



*Research article***Exact soliton, periodic and rational solutions of the M -fractional Nurshuak-Tolkynay-Myrzakulov system****Aigul Taishiyeva¹, Gulgassyl Nugmanova^{2,*}, Aidana Azhikhan^{2,3}, Akbota Myrzakul^{3,*} and Ratbay Myrzakulov⁴**¹ Kh. Dosmukhamedov Atyrau University, Atyrau, Kazakhstan² Eurasian National University, Astana, Kazakhstan³ Astana IT University, Astana, Kazakhstan⁴ Ratbay Myrzakulov Eurasian International Center for Theoretical Physics*** Correspondence:** Email: nugmanovagn@gmail.com, a.myrzakul@astanait.edu.kz.

Abstract: The object of this study was the Nurshuak-Tolkynay-Myrzakulov (NTM) system with M -fractional in both time and space. Using appropriate traveling-wave transformations, the fractional nonlinear system was reduced to a nonlinear ordinary differential equation. Exact analytical solutions were constructed using the generalized Kudryashov method and the Sardar subequation method. The obtained solutions contained hyperbolic, trigonometric, rational, and Jacobi elliptic (cnoidal) wave structures that described localized solitons, periodic waves and algebraically decaying profiles. The influence of the fractional-order parameter η on the amplitude and localization width and propagation characteristics of the nonlinear waves used two- and three-dimensional graphical representations. In the limit $\eta \rightarrow 1$, all solutions recovered the corresponding exact solutions of the classical integer-order NTM system. This was the first study of the M -fractional NTM system and its exact analytical solutions.

Keywords: M -fractional; Nurshuak-Tolkynay-Myrzakulov equation; exact soliton solutions; Kudryashov method; Sardar subequation method; Jacobi elliptic solution solutions

Mathematics Subject Classification: 35A30, 35C08

1. Introduction

Nonlinear evolution equations play a fundamental role in the mathematical modeling of complex physical phenomena arising in plasma physics, nonlinear optics, hydrodynamics, and wave propagation. In particular, integrable nonlinear equations are widely used to describe nonlinear wave interactions, energy transfer, and stability mechanisms in continuous media [1–3]. Exact analytical

solutions of such systems provide a deeper understanding of the internal dynamics of the modeled processes and serve as model solutions for numerical and approximate methods [4–6].

In recent years, fractional calculus has attracted increasing attention in the field of nonlinear equations in mathematical physics due to its ability to more accurately describe the properties of various physical systems than classical integer-order models [7, 8]. Nonlinear fractional partial differential equations are successfully applied in visco-elasticity, anomalous diffusion, and electromagnetic theory [9, 10]. As a result, extending well-known nonlinear evolution equations to their fractional counterparts has become an active area of research in applied mathematics and mathematical physics [11–13].

The Nurshuak-Tolkynay-Myrzakulov (NTM) system represents an important class of coupled nonlinear evolution equations describing complex wave interactions in nonlinear media [14–16]. The NTM system describes the propagation and interaction of nonlinear wave envelopes in dispersive media. The complex field q represents the slowly varying envelope of the wave, while the auxiliary fields v and w account for nonlinear interactions and density perturbations. Such models arise in nonlinear optics, plasma physics, and hydrodynamics, where the interplay between dispersion and nonlinearity leads to the formation of localized solitons and periodic wave structures.

The M -fractional is a local fractional operator that preserves the chain rule and the classical traveling-wave reduction [17–19]. It allows one to incorporate memory effects and anomalous dispersion while remaining computationally tractable. The two-parameter flexibility of the M -fractional (η, γ) provides additional freedom in fitting experimental data compared with single-parameter fractional operators. Although the classical form of this system has been studied using various analytical and numerical methods, its generalization to fractional order remains understudied. Exact analytical solutions for the fractional NTM system is limited, motivating further research.

To fill this gap, effective analytical methods are needed. Among them, the Kudryashov method and Sardar subequation method have proven to be powerful and systematic approaches for obtaining exact solutions to nonlinear fractional differential equations [20–22]. Both the generalized Kudryashov method and the Sardar subequation method are effective tools for constructing exact solutions of nonlinear evolution equations. The Kudryashov method, based on a Riccati-type auxiliary equation, is particularly efficient for obtaining rational and mixed hyperbolic-trigonometric solutions. The Sardar subequation method, relying on a first-order elliptic auxiliary equation, naturally generates a wider class of solutions that includes Jacobi elliptic (cnoidal) waves in addition to the hyperbolic and trigonometric limits. By applying both techniques we obtain complementary families of solutions and gain a more complete picture of the nonlinear wave structures admitted by the M -fractional NTM system. Neither method is claimed to be new; their joint use allows a systematic comparison and cross-verification of the results.

Although the classical (integer-order) NTM system has been studied by analytical and numerical methods, its generalization to the M -fractional case remains unexplored. The main contributions of the present paper are as follows:

- The M -fractional is introduced into the NTM system for the first time and the corresponding fractional model that accounts for memory effects is derived.
- By applying the generalized Kudryashov method and the Sardar subequation method, several new families of exact analytical solutions, including hyperbolic, trigonometric, rational, and Jacobi elliptic (cnoidal) wave structures, are obtained.

- The influence of the fractional-order parameter η on the dynamical properties of the nonlinear waves (amplitude, width, and propagation characteristics) is investigated systematically.
- We demonstrate that in the limit $\eta \rightarrow 1$, the fractional solutions reduce exactly to the known solutions of the classical NTM system.

These results extend the theory of the NTM system to the fractional setting and provide new analytical insight into nonlinear wave dynamics with memory effects.

Based on the above considerations, the main goal of this paper is to study the M -fractional NTM system using the Kudryashov method and Sardar subequation technique. We will construct several new exact analytical solutions explicitly and conduct a comparative analysis with solutions of the integer-order NTM system.

We will consider the NTM system [14]:

$$\begin{aligned} q_t + q_{xxt} - vq - (wq)_x &= 0, \\ v_x + 2\varepsilon(\bar{q}_{xt}q - \bar{q}q_{xt}) &= 0, \\ w_x - 2\varepsilon(|q|^2)_t &= 0. \end{aligned}$$

Here, $q(x, t)$ represents a complex wave envelope, while $v(x, t)$ and $w(x, t)$ are auxiliary fields that describe nonlinear interactions in the physical medium. As will be shown in Section 3, under the traveling-wave reduction and the compatibility condition (3.9), the field v turns out to be purely imaginary, whereas the field w remains real-valued. The physically observable quantities are primarily the modulus $|q|$ and the real auxiliary field w . The indices x and t denote partial derivatives with respect to space and time, respectively, and the overbar denotes complex conjugation. The parameter ε characterizes the strength of nonlinear effects.

The NTM system describes the propagation and interaction of nonlinear wave envelopes associated with auxiliary fields in dispersive nonlinear media. Such models arise in the study of nonlinear optics, plasma dynamics, and wave propagation phenomena, where the balance between dispersion and nonlinearity leads to the formation of localized structures and soliton-type waves. Due to these properties, the NTM system has become an important mathematical model for studying nonlinear wave dynamics. By replacing integer-order derivatives in the classical NTM system with fractional derivatives, as defined by M -derivatives [23–25], we obtain a fractional NTM model that allows for the correct accounting of memory effects of previous processes in complex physical systems.

Consider the M -fractional NTM system:

$$D_{M,t}^{\eta,\gamma}q + D_{M,xx}^{\eta,\gamma}q - vq - D_{M,x}^{\eta,\gamma}(wq) = 0, \quad (1.1)$$

$$D_{M,t}^{\eta,\gamma}v + 2\varepsilon(qD_{M,xt}^{\eta,\gamma}\bar{q} - \bar{q}D_{M,xt}^{\eta,\gamma}q) = 0, \quad (1.2)$$

$$D_{M,x}^{\eta,\gamma}w - 2\varepsilon D_{M,t}^{\eta,\gamma}|q|^2 = 0. \quad (1.3)$$

In Eqs (1.1)–(1.3), $D_{M,t}^{\eta,\gamma}$ and $D_{M,x}^{\eta,\gamma}$ denote the fractional derivatives of order η with respect to time and space, respectively, where $0 < \eta \leq 1$.

2. Adapting existing methods to solve M -fractional NTM system

2.1. The Kudryashov method

Consider the following nonlinear partial differential equation in its general form:

$$S(h, h^2 h_x, h_t, h_{xx}, h_{tt}, h_{xt}, \dots) = 0. \quad (2.1)$$

To reduce Eq (2.1) into an ordinary differential equation, a traveling wave transformation is introduced in the form

$$h(x, t) = H(\omega), \quad \omega = \delta x + \lambda t, \quad (2.2)$$

where δ and λ are nonzero constants. Substituting Eq (2.2) into Eq (2.1) yields the following nonlinear ordinary differential equation:

$$T(H, H', H'', \dots) = 0, \quad (2.3)$$

where the prime denotes differentiation with respect to ω .

The solution of Eq (2.3) is assumed in the form

$$H(\omega) = a_0 + \sum_{j=1}^M \frac{a_j}{(1 + \phi(\omega))^j}, \quad (2.4)$$

where a_j , ($j = 0, 1, 2, \dots, M$) are constants to be determined and M is a positive integer obtained using the homogeneous balance principle.

The auxiliary function $\phi(\omega)$ satisfies the following first-order nonlinear ordinary differential equation:

$$\phi'(\omega) = \rho + \sigma\phi(\omega) + \Omega\phi^2(\omega), \quad (2.5)$$

where ρ , σ , and Ω are arbitrary constants. Depending on the parametric values of these constants, different types of exact solutions can be obtained.

For instance, if $\rho \neq 0$, $\sigma \neq 0$, and $\Omega \neq 0$, the auxiliary Eq (2.5) admits the following families of solutions.

The structure of solutions depends on the discriminant $\Delta = 4\rho\Omega - \sigma^2$.

Case 1: Trigonometric solutions ($\Delta > 0$).

If $4\rho\Omega > \sigma^2$, the auxiliary equation admits the trigonometric solution

$$\phi(\omega) = \frac{1}{2\Omega} \left(\sqrt{4\rho\Omega - \sigma^2} \tan\left(\frac{1}{2} \sqrt{4\rho\Omega - \sigma^2} (\omega + d_0)\right) - \sigma \right), \quad (2.6)$$

where d_0 is an arbitrary integration constant.

Case 2: Hyperbolic solutions ($\Delta < 0$).

If $4\rho\Omega < \sigma^2$, the hyperbolic solutions are obtained as

$$\phi(\omega) = \frac{1}{2\Omega} \left(\sqrt{\sigma^2 - 4\rho\Omega} \tanh\left(\frac{1}{2} \sqrt{\sigma^2 - 4\rho\Omega} (\omega + d_0)\right) - \sigma \right). \quad (2.7)$$

Case 3: Rational solution ($\Delta = 0$).

If $4\rho\Omega = \sigma^2$, the auxiliary equation yields the rational solution

$$\phi(\omega) = -\frac{\sigma}{2\Omega} - \frac{1}{\Omega(\omega + d_0)}.$$

Remark. If $\Omega = 0$, Eq (2.5) reduces to the linear ordinary differential equation

$$\phi'(\omega) = \rho + \sigma\phi(\omega),$$

whose solution is given by

$$\phi(\omega) = -\frac{\rho}{\sigma} + d_0 e^{\sigma\omega}, \quad (\sigma \neq 0).$$

In the special case $\sigma = 0$, the solution becomes

$$\phi(\omega) = \rho\omega + d_0.$$

The obtained solution families of the auxiliary equation play a fundamental role in constructing exact wave solutions of the M -fractional NTM system. In particular, the sign of the discriminant $\Delta = 4\rho\Omega - \sigma^2$ determines the qualitative nature of the resulting nonlinear wave structures.

The case $\Delta > 0$ generates periodic (trigonometric-type) solutions, which correspond to oscillatory wave patterns. For $\Delta < 0$, hyperbolic solutions arise, leading to localized solitary wave structures of bright or dark type depending on the parameter configuration. When $\Delta = 0$, rational solutions are obtained, which describe algebraically decaying wave profiles.

2.2. The Sardar subequation technique

The Sardar subequation technique is an efficient analytical approach for constructing exact solutions of nonlinear fractional partial differential equations. Consider the following nonlinear fractional partial differential equation:

$$J(g, g_x, g_t, g_{xx}, g_{xt}, g_{tt}, \dots) = 0, \quad (2.8)$$

where $g = g(x, t)$ represents the wave profile.

To reduce Eq (2.8) into an ordinary differential equation, the traveling wave transformation

$$g(x, t) = G(\omega), \quad \omega = \lambda x + \mu t, \quad (2.9)$$

is employed, where λ and μ are nonzero constants. Substituting Eq (2.9) into Eq (2.8) yields the following nonlinear ordinary differential equation:

$$Y(G, G', G'', GG', G^2, \dots) = 0, \quad (2.10)$$

where the prime denotes differentiation with respect to ω .

The solution of Eq (2.10) is assumed in the form

$$G(\omega) = \sum_{i=0}^M b_i \psi^i(\omega), \quad (2.11)$$

where b_i ($i = 0, 1, 2, \dots, M$) are constants to be determined and M is a positive integer obtained via the homogeneous balance principle.

The auxiliary function $\psi(\omega)$ satisfies the following nonlinear ordinary differential equation:

$$\psi'(\omega) = \sqrt{\sigma + \kappa\psi^2(\omega) + \psi^4(\omega)}, \quad (2.12)$$

where σ and κ are real parameters.

Substituting Eq (2.11) together with Eq (2.12) into Eq (2.10) and equating the coefficients of like powers of $\psi(\omega)$, a system of algebraic equations involving the unknowns b_i , λ , and μ is obtained. Solving this algebraic system yields the exact solutions of the main equation.

The qualitative behavior of the solutions depends mainly on the sign of κ and the relation between κ and σ .

Case 1: Hyperbolic-type solutions ($\kappa > 0$).

If $\kappa > 0$ and $\sigma < 0$, Eq (2.12) admits localized hyperbolic solutions of the form

$$\psi(\omega) = \pm \sqrt{\frac{\kappa}{-\sigma}} \operatorname{sech}(\sqrt{\kappa} \omega),$$

or

$$\psi(\omega) = \pm \sqrt{\frac{\kappa}{-\sigma}} \operatorname{csch}(\sqrt{\kappa} \omega).$$

Case 2: Trigonometric-type solutions ($\kappa < 0$).

If $\kappa < 0$ and $\sigma > 0$, the solutions take the periodic form

$$\psi(\omega) = \pm \sqrt{\frac{-\kappa}{\sigma}} \sec(\sqrt{-\kappa} \omega),$$

or

$$\psi(\omega) = \pm \sqrt{\frac{-\kappa}{\sigma}} \csc(\sqrt{-\kappa} \omega).$$

Case 3: Special parametric reduction.

For particular parameter relations, Eq (2.12) reduces to integrable forms yielding

$$\psi(\omega) = \pm \sqrt{\frac{\kappa}{2\sigma}} \tanh\left(\sqrt{\frac{\kappa}{2}} \omega\right),$$

or

$$\psi(\omega) = \pm \sqrt{\frac{\kappa}{2\sigma}} \coth\left(\sqrt{\frac{\kappa}{2}} \omega\right).$$

Case 4: Jacobi elliptic solutions (sn-family).

The auxiliary Eq (2.10) is elliptic. For $\kappa < 0$ and $0 < m < 1$, it admits the real solution

$$\psi(\omega) = \pm \sqrt{\frac{-\kappa m}{1+m}} \operatorname{sn}\left(\sqrt{\frac{-\kappa}{1+m}} \omega; m\right),$$

if, and only if,

$$\sigma = \frac{\kappa^2 m}{(1+m)^2}.$$

This solution, expressed through Jacobi elliptic functions, describes cnoidal waves and significantly expands the spectrum of periodic nonlinear structures allowed by the model under consideration.

Profiles of the form $A \operatorname{cn}(B\omega; m)$ and $A \operatorname{dn}(B\omega; m)$ produce a negative coefficient of ψ^4 and do not solve (2.10); they are not used. In the limit $m \rightarrow 1$, one has $\operatorname{sn}(u; 1) = \tanh u$, and the formula reduces to Case 3.

3. Mathematical treatment of the main model

To derive exact analytical solutions of the fractional NTM system, the following traveling wave transformations are introduced:

$$\begin{aligned} q(x, t) &= Q(\omega)e^{i\psi}, \\ v(x, t) &= V(\omega), \\ w(x, t) &= W(\omega), \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} \psi &= \frac{\Gamma(1+\gamma)}{\eta} (\lambda t^\eta - a x^\eta), \\ \omega &= \frac{\Gamma(1+\gamma)}{\eta} (x^\eta - \tau t^\eta). \end{aligned} \quad (3.2)$$

Here, a denotes the soliton frequency, λ is the soliton wave number, and τ represents the soliton velocity.

Substituting the transformations (3.1) and (3.2) into the main fractional system and separating the real and imaginary parts, the following system of ordinary differential equations is obtained.

$$\begin{aligned} \tau Q''' - i(2a\tau + \lambda)Q'' - (-\tau + a^2\tau + 2a\lambda)Q' + i(a^2\lambda - \lambda)Q + WQ' + (V + W' - iaW)Q &= 0, \\ V' - 4i\varepsilon(\lambda + a\tau)QQ' &= 0, \\ W' + 4\varepsilon\tau QQ' &= 0. \end{aligned} \quad (3.3)$$

Integrating Eq (3.3) with respect to ω , we obtain

$$W(\omega) = -2\varepsilon\tau Q^2(\omega),$$

where the integration constant is taken to be zero for simplicity.

Substituting the obtained expressions of $V(\omega)$ and $W(\omega)$ into the reduced system, we obtain

$$V(\omega) = 2\varepsilon i(\lambda + a\tau)Q^2(\omega).$$

We emphasize that the auxiliary field $V(\omega)$ is purely imaginary under the compatibility condition (3.9). Indeed, this condition implies $\lambda + a\tau = \pm\tau$ (assuming $\tau \neq 0$), so the factor $i(\lambda + a\tau)$ cannot be eliminated without degenerating the wave velocity. This feature is intrinsic to the traveling-wave reduction of the system. Nevertheless, the modulus $|q|$ and the field w remain real. In the limiting

case $\eta \rightarrow 1$, the obtained solutions recover the corresponding solutions of the classical (integer-order) NTM system.

As a result, the main equation for $Q(\omega)$ can be written as

$$\tau Q''' - i(2a\tau + \lambda)Q'' - (a^2\tau - \tau + 2a\lambda)Q' + i(a^2\lambda - \lambda)Q - 6\varepsilon\tau Q^2 Q' + 2i\varepsilon(\lambda + 2a\tau)Q^3 = 0. \quad (3.4)$$

Separating the real and imaginary parts of Eq (3.4), we obtain the following equations:

$$\tau Q''' - (a^2\tau - \tau + 2a\lambda)Q' - 6\varepsilon\tau Q^2 Q' = 0, \quad (3.5)$$

$$(2a\tau + \lambda)Q'' - (a^2\lambda - \lambda)Q - 2\varepsilon(\lambda + 2a\tau)Q^3 = 0. \quad (3.6)$$

Integrating Eq (3.5) with respect to ω , we obtain

$$\tau Q'' - (a^2\tau - \tau + 2a\lambda)Q - 2\varepsilon\tau Q^3 + C = 0, \quad (3.7)$$

where C is the constant of integration; further, we assume $C = 0$.

Dividing Eq (3.7) by τ and rearranging terms yields

$$Q'' - \left(a^2 - 1 + \frac{2a\lambda}{\tau}\right)Q - 2\varepsilon Q^3 = 0. \quad (3.8)$$

To combine Eqs (3.6) and (3.8), we define the coefficients. This implies that the parameters must satisfy the following condition:

$$\tau^2(a^2 - 1) + 2a\lambda\tau + \lambda^2 = 0,$$

or equivalently

$$(a\tau + \lambda)^2 = \tau^2 \Rightarrow \lambda + a\tau = \pm\tau. \quad (3.9)$$

When this condition is satisfied, Eqs (3.6) and (3.8) are compatible and reduce to the following nonlinear ordinary differential Eq (3.8), we will solve this equation using the generalized Kudryashov method and the Sardar subequation method [20–22].

4. Analytical soliton solutions

4.1. Solutions via the generalized Kudryashov method

Equation (3.8) contains a second-order derivative and a quadratic nonlinear term. To determine the structure of solution (2.4), using the balancing principle, the leading degree of the highest-order derivative and the highest-degree nonlinear term are equated [6].

We look for a solution in the following form:

$$Q(\omega) \sim \omega^{-M},$$

where $M > 0$ is to be determined. Then,

$$Q''(\omega) \sim \omega^{-M-2},$$

and for the nonlinear term,

$$Q^3(\omega) \sim \omega^{-3M}.$$

Since the leading terms of the equation are determined by these two expressions, we equate their powers:

$$-M - 2 = -3M.$$

Hence, $M = 1$.

For $M = 1$ in the formula (2.4), the reduced ordinary differential Eq (3.8) admits the following solution form:

$$Q(\omega) = a_0 + \frac{a_1}{1 + \phi(\omega)}, \quad (4.1)$$

where the auxiliary function $\phi(\omega)$ satisfies (2.5).

Raising the expression (4.1) to the third power and find its second derivative:

$$Q^3(\omega) = a_0^3 + \frac{3a_0^2a_1}{1 + \phi(\omega)} + \frac{3a_0a_1^2}{(1 + \phi(\omega))^2} + \frac{a_1^3}{(1 + \phi(\omega))^3}$$

and

$$Q''(\omega) = -\frac{a_1(\sigma + 2\Omega\phi(\omega))\phi'(\omega)}{(1 + \phi(\omega))^2} + \frac{2a_1(\phi'(\omega))^2}{(1 + \phi(\omega))^3}.$$

Then, Eq (3.8) can be written in the following form:

$$\begin{aligned} & -\frac{a_1(\sigma + 2\Omega\phi(\omega))\phi'(\omega)}{(1 + \phi(\omega))^2} + \frac{2(\phi'(\omega))^2}{(1 + \phi(\omega))^3}a_1 \\ & - \left(a^2 - 1 + \frac{2a\lambda}{\tau}\right)\left(a_0 + \frac{a_1}{1 + \phi(\omega)}\right) \\ & - 2\varepsilon\left(a_0^3 + \frac{3a_0^2a_1}{1 + \phi(\omega)} + \frac{3a_0a_1^2}{(1 + \phi(\omega))^2} + \frac{a_1^3}{(1 + \phi(\omega))^3}\right) = 0. \end{aligned} \quad (4.2)$$

To determine the types a_0, a_1 and unknown coefficients, we equate all coefficients of $\phi(\omega)$ for different degrees to zero. This results in the following system of algebraic equations:

$$\begin{cases} -Aa_0 - 2\Omega^2a_1 + \Omega\sigma a_1 - 2\varepsilon a_0^3 = 0, \\ -3Aa_0 - Aa_1 + 2\Omega\rho a_1 - 3\Omega\sigma a_1 + \sigma^2a_1 - 6\varepsilon a_0^3 - 6\varepsilon a_0^2a_1 = 0, \\ -3Aa_0 - 2Aa_1 - 2\Omega\rho a_1 + 3\rho\sigma a_1 - \sigma^2a_1 - 6\varepsilon a_0^3 - 12\varepsilon a_0^2a_1 - 6\varepsilon a_0a_1^2 = 0, \\ -Aa_0 - Aa_1 + 2\rho^2a_1 - \rho\sigma a_1 - 2\varepsilon a_0^3 - 6\varepsilon a_0^2a_1 - 6\varepsilon a_0a_1^2 - 2\varepsilon a_1^3 = 0, \end{cases} \quad (4.3)$$

where

$$A = a^2 - 1 + \frac{2a\lambda}{\tau}.$$

Solving the algebraic system (4.3) under the compatibility condition (3.9) yields

$$a_0 = 0, \quad \sigma = 2\Omega, \quad \rho = \frac{A + 2\Omega^2}{2\Omega}, \quad a_1 = \pm \sqrt{\frac{2\rho^2 - 2\Omega\rho - A}{2\varepsilon}},$$

where $A = -(a \pm 1)^2$.

The corresponding exact solutions of the M -fractional NTM system are given below.

Case 1: Trigonometric solutions, $\Delta > 0$.

$$\begin{aligned}
 q(x, t) &= \pm \sqrt{\frac{2\rho^2 - 2\Omega\rho - A}{2\varepsilon}} \frac{2\Omega}{2\Omega + \sqrt{4\rho\Omega - \sigma^2} \tan\left(\frac{1}{2} \sqrt{4\rho\Omega - \sigma^2} (\omega + d_0)\right) - \sigma} \\
 &\quad \times \exp\left[\frac{i\Gamma(1+\gamma)}{\eta} (\lambda t^\eta - ax^\eta)\right], \\
 v(x, t) &= \frac{4i\Omega^2(\lambda + a\tau)(2\rho^2 - 2\Omega\rho - A)}{\left(2\Omega + \sqrt{4\rho\Omega - \sigma^2} \tan\left(\frac{1}{2} \sqrt{4\rho\Omega - \sigma^2} (\omega + d_0)\right) - \sigma\right)^2}, \\
 w(x, t) &= \frac{-4\tau\Omega^2(2\rho^2 - 2\Omega\rho - A)}{\left(2\Omega + \sqrt{4\rho\Omega - \sigma^2} \tan\left(\frac{1}{2} \sqrt{4\rho\Omega - \sigma^2} (\omega + d_0)\right) - \sigma\right)^2}.
 \end{aligned}$$

Case 2: Hyperbolic solutions, $\Delta < 0$.

$$\begin{aligned}
 q(x, t) &= \pm \sqrt{\frac{2\rho^2 - 2\Omega\rho - A}{2\varepsilon}} \frac{2\Omega}{2\Omega + \sqrt{\sigma^2 - 4\rho\Omega} \tanh\left(\frac{1}{2} \sqrt{\sigma^2 - 4\rho\Omega} (\omega + d_0)\right) - \sigma} \\
 &\quad \times \exp\left[\frac{i\Gamma(1+\gamma)}{\eta} (\lambda t^\eta - ax^\eta)\right], \\
 v(x, t) &= \frac{4i\Omega^2(\lambda + a\tau)(2\rho^2 - 2\Omega\rho - A)}{\left(2\Omega + \sqrt{\sigma^2 - 4\rho\Omega} \tanh\left(\frac{1}{2} \sqrt{\sigma^2 - 4\rho\Omega} (\omega + d_0)\right) - \sigma\right)^2}, \\
 w(x, t) &= \frac{-4\tau\Omega^2(2\rho^2 - 2\Omega\rho - A)}{\left(2\Omega + \sqrt{\sigma^2 - 4\rho\Omega} \tanh\left(\frac{1}{2} \sqrt{\sigma^2 - 4\rho\Omega} (\omega + d_0)\right) - \sigma\right)^2}.
 \end{aligned}$$

Case 3: Rational solution, $\Delta = 0$.

$$q(x, t) = \pm \sqrt{\frac{2\rho^2 - 2\Omega\rho - A}{2\varepsilon}} \frac{1}{1 - \frac{\sigma}{2\Omega} - \frac{1}{\Omega(\omega + d_0)}} \times \exp\left[\frac{i\Gamma(1 + \gamma)}{\eta} (\lambda t^\eta - ax^\eta)\right],$$

$$v(x, t) = \frac{i(\lambda + a\tau)(2\rho^2 - 2\Omega\rho - A)}{\left(1 - \frac{\sigma}{2\Omega} - \frac{1}{\Omega(\omega + d_0)}\right)^2},$$

$$w(x, t) = \frac{-\tau(2\rho^2 - 2\Omega\rho - A)}{\left(1 - \frac{\sigma}{2\Omega} - \frac{1}{\Omega(\omega + d_0)}\right)^2}.$$

4.2. Sardar subequation technique

For $M = 1$, Eq (2.11) reduces to the quadratic expansion

$$Q(\omega) = b_0 + b_1\psi(\omega). \quad (4.4)$$

Substituting expression (4.4) into Eq (3.8) and using relation (2.12), we obtain the following solutions:

$$b_0 = 0, \quad b_1 = \pm \frac{1}{\sqrt{\varepsilon}}.$$

The exact solutions obtained for the M -fractional NTM system can be classified as follows.

Case 1: Bright soliton solutions.

$$q(x, t) = \sqrt{\frac{\kappa}{-\varepsilon\sigma}} \operatorname{sech}(\sqrt{\kappa}\omega) \exp\left[\frac{i\Gamma(1 + \gamma)}{\eta} (\lambda t^\eta - ax^\eta)\right],$$

$$v(x, t) = \frac{-2i(\lambda + a\tau)\kappa}{\sigma} \operatorname{sech}^2(\sqrt{\kappa}\omega), \quad (4.5)$$

$$w(x, t) = \frac{2\tau\kappa}{\sigma} \operatorname{sech}^2(\sqrt{\kappa}\omega),$$

or

$$q(x, t) = \sqrt{\frac{\kappa}{-\varepsilon\sigma}} \operatorname{csch}(\sqrt{\kappa}\omega) \exp\left[\frac{i\Gamma(1 + \gamma)}{\eta} (\lambda t^\eta - ax^\eta)\right],$$

$$v(x, t) = \frac{-2i(\lambda + a\tau)\kappa}{\sigma} \operatorname{csch}^2(\sqrt{\kappa}\omega), \quad (4.6)$$

$$w(x, t) = \frac{2\tau\kappa}{\sigma} \operatorname{csch}^2(\sqrt{\kappa}\omega).$$

Remark. Under the compatibility condition (3.9), the auxiliary field v is purely imaginary. This does not affect the real-valued character of $|q|$ and w . The same observation holds for the other solution families presented below.

Case 2: Periodic solutions.

$$\begin{aligned}
q(x, t) &= \sqrt{\frac{-\kappa}{\varepsilon\sigma}} \sec(\sqrt{-\kappa}\omega) \exp\left[\frac{i\Gamma(1+\gamma)}{\eta}(\lambda t^\eta - ax^\eta)\right], \\
v(x, t) &= \frac{-2i(\lambda + a\tau)\kappa}{\sigma} \sec^2(\sqrt{-\kappa}\omega), \\
w(x, t) &= \frac{2\tau\kappa}{\sigma} \sec^2(\sqrt{-\kappa}\omega),
\end{aligned} \tag{4.7}$$

or

$$\begin{aligned}
q(x, t) &= \sqrt{\frac{-\kappa}{\varepsilon\sigma}} \csc(\sqrt{-\kappa}\omega) \exp\left[\frac{i\Gamma(1+\gamma)}{\eta}(\lambda t^\eta - ax^\eta)\right], \\
v(x, t) &= \frac{-2i(\lambda + a\tau)\kappa}{\sigma} \csc^2(\sqrt{-\kappa}\omega), \\
w(x, t) &= \frac{2\tau\kappa}{\sigma} \csc^2(\sqrt{-\kappa}\omega).
\end{aligned} \tag{4.8}$$

Case 3: Dark soliton solutions.

$$\begin{aligned}
q(x, t) &= \sqrt{\frac{\kappa}{2\varepsilon\sigma}} \tanh\left(\sqrt{\frac{\kappa}{2}}\omega\right) \exp\left[\frac{i\Gamma(1+\gamma)}{\eta}(\lambda t^\eta - ax^\eta)\right], \\
v(x, t) &= \frac{i(\lambda + a\tau)\kappa}{\sigma} \tanh^2\left(\sqrt{\frac{\kappa}{2}}\omega\right), \\
w(x, t) &= \frac{-\tau\kappa}{\sigma} \tanh^2\left(\sqrt{\frac{\kappa}{2}}\omega\right),
\end{aligned} \tag{4.9}$$

or

$$\begin{aligned}
q(x, t) &= \sqrt{\frac{\kappa}{2\varepsilon\sigma}} \coth\left(\sqrt{\frac{\kappa}{2}}\omega\right) \exp\left[\frac{i\Gamma(1+\gamma)}{\eta}(\lambda t^\eta - ax^\eta)\right], \\
v(x, t) &= \frac{i(\lambda + a\tau)\kappa}{\sigma} \coth^2\left(\sqrt{\frac{\kappa}{2}}\omega\right), \\
w(x, t) &= \frac{-\tau\kappa}{\sigma} \coth^2\left(\sqrt{\frac{\kappa}{2}}\omega\right).
\end{aligned} \tag{4.10}$$

Case 4: Jacobi elliptic solutions, sn-family.Assume $\kappa < 0$, $0 < m < 1$.

$$\sigma = \frac{\kappa^2 m}{(1+m)^2}, \quad \kappa = a^2 - 1 + \frac{2a\lambda}{\tau}, \tag{4.11}$$

and the compatibility condition (3.9). Then,

$$\begin{aligned} q(x, t) &= \pm \sqrt{\frac{-\kappa m}{\epsilon(1+m)}} \operatorname{sn} \left(\sqrt{\frac{-\kappa}{1+m}} \omega; m \right) \exp \left(\frac{i\Gamma(1+\gamma)}{\eta} (\lambda t^\eta - ax^\eta) \right), \\ v(x, t) &= -2i(\lambda + a\tau) \frac{\kappa m}{1+m} \operatorname{sn}^2 \left(\sqrt{\frac{-\kappa}{1+m}} \omega; m \right), \\ w(x, t) &= 2\tau \frac{\kappa m}{1+m} \operatorname{sn}^2 \left(\sqrt{\frac{-\kappa}{1+m}} \omega; m \right), \end{aligned} \quad (4.12)$$

where

$$\omega = \frac{\Gamma(1+\gamma)}{\eta} (x^\eta - \tau t^\eta).$$

Under (3.9) the field v is purely imaginary, while $|q|$ and w remain real. As $m \rightarrow 1$, the profile reduces to Case 3.

4.3. Verification of the solutions

To confirm the correctness of the derived exact solutions, representative members from each family (hyperbolic, trigonometric, rational, and Jacobi elliptic) were substituted back into the reduced ordinary differential Eq (3.8) and, subsequently, into the original M -fractional system (1.1)–(1.3). A representative of the sn-family was substituted into (2.10) and into the reduced Eq (3.8). Both identities hold if, and only if, $\sigma = \kappa^2 m / (1+m)^2$ and $\kappa = a^2 - 1 + 2a\lambda/\tau$. The substitutions were repeated in a computer-algebra session. The profiles A_{cn} and A_{dn} do not satisfy (2.10) and are not used. In the limit $\eta \rightarrow 1$ the solutions recover the corresponding exact solutions of the classical NTM system reported in Ref. [14].

5. Graphical analysis / discussion

To study the characteristics of the obtained solutions, numerical simulations were conducted using solutions derived using the Sardar subequation method. The analysis was conducted for three main types of solutions: hyperbolic, periodic, and rational. For each case, both 2D and 3D visualizations were examined.

Figure 1(a) shows two-dimensional profiles of the hyperbolic solution (4.5). The quantity $|q(x, t)|$ is plotted for different values of the fractional parameter ($\eta = 0.25, 0.50, 0.75, 1.00$). The plots clearly reveal a localized bright soliton structure. As η increases, both the amplitude and width of the soliton increase. This indicates that the fractional order parameter plays a significant role in controlling the localization properties of the wave. As can be seen from Figure 1, the soliton solution of the fractional NTM system (1.1)–(1.3) exactly coincides with the plot of the fundamental NTM system at $\eta = 1$.

Figure 1(b) shows the corresponding three-dimensional surface of the hyperbolic solution (4.5). A spatially localized soliton structure is observed that maintains its shape over time. The wave propagates without distortion, confirming its soliton behavior.

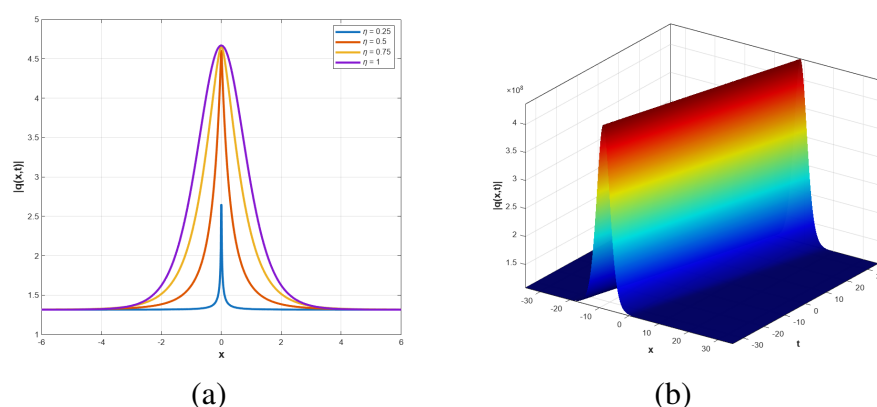


Figure 1. (a) Two-dimensional profiles of the hyperbolic solution of the M -fractional NTM system for different values of the fractional-order parameter $\eta = 0.25, 0.50, 0.75, 1$; (b) three-dimensional surface of the solution at $\eta = 1$, illustrating a localized bright soliton structure. The parameters are taken as $\lambda = 0.35, a = -0.4, \tau = 0.25, \varepsilon = 1, \kappa = 1.96, \gamma = 0.8$, and $\sigma = -1$ for both subfigures.

Figure 2(a) shows the 2D representations of the periodic solution (4.7). In this case, the solution exhibits a periodic pattern with singular features. Variations in the parameter η affect both the spacing between successive peaks and the amplitude of the wave. For smaller values of η , the structure becomes more intricate and densely distributed.

Figure 2(b) displays the 3D visualization of the periodic solution (4.7). Repeating wave patterns are clearly visible in space, and the wave maintains its periodic nature as it evolves over time, which is consistent with the properties of trigonometric-type solutions.

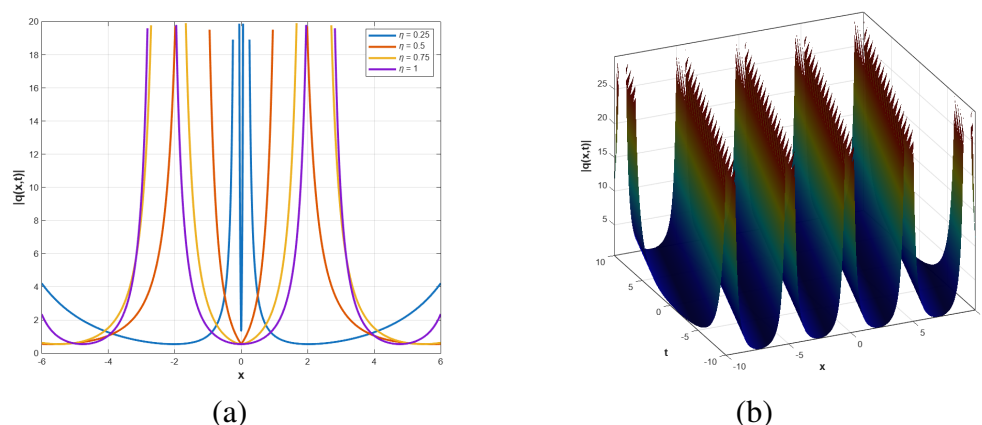


Figure 2. (a) Two-dimensional plots of the periodic solution for different values of the fractional-order parameter η ; (b) three-dimensional representation at $\eta = 1$, showing the spatial-temporal periodic behavior along with singular wave structures. The parameters used in both subfigures are $\lambda = 0.35, a = -0.4, \tau = 0.25, \varepsilon = 1, \kappa = -1.96, \sigma = 0.15$, and $\gamma = 0.8$.

Figure 3(a) depicts the 2D profiles of the rational solution (4.9). Here, the wave remains localized, but its decay is slower compared to the hyperbolic soliton case. As η increases, the amplitude becomes larger and the localization becomes more pronounced.

Figure 3(b) presents the 3D surface of the rational solution (4.9). The wave retains its localized structure in space, while its decay occurs more gradually than in the hyperbolic case. This behavior confirms the algebraic decay characteristic of rational solutions.

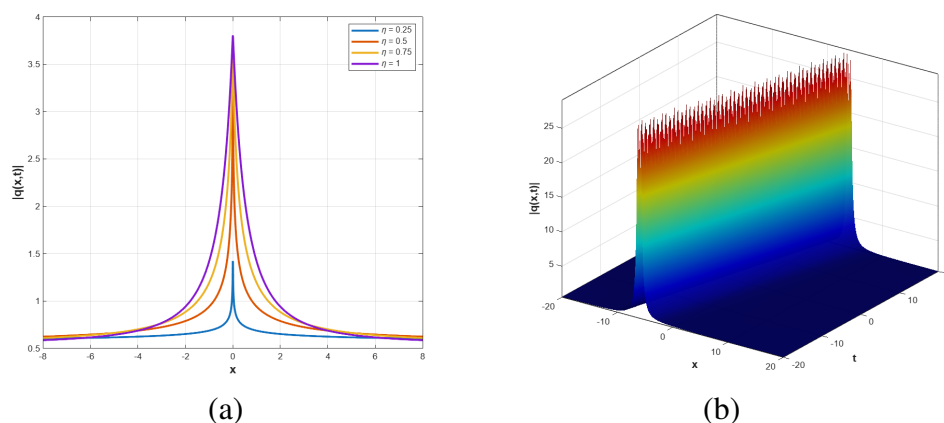


Figure 3. (a) Two-dimensional profiles of the rational solution for different values of the fractional-order parameter η ; (b) three-dimensional surface at $\eta = 1$, demonstrating a localized wave with algebraic decay characteristics. The parameters used in both subfigures are $\lambda = 0.35$, $a = -0.4$, $\tau = 0.25$, $\varepsilon = 1$, $\kappa = 1.96$, $\sigma = 0.96$, and $\gamma = 0.8$.

The graphical analysis shows that a decrease of the fractional order η leads to a reduction of the soliton amplitude and a broadening of the wave profile. This behavior is consistent with the presence of memory effects: smaller values of η correspond to stronger hereditary influence, which dissipates energy and weakens localization. In the limit $\eta \rightarrow 1$, the classical soliton dynamics are fully recovered. The Jacobi elliptic solutions describe cnoidal waves that can be interpreted as nonlinear periodic trains of pulses, while the rational solutions correspond to algebraically decaying localized structures.

Overall, the graphical results show that the fractional order parameter η has a strong influence on the wave dynamics. As η approaches unity, the solutions tend toward the classical case, with increasing amplitude and more distinct structures. When $\eta < 1$, the fractional order effects become more pronounced, leading to noticeable changes in the wave profile.

6. Conclusions

In this paper we have investigated the M -fractional NTM system. Using the generalized Kudryashov method and the Sardar subequation method, several families of exact analytical solutions, including hyperbolic solitons, trigonometric, and Jacobi elliptic (cnoidal) periodic waves, and rational solutions with algebraic decay are obtained. The influence of the fractional-order parameter η on the wave amplitude, width, and propagation characteristics has been analyzed graphically. In the limit $\eta \rightarrow 1$, all solutions reduce to the corresponding solutions of the classical integer-order NTM system.

The present study provides the first analytical treatment of the M -fractional NTM system and demonstrates the rich variety of nonlinear wave structures admitted by the model.

Future work

Possible extensions of this research include the investigation of multi-soliton solutions and soliton interactions, the stability analysis of the obtained wave structures under small perturbations, and the application of other fractional operators (Caputo, Atangana-Baleanu, etc.) to the NTM system. Numerical simulations of the fractional system under realistic initial and boundary conditions would also be of interest.

Author contributions

Ratbay Myrzakulov: Conceptualization, methodology, supervision; Gulgassyl Nugmanova: Paper writing and overall coordination; Aigul Taishiyeva: Formal analysis, validation, methodology, writing-original draft; Aidana Azhikhan: formal analysis, methodology. Akbota Myrzakul: Conceptualization, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors used AI tools to improve the grammatical correctness and linguistic coherence of the manuscript. All scientific content and results were produced solely by the authors.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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