhibit oscillatory behavior with amplitudes that decrease over time, indicating dispersion wave propagation. Near the excitation region, the stresses are more localized and have higher amplitudes, while at larger distances the amplitude the waves decrease. In general, the first constituent exhibits higher stress amplitudes than the second, which may be interpreted due to the effect of material properties and coupling effects of the mathematical model.

Figures 13 and 14 show the spatial evolution of the stress for the first and second components of the binary mixture at three different time instances. The stress fields exhibit oscillatory waves that propagate along the spatial domain with amplitudes that gradually decrease due to dispersion and

energy dissipation. In the vicinity of the excitation region, the stresses are more localized and of lower amplitude, while farther away the wave propagates spatially and gradually decays until the wave motion vanishes. The first constituent shows higher stress amplitudes than the second, emphasizing the role of material contrast and nonlinear coupling in the binary mixture.

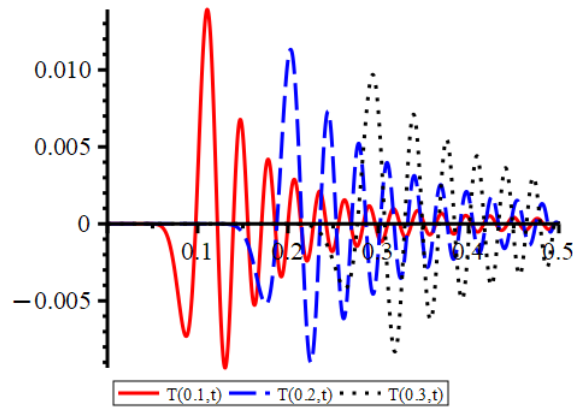


Figure 11. $T(\cdot, t)$ at different positions.

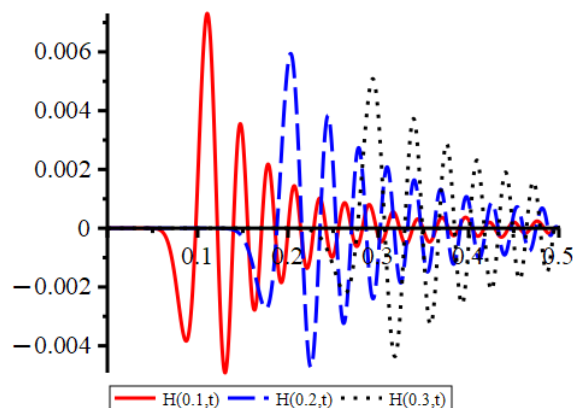


Figure 12. $H(\cdot, t)$ at different positions.

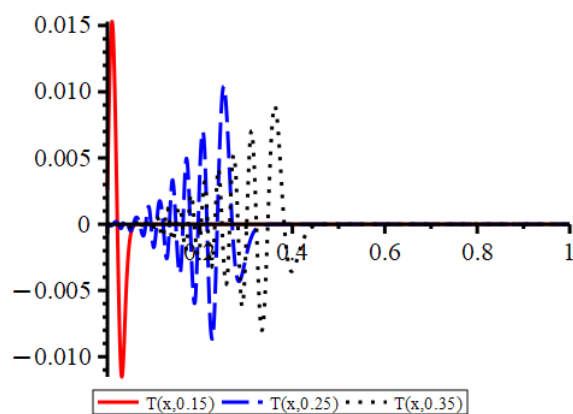


Figure 13. $T(x, \cdot)$ at different times.

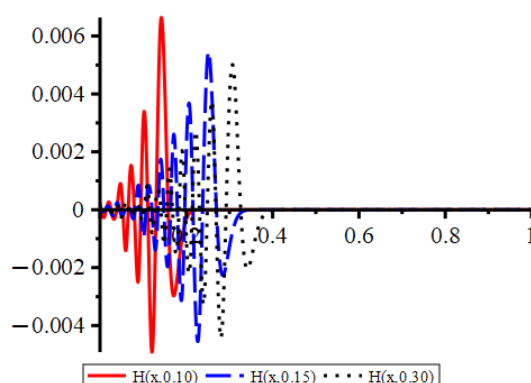


Figure 14. $H(x, :)$ at different times.

Figure 15 shows the temporal response of the diffusive vector of the binary mixture at three positions, while Figure 16 presents its spatial evolution at three time instances. The temporal profiles exhibit oscillations with amplitudes decreasing away from the excitation region, indicating diffusion-driven wave propagation. Geometrical non-linearity is manifest in the wave characteristics. The spatial profile of the waves reveals a slight amplitude increase due to weak excitation near the source, followed by decay to zero at larger distances, while the overall amplitude diminishes over time due to dissipative effects.

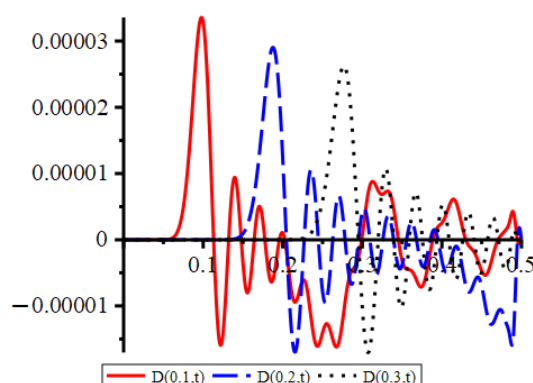


Figure 15. $D(:, t)$ at different positions.

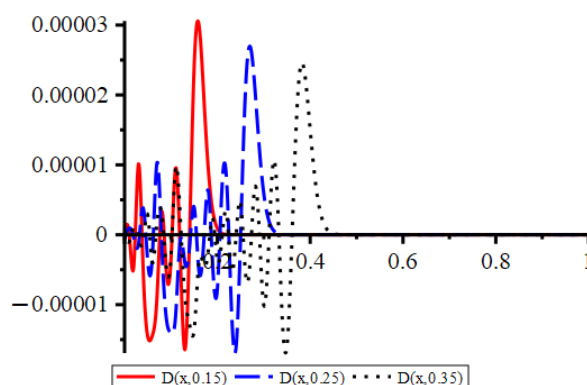


Figure 16. $D(x, :)$ at different times.

6. Conclusions

In this study, a numerical solution is derived for a coupled system of nonlinear partial differential equations used to describe the behavior of a binary mixture consisting of two elastic solids, where the weak non-linearity results from employing the Green-Lagrange strain tensor in a Lagrangian coordinate framework. The finite difference method is employed to discretize the space-time domain, yielding discrete forms of both the governing field equations and the constitutive relations. The initial-boundary conditions specified at the origin are treated carefully to determine the values of the displacement fields at the initial time level of the nodal grid. In addition, different sets of initial-boundary conditions may be considered, needing a different treatment of the first time or spatial levels. Three-dimensional figures are presented to interpret the displacement fields and velocities of the two components of the mixture, while two-dimensional plots are introduced to describe the displacements and constitutive relations of wave propagation at various spatial locations and time instances. The results show that the nonlinear wave propagation does not completely dissipate with time, due to the absence of damping in the mathematical model; instead, the wave energy is redistributed spatially through dispersion and nonlinear interactions, resulting in a gradual decay of the wave amplitude as the spatial domain becomes sufficiently large.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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Appendix

We consider two different material points s_1 and s_2 , respectively located in space X_i , $i = 1, 2, 3$, where X_i refers to the Lagrangian coordinate system, as shown in Figure A.1.

The position vector of the material element s_1 is denoted by X_i ($i = 1, 2, 3$), while the position vector of the material element s_2 is denoted by Y_i . In the unreformed configuration, at $t = 0$, the two material elements coincide:

$$X_i = Y_i, \quad i = 1, 2, 3. \quad (\text{A.1})$$

When the medium is deformed under external action, the two material elements move to new positions described in the Eulerian coordinate system x_i ($i = 1, 2, 3$). Accordingly, the position vectors of the two material elements in the deformed configuration are denoted by x_i for the first material element and y_i for the second. Furthermore, the displacement field of the first material element is represented by u_i , while that of the second is represented by w_i .

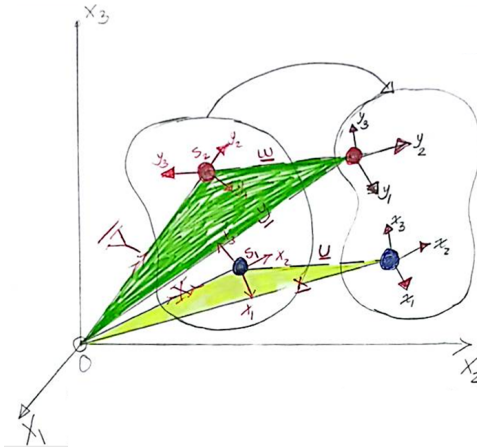


Figure A.1. Lagrangian coordinate system.

From Figure A.1 and vector calculus, the motion of the first material element s_1 is written as

$$X_i + u_i = x_i, \quad (\text{A.2})$$

and for the second material element s_2 :

$$Y_i + w_i = y_i. \quad (\text{A.3})$$

From Eq (A.1), Eq (A.3) can also be written as

$$X_i + w_i = y_i. \quad (\text{A.4})$$

Finally, the motion of the two material elements in both Lagrangian and Eulerian coordinate systems is expressed as

$$x_i = u_i + \delta_{iL}X_L, \quad y_i = w_i + \delta_{iL}X_L, \quad (\text{A.5})$$

where X_i represents a vector relative to the Lagrangian coordinate system but expressed in the Eulerian system. The deformation gradient tensor characterizes the local deformation of a material body and is defined as

$$F_{iK} = x_{i,K},$$

with the inverse deformation gradient tensor

$$F_{Ki}^{-1} = X_{K,i}.$$

By differentiating Eq (A.5), we have

$$x_{i,K} = u_{i,K} + \delta_{iL}X_{L,K}, \quad y_{i,K} = w_{i,K} + \delta_{iL}X_{L,K}, \quad (\text{A.6})$$

which can be simplified as

$$x_{i,K} = u_{i,K} + \delta_{iK}, \quad y_{i,K} = w_{i,K} + \delta_{iK}. \quad (\text{A.7})$$

The Green and Cauchy deformation tensors are defined by

$$C_{KL} = x_{i,K} x_{i,L}, \quad c_{kl} = X_{K,k} X_{K,l}. \quad (\text{A.8})$$

Hence, the Lagrangian and Euler strain tensors are

$$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL}), \quad e_{kl} = \frac{1}{2}(\delta_{kl} - c_{kl}). \quad (\text{A.9})$$

Since the medium consists of two distinct material elements, each point is associated with both Lagrangian and Eulerian strain tensors:

$$\begin{aligned} E_{KL} &= \frac{1}{2}(C_{KL} - \delta_{KL}), & \text{for the first material element } s_1, \\ G_{KL} &= \frac{1}{2}(C_{KL}^* - \delta_{KL}), & \text{for the second material element } s_2, \end{aligned} \quad (\text{A.10})$$

where C_{KL} and C_{KL}^* are the Green deformation tensors for s_1 and s_2 , respectively:

$$C_{KL} = x_{i,K} x_{i,L}, \quad C_{KL}^* = x_{i,K} y_{i,L}. \quad (\text{A.11})$$

Substituting Eq (A.11) into Eq (A.10) gives

$$E_{KL} = \frac{1}{2}(x_{i,K} x_{i,L} - \delta_{KL}), \quad G_{KL} = \frac{1}{2}(x_{i,K} y_{i,L} - \delta_{KL}). \quad (\text{A.12})$$

Finally, substituting Eq (A.7) into Eq (A.12), the strain tensors are expressed in terms of displacement gradients:

$$E_{KL} = \frac{1}{2}(u_{K,L} + u_{L,K} + u_{i,K} u_{i,L}), \quad G_{KL} = \frac{1}{2}(u_{L,K} + w_{K,L} + u_{i,K} w_{i,L}). \quad (\text{A.13})$$



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