



Research article

Hausdorff dimension of the graph of the unweighted error-sum function of continued fractions

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Abstract: We study the unweighted error-sum function $\mathcal{E}(x) := \sum_{n \geq 0} (x - p_n(x)/q_n(x))$, where $p_n(x)/q_n(x)$ denotes the n th convergent of the continued fraction expansion of $x \in \mathbb{R}$, and we prove that the graph of $\mathcal{E}$ has Hausdorff dimension 1. The key ingredient is an exact formula for the oscillation of $\mathcal{E}$ on fundamental intervals, which we obtain in a symbolic space of continued fraction sequences containing non-realizable sequences. Using the same framework, we also give a new derivation of the upper bound $3/2$ of Ridley and Petruska (2000) for the Hausdorff dimension of the graph of the relative error-sum function $P(x) := \sum_{n \geq 0} (q_n(x)x - p_n(x))$. Moreover, we present an alternative argument based on Möbius inversion and summation over coprime convergent denominators, and we explain why the square coverings used for P cannot yield an upper bound smaller than $3/2$.

Keywords: continued fraction; error-sum function; Hausdorff dimension; graph of a function; symbolic space

Mathematics Subject Classification: 11A55, 11K50, 26A18, 28A80, 33E20, 37E05

1. Introduction

Given $x \in \mathbb{R}$, we denote the fractional part of x by $\{x\} := x - \lfloor x \rfloor \in [0, 1)$, and set

$$d_1(x) := \begin{cases} \lfloor 1/\{x\} \rfloor, & \text{if } \{x\} \neq 0; \\ \infty, & \text{if } \{x\} = 0, \end{cases} \quad \text{and} \quad T(x) := \begin{cases} 1/\{x\} - d_1(x), & \text{if } \{x\} \neq 0; \\ 0, & \text{if } \{x\} = 0. \end{cases} \quad (1.1)$$

The map $T: \mathbb{R} \rightarrow [0, 1)$ is called the *continued fraction (CF) transformation*. For each $n \in \mathbb{N}$, we define

$$d_n(x) := d_1(T^{n-1}(x)), \quad (1.2)$$

where $d_n(x)$ is called the n th *digit* of the CF expansion of x . Setting $d_0(x) := \lfloor x \rfloor$, the algorithms (1.1) and (1.2) yield the unique expansion $x = [d_0(x); d_1(x), d_2(x), \dots]$. Here, for any sequence $(\sigma_0, \sigma_1, \sigma_2, \dots) \in \mathbb{Z} \times (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$, we define

$$[\sigma_0; \sigma_1, \sigma_2, \sigma_3, \dots] := \sigma_0 + \frac{1}{\sigma_1 + \frac{1}{\sigma_2 + \frac{1}{\sigma_3 + \ddots}}},$$

with the conventions $1/\infty = 0$, $\infty + \infty = \infty$, and $\infty + c = \infty$ for any $c \in \mathbb{R}$. If $x = [d_0(x); d_1(x), \dots, d_n(x), \infty, \infty, \dots]$ with $d_n(x) < \infty$, we say that x has a *finite* CF expansion of *length* n and write $x = [d_0(x); d_1(x), \dots, d_n(x)]$. In this case, $d_n(x) > 1$ whenever $n \geq 1$ (see Proposition 2.5 below).

We remark that the above construction ensures the uniqueness of the CF expansion. Without it, every rational number in $(0, 1)$ would admit two distinct finite continued fraction expansions in the classical sense. Indeed, if x has a finite CF expansion of length n , then its last digit satisfies $d_n(x) = (d_n(x) - 1) + \frac{1}{1}$, with $d_n(x) - 1 \in \mathbb{N}$, which produces an alternative longer expansion. Thus, the term “CF expansion” throughout this paper refers specifically to the expansion generated by Algorithms (1.1) and (1.2).

In 2000, Ridley and Petruska [15] studied the *error-sum function* for CF expansions $P: \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$P(x) := \sum_{n \geq 0} (q_n(x)x - p_n(x)),$$

where $(p_n(x)/q_n(x))_{n \geq 0}$ denotes the sequence of *convergents* to $x \in \mathbb{R}$, given by

$$\frac{p_n(x)}{q_n(x)} := [d_0(x); d_1(x), d_2(x), \dots, d_n(x), \infty, \infty, \dots],$$

with initial values $p_0(x) := d_0(x)$, $q_0(x) := 1$, and $\gcd(p_n(x), q_n(x)) = 1$ for all $n \geq 1$. They referred to P as an error-sum function because each term $q_n(x)x - p_n(x)$ measures the approximation error $x - p_n(x)/q_n(x)$ of the n th convergent, scaled relative to the spacing between adjacent rationals with denominator $q_n(x)$. Among other properties, they investigated the continuity structure of P and the Hausdorff dimension of its graph. They also showed that the graph of P exhibits a fractal-like appearance, although it lacks strict self-similarity (see Figure 1). Despite the passage of more than two decades, the exact Hausdorff dimension of the graph has not been determined, and their original upper bound has not been improved. Thus, determining the Hausdorff dimension of the graph of P remains a natural open problem at the intersection of number theory and fractal geometry. For further developments on error-sum functions in CF expansions, see, e.g., [6, 7] by Elsner and [8, 9] by Elsner and Stein.

Inspired by Shen and Wu’s work [17] on the unweighted error-sum function of the Lüroth series and our previous study [1] on the Pierce expansion, we consider the analogous unweighted error-sum function associated with the CF expansion. This function $\mathcal{E}: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\mathcal{E}(x) := \sum_{n \geq 0} \left(x - \frac{p_n(x)}{q_n(x)} \right)$$

for each $x \in \mathbb{R}$. To distinguish the two functions studied in this paper, we henceforth refer to P as the *relative error-sum function* and $\mathcal{E}$ as the *unweighted error-sum function*.

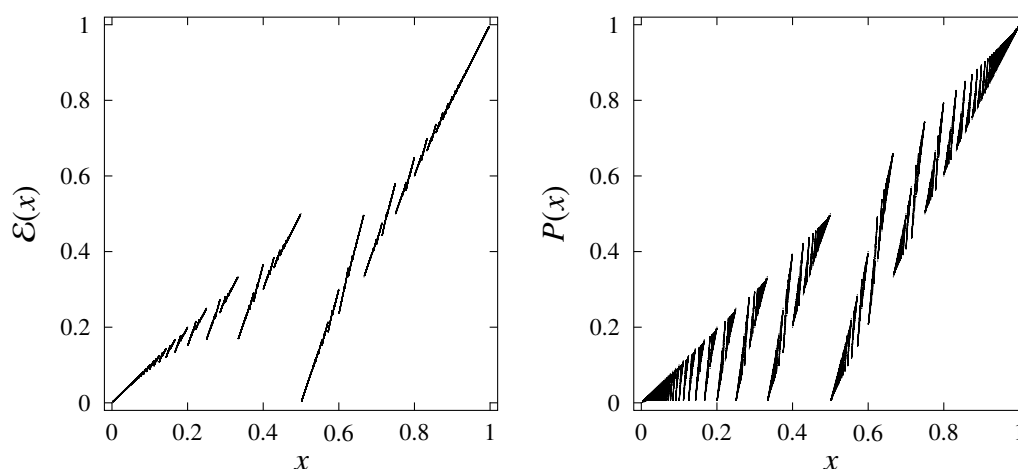


Figure 1. The graphs of the unweighted error-sum function $\mathcal{E}$ (left) and the relative error-sum function P (right) on $[0, 1)$, plotted at all rational points $p/q \in [0, 1)$ with $q \leq 300$. For comparison, Ridley and Petruska [15] plotted P on $(-1, 1)$ at rational points with denominators at most 60.

Error-sum functions have also been defined and studied for various other representations of real numbers. For example, see [5] for the tent map base series, [17] for the Lüroth series, [16] for the alternating Lüroth series, [4] for the α -Lüroth series, [13] for the alternating Sylvester series, and our previous work [1] for the Pierce expansion. In particular, the graph of the error-sum function was shown to have Hausdorff dimension 1 in [1, 4, 13, 16, 17], where, in the case of the α -Lüroth series, the partition α is assumed to have the eventually decreasing property [4]. For the Lüroth-type series, the length of a fundamental interval of order n is a product of factors each of which depends on a single digit, and this product structure facilitates the estimates of the covering sums. For continued fractions, however, the length of a fundamental interval depends on the digits only through the convergent denominators q_n and q_{n-1} , which are determined by a recurrence relation. Moreover, every rational number in $(0, 1)$ admits two symbolic representations of different lengths (see Proposition 2.4 below). These two features are the main sources of difficulty in the present setting.

In this paper, we determine the Hausdorff dimension of the graph of the unweighted error-sum function $\mathcal{E}$, and we provide a new derivation of a known upper bound for the graph of the relative error-sum function P . The main results are as follows.

Theorem 1.1. *The graph of the unweighted error-sum function $\mathcal{E}: \mathbb{R} \rightarrow \mathbb{R}$ has Hausdorff dimension 1.*

Theorem 1.2 ([15, Theorem 3.2]). *The graph of the relative error-sum function $P: \mathbb{R} \rightarrow \mathbb{R}$ has Hausdorff dimension at most $3/2$.*

Theorem 1.1 is the main result of this paper. On the other hand, Theorem 1.2 is due to Ridley and Petruska [15], and we do not improve their upper bound; instead, we provide a new derivation of it within the framework developed for $\mathcal{E}$.

The key ingredient in the proof of Theorem 1.1 is an exact formula for the oscillation of $\mathcal{E}$ on fundamental intervals. More precisely, for any $n \in \mathbb{N}$ and any digit block σ of length n , the oscillation

of $\mathcal{E}$ on the corresponding fundamental interval I_σ equals $n + 1$ times the length of I_σ (Lemma 3.9). Combining this formula with the Fibonacci lower bound for the convergent denominators, we obtain a short covering argument (Section 4). The same framework yields the exact oscillation of P on fundamental intervals, from which Theorem 1.2 follows (Section 5). In Section 6, we present an alternative argument, in which we convert the relevant covering sums into sums over coprime pairs of convergent denominators and evaluate them by means of Möbius inversion. This argument is not needed for the proofs of the main results, but it provides exact formulas for the covering series. Together with Remark 5.2, it also explains why the square coverings used for P in this paper cannot yield an upper bound smaller than $3/2$ for the Hausdorff dimension of the graph of P .

We remark that the statement of Theorem 1.1 also appears in [3, Theorem 5]. However, the proof given there is incomplete. The bound used there for the oscillation of $\mathcal{E}$ on fundamental intervals is not justified, and in the final step, the sum of the lengths of $I_{(a_1, \dots, a_i)}$ over all digit blocks $(a_1, \dots, a_n) \in \mathbb{N}^n$ is taken to be 1 for every $i \leq n$, although it is infinite for $i < n$. We give a complete proof using the exact oscillation formula in Lemma 3.9.

Our strategy follows our previous work [1] on Pierce expansions. Namely, we lift the error-sum function to a symbolic space containing finite, possibly non-realizable, digit sequences, prove the continuity of the lifted function, compute its oscillation on cylinder sets, and cover the graph by columns over fundamental intervals. The symbolic space of CF sequences and its topology were introduced in [2]. However, the following ingredients are specific to continued fractions. Firstly, every rational number in $(0, 1)$ has two symbolic representations, one of which is non-realizable. Consequently, the fundamental interval associated with a non-realizable sequence is open, and the supremum or infimum of $\mathcal{E}$ over a fundamental interval is not attained in general. The non-realizable sequences in the symbolic space provide the missing endpoint values (Lemmas 3.1 and 3.2), which allows us to compute the oscillations exactly (Lemmas 3.8 and 3.9). Secondly, for P , the weights $q_k(x)$ are unbounded, so that the continuity of the lifted function at finite sequences and the monotonicity of the scaled approximation errors require separate arguments (Lemmas 3.3 and 3.6). Thirdly, the alternative argument in Section 6 relies on the correspondence between finite CF sequences and pairs of coprime convergent denominators (Lemma 6.4).

We denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{N}_0$ the set of non-negative integers, and by $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ the set of extended positive integers. Let $\mathbb{I} := [0, 1) \setminus \mathbb{Q}$ denote the set of irrational numbers in the interval $[0, 1)$. The Fibonacci sequence $(F_n)_{n \in \mathbb{N}_0}$ is defined recursively by $F_0 := 0$, $F_1 := 1$, and $F_{n+2} := F_{n+1} + F_n$ for all $n \in \mathbb{N}_0$. Denote by $\gamma := (1 + \sqrt{5})/2$ the golden ratio. By Binet's formula, or by induction on n , we have

$$F_n \geq \gamma^{n-2} \quad \text{for each } n \in \mathbb{N}. \quad (1.3)$$

We denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$, by $\mathcal{H}^s$ the s -dimensional Hausdorff measure, and by $\dim_{\mathcal{H}}$ the Hausdorff dimension.

The paper is organized as follows. In Section 2, we recall some properties of continued fractions and of the symbolic space of CF sequences. In Section 3, we determine the oscillation of $\mathcal{E}$ and P on fundamental intervals. We prove Theorems 1.1 and 1.2 in Sections 4 and 5, respectively. In Section 6, we present the alternative argument via coprime summation and Möbius inversion. Finally, Section 7 contains some concluding remarks.

2. Preliminaries

In this section, we recall some basic properties of continued fractions. Arithmetic properties can be found in the classical references [11, 12, 14], while topological properties of the CF digit coding map $x \mapsto (d_1(x), d_2(x), \dots)$ can be found in [2].

The relative error-sum function P is periodic with period 1 (see [15, Proposition 1.2]). Likewise, the unweighted error-sum function $\mathcal{E}$ is also 1-periodic. We give a proof which works for both functions.

Proposition 2.1. *For any $x \in \mathbb{R}$, we have $\mathcal{E}(x+1) = \mathcal{E}(x)$ and $P(x+1) = P(x)$.*

Proof. Let $x \in \mathbb{R}$. Since the CF expansions of x and $\{x\}$ differ only in their integer parts, we have $p_n(x)/q_n(x) = \lfloor x \rfloor + p_n(\{x\})/q_n(\{x\})$ for each $n \geq 0$. Here, both fractions are in lowest terms, and hence $q_n(x) = q_n(\{x\})$ and $p_n(x) = \lfloor x \rfloor q_n(x) + p_n(\{x\})$ for each $n \geq 0$. It follows that

$$x - \frac{p_n(x)}{q_n(x)} = \{x\} - \frac{p_n(\{x\})}{q_n(\{x\})} \quad \text{and} \quad q_n(x)x - p_n(x) = q_n(\{x\})\{x\} - p_n(\{x\})$$

for each $n \geq 0$. Summing over $n \geq 0$, we obtain $\mathcal{E}(x) = \mathcal{E}(\{x\})$ and $P(x) = P(\{x\})$. Since $\{x+1\} = \{x\}$, the proposition follows. $\square$

In view of Proposition 2.1, we henceforth restrict our attention to the half-open unit interval $[0, 1)$. Moreover, since $d_0(x) = 0$ for all $x \in [0, 1)$, we shall write $[d_1(x), d_2(x), \dots]$ instead of $[0; d_1(x), d_2(x), \dots]$ for convenience.

Define the CF digit coding map $f: [0, 1) \rightarrow \mathbb{N}_\infty^\mathbb{N}$ by

$$f(x) := (d_k(x))_{k \in \mathbb{N}} = (d_1(x), d_2(x), d_3(x), \dots)$$

for each $x \in [0, 1)$.

We now introduce the symbolic space of CF digit sequences. This symbolic space, denoted by Σ , was extensively studied in [2] in connection with the topological properties of the map f . Let $\Sigma_0 := \{\infty\}^\mathbb{N}$, and for each $n \in \mathbb{N}$, define

$$\Sigma_n := \mathbb{N}^n \times \{\infty\}^{\mathbb{N} \setminus \{1, \dots, n\}}.$$

For notational convenience, we will occasionally write an element $(\sigma_k)_{k \in \mathbb{N}} \in \Sigma_n$ as the finite sequence $(\sigma_1, \dots, \sigma_n)$, with the understanding that it denotes the infinite sequence $(\sigma_1, \dots, \sigma_n, \infty, \infty, \dots)$. Define

$$\Sigma_\infty := \mathbb{N}^\mathbb{N},$$

and set

$$\Sigma := \bigcup_{n \in \mathbb{N}_0} \Sigma_n \cup \Sigma_\infty.$$

We refer to each $\sigma \in \Sigma$ as a CF sequence. A sequence $\sigma := (\sigma_k)_{k \in \mathbb{N}} \in \Sigma$ is called *realizable* if there exists $x \in [0, 1)$ such that $d_k(x) = \sigma_k$ for all $k \in \mathbb{N}$. Let $\Sigma_{\text{re}} \subseteq \Sigma$ denote the set of all realizable CF sequences.

We remark that the enlarged space Σ , rather than the set Σ_{re} of digit sequences of numbers in $[0, 1)$, is essential for our purposes. In addition to the realizable sequences, the space Σ contains

non-realizable finite sequences such as $(\sigma_1, \dots, \sigma_{n-1}, 1)$. Such sequences represent the endpoints of fundamental intervals which are not attained in the intervals themselves (see Proposition 2.6 and Example 2.7 below). They supply the missing endpoint values which are needed to compute the exact oscillation of the error-sum functions on fundamental intervals in Section 3.

For each $\sigma := (\sigma_k)_{k \in \mathbb{N}} \in \Sigma$ and $n \in \mathbb{N}$, define

$$\sigma^{(n)} := (\tau_k)_{k \in \mathbb{N}} \in \Sigma \quad \text{by} \quad \tau_k := \begin{cases} \sigma_k, & \text{if } 1 \leq k \leq n; \\ \infty, & \text{if } k > n. \end{cases}$$

Thus, $\sigma^{(n)} = (\sigma_1, \dots, \sigma_n, \infty, \infty, \dots)$. For notational consistency, we also define $\sigma^{(0)} := (\infty, \infty, \dots) \in \Sigma_0$.

Let $\sigma := (\sigma_k)_{k \in \mathbb{N}} \in \Sigma$. Define $\varphi_0(\sigma) := 0$, and for each $k \in \mathbb{N}$, define

$$\varphi_k(\sigma) := [\sigma_1, \sigma_2, \dots, \sigma_k, \infty, \infty, \dots].$$

For each $\sigma \in \bigcup_{n \in \mathbb{N}_0} \Sigma_n$, define

$$\varphi(\sigma) := \lim_{k \rightarrow \infty} \varphi_k(\sigma),$$

which exists because the sequence is eventually constant.

Proposition 2.2 ([12, Proposition 1.1.2]). *Let $\sigma \in \Sigma_\infty$. Then the limit $\varphi(\sigma) := \lim_{k \rightarrow \infty} \varphi_k(\sigma)$ exists and satisfies $\varphi(\sigma) \in (0, 1) \setminus \mathbb{Q}$. Moreover, the CF expansion digits of $\varphi(\sigma)$ are precisely given by σ , i.e., $\sigma = f(\varphi(\sigma))$.*

Thus, φ is well defined on Σ , and hence determines a map $\varphi: \Sigma \rightarrow [0, 1]$.

For each $\sigma \in \Sigma$ and $n \in \mathbb{N}$, define

$$\frac{p_n^*(\sigma)}{q_n^*(\sigma)} := \varphi_n(\sigma),$$

where $\gcd(p_n^*(\sigma), q_n^*(\sigma)) = 1$. We extend this notation by defining $p_0^*(\sigma) := 0$, $q_0^*(\sigma) := 1$, $p_{-1}^*(\sigma) := 1$, and $q_{-1}^*(\sigma) := 0$. Note that $p_n^*(\sigma)$ and $q_n^*(\sigma)$ depend only on the first n terms $\sigma_1, \dots, \sigma_n$ of σ .

We now summarize several standard properties of continued fraction convergents, which will be repeatedly used throughout the paper.

Proposition 2.3 (See [2, 11, 12, 14, 15]). *For any $\sigma := (\sigma_k)_{k \in \mathbb{N}} \in \Sigma$, the following hold.*

(i) *If $\sigma \in \Sigma_\infty$, then*

$$p_k^*(\sigma) = \sigma_k p_{k-1}^*(\sigma) + p_{k-2}^*(\sigma), \quad q_k^*(\sigma) = \sigma_k q_{k-1}^*(\sigma) + q_{k-2}^*(\sigma)$$

for each $k \in \mathbb{N}$.

If $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, then the same equalities hold for all $1 \leq k \leq n$, and

$$p_k^*(\sigma) = p_n^*(\sigma), \quad q_k^*(\sigma) = q_n^*(\sigma)$$

for all $k \geq n$. In particular, $1 = q_0^*(\sigma) \leq q_1^*(\sigma) \leq q_2^*(\sigma) \leq \dots$ for every $\sigma \in \Sigma$.

(ii) Suppose $\sigma := (\sigma_1, \dots, \sigma_{n-1}, \sigma_n) \in \Sigma_n$ for some $n \in \mathbb{N}$ with $\sigma_n > 1$, and define

$$\sigma' := (\sigma_1, \dots, \sigma_{n-1}, \sigma_n - 1, 1) \in \Sigma_{n+1}.$$

Then

$$\begin{cases} p_k^*(\sigma') = p_k^*(\sigma), & q_k^*(\sigma') = q_k^*(\sigma), & \text{for } 1 \leq k \leq n-1; \\ p_n^*(\sigma') = p_n^*(\sigma) - p_{n-1}^*(\sigma), & q_n^*(\sigma') = q_n^*(\sigma) - q_{n-1}^*(\sigma); \\ p_k^*(\sigma') = p_n^*(\sigma), & q_k^*(\sigma') = q_n^*(\sigma), & \text{for } k \geq n+1. \end{cases}$$

(iii) If $\sigma \in \Sigma_\infty$, then $q_k^*(\sigma) \geq F_{k+1}$ for each $k \in \mathbb{N}_0$.

If $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, then the same inequality holds for all $0 \leq k \leq n$.

(iv) If $\sigma \in \Sigma_\infty$, then

$$p_{k+1}^*(\sigma)q_k^*(\sigma) - p_k^*(\sigma)q_{k+1}^*(\sigma) = (-1)^k$$

for each $k \geq -1$.

If $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, then the same identity holds for all $-1 \leq k \leq n-1$.

(v) If $\sigma \in \Sigma_\infty$, then

$$\frac{1}{q_k^*(\sigma)(q_{k+1}^*(\sigma) + q_k^*(\sigma))} < (-1)^k(\varphi(\sigma) - \varphi_k(\sigma)) < \frac{1}{q_k^*(\sigma)q_{k+1}^*(\sigma)}$$

for each $k \in \mathbb{N}_0$.

If $\sigma \in \Sigma_n$ for some $n \geq 2$, then

$$\frac{1}{q_k^*(\sigma)(q_{k+1}^*(\sigma) + q_k^*(\sigma))} \leq (-1)^k(\varphi(\sigma) - \varphi_k(\sigma)) < \frac{1}{q_k^*(\sigma)q_{k+1}^*(\sigma)}$$

for each $0 \leq k \leq n-2$, where the first inequality is strict unless $k = n-2$ and $\sigma_n = 1$. If $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, then

$$\varphi(\sigma) - \varphi_{n-1}(\sigma) = \frac{(-1)^{n-1}}{q_{n-1}^*(\sigma)q_n^*(\sigma)} \quad \text{and} \quad \varphi(\sigma) - \varphi_k(\sigma) = 0 \quad \text{for all } k \geq n.$$

For every $\sigma \in \Sigma$,

$$0 \leq (-1)^k(\varphi(\sigma) - \varphi_k(\sigma))q_k^*(\sigma) \leq \frac{1}{F_{k+2}} \leq \frac{1}{\gamma^k}$$

for all $k \in \mathbb{N}_0$.

(vi) Let $n \in \mathbb{N}$ and $\sigma := (\sigma_1, \sigma_2, \dots) \in \Sigma$. If $\sigma \in \Sigma_k$ for some $k \geq n$ or $\sigma \in \Sigma_\infty$, then

$$\frac{q_{n-1}^*(\sigma)}{q_n^*(\sigma)} = [\sigma_n, \dots, \sigma_1].$$

We remark that the equality in the first inequality of part (v) can occur only for non-realizable sequences. Indeed, if $\sigma \in \Sigma_n$ with $n \geq 2$ and $\sigma_n = 1$, then $\varphi(\sigma) = \varphi(\tilde{\sigma})$ for $\tilde{\sigma} := (\sigma_1, \dots, \sigma_{n-2}, \sigma_{n-1} + 1) \in \Sigma_{n-1}$, and by part (i), we have $q_{n-1}^*(\tilde{\sigma}) = q_{n-1}^*(\sigma) + q_{n-2}^*(\sigma)$. Applying the identity for $\varphi(\tilde{\sigma}) - \varphi_{n-2}(\tilde{\sigma})$ in part (v), we obtain

$$(-1)^{n-2}(\varphi(\sigma) - \varphi_{n-2}(\sigma)) = \frac{1}{q_{n-2}^*(\sigma)(q_{n-1}^*(\sigma) + q_{n-2}^*(\sigma))}.$$

It is well known that every non-integer rational number has two distinct finite continued fraction expressions. Specifically, if $x \in (0, 1) \cap \mathbb{Q}$, then

$$[d_1(x), \dots, d_{n-1}(x), d_n(x)] = [d_1(x), \dots, d_{n-1}(x), d_n(x) - 1, 1]$$

(see, e.g., [11, Theorem 162]). The expression on the left is the CF expansion generated by (1.1) and (1.2), whereas the digit sequence corresponding to the expression on the right is not realizable.

The following proposition reformulates the preceding discussion in terms of the maps f and φ . It describes the fibers of φ in the rational and irrational cases.

Proposition 2.4 ([2, Proposition 2.4]). *Let $x \in [0, 1)$. Then the following hold.*

(i) *If $x \in (0, 1) \cap \mathbb{Q}$, then $\varphi^{-1}(\{x\}) = \{\sigma, \sigma'\}$, where for some $n \in \mathbb{N}$,*

$$\begin{aligned}\sigma &:= (d_1(x), d_2(x), \dots, d_{n-1}(x), d_n(x)) = f(x) \in \Sigma_n \cap \Sigma_{\text{re}}, \\ \sigma' &:= (d_1(x), d_2(x), \dots, d_{n-1}(x), d_n(x) - 1, 1) \in \Sigma_{n+1} \setminus \Sigma_{\text{re}}.\end{aligned}$$

(ii) *If $x \in \{0\} \cup \mathbb{I}$, then $\varphi^{-1}(\{x\}) = \{\sigma\}$, where $\sigma := f(x)$. Specifically, $\sigma \in \Sigma_\infty$ if $x \in \mathbb{I}$, and $\sigma = (\infty, \infty, \dots) \in \Sigma_0$ if $x = 0$.*

In particular, $\varphi(f(x)) = x$ for every $x \in [0, 1)$.

The next proposition provides a complete characterization of realizable CF sequences in the symbolic space Σ . The result follows directly from Proposition 2.4, so the proof is omitted.

Proposition 2.5. *Let $\sigma \in \Sigma$. Then $\sigma \in \Sigma_{\text{re}}$ if and only if one of the following holds.*

- (i) $\sigma \in \Sigma_0$.
- (ii) $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$ with $\sigma_n > 1$.
- (iii) $\sigma \in \Sigma_\infty$.

In particular, if $x \in [0, 1)$ has a finite CF expansion of length $n \in \mathbb{N}$, then $d_n(x) > 1$, as mentioned in Section 1.

Let $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$. The *cylinder set* determined by σ is defined by

$$\Upsilon_\sigma := \{v \in \Sigma : v^{(n)} = \sigma\}.$$

The corresponding *fundamental interval* is defined by

$$I_\sigma := \{x \in [0, 1) : d_k(x) = \sigma_k \text{ for all } 1 \leq k \leq n\} = f^{-1}(\Upsilon_\sigma).$$

For convenience, we also define the sequence $\widehat{\sigma} \in \Sigma_n$ by

$$\widehat{\sigma} = (\sigma_1, \dots, \sigma_{n-1}, \sigma_n + 1).$$

The following classical result gives an explicit description of the fundamental interval I_σ .

Proposition 2.6 ([12, Theorem 1.2.2]). *Let $n \in \mathbb{N}$ and $\sigma := (\sigma_k)_{k \in \mathbb{N}} \in \Sigma_n$. Then*

$$I_\sigma = \begin{cases} (\varphi(\widehat{\sigma}), \varphi(\sigma)], & \text{if } n \text{ is odd;} \\ [\varphi(\sigma), \varphi(\widehat{\sigma})), & \text{if } n \text{ is even,} \end{cases} \quad \text{or} \quad I_\sigma = \begin{cases} (\varphi(\widehat{\sigma}), \varphi(\sigma)), & \text{if } n \text{ is odd;} \\ (\varphi(\sigma), \varphi(\widehat{\sigma})), & \text{if } n \text{ is even,} \end{cases}$$

where the former case holds if $\sigma \in \Sigma_{\text{re}}$ and the latter if $\sigma \notin \Sigma_{\text{re}}$. Consequently, in either case,

$$\mathcal{L}(I_\sigma) = |\varphi(\sigma) - \varphi(\widehat{\sigma})| = \frac{1}{q_n^*(\sigma)(q_n^*(\sigma) + q_{n-1}^*(\sigma))}. \quad (2.1)$$

The following example illustrates why the endpoint $\varphi(\sigma)$ is excluded when $\sigma \notin \Sigma_{\text{re}}$.

Example 2.7. Consider the sequences $\sigma := (2) \in \Sigma_1 \cap \Sigma_{\text{re}}$ and $\sigma' := (1, 1) \in \Sigma_2 \setminus \Sigma_{\text{re}}$. Both are mapped to the same value under φ , with $\varphi(\sigma) = \varphi(\sigma') = \frac{1}{2}$. By definition, $I_\sigma = \left(\frac{1}{3}, \frac{1}{2}\right]$ and $I_{\sigma'} = \left(\frac{1}{2}, \frac{2}{3}\right)$, so I_σ contains the endpoint $\varphi(\sigma)$, whereas $I_{\sigma'}$ does not contain $\varphi(\sigma')$.

The following proposition summarizes the main results of [2] concerning the topology of the symbolic space Σ .

Proposition 2.8 ([2]). *There exists a topology $\mathcal{T}_\Sigma$ on Σ such that the topological space $(\Sigma, \mathcal{T}_\Sigma)$ is compact metrizable. With this topology, the following hold.*

- (i) *For each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, both the cylinder set Υ_σ and its complement $\Sigma \setminus \Upsilon_\sigma$ are closed and compact in $(\Sigma, \mathcal{T}_\Sigma)$.*
- (ii) *Equip $\mathbb{N}_\infty^\mathbb{N}$ with the product topology, where $\mathbb{N}_\infty$ denotes the one-point compactification of the discrete space $\mathbb{N}$. Define $g: \mathbb{N}_\infty^\mathbb{N} \rightarrow (\Sigma, \mathcal{T}_\Sigma)$ as follows. For $\sigma := (\sigma_k)_{k \in \mathbb{N}} \in \mathbb{N}_\infty^\mathbb{N}$, if there exists a smallest $n \in \mathbb{N}_0$ such that $\sigma_{n+1} = \infty$, set*

$$g(\sigma) := (\sigma_1, \dots, \sigma_n, \infty, \infty, \dots) \in \Sigma_n.$$

If no such n exists (that is, if $\sigma_k \in \mathbb{N}$ for all k), set $g(\sigma) := \sigma \in \Sigma_\infty$. Then g is a quotient map.

- (iii) *The map $\varphi: (\Sigma, \mathcal{T}_\Sigma) \rightarrow [0, 1]$ is continuous.*
- (iv) *The continued fraction digit coding map $f: [0, 1) \rightarrow (\Sigma, \mathcal{T}_\Sigma)$ satisfies the following.*
 - (a) *f is continuous at every irrational point and at the left endpoint 0.*
 - (b) *At each rational point $x \in (0, 1) \cap \mathbb{Q}$, the map f is discontinuous. More precisely, if $\varphi^{-1}(\{x\}) = \{\sigma, \tau\}$, then*

$$\lim_{\substack{t \rightarrow x \\ t \in I_\sigma}} f(t) = \sigma \quad \text{and} \quad \lim_{\substack{t \rightarrow x \\ t \notin I_\sigma}} f(t) = \tau.$$

The following lemma is an immediate consequence of the construction in [2]. Although it was not stated explicitly there, we record it for later reference.

Lemma 2.9. *For each $n \in \mathbb{N}_0$, the map $\varphi_n: (\Sigma, \mathcal{T}_\Sigma) \rightarrow [0, 1]$ is continuous.*

Proof. Since φ_0 is constantly 0 by definition, it is continuous.

Now, let $n \in \mathbb{N}$ and define

$$\widetilde{\varphi}_n(\sigma) := [\sigma_1, \dots, \sigma_n, \infty, \infty, \dots], \quad \sigma \in \mathbb{N}_\infty^\mathbb{N}.$$

Since finite continued fractions depend continuously on their digits (with the convention $1/\infty = 0$), the map $\widetilde{\varphi}_n$ is continuous. By definition, $\widetilde{\varphi}_n = \varphi_n \circ g$. Since g is a quotient map by Proposition 2.8(ii), the continuity of $\widetilde{\varphi}_n$ implies the continuity of φ_n . $\square$

3. Exact oscillation of the error-sum functions on fundamental intervals

In this section, we determine the oscillation of $\mathcal{E}$ and P on each fundamental interval (Lemma 3.9). In order to treat the two functions simultaneously, for each $\lambda \in \{0, 1\}$, we define $\mathcal{E}_\lambda: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\mathcal{E}_\lambda(x) := \sum_{n \geq 0} (q_n(x))^\lambda \left(x - \frac{p_n(x)}{q_n(x)} \right).$$

Then $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_1 = P$. Some of the results in this section are analogues of those established for Pierce expansions in [1]. (See Section 1 for the main differences.)

Lemma 3.1. *Let $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$. Then the following hold.*

(i) *We have the strict inclusion $f(I_\sigma) \subsetneq \Upsilon_\sigma$. More explicitly,*

$$\Upsilon_\sigma \setminus f(I_\sigma) = \bigcup_{m \geq n} \left\{ \tau := (\tau_k)_{k \in \mathbb{N}} \in \Sigma_m : \tau^{(n)} = \sigma \text{ and } \tau_m = 1 \right\}.$$

(ii) *We have $\overline{f(I_\sigma)} = \Upsilon_\sigma$; that is, the image $f(I_\sigma)$ is dense in the cylinder set Υ_σ .*

Proof. Part (i) follows directly from Proposition 2.5, so it remains to prove part (ii). Let $\sigma := (\sigma_k)_{k \in \mathbb{N}}$. By definition, $f(I_\sigma) \subseteq \Upsilon_\sigma$. Moreover, Proposition 2.8(i) shows that Υ_σ is closed in $(\Sigma, \mathcal{T}_\Sigma)$. Therefore, it suffices to show that every point in $\Upsilon_\sigma \setminus f(I_\sigma)$ is a limit point of $f(I_\sigma)$. Take any $v := (v_k)_{k \in \mathbb{N}} \in \Upsilon_\sigma \setminus f(I_\sigma)$. Then, by part (i), there exists $m \geq n$ such that

$$v = (v_1, \dots, v_m) \in \Sigma_m, \quad v^{(n)} = \sigma, \quad v_m = 1.$$

Define a sequence $(\tau_k)_{k \in \mathbb{N}}$ in Σ by

$$\tau_k := (v_1, \dots, v_m, k+1) \in \Sigma_{m+1}, \quad k \in \mathbb{N}.$$

By Proposition 2.5, each τ_k belongs to $f(I_\sigma)$, since its last digit $k+1 > 1$ ensures realizability. We have $\tau_k \rightarrow v$ as $k \rightarrow \infty$ in the product space $\mathbb{N}_\infty^\mathbb{N}$. Using the continuous map $g: \mathbb{N}_\infty^\mathbb{N} \rightarrow (\Sigma, \mathcal{T}_\Sigma)$ from Proposition 2.8(ii), we obtain $g(\tau_k) \rightarrow g(v)$ as $k \rightarrow \infty$ in $(\Sigma, \mathcal{T}_\Sigma)$. Moreover, $g(\tau_k) = \tau_k$ and $g(v) = v$. Thus, $v \in \overline{f(I_\sigma)}$, as required. $\square$

The following lemma describes the image of a cylinder set under φ . Notice that the image is the closure $\overline{I_\sigma}$ of the fundamental interval, not I_σ itself.

Lemma 3.2. *Let $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$. Then $\varphi(\Upsilon_\sigma) = \overline{I_\sigma}$. Consequently, for every $\tau \in \Upsilon_\sigma$, we have*

$$|\varphi(\tau) - \varphi(\sigma)| \leq \mathcal{L}(I_\sigma),$$

where the equality holds if $\tau = \sigma^+ := (\sigma_1, \dots, \sigma_n, 1) \in \Upsilon_\sigma \cap \Sigma_{n+1}$.

Proof. By Proposition 2.4, we have $\varphi(f(x)) = x$ for each $x \in [0, 1)$, and so $\varphi(f(I_\sigma)) = I_\sigma$. By Lemma 3.1(ii), Υ_σ is the closure of $f(I_\sigma)$ in $(\Sigma, \mathcal{T}_\Sigma)$. Since φ is continuous by Proposition 2.8(iii), it follows that

$$\varphi(\Upsilon_\sigma) = \varphi(\overline{f(I_\sigma)}) \subseteq \overline{\varphi(f(I_\sigma))} = \overline{I_\sigma}.$$

On the other hand, since Υ_σ is compact by Proposition 2.8(i), the set $\varphi(\Upsilon_\sigma)$ is a compact subset of $\mathbb{R}$ containing I_σ . Thus $\overline{I_\sigma} \subseteq \varphi(\Upsilon_\sigma)$, and we conclude that $\varphi(\Upsilon_\sigma) = \overline{I_\sigma}$.

Now, by Proposition 2.6, $\overline{I_\sigma}$ is a closed interval of length $\mathcal{L}(I_\sigma)$ with endpoints $\varphi(\sigma)$ and $\varphi(\widehat{\sigma})$. Hence $|\varphi(\tau) - \varphi(\sigma)| \leq \mathcal{L}(I_\sigma)$ for every $\tau \in \Upsilon_\sigma$. Finally, since $\varphi(\sigma^+) = [\sigma_1, \dots, \sigma_n, 1] = [\sigma_1, \dots, \sigma_{n-1}, \sigma_n + 1] = \varphi(\widehat{\sigma})$, the equality holds for $\tau = \sigma^+$. This completes the proof. $\square$

The following lemma shows the monotonicity of the approximation errors and of the scaled approximation errors for all CF sequences, including non-realizable ones. When $\lambda = 0$ and $\sigma \in \Sigma_\infty$, the lemma reduces to the classical fact that $|x - p_k(x)/q_k(x)|$ decreases in k (see [11, Theorem 164]).

Lemma 3.3. *Let $\sigma \in \Sigma$ and $\lambda \in \{0, 1\}$, and define*

$$r_k^{(\lambda)}(\sigma) := (-1)^k (\varphi(\sigma) - \varphi_k(\sigma)) (q_k^*(\sigma))^\lambda, \quad k \in \mathbb{N}_0.$$

Then $(r_k^{(\lambda)}(\sigma))_{k \in \mathbb{N}_0}$ is a non-increasing sequence of non-negative real numbers converging to 0.

Proof. Put $q_k := q_k^*(\sigma)$, $r_k := r_k^{(1)}(\sigma)$, and $a_k := r_k^{(0)}(\sigma) = r_k/q_k$ for each $k \in \mathbb{N}_0$. By Proposition 2.3(v), we have $0 \leq r_k \leq 1/F_{k+2}$, and so $r_k \rightarrow 0$ as $k \rightarrow \infty$. Since $q_k \geq 1$, the same holds for a_k .

We first show that $r_{k+1} \leq r_k$ for each $k \in \mathbb{N}_0$. If $\sigma \in \Sigma_m$ for some $m \leq k+1$, then $\varphi_{k+1}(\sigma) = \varphi(\sigma)$ by Proposition 2.3(v), so that $r_{k+1} = 0 \leq r_k$. Otherwise, either $\sigma \in \Sigma_\infty$ or $\sigma \in \Sigma_m$ for some $m \geq k+2$; in particular, $\sigma_{k+2} \in \mathbb{N}$. In either case, by Proposition 2.3(v), applied with the index $k+1$ for the upper bound and with the index k for the lower bound, we have

$$r_{k+1} \leq q_{k+1} \cdot \frac{1}{q_{k+1}q_{k+2}} = \frac{1}{q_{k+2}} \quad \text{and} \quad r_k \geq q_k \cdot \frac{1}{q_k(q_{k+1} + q_k)} = \frac{1}{q_{k+1} + q_k}.$$

Since $\sigma_{k+2} \geq 1$, Proposition 2.3(i) tells us that $q_{k+2} = \sigma_{k+2}q_{k+1} + q_k \geq q_{k+1} + q_k$. Therefore, $r_{k+1} \leq r_k$.

Next, by Proposition 2.3(i), we have $q_{k+1} \geq q_k \geq 1$ for each $k \in \mathbb{N}_0$. It follows that

$$a_{k+1} = \frac{r_{k+1}}{q_{k+1}} \leq \frac{r_k}{q_k} = a_k$$

for each $k \in \mathbb{N}_0$. This proves the lemma. $\square$

For $\lambda \in \{0, 1\}$, we define the *error-sum functions of CF sequences* $\mathcal{E}_\lambda^*: \Sigma \rightarrow \mathbb{R}$ by

$$\mathcal{E}_\lambda^*(\sigma) := \sum_{n \geq 0} (q_n^*(\sigma))^\lambda (\varphi(\sigma) - \varphi_n(\sigma))$$

for each $\sigma \in \Sigma$. We also write $\mathcal{E}^* := \mathcal{E}_0^*$ and $P^* := \mathcal{E}_1^*$, and call them the *unweighted* and the *relative error-sum functions of CF sequences*, respectively. By Proposition 2.3(v), we have

$$|(q_n^*(\sigma))^\lambda (\varphi(\sigma) - \varphi_n(\sigma))| \leq \frac{1}{F_{n+2}} \leq \frac{1}{\gamma^n} \quad (3.1)$$

for all $\sigma \in \Sigma$, $n \in \mathbb{N}_0$, and $\lambda \in \{0, 1\}$. Hence the series defining $\mathcal{E}_\lambda^*$ converges absolutely and uniformly on Σ , and $\mathcal{E}_\lambda^*$ is well defined.

Lemma 3.4. *For each $\lambda \in \{0, 1\}$, we have $\mathcal{E}_\lambda = \mathcal{E}_\lambda^* \circ f$ on $[0, 1]$; that is, the following diagram commutes:*

$$\begin{array}{ccc}
 [0, 1) & \xrightarrow{\mathcal{E}_\lambda} & \mathbb{R} \\
 & \searrow f & \nearrow \mathcal{E}_\lambda^* \\
 & \Sigma &
 \end{array}$$

In particular, $\mathcal{E} = \mathcal{E}^* \circ f$ and $P = P^* \circ f$ on $[0, 1)$.

Proof. Let $x \in [0, 1)$, and let $\sigma := f(x) = (d_1(x), d_2(x), \dots) \in \Sigma$. By Proposition 2.4, we have $\varphi(\sigma) = x$. By the definitions of φ_n and of the convergents, we have $\varphi_n(\sigma) = [d_1(x), \dots, d_n(x), \infty, \infty, \dots] = p_n(x)/q_n(x)$ for each $n \in \mathbb{N}_0$. Since $\gcd(p_n(x), q_n(x)) = 1$, the reduced denominator of $\varphi_n(\sigma)$ is $q_n(x)$. Hence $q_n^*(\sigma) = q_n(x)$. Therefore,

$$(\mathcal{E}_\lambda^* \circ f)(x) = \sum_{n \geq 0} (q_n^*(\sigma))^\lambda (\varphi(\sigma) - \varphi_n(\sigma)) = \sum_{n \geq 0} (q_n(x))^\lambda \left(x - \frac{p_n(x)}{q_n(x)} \right) = \mathcal{E}_\lambda(x),$$

as desired. $\square$

However, the converse identity $\mathcal{E} \circ \varphi = \mathcal{E}^*$ does not hold in general.

Example 3.5. Let $\sigma := (1, 1) \in \Sigma_2 \setminus \Sigma_{\text{re}}$. We have $\varphi(\sigma) = \frac{1}{1+\frac{1}{1}} = \frac{1}{2} = [2]$. Hence, $(\mathcal{E} \circ \varphi)(\sigma) = \mathcal{E}([2]) = \left(\frac{1}{2} - 0\right) + \left(\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}$. On the other hand, $\mathcal{E}^*(\sigma) = \left(\frac{1}{2} - 0\right) + \left(\frac{1}{2} - 1\right) = 0$. Thus, the identity $\mathcal{E} \circ \varphi = \mathcal{E}^*$ fails in general. The example shows that this can happen when $\sigma \notin \Sigma_{\text{re}}$. Similarly, $(P \circ \varphi)(\sigma) = P([2]) = \frac{1}{2} + \left(2 \cdot \frac{1}{2} - 1\right) = \frac{1}{2}$, whereas $P^*(\sigma) = \frac{1}{2} + \left(\frac{1}{2} - 1\right) \cdot 1 = 0$, since $q_1^*(\sigma) = 1$.

We now establish the continuity of $\mathcal{E}_\lambda^*$ on $(\Sigma, \mathcal{T}_\Sigma)$. For $\lambda = 0$, this is a CF analogue of [1, Lemma 3.15]. For $\lambda = 1$, however, some care is needed, since the weights q_k^* are not continuous on Σ . For instance, we have $(m) \rightarrow (\infty, \infty, \dots)$ as $m \rightarrow \infty$ in $(\Sigma, \mathcal{T}_\Sigma)$ by Proposition 2.8(ii), whereas $q_1^*((m)) = m$ for each $m \in \mathbb{N}$ and $q_1^*((\infty, \infty, \dots)) = 1$.

Lemma 3.6. Let $\lambda \in \{0, 1\}$. For each $k \in \mathbb{N}_0$, the map

$$u_k^{(\lambda)} := (q_k^*)^\lambda (\varphi - \varphi_k) : (\Sigma, \mathcal{T}_\Sigma) \rightarrow \mathbb{R}$$

is continuous. Consequently, $\mathcal{E}_\lambda^* : (\Sigma, \mathcal{T}_\Sigma) \rightarrow \mathbb{R}$ is continuous. In particular, both $\mathcal{E}^*$ and P^* are continuous.

Proof. If $\lambda = 0$, then $u_k^{(0)} = \varphi - \varphi_k$ is continuous by Proposition 2.8(iii) and Lemma 2.9. Assume $\lambda = 1$. Since $u_0^{(1)} = \varphi$ is continuous, we may assume that $k \in \mathbb{N}$. Since g is a quotient map by Proposition 2.8(ii), it suffices to show that $u_k^{(1)} \circ g$ is continuous on $\mathbb{N}_\infty^{\mathbb{N}}$. Moreover, since $\mathbb{N}_\infty^{\mathbb{N}}$ is metrizable, it is enough to establish the sequential continuity. Let $\rho_j := (\rho_{j,i})_{i \in \mathbb{N}}$, $j \in \mathbb{N}$, be a sequence in $\mathbb{N}_\infty^{\mathbb{N}}$ converging to $\rho := (\rho_i)_{i \in \mathbb{N}}$ in the product topology, i.e., $\rho_{j,i} \rightarrow \rho_i$ in $\mathbb{N}_\infty$ as $j \rightarrow \infty$ for each $i \in \mathbb{N}$. Put $\sigma := g(\rho)$ and $\sigma_{(j)} := g(\rho_j)$ for each $j \in \mathbb{N}$. Note that $\varphi \circ g$ and $\varphi_k \circ g$ are continuous by Proposition 2.8(ii),(iii) and Lemma 2.9. We consider two cases.

Case I. Assume $\rho_1, \dots, \rho_k \in \mathbb{N}$. Since each point of $\mathbb{N}$ is isolated in $\mathbb{N}_\infty$, there exists $J \in \mathbb{N}$ such that $\rho_{j,i} = \rho_i$ for all $1 \leq i \leq k$ and all $j \geq J$. For such j , the first k terms of $\sigma_{(j)}$ and σ coincide, and hence $q_k^*(\sigma_{(j)}) = q_k^*(\sigma)$. Therefore,

$$u_k^{(1)}(\sigma_{(j)}) = q_k^*(\sigma)((\varphi \circ g)(\rho_j) - (\varphi_k \circ g)(\rho_j)) \rightarrow q_k^*(\sigma)(\varphi(\sigma) - \varphi_k(\sigma)) = u_k^{(1)}(\sigma)$$

as $j \rightarrow \infty$.

Case II. Assume $\rho_i = \infty$ for some $1 \leq i \leq k$. Let $m+1$ be the smallest such i , so that $0 \leq m \leq k-1$ and $\sigma = g(\rho) \in \Sigma_m$. Then $\varphi_k(\sigma) = \varphi(\sigma)$, and so $u_k^{(1)}(\sigma) = 0$. Fix $j \in \mathbb{N}$. If $\sigma_{(j)} \in \Sigma_{m'}$ for some $m' \leq k$, then $u_k^{(1)}(\sigma_{(j)}) = 0$. Otherwise, either $\sigma_{(j)} \in \Sigma_\infty$ or $\sigma_{(j)} \in \Sigma_{m'}$ for some $m' \geq k+1$; in particular, the $(m+1)$ th term of $\sigma_{(j)}$ is $\rho_{j,m+1} \in \mathbb{N}$. In this case, by Proposition 2.3(v) and (i), we have

$$|u_k^{(1)}(\sigma_{(j)})| \leq q_k^*(\sigma_{(j)}) \cdot \frac{1}{q_k^*(\sigma_{(j)})q_{k+1}^*(\sigma_{(j)})} = \frac{1}{q_{k+1}^*(\sigma_{(j)})} \leq \frac{1}{q_{m+1}^*(\sigma_{(j)})} \leq \frac{1}{\rho_{j,m+1}},$$

where the last inequality follows from $q_{m+1}^*(\sigma_{(j)}) = \rho_{j,m+1}q_m^*(\sigma_{(j)}) + q_{m-1}^*(\sigma_{(j)}) \geq \rho_{j,m+1}$. Thus, with the convention $1/\infty = 0$, we have $|u_k^{(1)}(\sigma_{(j)})| \leq 1/\rho_{j,m+1}$ for all $j \in \mathbb{N}$. Since $\rho_{j,m+1} \rightarrow \rho_{m+1} = \infty$ in $\mathbb{N}_\infty$, we conclude that $u_k^{(1)}(\sigma_{(j)}) \rightarrow 0 = u_k^{(1)}(\sigma)$ as $j \rightarrow \infty$.

In either case, we have $u_k^{(1)}(g(\rho_j)) \rightarrow u_k^{(1)}(g(\rho))$ as $j \rightarrow \infty$, which proves the continuity of $u_k^{(1)}$.

Finally, we have $\mathcal{E}_\lambda^* = \sum_{k \geq 0} u_k^{(1)}$, where each $u_k^{(1)}$ is continuous and $\sup_\Sigma |u_k^{(1)}| \leq \gamma^{-k}$ by (3.1). By the Weierstrass M-test, the series converges uniformly on Σ , and hence $\mathcal{E}_\lambda^*$ is continuous. Thus the lemma follows. $\square$

Lemma 3.7. Let $\sigma := (\sigma_k)_{k=1}^n \in \Sigma_n \cap \Sigma_{\text{re}}$ for some $n \in \mathbb{N}$, and define $\sigma' := (\sigma_1, \dots, \sigma_{n-1}, \sigma_n - 1, 1) \in \Sigma_{n+1} \setminus \Sigma_{\text{re}}$. Then, for each $\lambda \in \{0, 1\}$,

$$\mathcal{E}_\lambda^*(\sigma') = \mathcal{E}_\lambda^*(\sigma) + \frac{(-1)^n}{q_n^*(\sigma)(q_n^*(\sigma) - q_{n-1}^*(\sigma))^{1-\lambda}}.$$

In particular,

$$\mathcal{E}^*(\sigma') = \mathcal{E}^*(\sigma) + \frac{(-1)^n}{q_n^*(\sigma)(q_n^*(\sigma) - q_{n-1}^*(\sigma))} \quad \text{and} \quad P^*(\sigma') = P^*(\sigma) + \frac{(-1)^n}{q_n^*(\sigma)},$$

where the latter identity is due to Ridley and Petruska [15, Lemma 1.1].

Proof. Observe that $\varphi(\sigma') = \varphi(\sigma)$, that $\varphi_k(\sigma') = \varphi_k(\sigma)$ and $q_k^*(\sigma') = q_k^*(\sigma)$ for $0 \leq k \leq n-1$ by Proposition 2.3(ii), that $\varphi_k(\sigma) = \varphi(\sigma)$ for $k \geq n$, and that $\varphi_k(\sigma') = \varphi(\sigma')$ for $k \geq n+1$. Hence all terms of $\mathcal{E}_\lambda^*(\sigma')$ and $\mathcal{E}_\lambda^*(\sigma)$ coincide except for the terms with index n , and

$$\mathcal{E}_\lambda^*(\sigma') - \mathcal{E}_\lambda^*(\sigma) = (q_n^*(\sigma'))^\lambda (\varphi_n(\sigma) - \varphi_n(\sigma')).$$

By Proposition 2.3(ii), we have $q_n^*(\sigma') = q_n^*(\sigma) - q_{n-1}^*(\sigma)$ and

$$\varphi_n(\sigma') = \frac{p_n^*(\sigma')}{q_n^*(\sigma')} = \frac{p_n^*(\sigma) - p_{n-1}^*(\sigma)}{q_n^*(\sigma) - q_{n-1}^*(\sigma)}.$$

Hence,

$$\begin{aligned} \varphi_n(\sigma) - \varphi_n(\sigma') &= \frac{p_n^*(\sigma)}{q_n^*(\sigma)} - \frac{p_n^*(\sigma) - p_{n-1}^*(\sigma)}{q_n^*(\sigma) - q_{n-1}^*(\sigma)} \\ &= \frac{-p_n^*(\sigma)q_{n-1}^*(\sigma) + p_{n-1}^*(\sigma)q_n^*(\sigma)}{q_n^*(\sigma)(q_n^*(\sigma) - q_{n-1}^*(\sigma))} = \frac{(-1)^n}{q_n^*(\sigma)(q_n^*(\sigma) - q_{n-1}^*(\sigma))}, \end{aligned}$$

where the last equality follows from Proposition 2.3(iv). Multiplying both sides by $(q_n^*(\sigma'))^\lambda = (q_n^*(\sigma) - q_{n-1}^*(\sigma))^\lambda$, we obtain the desired identity. $\square$

The following two lemmas are the main results of this section. For $n \in \mathbb{N}$, $\sigma \in \Sigma_n$, and $\lambda \in \{0, 1\}$, put

$$W_\lambda(\sigma) := \sum_{k=0}^n (q_k^*(\sigma))^\lambda, \quad \text{so that} \quad W_0(\sigma) = n + 1 \quad \text{and} \quad W_1(\sigma) = \sum_{k=0}^n q_k^*(\sigma). \quad (3.2)$$

Lemma 3.8. *Let $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, and let $\lambda \in \{0, 1\}$. Then, for every $\tau \in \Upsilon_\sigma$, we have*

$$0 \leq (-1)^n (\mathcal{E}_\lambda^*(\tau) - \mathcal{E}_\lambda^*(\sigma)) \leq W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma), \quad (3.3)$$

where the lower bound is attained at $\tau = \sigma$ and the upper bound is attained at $\tau = \sigma^+ := (\sigma_1, \dots, \sigma_n, 1) \in \Upsilon_\sigma \cap \Sigma_{n+1}$. Consequently, the following hold.

(i) *If n is odd, then*

$$\max_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) = \mathcal{E}_\lambda^*(\sigma) \quad \text{and} \quad \min_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) = \mathcal{E}_\lambda^*(\sigma) - W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma).$$

(ii) *If n is even, then*

$$\max_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) = \mathcal{E}_\lambda^*(\sigma) + W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma) \quad \text{and} \quad \min_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) = \mathcal{E}_\lambda^*(\sigma).$$

Proof. Fix $\tau \in \Upsilon_\sigma$, and put $r_k := r_k^{(\lambda)}(\tau)$ for each $k \in \mathbb{N}_0$, as in Lemma 3.3. Then the k th term of the series defining $\mathcal{E}_\lambda^*(\tau)$ equals $(-1)^k r_k$. For each $m \in \mathbb{N}_0$, we define the tail

$$R_m := \sum_{k \geq m} (-1)^k r_k.$$

By Lemma 3.3, the sequence $(r_k)_{k \in \mathbb{N}_0}$ is non-negative and non-increasing, and it converges to 0. Hence, by the standard estimate for alternating series, we have

$$0 \leq (-1)^m R_m = r_m - r_{m+1} + r_{m+2} - \dots \leq r_m \quad \text{for each } m \in \mathbb{N}_0. \quad (3.4)$$

Since $\tau^{(n)} = \sigma$, the sequences τ and σ share the first n terms, so that $\varphi_k(\tau) = \varphi_k(\sigma)$ and $q_k^*(\tau) = q_k^*(\sigma)$ for $0 \leq k \leq n$. Moreover, since $\sigma \in \Sigma_n$, we have $\varphi_k(\sigma) = \varphi(\sigma)$ for all $k \geq n$, i.e., the terms of $\mathcal{E}_\lambda^*(\sigma)$ with index $k \geq n$ vanish. It follows that

$$\mathcal{E}_\lambda^*(\tau) - \mathcal{E}_\lambda^*(\sigma) = \sum_{k=0}^{n-1} (q_k^*(\sigma))^\lambda (\varphi(\tau) - \varphi(\sigma)) + R_n. \quad (3.5)$$

Notice that the first term of R_n equals $(q_n^*(\sigma))^\lambda (\varphi(\tau) - \varphi(\sigma)) = (q_n^*(\sigma))^\lambda (\varphi(\tau) - \varphi(\sigma))$. Hence, (3.5) can be rewritten as

$$\mathcal{E}_\lambda^*(\tau) - \mathcal{E}_\lambda^*(\sigma) = W_\lambda(\sigma) (\varphi(\tau) - \varphi(\sigma)) + R_{n+1}. \quad (3.6)$$

By Proposition 2.3(v), applied to τ with the index n , we also have

$$(-1)^n (\varphi(\tau) - \varphi(\sigma)) = (-1)^n (\varphi(\tau) - \varphi_n(\tau)) \geq 0. \quad (3.7)$$

We first prove the lower bound. Multiplying both sides of (3.5) by $(-1)^n$, and using (3.7) and the inequality $(-1)^n R_n \geq 0$ from (3.4), we obtain $(-1)^n(\mathcal{E}_\lambda^*(\tau) - \mathcal{E}_\lambda^*(\sigma)) \geq 0$.

We next prove the upper bound. Multiplying both sides of (3.6) by $(-1)^n$, and using the inequality $(-1)^n R_{n+1} = -(-1)^{n+1} R_{n+1} \leq 0$ from (3.4), we obtain

$$(-1)^n(\mathcal{E}_\lambda^*(\tau) - \mathcal{E}_\lambda^*(\sigma)) \leq W_\lambda(\sigma) |\varphi(\tau) - \varphi(\sigma)| \leq W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma),$$

where the last inequality follows from Lemma 3.2. This proves (3.3).

It remains to consider the equality cases. If $\tau = \sigma$, the middle term of (3.3) is clearly 0. If $\tau = \sigma^+$, then $\varphi_k(\sigma^+) = \varphi(\sigma^+)$ for all $k \geq n+1$, so that $R_{n+1} = 0$ in (3.6). Moreover, we have $|\varphi(\sigma^+) - \varphi(\sigma)| = \mathcal{L}(I_\sigma)$ by Lemma 3.2, and so $(-1)^n(\varphi(\sigma^+) - \varphi(\sigma)) = \mathcal{L}(I_\sigma)$ by (3.7). Thus,

$$(-1)^n(\mathcal{E}_\lambda^*(\sigma^+) - \mathcal{E}_\lambda^*(\sigma)) = W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma).$$

Now, parts (i) and (ii) follow immediately from (3.3) and the equality cases. This finishes the proof of the lemma. $\square$

It is worth noting where the weight $(q_k^*)^\lambda$ plays a role in the above argument. In the unweighted case $\lambda = 0$, the factor $W_0(\sigma) = n+1$ in Lemma 3.8 simply counts the terms with index at most n , whereas, in the relative case $\lambda = 1$, this factor is replaced by $W_1(\sigma) = \sum_{k=0}^n q_k^*(\sigma)$. Apart from this, the weight matters only in the monotonicity of the scaled errors (Lemma 3.3) and in the continuity at finite sequences (Lemma 3.6).

Lemma 3.9. *Let $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, and let $\lambda \in \{0, 1\}$. Then*

$$\sup_{t, u \in I_\sigma} |\mathcal{E}_\lambda(t) - \mathcal{E}_\lambda(u)| = W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma).$$

In particular,

$$\sup_{t, u \in I_\sigma} |\mathcal{E}(t) - \mathcal{E}(u)| = (n+1) \cdot \mathcal{L}(I_\sigma) \quad \text{and} \quad \sup_{t, u \in I_\sigma} |P(t) - P(u)| = \mathcal{L}(I_\sigma) \sum_{k=0}^n q_k^*(\sigma).$$

Proof. By Lemma 3.4, we have $\mathcal{E}_\lambda(t) = \mathcal{E}_\lambda^*(f(t))$ for all $t \in I_\sigma$. Since $\mathcal{E}_\lambda^*$ is continuous by Lemma 3.6 and $f(I_\sigma)$ is dense in Υ_σ by Lemma 3.1(ii), the supremum and infimum of $\mathcal{E}_\lambda^*$ over $f(I_\sigma)$ coincide with those over Υ_σ . Therefore,

$$\sup_{t \in I_\sigma} \mathcal{E}_\lambda(t) = \max_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) \quad \text{and} \quad \inf_{t \in I_\sigma} \mathcal{E}_\lambda(t) = \min_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau),$$

where the maximum and the minimum exist by Lemma 3.8. Thus, again by Lemma 3.8, we have

$$\sup_{t, u \in I_\sigma} |\mathcal{E}_\lambda(t) - \mathcal{E}_\lambda(u)| = \max_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) - \min_{\tau \in \Upsilon_\sigma} \mathcal{E}_\lambda^*(\tau) = W_\lambda(\sigma) \cdot \mathcal{L}(I_\sigma),$$

as desired. $\square$

4. Hausdorff dimension of the graph of $\mathcal{E}$

In this section, we prove Theorem 1.1. We begin with some elementary facts on the fundamental intervals of a fixed order, which will also be used in Section 5.

Lemma 4.1. *Let $n \in \mathbb{N}$. Then the following hold.*

- (i) *The intervals I_σ , $\sigma \in \Sigma_n$, are pairwise disjoint subsets of $[0, 1)$ whose union contains $\mathbb{I}$.*
- (ii) $\sum_{\sigma \in \Sigma_n} \mathcal{L}(I_\sigma) = 1$.
- (iii) $\sup_{\sigma \in \Sigma_n} \mathcal{L}(I_\sigma) \leq 1/(F_{n+1}F_{n+2}) \leq \gamma^{-(2n-1)}$.

Proof. (i) The cylinder sets Υ_σ , $\sigma \in \Sigma_n$, are pairwise disjoint, and hence so are the intervals $I_\sigma = f^{-1}(\Upsilon_\sigma)$. Given $x \in \mathbb{I}$, we have $\tau := f(x) \in \Sigma_\infty$ by Proposition 2.4(ii), and $\tau^{(n)} \in \Sigma_n$ satisfies $x \in f^{-1}(\Upsilon_{\tau^{(n)}}) = I_{\tau^{(n)}}$.

(ii) Since Σ_n is countable and $\mathcal{L}(\mathbb{I}) = 1$, part (i) and the countable additivity of $\mathcal{L}$ give $\sum_{\sigma \in \Sigma_n} \mathcal{L}(I_\sigma) = \mathcal{L}(\bigcup_{\sigma \in \Sigma_n} I_\sigma) = 1$.

(iii) By Proposition 2.3(iii), we have $q_n^*(\sigma) \geq F_{n+1}$ and $q_n^*(\sigma) + q_{n-1}^*(\sigma) \geq F_{n+1} + F_n = F_{n+2}$ for every $\sigma \in \Sigma_n$. Hence the first inequality follows from (2.1), and the second from (1.3). $\square$

Define $G_\mathcal{E}: \mathbb{R} \rightarrow \mathbb{R}^2$ by $G_\mathcal{E}(x) := (x, \mathcal{E}(x))$ for $x \in \mathbb{R}$. Then the graph of $\mathcal{E}$ is given by $G_\mathcal{E}(\mathbb{R}) = \{(x, \mathcal{E}(x)) : x \in \mathbb{R}\}$.

Lemma 4.2. *We have $\dim_H G_\mathcal{E}([0, 1)) = 1$.*

Proof. For the lower bound, consider the projection map onto the first coordinate, $p: \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by $p(x, y) := x$ for each $(x, y) \in \mathbb{R}^2$. Since p is Lipschitz, $\dim_H p(F) \leq \dim_H F$ for every $F \subseteq \mathbb{R}^2$. Hence,

$$1 = \dim_H [0, 1) = \dim_H p(G_\mathcal{E}([0, 1))) \leq \dim_H G_\mathcal{E}([0, 1)),$$

as claimed.

For the upper bound, we construct an appropriate covering of $G_\mathcal{E}([0, 1))$. For each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, define the closed interval

$$J_\sigma := \left[\inf_{t \in I_\sigma} \mathcal{E}(t), \sup_{t \in I_\sigma} \mathcal{E}(t) \right].$$

Then $I_\sigma \times \mathcal{E}(I_\sigma) \subseteq I_\sigma \times J_\sigma$. By Lemma 4.1(i), for each n , the collection

$$\mathcal{J}_n := \{I_\sigma \times J_\sigma : \sigma \in \Sigma_n\}$$

forms a covering of $F := G_\mathcal{E}(\mathbb{I})$, which differs from $G_\mathcal{E}([0, 1))$ only by a countable set. By Lemma 3.9, we have $\mathcal{L}(J_\sigma) = (n+1)\mathcal{L}(I_\sigma)$. Each rectangle $I_\sigma \times J_\sigma$ is thus covered by $n+1$ axis-aligned squares of side length $\mathcal{L}(I_\sigma)$ and diagonal $\sqrt{2}\mathcal{L}(I_\sigma)$. By Lemma 4.1(iii), the diameters of these covering squares tend uniformly to zero as $n \rightarrow \infty$.

Let $\varepsilon > 0$. Then the $(1 + \varepsilon)$ -dimensional Hausdorff measure of F is bounded above as

$$\mathcal{H}^{1+\varepsilon}(F) \leq \liminf_{n \rightarrow \infty} \sum_{\sigma \in \Sigma_n} (n+1) \left(\sqrt{2}\mathcal{L}(I_\sigma) \right)^{1+\varepsilon} = (\sqrt{2})^{1+\varepsilon} \liminf_{n \rightarrow \infty} (n+1)a_n, \quad (4.1)$$

where

$$a_n := \sum_{\sigma \in \Sigma_n} (\mathcal{L}(I_\sigma))^{1+\varepsilon} = \sum_{\sigma \in \Sigma_n} \left(\frac{1}{q_n^*(\sigma)(q_n^*(\sigma) + q_{n-1}^*(\sigma))} \right)^{1+\varepsilon}. \quad (4.2)$$

By Lemma 4.1(ii) and (iii), we have

$$a_n = \sum_{\sigma \in \Sigma_n} \mathcal{L}(I_\sigma) \cdot (\mathcal{L}(I_\sigma))^\varepsilon \leq \gamma^{-(2n-1)\varepsilon} \sum_{\sigma \in \Sigma_n} \mathcal{L}(I_\sigma) = \gamma^{-(2n-1)\varepsilon}.$$

Hence $(n+1)a_n \leq (n+1)\gamma^{-(2n-1)\varepsilon} \rightarrow 0$ as $n \rightarrow \infty$, and so (4.1) yields $\mathcal{H}^{1+\varepsilon}(F) = 0$. Then $\dim_{\mathbb{H}} F \leq 1+\varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\dim_{\mathbb{H}} F \leq 1$. Since $F = G_{\mathcal{E}}(\mathbb{I})$ differs from $G_{\mathcal{E}}([0, 1])$ only by the countable set $G_{\mathcal{E}}([0, 1] \cap \mathbb{Q})$, and since any countable set has Hausdorff dimension 0, the countable stability property of the Hausdorff dimension (see [10, pp. 48–49]) implies that $\dim_{\mathbb{H}} G_{\mathcal{E}}([0, 1]) = \dim_{\mathbb{H}} F \leq 1$. This completes the proof of the lemma. $\square$

Proof of Theorem 1.1. Since $\mathcal{E}$ is periodic with period 1 by Proposition 2.1, each set $G_{\mathcal{E}}([n, n+1])$ is a horizontal translation of $G_{\mathcal{E}}([0, 1])$. Therefore, by the countable stability property of Hausdorff dimension (see [10, pp. 48–49]),

$$\dim_{\mathbb{H}} G_{\mathcal{E}}(\mathbb{R}) = \sup_{n \in \mathbb{Z}} \dim_{\mathbb{H}} G_{\mathcal{E}}([n, n+1]) = \dim_{\mathbb{H}} G_{\mathcal{E}}([0, 1]) = 1,$$

where the last equality follows from Lemma 4.2. $\square$

Remark 4.3. We remark that the above proof relies only on the exact oscillation formula in Lemma 3.9, the identity $\sum_{\sigma \in \Sigma_n} \mathcal{L}(I_\sigma) = 1$, and the Fibonacci lower bound for the convergent denominators. In Section 6, we present an alternative argument, which shows that the series $\sum_{n \geq 1} a_n$ converges and evaluates it exactly in terms of the Riemann zeta function and an explicit double series.

5. An upper bound for the Hausdorff dimension of the graph of P

In this section, we prove Theorem 1.2, that is, we give a new derivation of the upper bound $3/2$ of Ridley and Petruska [15] by using the oscillation formula for P in Lemma 3.9. We need the following injectivity property of the convergent denominators.

Lemma 5.1. *Let $n \in \mathbb{N}$. Then the map $\Sigma_n \rightarrow \mathbb{N}^2$ given by $\sigma \mapsto (q_n^*(\sigma), q_{n-1}^*(\sigma))$ is injective. Moreover, if $n \geq 2$, then $1 \leq q_{n-1}^*(\sigma) < q_n^*(\sigma)$ for every $\sigma \in \Sigma_n$.*

Proof. If $n = 1$, then $(q_1^*(\sigma), q_0^*(\sigma)) = (\sigma_1, 1)$ determines $\sigma = (\sigma_1)$. Let $n \geq 2$ and $\sigma \in \Sigma_n$. By Proposition 2.3(i), we have $q_{n-2}^*(\sigma) \geq q_0^*(\sigma) = 1$, and hence $q_n^*(\sigma) = \sigma_n q_{n-1}^*(\sigma) + q_{n-2}^*(\sigma) > q_{n-1}^*(\sigma) \geq 1$. This proves the second assertion.

Now, suppose that $\sigma, \tau \in \Sigma_n$ satisfy $(q_n^*(\sigma), q_{n-1}^*(\sigma)) = (q_n^*(\tau), q_{n-1}^*(\tau))$, and put $x := q_{n-1}^*(\sigma)/q_n^*(\sigma) \in (0, 1) \cap \mathbb{Q}$. By the reversal identity in Proposition 2.3(vi), we have

$$\varphi((\sigma_n, \dots, \sigma_1)) = [\sigma_n, \dots, \sigma_1] = x = [\tau_n, \dots, \tau_1] = \varphi((\tau_n, \dots, \tau_1)),$$

where both reversed sequences $(\sigma_n, \dots, \sigma_1)$ and $(\tau_n, \dots, \tau_1)$ belong to Σ_n . By Proposition 2.4(i), the fiber $\varphi^{-1}(\{x\})$ consists of exactly two sequences, which belong to Σ_m and Σ_{m+1} for some $m \in \mathbb{N}$. In other words, the two sequences in $\varphi^{-1}(\{x\})$ have different lengths. Since both reversed sequences lie in $\varphi^{-1}(\{x\}) \cap \Sigma_n$, they must coincide. Hence $\sigma = \tau$, as desired. $\square$

Define $G_P: \mathbb{R} \rightarrow \mathbb{R}^2$ by $G_P(x) := (x, P(x))$ for each $x \in \mathbb{R}$, so that $G_P(\mathbb{R}) = \{(x, P(x)) : x \in \mathbb{R}\}$ is the graph of P . We now proceed to prove that $\dim_H G_P(\mathbb{R}) \leq 3/2$.

Proof of Theorem 1.2. As in the proof of Theorem 1.1, by Proposition 2.1 and the countable stability property of the Hausdorff dimension (see [10, pp. 48–49]), it suffices to show that $\dim_H G_P([0, 1]) \leq 3/2$.

To establish this upper bound, we construct a suitable cover for $G_P([0, 1])$. For each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, we define the closed interval

$$K_\sigma := \left[\inf_{t \in I_\sigma} P(t), \sup_{t \in I_\sigma} P(t) \right].$$

By Lemma 4.1(i), for any $n \in \mathbb{N}$, the collection $\mathcal{K}_n := \{I_\sigma \times K_\sigma : \sigma \in \Sigma_n\}$ covers the set $G := G_P(\mathbb{I})$. By Lemma 3.9,

$$\mathcal{L}(K_\sigma) = \mathcal{L}(I_\sigma) \sum_{0 \leq k \leq n} q_k^*(\sigma).$$

Therefore, $I_\sigma \times K_\sigma$ can be covered by $\sum_{0 \leq k \leq n} q_k^*(\sigma)$ axis-aligned squares of side length $\mathcal{L}(I_\sigma)$ and diameter $\sqrt{2}\mathcal{L}(I_\sigma)$. Moreover, by Lemma 4.1(iii), the diameters of these covering squares tend uniformly to zero.

Now, let $\varepsilon > 1/2$ be given. Since $q_0^*(\sigma) \leq q_1^*(\sigma) \leq \dots \leq q_n^*(\sigma)$ for any $\sigma \in \Sigma_n$ and $n \in \mathbb{N}$ by Proposition 2.3(i), it follows that

$$\begin{aligned} \mathcal{H}^{1+\varepsilon}(G) &\leq \liminf_{n \rightarrow \infty} \sum_{\sigma \in \Sigma_n} \left[(\sqrt{2}\mathcal{L}(I_\sigma))^{1+\varepsilon} \sum_{0 \leq k \leq n} q_k^*(\sigma) \right] \\ &\leq \liminf_{n \rightarrow \infty} \sum_{\sigma \in \Sigma_n} \left[(\sqrt{2}\mathcal{L}(I_\sigma))^{1+\varepsilon} (n+1)q_n^*(\sigma) \right] \\ &= (\sqrt{2})^{1+\varepsilon} \liminf_{n \rightarrow \infty} (n+1)b_n, \end{aligned} \quad (5.1)$$

where

$$b_n := \sum_{\sigma \in \Sigma_n} q_n^*(\sigma)(\mathcal{L}(I_\sigma))^{1+\varepsilon} = \sum_{\sigma \in \Sigma_n} \frac{1}{(q_n^*(\sigma))^\varepsilon (q_n^*(\sigma) + q_{n-1}^*(\sigma))^{1+\varepsilon}}, \quad n \in \mathbb{N}. \quad (5.2)$$

We now estimate b_n for $n \geq 2$. By Lemma 5.1, distinct sequences $\sigma \in \Sigma_n$ give distinct pairs $(j, k) := (q_n^*(\sigma), q_{n-1}^*(\sigma))$ with $1 \leq k < j$. Moreover, $j \geq F_{n+1}$ by Proposition 2.3(iii). It follows that

$$\begin{aligned} b_n &\leq \sum_{j \geq F_{n+1}} \sum_{1 \leq k < j} \frac{1}{j^\varepsilon (j+k)^{1+\varepsilon}} \leq \sum_{j \geq F_{n+1}} \frac{1}{j^\varepsilon} \cdot \frac{j-1}{(j+1)^{1+\varepsilon}} \leq \sum_{j \geq F_{n+1}} \frac{1}{j^{2\varepsilon}} \\ &\leq \int_{F_{n+1}-1}^{\infty} \frac{dx}{x^{2\varepsilon}} \leq \int_{F_n}^{\infty} \frac{dx}{x^{2\varepsilon}} = \frac{1}{2\varepsilon-1} \cdot \frac{1}{F_n^{2\varepsilon-1}}, \end{aligned}$$

where we used $F_{n+1} - 1 \geq F_n \geq 1$ for $n \geq 2$ and $2\varepsilon > 1$. Hence, by (1.3), we obtain

$$(n+1)b_n \leq \frac{n+1}{2\varepsilon-1} \cdot \frac{1}{\gamma^{(n-2)(2\varepsilon-1)}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Substituting this into (5.1), we deduce that $\mathcal{H}^{1+\varepsilon}(G) = 0$, and therefore $\dim_{\mathbb{H}} G \leq 1 + \varepsilon$. Since $\varepsilon > 1/2$ was arbitrary, we conclude that $\dim_{\mathbb{H}} G \leq 1 + 1/2 = 3/2$. Since $G = G_P(\mathbb{I})$ differs from $G_P([0, 1))$ only by the countable set $G_P([0, 1) \cap \mathbb{Q})$, countable stability of Hausdorff dimension yields

$$\dim_{\mathbb{H}} G_P([0, 1)) = \max \{\dim_{\mathbb{H}} G_P(\mathbb{I}), \dim_{\mathbb{H}} G_P([0, 1) \cap \mathbb{Q})\} \leq \frac{3}{2}.$$

□

Remark 5.2 (On the exponent $3/2$). We explain why the covering argument in the above proof cannot yield an upper bound smaller than $3/2$. Let $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, and put $q_k := q_k^*(\sigma)$ and $L := \mathcal{L}(I_\sigma)$. By (2.1), we have

$$q_n L = \frac{1}{q_n + q_{n-1}} = \left(\frac{q_n}{q_n + q_{n-1}} \right)^{1/2} L^{1/2},$$

where $1/2 \leq q_n/(q_n + q_{n-1}) \leq 1$. Since $q_n \leq W_1(\sigma) \leq (n+1)q_n$, it follows from Lemma 3.9 that

$$2^{-1/2} L^{1/2} \leq \sup_{t, u \in I_\sigma} |P(t) - P(u)| \leq (n+1) L^{1/2}. \quad (5.3)$$

On the other hand, the oscillation of $\mathcal{E}$ on I_σ equals $(n+1)L$, where $n+1 \leq \frac{1}{2} \log_\gamma(1/L) + \frac{3}{2}$ by Lemma 4.1(iii). In other words, at the scale of fundamental intervals, $\mathcal{E}$ behaves like a Lipschitz function up to a logarithmic factor, whereas P behaves like a Hölder function of exponent $1/2$. Recall that the graph of a continuous function satisfying a Hölder condition of exponent $\alpha \in (0, 1]$ has dimension at most $2 - \alpha$ (see [10, Chapter 11]), and note that $2 - 1/2 = 3/2$. Since P is not continuous, this is merely a heuristic analogy. Nevertheless, the lower bound in (5.3) shows that the coverings used in the above proof, in which each full column $I_\sigma \times K_\sigma$ with $\sigma \in \Sigma_n$ is covered by squares of side length $\mathcal{L}(I_\sigma)$, cannot yield an upper bound smaller than $3/2$. To see this, let $0 < \varepsilon \leq 1/2$. Since at least $W_1(\sigma) \geq q_n$ squares of side length L are needed to cover the column $I_\sigma \times K_\sigma$, we have

$$\sum_{\sigma \in \Sigma_n} W_1(\sigma) (\sqrt{2} \mathcal{L}(I_\sigma))^{1+\varepsilon} \geq b_n = \sum_{\sigma \in \Sigma_n} (q_n^*(\sigma) \mathcal{L}(I_\sigma)) (\mathcal{L}(I_\sigma))^\varepsilon \geq 2^{-1/2} \sum_{\sigma \in \Sigma_n} (\mathcal{L}(I_\sigma))^{1/2+\varepsilon} \geq 2^{-1/2}$$

for every $n \in \mathbb{N}$, where the last inequality follows from $\mathcal{L}(I_\sigma) \leq 1$, $1/2 + \varepsilon \leq 1$, and Lemma 4.1(ii). Therefore, such coverings can never show that $\mathcal{H}^{1+\varepsilon}(G_P(\mathbb{I})) = 0$ for $\varepsilon \leq 1/2$, no matter how the factor $W_1(\sigma) \leq (n+1)q_n^*(\sigma)$ is estimated. We emphasize that this observation concerns only these particular coverings, and it does not exclude the possibility that other coverings yield a smaller upper bound. For instance, one may use coverings adapted to the distribution of the graph of P within each column $I_\sigma \times K_\sigma$, or coverings whose members meet several columns at once.

6. An alternative approach via coprime summation and Möbius inversion

In this section, we present an alternative argument for the convergence of the covering sums in Sections 4 and 5. Instead of estimating a_n and b_n for each n separately, we evaluate the series $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} b_n$ exactly. To this end, we convert sums over finite CF sequences into sums over coprime pairs of convergent denominators, and then apply Möbius inversion. Although this argument is not

needed for the proofs of Theorems 1.1 and 1.2, it provides exact formulas for these series (Proposition 6.7) and shows that the series $\sum_{n \geq 1} b_n$ diverges if and only if $\varepsilon \leq 1/2$.

We begin by recalling two classical identities involving the Möbius function. Recall that the Riemann zeta function ζ is defined by $\zeta(s) := \sum_{n \geq 1} 1/n^s$ for each complex number s with $\operatorname{Re}(s) > 1$. The Möbius function $\mu: \mathbb{N} \rightarrow \{-1, 0, 1\}$ is defined for each $n \in \mathbb{N}$ by

$$\mu(n) := \begin{cases} 0, & \text{if } n \text{ has a squared prime factor;} \\ 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

Proposition 6.1 (See [11, Theorem 263]). *For each $n \in \mathbb{N}$,*

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1. \end{cases}$$

Proposition 6.2 (See [11, Theorem 287]). *For each real number $s > 1$,*

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

To evaluate the series $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} b_n$, we analyze the double sum

$$S := \sum_{n \geq 1} \sum_{\sigma \in \Sigma_n} \Phi(q_n^*(\sigma), q_{n-1}^*(\sigma)),$$

where Φ is a non-negative function on $\mathbb{R}^2$. For every pair of coprime integers $1 \leq k < j$, the pair (j, k) occurs exactly twice among

$$(q_n^*(\sigma), q_{n-1}^*(\sigma)), \quad n \geq 1, \sigma \in \Sigma_n.$$

The exceptional pair $(1, 1)$ occurs exactly once, whereas the pair $(1, 0)$ does not occur. We illustrate this fact with an example below; see also [6, p. 9].

Example 6.3. *Let $q_n^*(\sigma) = 10$ for some $\sigma \in \Sigma_n$, and consider $k = 3$, which satisfies $\gcd(10, 3) = 1$ and lies in $\{0, \dots, 9\}$. Since*

$$\frac{3}{10} = [0; 3, 3, \infty, \infty, \dots] = [0; 3, 2, 1, \infty, \infty, \dots],$$

the two finite CF representations of $3/10$ are $(3, 3)$ and $(3, 2, 1)$. Reversing these digit sequences, we obtain $(3, 3)$ and $(1, 2, 3)$. By the reversal identity in Proposition 2.3(vi), we therefore have

$$q_2^*((3, 3)) = 10, \quad q_1^*((3, 3)) = 3, \quad \text{and} \quad q_3^*((1, 2, 3)) = 10, \quad q_2^*((1, 2, 3)) = 3.$$

Thus, the Φ -value corresponding to the pair $(10, 3)$ appears exactly twice in the double sum.

The following lemma justifies this counting claim.

Lemma 6.4. *The following hold.*

(i) For every pair of coprime integers $1 \leq k < j$, there exist some $n \in \mathbb{N}$ and exactly two finite CF sequences $\sigma \in \Sigma_n$ and $\sigma' \in \Sigma_{n+1}$ satisfying

$$(q_n^*(\sigma), q_{n-1}^*(\sigma)) = (q_{n+1}^*(\sigma'), q_n^*(\sigma')) = (j, k).$$

(ii) There exists exactly one finite CF sequence σ satisfying

$$(q_n^*(\sigma), q_{n-1}^*(\sigma)) = (1, 1),$$

namely $\sigma = (1) \in \Sigma_1$.

(iii) There is no finite CF sequence σ satisfying

$$(q_n^*(\sigma), q_{n-1}^*(\sigma)) = (1, 0).$$

Here, in parts (ii) and (iii), $n \in \mathbb{N}$ denotes the length of σ , i.e., $\sigma \in \Sigma_n$.

Proof. (i) Let $1 \leq k < j$ with $\gcd(j, k) = 1$. Since $k/j \in (0, 1) \cap \mathbb{Q}$, its CF expansion generated by (1.1) and (1.2) is finite. Write

$$\frac{k}{j} = [b_1, \dots, b_n], \quad \text{where } b_n > 1.$$

By Proposition 2.4(i), the only other finite CF representation of k/j is

$$[b_1, \dots, b_n - 1, 1].$$

Consider the reversed digit lists

$$\sigma := (b_n, \dots, b_1) \in \Sigma_n, \quad \sigma' := (1, b_n - 1, b_{n-1}, \dots, b_1) \in \Sigma_{n+1}.$$

By the reversal identity in Proposition 2.3(vi),

$$\frac{q_{n-1}^*(\sigma)}{q_n^*(\sigma)} = [b_1, \dots, b_n] = \frac{k}{j}.$$

Since the numerator and denominator on each side are relatively prime, $(q_n^*(\sigma), q_{n-1}^*(\sigma)) = (j, k)$. Similarly,

$$\frac{q_n^*(\sigma')}{q_{n+1}^*(\sigma')} = [b_1, \dots, b_n - 1, 1] = \frac{k}{j},$$

and hence $(q_{n+1}^*(\sigma'), q_n^*(\sigma')) = (j, k)$.

For uniqueness, suppose that $\tau := (c_1, \dots, c_\ell) \in \Sigma_\ell$, $\ell \in \mathbb{N}$, also satisfies $(q_\ell^*(\tau), q_{\ell-1}^*(\tau)) = (j, k)$. By the reversal identity,

$$\frac{k}{j} = \frac{q_{\ell-1}^*(\tau)}{q_\ell^*(\tau)} = [c_\ell, \dots, c_1].$$

Proposition 2.4(i) then implies that either

$$(c_\ell, \dots, c_1) = (b_1, \dots, b_n) \quad \text{or} \quad (c_\ell, \dots, c_1) = (b_1, \dots, b_n - 1, 1).$$

Reversing the digit lists, we obtain $\tau = \sigma$ or $\tau = \sigma'$. Thus, there are exactly two such sequences.

(ii) For $\sigma := (1) \in \Sigma_1$, we have $q_1^*(\sigma) = q_0^*(\sigma) = 1$.

Conversely, if $\tau := (c_1) \in \Sigma_1$ satisfies $(q_1^*(\tau), q_0^*(\tau)) = (1, 1)$, then $q_1^*(\tau) = c_1 = 1$, and hence $\tau = (1)$. No sequence of length $n \geq 2$ can produce $(1, 1)$ because $q_n^* > q_{n-1}^* \geq 1$ by Lemma 5.1. Therefore, $(1) \in \Sigma_1$ is the unique such sequence.

(iii) Since $q_{n-1}^* \geq 1$ for every $n \geq 1$, there is no finite CF sequence σ satisfying $(q_n^*(\sigma), q_{n-1}^*(\sigma)) = (1, 0)$. $\square$

This explains precisely why each coprime pair (j, k) with $1 \leq k < j$ occurs twice in the double sum, while the exceptional pair $(1, 1)$ occurs only once.

The next lemma reformulates the double sum S using the correspondence established in Lemma 6.4; it generalizes equation (9) in [6, p. 9].

Lemma 6.5. *Let $\Phi: \mathbb{R}^2 \rightarrow [0, \infty)$ be a non-negative function. Then, as an identity in $[0, \infty]$,*

$$S = -2\Phi(1, 0) + \Phi(1, 1) + 2 \sum_{j \geq 1} \sum_{\substack{0 \leq k < j \\ \gcd(j, k) = 1}} \Phi(j, k). \quad (6.1)$$

Proof. Define

$$T := \sum_{j \geq 1} \sum_{\substack{0 \leq k < j \\ \gcd(j, k) = 1}} \Phi(j, k).$$

By Lemma 6.4, each coprime pair (j, k) with $1 \leq k < j$ occurs exactly twice among the pairs

$$(q_n^*(\sigma), q_{n-1}^*(\sigma)), \quad n \geq 1, \sigma \in \Sigma_n,$$

whereas the pair $(1, 1)$ occurs exactly once and the pair $(1, 0)$ does not occur.

On the other hand, T contains each coprime pair (j, k) with $1 \leq k < j$ exactly once, contains the pair $(1, 0)$, and does not contain the pair $(1, 1)$. Therefore, since all terms are non-negative,

$$S = 2T - 2\Phi(1, 0) + \Phi(1, 1),$$

which is precisely (6.1). $\square$

If Φ satisfies the scaling relation below, the coprimality condition in (6.1) can be removed, as the following lemma shows.

Lemma 6.6. *Let $s > 1$, and let $\Phi: \mathbb{R}^2 \rightarrow [0, \infty)$ satisfy $\Phi(dj, dk) = d^{-s}\Phi(j, k)$ for all $d \in \mathbb{N}$ and all integers $0 \leq k < j$. Then, as an identity in $[0, \infty]$, we have*

$$\sum_{j \geq 1} \sum_{\substack{0 \leq k < j \\ \gcd(j, k) = 1}} \Phi(j, k) = \frac{1}{\zeta(s)} \sum_{j \geq 1} \sum_{0 \leq k < j} \Phi(j, k). \quad (6.2)$$

Proof. Every pair of integers (j, k) with $0 \leq k < j$ can be uniquely written as $(j, k) = (dj', dk')$, where $d := \gcd(j, k) \in \mathbb{N}$, $\gcd(j', k') = 1$, and $0 \leq k' < j'$. By non-negativity and the assumed scaling relation, it follows that

$$\sum_{j \geq 1} \sum_{0 \leq k < j} \Phi(j, k) = \sum_{d \geq 1} \frac{1}{d^s} \sum_{j' \geq 1} \sum_{\substack{0 \leq k' < j' \\ \gcd(j', k') = 1}} \Phi(j', k') = \zeta(s) \sum_{j \geq 1} \sum_{\substack{0 \leq k < j \\ \gcd(j, k) = 1}} \Phi(j, k)$$

in $[0, \infty]$, which is equivalent to (6.2).

If the right-hand side of (6.2) is finite, the identity can also be derived directly from Möbius inversion as follows. By Proposition 6.1, for each $j \in \mathbb{N}$,

$$\sum_{\substack{0 \leq k < j \\ \gcd(j,k)=1}} \Phi(j, k) = \sum_{0 \leq k < j} \Phi(j, k) \sum_{d \mid \gcd(j,k)} \mu(d) = \sum_{d \mid j} \mu(d) \sum_{\substack{0 \leq k < j \\ d \mid k}} \Phi(j, k).$$

The resulting triple series is absolutely convergent. Indeed, using non-negativity to interchange the sums and then writing $j = dj'$ and $k = dk'$, we obtain

$$\sum_{j \geq 1} \sum_{d \mid j} \sum_{\substack{0 \leq k < j \\ d \mid k}} |\mu(d)| \Phi(j, k) = \sum_{d \geq 1} \frac{|\mu(d)|}{d^s} \sum_{j \geq 1} \sum_{0 \leq k < j} \Phi(j, k) < \infty.$$

Hence, the order of summation can be interchanged. Rewriting the double sum as a sum over common divisors d , and noting that $0 \leq dk' < dj'$ if and only if $0 \leq k' < j'$, we obtain

$$\begin{aligned} \sum_{j \geq 1} \sum_{\substack{0 \leq k < j \\ \gcd(j,k)=1}} \Phi(j, k) &= \sum_{d \geq 1} \mu(d) \sum_{j \geq 1} \sum_{\substack{0 \leq k < j \\ d \mid j \\ d \mid k}} \Phi(j, k) = \sum_{d \geq 1} \mu(d) \sum_{j' \geq 1} \sum_{0 \leq k' < j'} \Phi(dj', dk') \\ &= \sum_{d \geq 1} \frac{\mu(d)}{d^s} \sum_{j \geq 1} \sum_{0 \leq k < j} \Phi(j, k) = \frac{1}{\zeta(s)} \sum_{j \geq 1} \sum_{0 \leq k < j} \Phi(j, k), \end{aligned}$$

where the last equality holds by Proposition 6.2. $\square$

We are now in a position to evaluate the series $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} b_n$, where a_n and b_n are defined by (4.2) and (5.2), respectively. Here, we regard $a_n = a_n(\varepsilon)$ and $b_n = b_n(\varepsilon)$ as defined for every $\varepsilon > 0$.

Proposition 6.7. *Let $\varepsilon > 0$. Then*

$$\sum_{n \geq 1} a_n = -2 + \frac{1}{2^{1+\varepsilon}} + \frac{2}{\zeta(2+2\varepsilon)} \sum_{j \geq 1} \frac{1}{j^{1+\varepsilon}} \sum_{j \leq m < 2j} \frac{1}{m^{1+\varepsilon}} \leq -2 + \frac{1}{2^{1+\varepsilon}} + \frac{2\zeta(1+2\varepsilon)}{\zeta(2+2\varepsilon)} < \infty, \quad (6.3)$$

$$\sum_{n \geq 1} b_n = -2 + \frac{1}{2^{1+\varepsilon}} + \frac{2}{\zeta(1+2\varepsilon)} \sum_{j \geq 1} \frac{1}{j^\varepsilon} \sum_{j \leq m < 2j} \frac{1}{m^{1+\varepsilon}}, \quad (6.4)$$

where (6.4) is an identity in $[0, \infty]$. Moreover, $\sum_{n \geq 1} b_n < \infty$ if and only if $\varepsilon > 1/2$, in which case $\sum_{n \geq 1} b_n \leq -2 + 2^{-1-\varepsilon} + 2\zeta(2\varepsilon)/\zeta(1+2\varepsilon)$.

Proof. By (2.1), we have $a_n = \sum_{\sigma \in \Sigma_n} \Phi_a(q_n^*(\sigma), q_{n-1}^*(\sigma))$ and $b_n = \sum_{\sigma \in \Sigma_n} \Phi_b(q_n^*(\sigma), q_{n-1}^*(\sigma))$, where

$$\Phi_a(j, k) := \frac{1}{j^{1+\varepsilon}(j+k)^{1+\varepsilon}} \quad \text{and} \quad \Phi_b(j, k) := \frac{1}{j^\varepsilon(j+k)^{1+\varepsilon}}$$

for $j \geq 1$ and $k \geq 0$, and $\Phi_a := \Phi_b := 0$ elsewhere. Notice that both functions take the value 1 at $(1, 0)$ and the value $2^{-1-\varepsilon}$ at $(1, 1)$. They satisfy the scaling condition in Lemma 6.6 for all integers $0 \leq k < j$ and all $d \in \mathbb{N}$, with $s = 2 + 2\varepsilon$ and $s = 1 + 2\varepsilon$, respectively. Moreover, for each $j \in \mathbb{N}$, we have

$$\sum_{0 \leq k < j} \Phi_a(j, k) = \frac{1}{j^{1+\varepsilon}} \sum_{j \leq m < 2j} \frac{1}{m^{1+\varepsilon}} \quad \text{and} \quad \sum_{0 \leq k < j} \Phi_b(j, k) = \frac{1}{j^\varepsilon} \sum_{j \leq m < 2j} \frac{1}{m^{1+\varepsilon}}.$$

Applying Lemma 6.5 and then Lemma 6.6 with $s = 2 + 2\varepsilon$ and $s = 1 + 2\varepsilon$, respectively, we obtain the identities in (6.3) and (6.4). Now, since

$$\frac{1}{2^{1+\varepsilon}} \cdot \frac{1}{j^\varepsilon} = j \cdot \frac{1}{(2j)^{1+\varepsilon}} \leq \sum_{j \leq m < 2j} \frac{1}{m^{1+\varepsilon}} \leq j \cdot \frac{1}{j^{1+\varepsilon}} = \frac{1}{j^\varepsilon} \quad (6.5)$$

for each $j \in \mathbb{N}$, the double sum in (6.3) is at most $\zeta(1 + 2\varepsilon) < \infty$, and the double sum in (6.4) lies between $2^{-1-\varepsilon} \sum_{j \geq 1} j^{-2\varepsilon}$ and $\sum_{j \geq 1} j^{-2\varepsilon}$. Since the latter series converges if and only if $2\varepsilon > 1$, in which case its sum is $\zeta(2\varepsilon)$, the proof is complete. $\square$

Proposition 6.7 provides alternative proofs of Theorems 1.1 and 1.2. Indeed, since $\sum_{n \geq 1} a_n$ converges for every $\varepsilon > 0$, we have $\liminf_{n \rightarrow \infty} (n + 1)a_n = 0$. To see this, suppose for contradiction that $\liminf_{n \rightarrow \infty} (n + 1)a_n > 0$. Then there exist $c > 0$ and $N \in \mathbb{N}$ such that $a_n > c/(n + 1)$ for all $n \geq N$, which contradicts the convergence of $\sum_{n \geq 1} a_n$. Substituting $\liminf_{n \rightarrow \infty} (n + 1)a_n = 0$ into (4.1), we obtain $\mathcal{H}^{1+\varepsilon}(G_\varepsilon(\mathbb{I})) = 0$ for every $\varepsilon > 0$, which yields Lemma 4.2 and hence Theorem 1.1. In the same way, the convergence of $\sum_{n \geq 1} b_n$ for $\varepsilon > 1/2$, together with (5.1), yields Theorem 1.2.

Remark 6.8. *If $\varepsilon \leq 1/2$, then the series $\sum_{n \geq 1} b_n$ diverges by Proposition 6.7, and so the above argument cannot yield an upper bound smaller than $3/2$ for the Hausdorff dimension of the graph of P . We note that the divergence of the series alone does not exclude the possibility that $\liminf_{n \rightarrow \infty} (n + 1)b_n = 0$. However, Remark 5.2 shows the stronger statement that $b_n \geq 2^{-1/2}$ for every $n \in \mathbb{N}$ when $\varepsilon \leq 1/2$. Hence, both the argument in this section and that in Section 5 fail for every $0 < \varepsilon \leq 1/2$ for the same reason, namely, the lower bound (5.3) for the oscillation of P .*

7. Conclusions

In this paper, we have shown that the graph of the unweighted error-sum function

$$\mathcal{E}(x) = \sum_{n \geq 0} \left(x - \frac{p_n(x)}{q_n(x)} \right)$$

of continued fractions has Hausdorff dimension 1 (Theorem 1.1). The key step is the exact oscillation formula

$$\sup_{t, u \in I_\sigma} |\mathcal{E}(t) - \mathcal{E}(u)| = (n + 1) \cdot \mathcal{L}(I_\sigma), \quad \sigma \in \Sigma_n,$$

in Lemma 3.9. We obtained this formula by lifting $\mathcal{E}$ to the symbolic space Σ of CF sequences, which contains the non-realizable finite sequences representing the missing endpoints of fundamental intervals, and by analyzing the alternating tails of the lifted series. Once the formula is established, Theorem 1.1 follows from a short covering argument based on the Fibonacci lower bound for the convergent denominators.

As a consistency check of our framework, we applied the same method to the relative error-sum function P , whose oscillation on I_σ equals $\mathcal{L}(I_\sigma) \sum_{k=0}^n q_k^*(\sigma)$, and recovered the upper bound $3/2$ of Ridley and Petruska [15] for the Hausdorff dimension of the graph of P (Theorem 1.2). The alternative approach in Section 6, which is based on the correspondence between finite CF sequences and coprime pairs of convergent denominators together with Möbius inversion, evaluates the relevant covering series exactly.

Our analysis also clarifies the limitation of the method. By Remark 5.2, the oscillation of P on every fundamental interval I_σ is at least $2^{-1/2} \mathcal{L}(I_\sigma)^{1/2}$. Consequently, the coverings used in the proof of Theorem 1.2, in which each full column over a fundamental interval of order n is covered by squares of side length $\mathcal{L}(I_\sigma)$, cannot yield an upper bound smaller than $3/2$, no matter how the factor $\sum_{k=0}^n q_k^*(\sigma)$ is estimated. In accordance with this, the covering series $\sum_{n \geq 1} b_n$ diverges if and only if $\varepsilon \leq 1/2$ (Proposition 6.7). Figure 1 suggests that the graph of P is much more irregular than that of $\mathcal{E}$, but we do not have enough evidence to conjecture the exact value of its dimension. Therefore, the exact Hausdorff dimension of $G_P(\mathbb{R})$ remains unknown. In particular, it would be interesting to determine whether the upper bound $3/2$ is sharp. Any improvement through the present covering approach would require different coverings, for instance, coverings adapted to the internal structure of the graph of P within each column. Of course, other kinds of arguments for the dimension of the graph are not ruled out by our analysis.

Use of Generative-AI tools declaration

The author declares that Artificial Intelligence (AI) tools have been used in the preparation of this article. ChatGPT was used for English editing, including grammar corrections, improving readability, and refining sentence structure. Claude assisted in writing the TeX code used to generate Figure 1. The AI-assisted English editing applies to the overall manuscript.

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Conflict of interest

The author declares no conflict of interest.

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