



Research article**On Ore-type conditions for oriented discrepancy of Hamilton cycles****Akbar Ali¹ and Abdulaziz Mutlaq Alotaibi^{2,*}**¹ Department of Mathematics, College of Science, University of Ha'il, Ha'il, Saudi Arabia² Department of Mathematics, College of Science and Humanities in AlKharj, Prince Sattam Bin Abdulaziz University, AlKharj 11942, Saudi Arabia* **Correspondence:** Email: am.alotaibi@psau.edu.sa.

Abstract: The oriented discrepancy of Hamiltonian cycles in a Hamiltonian graph G of order n is the largest integer ℓ such that every orientation D of G contains a Hamiltonian oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil (n + \ell)/2 \rceil$, where $\sigma_{\max}(C)$ denotes the maximum of the numbers of forward and backward arcs of C . For a vertex w in an oriented graph D , let $d(w)$ denote the degree of w in the underlying graph of D . Ai, Guo, Gutin, Lan, Shao, Yeo and Zhou [*AIMS Math.* **11** (2026), 24106–24119] proved that, for any positive integer k , every oriented graph D of order $n \geq 30 + 4(k - 1)$ with $s^*(D) \geq 8k$ contains a Hamilton oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil (n + k)/2 \rceil$, where $s^*(D)$ is the minimum of $d(u) + d(v) - n$ over all distinct non-adjacent vertices u, v of D when D is not a tournament, while $s^*(D) = n - 2$ when D is a tournament. They asked whether the factor 8 in the bound $s^*(D) \geq 8k$ can be reduced. For every $k \geq 6$, we prove that the aforementioned conclusion follows from $s^*(D) \geq 4k - 3$; however, when $n + k$ is even and $k \geq 7$, then the condition $s^*(D) \geq 4k - 7$ is sufficient. We also obtain results for $1 \leq k \leq 5$. Moreover, for every fixed positive integer k and all sufficiently large n , we prove that the same conclusion follows from $s^*(D) \geq 3k$ when $n + k$ is odd and from $s^*(D) \geq 3k - 3$ when $n + k$ is even.

Keywords: oriented graph; Hamilton cycle; oriented discrepancy; ore-type condition; good diamond**Mathematics Subject Classification:** 05C20, 05C45

1. Introduction

All graphs considered in this paper are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively, and we use the notation $m(G) = |E(G)|$ for the size of G . Denote by P_n , K_n and K_{n_1, n_2} , respectively, the path graph of order n , the complete graph of order n and the complete bipartite graph with partite sets of sizes n_1 and n_2 . The complement of a graph G is denoted by $\overline{G}$. For a vertex $u \in V(G)$, the set of all those vertices of G that are adjacent to u is

denoted by $N_G(u)$. Every element of $N_G(u)$ is called a neighbor of u . Throughout the paper, degrees in an oriented graph are the degrees in the corresponding underlying graph. For a vertex w in an oriented graph D , its degree is denoted by $d_D(w)$, or simply by $d(w)$ when there is no danger of confusion. The minimum degree of D is denoted by $\delta(D)$.

For an oriented graph D , an oriented cycle is a cycle of the underlying graph together with the orientations inherited from D . After one of the two directions around the cycle is fixed, an arc is called forward if its direction agrees with that traversal and backward otherwise. If C is an oriented cycle, let $\sigma^+(C)$ and $\sigma^-(C)$ be its numbers of forward and backward arcs, respectively. Let $\sigma_{\max}(C) = \max\{\sigma^+(C), \sigma^-(C)\}$. Reversing the traversal interchanges $\sigma^+(C)$ and $\sigma^-(C)$, so $\sigma_{\max}(C)$ does not depend on the chosen traversal.

The oriented discrepancy of Hamiltonian cycles in a Hamiltonian graph G is the largest integer ℓ such that every orientation of G contains a Hamiltonian oriented cycle C such that

$$|\sigma^+(C) - \sigma^-(C)| \geq \ell; \quad (1.1)$$

if G has n vertices, then inequality (1.1) is equivalent to $\sigma_{\max}(C) \geq \lceil (n + \ell)/2 \rceil$. The study of oriented discrepancy of Hamilton cycles was initiated by Gishboliner, Krivelevich and Michaeli in [8], where they conjectured the following generalized version of Dirac's theorem [6]: if an oriented graph D of order $n \geq 3$ satisfies $\delta(D) \geq n/2$, then D contains a Hamilton oriented cycle C with $\sigma_{\max}(C) \geq \delta(D)$. This conjecture was proved by Freschi and Lo [7], who also asked for an Ore-type version of this result. Ore-type versions of the aforementioned generalization of Dirac's theorem were studied in [1, 5].

Following Ai et al. [1], we now define $s^*(D)$ for an oriented graph D of order n . If D is not a tournament, then $s^*(D) = \min\{d(u) + d(v) - n : \text{the vertices } u, v \in V(D) \text{ are distinct and non-adjacent}\}$, while if D is a tournament, then we set $s^*(D) = n - 2$. Thus, $s^*(D) \geq 0$ is exactly Ore's degree-sum condition. Ai et al. [1] conjectured that every oriented graph D of order $n \geq 3$ with $s^*(D) \geq 0$ has a Hamilton oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil (n + s^*(D))/2 \rceil$. (Chang et al. [5] provided an asymptotic affirmative answer to this conjecture for all sufficiently large n .) Among their supporting results, Ai et al. [1] proved the following result.

Theorem 1.1 ([1]). *Let D be an oriented graph of order n such that $n \geq 30 + 4(k - 1)$, where k is a positive integer. If $s^*(D) \geq 8k$, then D contains a Hamilton oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil \frac{n+k}{2} \rceil$.*

Ai et al. [1] asked whether the factor 8 in the bound $s^*(D) \geq 8k$, given in Theorem 1.1, can be reduced. We answer this question partially in two ways by providing two improvements. The first one has no additional lower bound on n depending on k and reduces the coefficient essentially from 8 to 4. Particularly, for every $k \geq 6$ and $n \geq 3$, it is shown that the conclusion of Theorem 1.1 follows from $s^*(D) \geq 4k - 3$; however, when $n + k$ is even and $k \geq 7$ and $n \geq 3$, then the condition $s^*(D) \geq 4k - 7$ is sufficient. The second improvement applies when k is fixed and n is sufficiently large; in this case, the aforementioned coefficient can be reduced further to 3. More precisely, for every positive fixed integer k and all sufficiently large n , it is shown that the conclusion of Theorem 1.1 follows from $s^*(D) \geq 3k$ when $n + k$ is odd and from $s^*(D) \geq 3k - 3$ when $n + k$ is even.

2. Preliminaries

We first state the following theorem of Freschi and Lo [7] used in the two subsequent sections.

Theorem 2.1 ([7]). *Let D be an oriented graph of order $n \geq 3$. If $\delta(D) \geq n/2$, then D contains a Hamilton oriented cycle C such that $\sigma_{\max}(C) \geq \delta(D)$.*

The underlying graph of an oriented graph D is denoted by $U(D)$. We also use the following lemma established by Ai et al. [1].

Lemma 2.2 ([1]). *Let D be an oriented graph of order n and let t be a non-negative integer. If $s^*(D) \geq t$, then $m(U(D)) \geq n(n+t)/4$.*

The graph obtained from the complete graph K_4 of order 4 by removing an edge is called the *diamond*. The two vertices of degree 2 in the diamond are called its ends. An orientation of the diamond is *good* if, for each of its two vertices of degree 3, the two arcs joining that vertex to the ends of the diamond are oriented in the same direction; that is, they are either both directed toward the vertex of degree 3 or both directed away from it. A good orientation of the diamond is called a *good diamond* (see Figure 1).

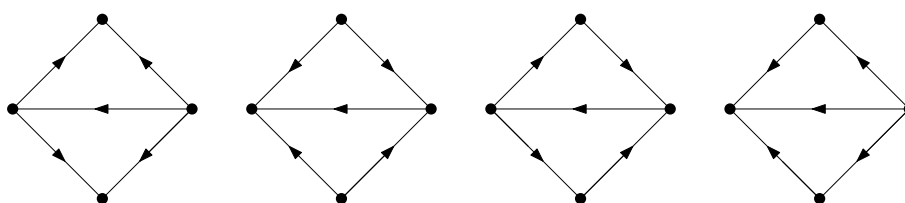


Figure 1. Good diamonds.

We use two facts about good diamonds from [8]. The following fact is Observation 2.1 in [8].

Observation 2.1 ([8]). *Let D be a good diamond with ends u and v . Then, D has a spanning path of length 3 from u to v containing at least two forward arcs, and also such a path from v to u .*

The following observation follows from the proof of Lemma 2.2 in [8].

Observation 2.2 ([8]). *If an edge of a graph G belongs to at least five triangles, then every orientation of G contains a good diamond using that edge.*

A graph in which every component is a path is known as a path forest. We also need the following result due to Ai et al. [1].

Lemma 2.3 ([1]). *Let t be a non-negative integer. Let G be a graph of order $n \geq 3$ such that the inequality $d(u) + d(v) \geq n + t$ holds for every pair of distinct non-adjacent vertices $u, v \in V(G)$. If $F \subset E(G)$ is the edge set of a path forest and $|F| \leq t$, then G has a Hamilton cycle containing every edge of F .*

Using Observation 2.1 and Lemma 2.3, we now establish the following result.

Lemma 2.4. *Let q be any positive integer. Let D be an oriented graph of order n . If $s^*(D) \geq 3q$ and D contains q pairwise vertex-disjoint good diamonds, then D contains a Hamilton oriented cycle C satisfying*

$$\sigma_{\max}(C) \geq \left\lceil \frac{n+q}{2} \right\rceil.$$

Proof. Let $D_1, \dots, D_q$ be pairwise vertex-disjoint good diamonds in D . For every $i \in \{1, \dots, q\}$, let c_i and d_i be the two ends of D_i . By Observation 2.1, D_i contains a spanning path P_i (respectively, Q_i) of length 3 from c_i to d_i (respectively, from d_i to c_i) having at least two forward arcs when it is traversed from c_i to d_i (respectively, from d_i to c_i). The set $F = \bigcup_{i=1}^q E(U(P_i))$ is therefore the edge set of a path forest with $|F| = 3q$, where $U(P_i)$ is the underlying graph of P_i .

If D is a tournament, then $U(D) = K_n$. Hence, the paths $U(P_1), \dots, U(P_q)$, together with the remaining vertices of $U(D)$ regarded as trivial paths, form a spanning path forest of $U(D)$. (This forest has $n - 3q \geq 2$ components because $n - 2 = s^*(D) \geq 3q$.) Joining these paths cyclically in $U(D)$ gives a Hamilton cycle containing every edge of F . Now, we consider the case where D is not a tournament. Since $n \geq 4q \geq 4$ and $s^*(D) \geq 3q$ (which implies that $d(u) + d(v) \geq n + 3q$ for every pair of distinct non-adjacent vertices u and v of D), from Lemma 2.3 it follows that $U(D)$ has a Hamilton cycle containing every edge of F . Therefore, in either case, $U(D)$ has a Hamilton cycle H containing every edge of F . Now, we choose a direction around H in which at least $\lceil (n - 3q)/2 \rceil$ of the edges in $E(H) \setminus F$ are forward.

For $i \in \{1, \dots, q\}$, if the chosen direction around H traverses P_i from c_i to d_i , we leave P_i unchanged; however, if the chosen direction around H traverses P_i from d_i to c_i , we replace P_i with Q_i .

Since P_i and Q_i have the same endvertices and vertex set, and the diamonds $D_1, \dots, D_q$ are pairwise vertex-disjoint, these replacements yield a Hamilton oriented cycle C in D and preserve all edges in $E(H) \setminus F$, together with their traversal directions. Also, we recall that each of P_i and Q_i has at least two forward arcs in its specified traversal direction. Consequently, we have

$$\sigma_{\max}(C) \geq \sigma^+(C) \geq \left\lceil \frac{n - 3q}{2} \right\rceil + 2q = \left\lceil \frac{n + q}{2} \right\rceil.$$

□

For a graph G , the booksize of G is denoted by $bk(G)$ and is defined [3] as the maximum number of triangles having a common edge in G . We also need the following result, due to Edwards [10] (see [3] for its proof).

Lemma 2.5 ([3, 10]). *If G is a graph of order n and size m satisfying $m > n^2/4$, then*

$$bk(G) \geq \frac{2m}{n} - \frac{n}{3}. \quad (2.1)$$

3. Auxiliary results

We start this section with the following elementary but important lemma for our first main result.

Lemma 3.1. *Let D be an oriented graph and let $X \subset V(D)$. Then $s^*(D - X) \geq s^*(D) - |X|$.*

Proof. Let $n = |V(D)|$ and let $q = |X|$. We consider the two possibilities in the definition of $s^*(D - X)$.

First, suppose that $D - X$ is not a tournament. We choose any two distinct non-adjacent vertices u and v of $D - X$. Then, $d_{D-X}(u) \geq d_D(u) - q$ and $d_{D-X}(v) \geq d_D(v) - q$, and hence, we have

$$d_{D-X}(u) + d_{D-X}(v) - (n - q) \geq d_D(u) + d_D(v) - n - q \geq s^*(D) - q,$$

which implies that $s^*(D - X) \geq s^*(D) - q$.

Now, we assume that $D - X$ is a tournament. By definition, $s^*(D - X) = n - q - 2$ and $s^*(D) \leq n - 2$, and hence, we have $s^*(D) - q \leq n - q - 2 = s^*(D - X)$. $\square$

Lemma 3.2. *Let D be an oriented graph containing no good diamond. If $s^*(D) \geq 4$, then $s^*(D) = 4$ and the underlying graph $U(D)$ is isomorphic to one of the following three graphs: K_6 , $K_{2,2,2,2}$, $K_{4,4,4}$.*

Proof. Let $G = U(D)$ and $n = |V(G)|$. Since $s^*(D) \geq 4$, and always $s^*(D) \leq n - 2$, we have $n \geq 6$.

Because D contains no good diamond, Observation 2.2 implies that no edge of G can belong to five triangles. Thus, every edge $uv \in E(G)$ has at most four common neighbors, which implies that $|N_G(u) \cap N_G(v)| \leq 4$, and hence, we have

$$d_G(u) + d_G(v) = |N_G(u) \cup N_G(v)| + |N_G(u) \cap N_G(v)| \leq n + 4. \quad (3.1)$$

For each edge $uv \in E(G)$, equality throughout in (3.1) holds if and only if $|N_G(u) \cap N_G(v)| = 4$ and $N_G(u) \cup N_G(v) = V(G)$. Now, by Lemma 2.2, we have

$$m \geq \frac{n(n+4)}{4}, \quad (3.2)$$

where m is the size of G . By the Cauchy–Schwarz inequality and the identity $\sum_{v \in V(G)} d_G(v) = 2m$, we have

$$\sum_{v \in V(G)} d_G(v)^2 \geq \frac{4m^2}{n}, \quad (3.3)$$

where equality in (3.3) holds if and only if G is regular. (The left-hand side of (3.3) is known as the first Zagreb index in chemical graph theory and its mathematical properties are well studied (particularly, the trivial inequality (3.3) has been derived in the literature many times); for instance, see [2, 4].) Therefore, from (3.1) and (3.3), it follows that

$$\frac{4m^2}{n} \leq \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) \leq m(n+4), \quad (3.4)$$

which yields $m \leq n(n+4)/4$, and hence, (3.2) gives $m = n(n+4)/4$, and equality throughout must hold in each of (3.1), (3.3) and (3.4). Therefore, G is regular with average degree $\frac{2m}{n} = \frac{n+4}{2}$, $|N_G(u) \cap N_G(v)| = 4$ and $N_G(u) \cup N_G(v) = V(G)$ for every edge $uv \in E(G)$.

Now, we consider the complement $\overline{G}$ of G . Since G is $(n+4)/2$ -regular, $\overline{G}$ is $(n-6)/2$ -regular. Suppose that u and v are distinct non-adjacent vertices in $\overline{G}$. Then, $uv \in E(G)$, and from the equality $N_G(u) \cup N_G(v) = V(G)$, it follows that u and v cannot have a common neighbor in $\overline{G}$. Consequently, every component of $\overline{G}$ is complete; otherwise, if a component of $\overline{G}$ is not complete, then the first and third vertices of a shortest path P between two distinct non-adjacent vertices in the considered component are non-adjacent in $\overline{G}$ and have the second vertex of P as a common neighbor in $\overline{G}$, a contradiction.

Since $\overline{G}$ is $(n-6)/2$ -regular, every component of $\overline{G}$ has order $(n-6)/2 + 1 = (n-4)/2$. Thus, n is even, and the number of components of $\overline{G}$ is

$$\frac{n}{(n-4)/2} = 2 + \frac{8}{n-4},$$

and hence, $n - 4 \in \{2, 4, 8\}$ (because $n \geq 6$), or equivalently, $n \in \{6, 8, 12\}$. If $n = 6$, then $\overline{G}$ consists of six isolated vertices, and hence, $G \cong K_6$. If $n = 8$, then $\overline{G}$ consists of four copies of K_2 , and hence, $G \cong K_{2,2,2,2}$. If $n = 12$, then $\overline{G}$ consists of three copies of K_4 , and hence, $G \cong K_{4,4,4}$.

Finally, if $G \cong K_6$, then D is a tournament, and hence, $s^*(D) = n - 2 = 4$. In the other two cases, D is not a tournament and every vertex of D has degree $(n + 4)/2$, and hence, $s^*(D) = 2(n + 4)/2 - n = 4$. $\square$

Lemma 3.3. *Let D be an oriented graph of order n containing no good diamond.*

- (i) *If $s^*(D) \geq 4$, then $n \leq 12$.*
- (ii) *If $s^*(D) \geq 3$, then $n \leq 14$.*
- (iii) *If $s^*(D) \geq 2$, then $n \leq 18$.*
- (iv) *If $s^*(D) \geq 1$, then $n \leq 20$.*

Proof. The first part is an immediate consequence of Lemma 3.2.

For the remaining three parts, we let $G = U(D)$ and let $t \in \{1, 2, 3\}$ such that $s^*(D) \geq t$. Also, let $m = |E(G)|$. Since D contains no good diamond, Observation 2.2 implies that every edge of G is contained in at most four triangles, and hence, we have

$$bk(G) \leq 4, \quad (3.5)$$

where $bk(G)$ is defined right before Lemma 2.5. Since $t \in \{1, 2, 3\}$, by Lemma 2.2, we have $m \geq n(n + t)/4 > n^2/4$, and hence, using Lemma 2.5, we obtain

$$4 \geq bk(G) \geq \frac{2m}{n} - \frac{n}{3} \geq \frac{n + t}{2} - \frac{n}{3} = \frac{n}{6} + \frac{t}{2},$$

which gives

$$n \leq 24 - 3t. \quad (3.6)$$

If $t = 3$, then (3.6) yields $n \leq 15$, and hence, $n \leq 14$ because for the choice $n = 15$, Lemma 2.2 gives $m \geq 68$, and hence, Lemma 2.5 gives $bk(G) > 4$, which contradicts (3.5). This completes the proof of the second part of the lemma.

If $t = 2$, then (3.6) yields $n \leq 18$, which proves the third part of the lemma.

If $t = 1$, then (3.6) yields $n \leq 21$, and hence, $n \leq 20$ because for the choice $n = 21$, Lemma 2.2 gives $m \geq 116$, and hence, Lemma 2.5 gives $bk(G) > 4$, which contradicts (3.5). This completes the proof of the fourth part of the lemma. $\square$

4. First main result

The following result can be considered as an improved version of Theorem 1.1.

Theorem 4.1. *Let k be any positive integer. Let D be an oriented graph of order $n \geq 3$. Then, D contains a Hamilton oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil (n + k)/2 \rceil$ if either of the following holds:*

- (i) *$n + k$ is even and*

$$s^*(D) \geq \begin{cases} 0 & \text{if } k = 1, \\ \lfloor 7(k - 1)/2 \rfloor + 1 & \text{if } 2 \leq k \leq 6, \\ 4k - 7 & \text{if } k \geq 7; \end{cases}$$

(ii) $n + k$ is odd and

$$s^*(D) \geq \begin{cases} \lfloor 7k/2 \rfloor + 1 & \text{if } 1 \leq k \leq 5, \\ 4k - 3 & \text{if } k \geq 6. \end{cases}$$

Proof. First, we consider the case where $n + k$ is even and $k = 1$. Then, n is odd. The condition $s^*(D) \geq 0$ is exactly Ore's condition for the underlying graph $U(D)$, and hence, Ore's theorem [9] gives a Hamilton cycle in $U(D)$. We regard this cycle with the orientations inherited from D . Among these n arcs, one of the two directions around the cycle contains at least $\lceil n/2 \rceil = (n+1)/2 = \lceil (n+k)/2 \rceil$ forward arcs. Thus, the desired conclusion holds in the considered case.

For all remaining cases, let

$$r = \begin{cases} k & \text{if } n + k \text{ is odd,} \\ k - 1 & \text{if } n + k \text{ is even.} \end{cases}$$

Then, we have $r \geq 1$ and $\lceil (n+r)/2 \rceil = \lceil (n+k)/2 \rceil$. Also, the hypothesis becomes

$$s^*(D) \geq \begin{cases} \lfloor 7r/2 \rfloor + 1 & \text{if } 1 \leq r \leq 5, \\ 4r - 3 & \text{if } r \geq 6. \end{cases} \quad (4.1)$$

In particular, (4.1) yields $s^*(D) > 3r$.

Suppose that D is a tournament. Then, $s^*(D) = n - 2$. Since $s^*(D) > 3r$, we have $n - 2 > r$, which yields $n - 1 > (n+r)/2$, and hence $\delta(D) = n - 1 \geq \lceil (n+r)/2 \rceil > n/2$. Therefore, by Theorem 2.1, we have the desired conclusion when D is a tournament.

In the rest of the proof, we assume that D is not a tournament. Let

$$t = \begin{cases} 4 - \lfloor (r-1)/2 \rfloor & \text{if } 1 \leq r \leq 5, \\ 1 & \text{if } r \geq 6. \end{cases} \quad (4.2)$$

Then, the hypothesis (4.1) can be written as

$$s^*(D) \geq 4(r-1) + t. \quad (4.3)$$

Since $s^*(D) \leq n - 2$, from (4.3) we have $n \geq 4(r-1) + t + 2$. Thus, deleting at most $4(r-1)$ vertices of D leaves an oriented graph of order at least 3.

First, we find $r - 1$ pairwise vertex-disjoint good diamonds in D for $r \geq 2$. Suppose that j such diamonds have been chosen and deleted, where $0 \leq j \leq r - 2$, and let D_j be the remaining oriented graph. Since a good diamond has four vertices, by (4.3) and Lemma 3.1, we have

$$s^*(D_j) \geq 4(r-1) + t - 4j. \quad (4.4)$$

We note that the minimum value of the right-hand side of (4.4) is $t + 4$, which occurs at $j = r - 2$. Since $t \geq 1$, from (4.4) it follows that $s^*(D_j) \geq 5$, and hence, by Lemma 3.2, D_j contains a good diamond. Consequently, we conclude that D contains $r - 1$ pairwise vertex-disjoint good diamonds for $r \geq 2$. We delete the vertices of these $r - 1$ good diamonds and denote the remaining oriented graph by H ; when $r = 1$, then we take $H = D$. Then, by Lemma 3.1 and (4.3), we have

$$s^*(H) \geq t. \quad (4.5)$$

If H contains a good diamond, then D has r pairwise vertex-disjoint good diamonds, and hence from Lemma 2.4 it follows that D contains a Hamilton oriented cycle C with $\sigma_{\max}(C) \geq \lceil (n+r)/2 \rceil = \lceil (n+k)/2 \rceil$ because $s^*(D) > 3r$. Thus, in what follows, we assume that H has no good diamond.

By using (4.2), (4.5) and Lemma 3.3, we have

$$|V(H)| \leq \begin{cases} 12 + 2 \lfloor (r-1)/2 \rfloor + 2 \lfloor (r-1)/4 \rfloor & \text{if } 1 \leq r \leq 5, \\ 20 & \text{if } r \geq 6. \end{cases} \quad (4.6)$$

Since $n = 4(r-1) + |V(H)|$, from (4.6) it follows that

$$n \leq \begin{cases} 4r + 2 \lfloor (r-1)/2 \rfloor + 2 \lfloor (r-1)/4 \rfloor + 8 & \text{if } 1 \leq r \leq 5, \\ 4r + 16 & \text{if } r \geq 6. \end{cases} \quad (4.7)$$

Let v be a vertex of minimum degree $\delta(D)$. Since D is not a tournament, we have $\delta(D) \leq n-2$. Thus, we choose a vertex $u \neq v$ such that it is not a neighbor of v . Then, $d_D(u) \leq n-2$. By the definition of $s^*(D)$, we have $\delta(D) + d_D(u) \geq n + s^*(D)$, which together with $d_D(u) \leq n-2$ gives $\delta(D) \geq s^*(D) + 2$, which together with (4.2) and (4.3) yields

$$\delta(D) \geq \begin{cases} 4r - \lfloor (r-1)/2 \rfloor + 2 & \text{if } 1 \leq r \leq 5, \\ 4r - 1 & \text{if } r \geq 6. \end{cases} \quad (4.8)$$

First, we suppose that $r \geq 2$. From (4.7) and (4.8), it follows that

$$2\delta(D) - n - r \geq \begin{cases} 3r - 4 \lfloor (r-1)/2 \rfloor - 2 \lfloor (r-1)/4 \rfloor - 4 & \text{if } 2 \leq r \leq 5, \\ 3r - 18 & \text{if } r \geq 6, \end{cases}$$

which yields $2\delta(D) - n - r \geq 0$ for every $r \geq 2$. Thus, $\delta(D) \geq \lceil (n+r)/2 \rceil > n/2$, and hence, from Theorem 2.1 it follows that D contains a Hamilton oriented cycle C satisfying

$$\sigma_{\max}(C) \geq \delta(D) \geq \left\lceil \frac{n+r}{2} \right\rceil = \left\lceil \frac{n+k}{2} \right\rceil.$$

Finally, we suppose that $r = 1$. Then, $H = D$ and from (4.2), we have $t = 4$. Since D is not a tournament and contains no good diamond, from Lemma 3.2 it follows that $U(D) \cong K_{2,2,2,2}$ or $K_{4,4,4}$. The minimum degrees of these two graphs are 6 and 8, respectively, whereas the corresponding values of $\lceil (n+1)/2 \rceil$ are 5 and 7. Hence, by Theorem 2.1, we have the required cycle. This completes the proof. $\square$

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let k be any integer larger than 5. Let D be an oriented graph of order n . If the inequality $s^*(D) \geq 4k - 3$ holds, then D contains a Hamilton oriented cycle C satisfying*

$$\sigma_{\max}(C) \geq \left\lceil \frac{n+k}{2} \right\rceil.$$

If $n+k$ is even and $k \geq 7$, then the condition $s^*(D) \geq 4k - 3$ can be replaced with $s^*(D) \geq 4k - 7$.

Remark 4.3. *For $k \geq 6$, Corollary 4.2, and therefore Theorem 4.1, strengthens Theorem 1.1: the conditions $s^*(D) \geq 8k$ and $n \geq 30 + 4(k-1)$ are replaced by $s^*(D) \geq 4k - 3$. If $n+k$ is even and $k \geq 7$, then the sufficient bound on $s^*(D)$ is reduced further to $4k - 7$.*

5. Second main result

In this section (particularly, in Theorem 5.3), for fixed k and for every sufficiently large n , we reduce the coefficient of k in the sufficient bound on $s^*(D)$ in Theorem 4.1 from 4 to 3.

For a given graph L , a graph G is said to be L -free if G contains no subgraph isomorphic to L . The maximum number of edges in an L -free graph of order n is called the Turán number, denoted by $\text{ex}(n, L)$. An L -free graph of order n and size $\text{ex}(n, L)$ is called an extremal n -vertex L -free graph.

The union of two vertex-disjoint graphs G_1 and G_2 is denoted by $G_1 \cup G_2$ and is defined as the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Their join, denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

The chromatic number of a graph G is denoted by $\chi(G)$ and is defined as the minimum number of colors required to color the vertices of G such that no two adjacent vertices have the same color. A graph G is said to be p -chromatic if $\chi(G) = p$.

We need the following special case of Theorem 2.2 of [11].

Theorem 5.1 ([11]). *Let L be a fixed graph of order $7q$ such that $\chi(L) = 3$, where $q \geq 2$ is an integer. Assume that L is a subgraph of $P_{7q} + \overline{K}_{7q}$. Suppose further that the following conditions hold:*

- (i) *For every set $X \subset V(L)$ with $|X| = q - 1$, it holds that $\chi(L - X) = 3$.*
- (ii) *There is a set $F \subset E(L)$ with $|F| = q$ such that $\chi(L - F) = 2$.*

Then, for every sufficiently large n , the graph $H(n, 2, q)$ is the unique extremal n -vertex L -free graph, where

$$H(n, 2, q) = K_{q-1} + K_{\lfloor (n-q+1)/2 \rfloor, \lceil (n-q+1)/2 \rceil}. \quad (5.1)$$

Let $B_5 = K_2 + \overline{K}_5$ (see Figure 2), which is the graph formed by five triangles sharing one common edge, and is sometimes referred [3] to as the book of size 5.

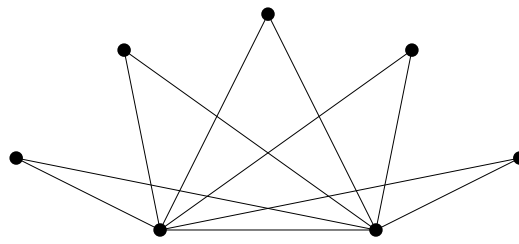


Figure 2. The graph B_5 .

Corollary 5.2. *Let q be a fixed integer larger than 1. Then, for every sufficiently large n , every graph of order n with more than $m(H(n, 2, q))$ edges contains qB_5 (the disjoint union of q copies of B_5), where $H(n, 2, q)$ is defined via (5.1) and*

$$m(H(n, 2, q)) = \binom{q-1}{2} + (q-1)(n-q+1) + \left\lfloor \frac{(n-q+1)^2}{4} \right\rfloor. \quad (5.2)$$

Proof. We note that the graph B_5 , and hence qB_5 , is 3-chromatic. Since the graph G' obtained from qB_5 by deleting any $q-1$ vertices contains B_5 , it follows that the chromatic number of G' remains

3. Also, deleting the common edge from each copy of B_5 in qB_5 gives $qK_{2,5}$, which is 2-chromatic. Moreover, we note that qB_5 is a subgraph of $P_{7q} + \overline{K}_{7q}$. Thus, all hypotheses of Theorem 5.1 hold with $L = qB_5$. Therefore, by Theorem 5.1, for every sufficiently large n , the graph $H(n, 2, q)$ is the unique extremal n -vertex qB_5 -free graph, and hence, every graph of order n and size larger than $m(H(n, 2, q))$ contains qB_5 . $\square$

Now, we are ready to prove the main result of this section.

Theorem 5.3. *For every fixed positive integer k , there exists an integer $n_0(k)$ such that every oriented graph D of order $n \geq n_0(k)$ contains a Hamilton oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil (n+k)/2 \rceil$ provided that one of the following holds:*

- (i) $n+k$ is odd and $s^*(D) \geq 3k$;
- (ii) $n+k$ is even and $s^*(D) \geq 3k-3$.

For $k=1$, in the second case, the conclusion holds for every odd $n \geq 3$ under the condition $s^*(D) \geq 0$.

Proof. The claim made in the last sentence of the theorem follows from the first paragraph of the proof of Theorem 4.1. Hence, in the rest of the proof, we assume that either $n+k$ is odd, or that $n+k$ is even and $k \geq 2$. Let

$$q = \begin{cases} k & \text{if } n+k \text{ is odd,} \\ k-1 & \text{if } n+k \text{ is even.} \end{cases}$$

Then, $q \geq 1$, $\lceil (n+q)/2 \rceil = \lceil (n+k)/2 \rceil$ and the hypothesis becomes $s^*(D) \geq 3q$. We choose $n_0(k) \geq 15$ large enough in such a way that the conclusion of Corollary 5.2 holds for every $q \in \{k-1, k\}$ with $q \geq 2$, whenever $n \geq n_0(k)$. Let $n \geq n_0(k)$.

First, we suppose that $q=1$. Since $n \geq 15$ and $s^*(D) \geq 3$, Lemma 3.3(ii) implies that D contains a good diamond. Hence, by Lemma 2.4, D contains a Hamilton oriented cycle C satisfying

$$\sigma_{\max}(C) \geq \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{n+k}{2} \right\rceil.$$

In the rest of the proof, we assume that $q \geq 2$. Since $s^*(D) \geq 3q$, from Lemma 2.2, it follows that $m(U(D)) \geq n(n+3q)/4$, which together with (5.2) yields

$$\begin{aligned} m(U(D)) - m(H(n, 2, q)) &\geq \frac{n(n+3q)}{4} - \left[\binom{q-1}{2} + (q-1)(n-q+1) + \frac{(n-q+1)^2}{4} \right] \\ &= \frac{nq + 2n + q^2 - 1}{4} > 0. \end{aligned}$$

Thus, by Corollary 5.2, $U(D)$ contains qB_5 . By Observation 2.2, every oriented copy of B_5 in D contains a good diamond. Since these q diamonds are pairwise vertex-disjoint in D and $s^*(D) \geq 3q$, from Lemma 2.4 it follows that D contains a Hamilton oriented cycle C satisfying

$$\sigma_{\max}(C) \geq \left\lceil \frac{n+q}{2} \right\rceil = \left\lceil \frac{n+k}{2} \right\rceil,$$

as required. $\square$

6. Conclusions

Ai, Guo, Gutin, Lan, Shao, Yeo and Zhou [1] proved that, for any positive integer k , every oriented graph D of order $n \geq 30 + 4(k - 1)$ with $s^*(D) \geq 8k$ contains a Hamilton oriented cycle C satisfying $\sigma_{\max}(C) \geq \lceil (n + k)/2 \rceil$. They asked whether the factor 8 in the bound $s^*(D) \geq 8k$ can be reduced. For every $k \geq 6$, we have shown that the aforementioned conclusion of Ai, Guo, Gutin, Lan, Shao, Yeo and Zhou [1] follows from $s^*(D) \geq 4k - 3$; however, when $n + k$ is even and $k \geq 7$, then the condition $s^*(D) \geq 4k - 7$ is sufficient (see Corollary 4.2). We also obtain results for $1 \leq k \leq 5$; see Theorem 4.1. Moreover, for every fixed positive integer k and all sufficiently large n , we prove that the same conclusion follows from $s^*(D) \geq 3k$ when $n + k$ is odd and from $s^*(D) \geq 3k - 3$ when $n + k$ is even; see Theorem 5.3.

Use of Generative-AI tools declaration

The authors used ChatGPT to assist with drafting and language editing during the preparation of this manuscript. The authors subsequently reviewed, substantially revised, and corrected the AI-assisted text, verified its accuracy, and take full responsibility for the final manuscript.

Author contributions

Investigation: A.A. and A.M.A.; validation: A.A. and A.M.A.; writing original draft preparation: A.A. and A.M.A.; writing review and editing: A.A. and A.M.A.; project administration: A.M.A. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare that they have no competing interests.

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