



*Research article***Rogue-wave patterns in the (3+1)-dimensional generalized Camassa–Holm Kadomtsev–Petviashvili equation: true vs. predicted****Majid Madadi^{1,2,3}, Kamyar Hosseini^{2,3,4,*}, Evren Hincal² and Sekson Sirisubtawee^{5,6,*}**

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Abstract: Rogue waves are rare, high-amplitude events that can appear unexpectedly in many physical settings, yet the mechanisms behind their complex structures, especially at higher orders, are not fully understood. In this work, we study higher-order rogue waves in the $(3 + 1)$ -dimensional generalized Camassa–Holm Kadomtsev–Petviashvili (gCH–KP) equation. Using the Kadomtsev–Petviashvili (KP) hierarchy reduction method, we obtain general rational solutions and reveal how their internal parameters shape a wide range of wave patterns. A central result of our study is an analytical characterization of the higher-order rogue-wave patterns through the root geometry of the Yablonskii–Vorob’ev (YV) polynomial hierarchy, which traditionally arise in the theory of Painlevé equations. In the regime where one internal parameter becomes large, we derive an asymptotic decomposition of the exact solution into fundamental rogue-wave components and obtain explicit formulas that map the relevant YV roots to their physical locations. This provides a quantified extension of the previously known YV pattern mechanism to the reduced $(3 + 1)$ -dimensional gCH–KP equation and avoids the direct evaluation of increasingly large determinant representations. Numerical tests confirm these predictions and demonstrate a unified algebraic–geometric mechanism for the formation of higher-order rogue wave patterns.

Keywords: CH–KP equation; rogue wave; YV polynomials; KP-hierarchy reduction method

1. Introduction

Nonlinear evolution equations can generally be investigated using analytical [1], numerical [2], and semi-analytical methods [3]. Computational methods [4, 5] provide additional tools for exploring their solution structures and dynamics. Recent studies have demonstrated the effectiveness of modern computational approaches in investigating complex nonlinear wave phenomena, including data-driven prediction of internal tides [6] and fully nonlinear numerical modeling of wave–current interactions [7]. Numerical methods have also been widely used to analyze the stability and propagation of localized nonlinear structures in nonlinear Schrödinger (NLS) systems [8]. These developments highlight the complementary role of computational and analytical approaches in the study of nonlinear wave dynamics. Depending on the balance between nonlinearity, dispersion, and dimensionality, their solutions may exhibit various types of wave structures, including solitons, solitary waves, rogue waves (RWs), lump waves, peakons, and line solitons.

A soliton is a localized, shape-preserving nonlinear wave that preserves its identity after interacting with other solitons, typically experiencing only a phase or position shift [9, 10]. A solitary wave is a localized wave that propagates without changing its overall shape but does not necessarily exhibit the characteristic collision properties of a soliton [11, 12]. A kink wave is a topological wave characterized by a smooth or monotonic transition between two distinct asymptotic states as the spatial coordinate varies from one side of the wave to the other [13]. A lump wave is localized in all spatial directions and is typically described by a rational function with algebraic decay [14]. A peakon is a peaked solitary wave with a continuous profile but a discontinuous first derivative at its crest [15].

Once considered maritime folklore, RWs are now recognized as extreme oceanic events that can appear suddenly and from unexpected directions, posing serious risks to ships and offshore structures. The first confirmed observation occurred on January 1, 1995, at the Draupner oil platform in the North Sea, where a 25.6-meter wave exceeded predictions from conventional models [16]. The mechanisms behind RWs are still under study, with explanations including the constructive interference of wave trains and wave energy concentrating in opposing currents [17]. RWs may also occur naturally as rare, extreme events exceeding twice the significant wave height [18]. RWs are not limited to the ocean; they also appear in systems such as plasma, optical fibers, and Bose–Einstein condensates [19]. Mathematically, they are modeled as space- and time-localized rational solutions. A landmark example is the Peregrine wave, the first analytical RW solution of the NLS equation [20], representing a single isolated wave of exceptional amplitude on a constant background and serving as the foundation for higher-order RW structures.

To construct and analyze RW solutions, several analytical techniques have been developed. These include the Darboux transformation [21], the inverse scattering transform [22], and Bäcklund transformation [23]. The ansatz/symbolic computation method has also been used [24, 25]. In addition, the KP-hierarchy reduction method has been developed for this purpose [26]. These analytical approaches have different requirements and advantages. In transformation-based methods, such as the Darboux and Bäcklund transformations, an appropriate transformation must first be

constructed and then applied, often repeatedly, to generate higher-order solutions. The construction of suitable transformations can become increasingly complicated for higher-order solutions. Similarly, ansatz-based approaches generally require an appropriate solution form to be identified and the resulting algebraic conditions to be solved. In contrast, once the target equation is transformed into an appropriate KP-hierarchy reduction, one can directly use the established hierarchy of τ -functions and determinant representations to construct higher-order rational solutions in a systematic manner [26]. The determinant representation incorporates the higher-order structure and multiple internal parameters in a unified form, avoiding the need to derive a new transformation for each higher-order solution. Moreover, the use of Schur polynomials provides compact algebraic representations of the corresponding differential operators [27]. Thus, the main advantage of the KP-hierarchy reduction in the present work is that it provides a systematic and scalable analytical framework for generating higher-order rational solutions from established τ -function structures, while also avoiding the increasingly complicated construction of transformation formulas for each higher order.

In the theory of integrable systems, two central research directions involve understanding the Painlevé equations and uncovering the pattern formation mechanisms of RWs. The Painlevé hierarchy characterizes the analytic foundations of integrability, while the rich spatial organization of high-order RWs reflects their geometric and algebraic manifestations. These two aspects are deeply interconnected: the root structures of the YV polynomials, which arise in the rational solutions of the second Painlevé equation, mirror the regular arrangements observed in high-order RW patterns.

The second Painlevé equation is one of the six canonical Painlevé equations introduced by Painlevé and Gambier [28, 29]. While its general solutions are transcendental and cannot be expressed in terms of elementary functions, it admits a hierarchy of rational solutions. These rational solutions, first derived by Yablonskii and Vorob'ev, are represented through the logarithmic derivative of a sequence of special polynomials $Q_n(z)$, known as the YV polynomials. The $Q_n(z)$ are monic polynomials of degree $\frac{1}{2}n(n+1)$ with integer coefficients. Their zeros form highly regular and symmetric triangular configurations, with each polynomial having simple roots and no shared roots with adjacent members of the sequence [30].

Understanding how RW patterns form is key to predicting extreme wave events from their early stages. While simpler, low-order RWs have been studied in detail, the more complex structures that appear in higher-order waves have received less attention. Initial work on RW patterns in the NLS equation [31, 32] combined Darboux transformations with numerical simulations to explore these phenomena. Related numerical studies of NLS models have further shown that spatially modulated nonlinearities can generate and stabilize complex multi-peak localized structures, emphasizing the importance of numerical stability and propagation analysis in understanding nonlinear wave patterns [8].

One striking observation is that an N th-order RW with a ring-like structure often contains a smaller $(N-2)$ nd-order RW at its center [31]. Researchers classified these patterns according to wave order and shifts in the parameters within the Akhmediev breather framework [32]. A major breakthrough came when it was discovered that as certain internal parameters grow large, high-order RWs naturally form universal geometric shapes, such as triangular, pentagonal, and heptagonal arrangements of fundamental Peregrine waves. For example, when the parameter a_3 becomes large, the individual peaks of a higher-order RW arrange themselves approximately in a triangular pattern, with their locations determined by the corresponding YV polynomial roots. These striking arrangements are not

random: they correspond analytically to the root patterns of the YV polynomials, linked through simple operations like scaling and rotation. This insight provides a clear, unified picture, explaining previous numerical observations and extending our understanding of how complex RW structures emerge.

YV polynomials provide important theoretical and practical insights into higher-order RW patterns. Their roots can determine the locations of individual RW peaks, avoiding the increasingly complicated determinant expressions of higher-order τ -functions [33]. They also reveal characteristic geometric configurations, including triangular, pentagonal, and heptagonal patterns, highlighting their universality across integrable systems [34]. This predictive capability is relevant to nonlinear optics and photonic systems for designing high-intensity pulses and anticipating noise spikes [35], as well as to oceanography for forecasting extreme waves and improving maritime safety [36]. From a theoretical perspective, YV polynomials simplify the analysis and provide a unified description of RW structures across different nonlinear wave equations [37]. These results highlight the importance of YV polynomials in understanding higher-order RW patterns. Motivated by this, in this work, we aim to clarify whether a similar connection between YV-polynomial roots and RW patterns exists for a $(3 + 1)$ -dimensional equation.

The Camassa–Holm (CH) equation is a nonlinear, integrable partial differential equation originally derived to describe the unidirectional propagation of shallow water waves. This equation was introduced by Camassa and Holm as a double Hamiltonian model in the study of shallow water waves [38]. In its $(1+1)$ -dimensional form, it is

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (1.1)$$

where $u(x, t)$ represents the fluid velocity along the primary propagation in x -direction and κ is related to the critical shallow water wave speed. The equation exhibits both smooth solitary waves and peaked solitary waves or peakons [39] of the form $u(x, t) = ce^{-|x-ct|}$. While the $(1+1)$ -dimensional CH equation captures essential features of unidirectional wave propagation, real fluid systems often exhibit weak variations in transverse directions. To account for this, the CH equation is extended to the $(2 + 1)$ -dimensional CH–KP form [40]

$$(u_t + \alpha u_x + \beta uu_x + \delta u_{xxt})_x + \lambda u_{yy} = 0, \quad (1.2)$$

where a transverse coordinate y is included. This extension allows for the study of line solitons, transverse modulations, and weakly multidimensional interactions, while preserving the fundamental balance between nonlinearity and dispersion along the primary direction. Several studies on Eq (1.2) and its extended form have explored RWs, line solitons, lump solutions, and breather waves, highlighting the role of transverse effects in multidimensional wave dynamics [41, 42].

The full $(3 + 1)$ -dimensional gCH–KP equation is derived by further including a second transverse coordinate z and considering the contributions of nonlinear, dispersive, and transverse effects in three spatial dimensions [43] as follows

$$(u_t + \alpha u_x + \beta uu_x + \delta u_{xxt})_x + \lambda u_{yy} + \mu u_{zz} = 0. \quad (1.3)$$

The derivation often begins with introducing a potential function $v(x, y, z, t)$ such that $u = v_x$, allowing the equation to be expressed in variational form. A corresponding Lagrangian functional is then constructed, and the Euler–Lagrange equation is applied, incorporating the effects of linear

propagation, nonlinearity, and dispersion along the primary direction, as well as weak transverse modulations in y and z . By carefully integrating each term by parts and selecting appropriate coefficients, one obtains the gCH–KP equation in a consistent, physically meaningful form.

Physically, the $(3 + 1)$ -dimensional gCH–KP equation models waves that propagate primarily in one direction while exhibiting weak but significant variations in two transverse directions. This allows the study of extended multidimensional wave phenomena, including RWs [44], lump waves [45], and multidimensional soliton interactions [46]. Other nonlinear wave structures can also be investigated within this framework [47]. Compared with the $(1 + 1)$ -dimensional CH equation, which describes unidirectional wave propagation, and the $(2 + 1)$ -dimensional CH–KP equation, which incorporates a single transverse modulation, the $(3 + 1)$ -dimensional gCH–KP equation provides an extended framework for investigating complex wave patterns involving multiple transverse variables.

Although the gCH–KP Eq (1.3) involves three spatial variables and one temporal variable, the coefficients α , β , δ , λ , and μ describe the relative effects of propagation, nonlinearity, longitudinal dispersion, and transverse modulation. These coefficients depend on the scaling and physical assumptions of the underlying wave system, such as shallow-water dynamics. In particular, α and β characterize propagation and nonlinearity, while δ , λ , and μ describe longitudinal and transverse dispersive effects [38].

Equation (1.3) contains five coefficients. They are not all independent under suitable rescaling of the variables. In this work, we retain the general form of the equation and consider the representative parameter regime $\alpha = -6\sqrt{2} - 1$, $\beta = 15$, $\delta = \frac{1}{2}$, $\lambda = -2$, and $\mu = 1$, which represents a strongly nonlinear dispersive regime of the gCH–KP equation. For this regime, the fundamental RW exhibits an amplification of approximately eight times the background level. Moreover, the asymptotic analysis shows that when a higher-order internal parameter becomes large, the solution decomposes into multiple fundamental RWs whose spatial arrangement is determined by the associated YV polynomial roots. The separation distance between these peaks scales as $O(|a_{2m+1}|^{1/(2m+1)})$, leading to triangular, pentagonal, and higher polygonal RW patterns. These results provide a qualitative description of possible extreme-wave structures in multidimensional nonlinear dispersive systems.

Although the polygonal organization of higher-order RWs and its connection with YV root structures have been established in several wave systems, it remains to clarify how this algebraic mechanism emerges from the exact higher-order rational solutions of the reduced $(3 + 1)$ -dimensional gCH–KP equation and how the corresponding YV roots determine the physical locations of the separated RW components.

In this work, we establish an explicit analytical realization of the YV root mechanism within the higher-order rational solutions of the reduced $(3 + 1)$ -dimensional gCH–KP equation. Starting from the KP-hierarchy reduction, we derive the large- $|a_{2m+1}|$ asymptotic decomposition of the exact τ -function and obtain explicit formulas mapping the relevant roots of the generalized YV polynomial hierarchy to the physical locations of the separated fundamental RW components. Thus, rather than claiming the previously reported triangular, pentagonal, or heptagonal patterns as new, we establish their analytical and quantitative realization for the present gCH–KP solution family. In particular, the resulting framework applies to arbitrary admissible odd dominant parameters a_{2m+1} and provides a quantified asymptotic approximation, including the corresponding polygonal configurations and central lower-order RW.

The KP-hierarchy reduction is also useful for revealing the broader structure of the solutions: it

reduces the problem to a $(1 + 1)$ -dimensional equation, allowing both RW solutions and parallel soliton lines to be obtained in suitable coordinates. Moreover, the resulting solution family contains the RW solutions previously derived using the test-function and ansatz methods in Ref. [44]. For $\mu = 0$, it further reduces to the solutions of the extended $(2 + 1)$ -dimensional CH–KP Eq (1.2) considered in [41]. Overall, the main contribution is the analytical and quantitative establishment of the YV-root-controlled asymptotic mechanism and its explicit mapping to physical RW locations for the gCH–KP equation.

The choice of the $(3 + 1)$ -dimensional gCH–KP model is motivated by its ability to describe wave propagation with nonlinear, dispersive, and two transverse effects, providing a more general framework than the lower-dimensional CH and CH–KP models. This makes it suitable for studying complex multidimensional RW structures that may arise in physical systems such as shallow-water and nonlinear optical settings. The KP-hierarchy reduction provides a systematic way to construct higher-order rational solutions, while the YV polynomials help characterize RW patterns and peak locations without handling increasingly large τ -function determinants. Together, this approach offers a physically relevant and efficient framework for analyzing complex RW structures.

The Draupner wave and other observed RW phenomena are mentioned as a physical motivation for studying localized extreme-wave structures. The generalized $(3 + 1)$ -dimensional equation considered here should be regarded as a mathematical model for investigating such localized wave dynamics, rather than as a direct quantitative model derived specifically from the Draupner-wave experiment. The present work focuses on the mathematical construction and dynamics of RW solutions within this generalized model.

The paper proceeds as follows. Section 2 presents the construction of higher-order rational solutions to Eq (1.3) within the KP-hierarchy framework and examines the dependence of their spatial configurations on the internal parameters. Section 3 establishes the analytical connection between large-parameter RW patterns and the root geometry of the YV polynomial hierarchy, leading to explicit formulas for predicting RWs' locations. Section 4 presents numerical validations, compares exact and predicted patterns, and quantifies the asymptotic accuracy of the theoretical results. Finally, Section 5 concludes our findings.

We must mention that all numerical quantities used in the calculations were evaluated in Maple before rounding or truncation for presentation. To improve readability and facilitate the presentation of the subsequent results, we summarize the main notation used throughout the paper in Table 1.

Table 1. Main notation used in this article.

Symbol	Meaning
x, y, z, t	Spatial variables and time variable of the gCH–KP equation
α, β, δ	Parameters describing propagation, nonlinearity, and dispersion effects
λ, μ	Transverse dispersion parameters
σ, ς	Parameters related to velocity shifts in reduced variables
$\kappa, \mathscr{Y}$	Reduced spatial variables used in RW solutions
τ_n	τ -function generating the n th-order RW
S_j	Schur polynomial from the KP hierarchy reduction
a_{2j+1}	Complex internal parameters controlling RW patterns
h_k	Constants from the hierarchy reduction and parameter shifts
Λ	Scaling parameter associated with large a_{2m+1}
$Q_n^{[m]}(z)$	YV hierarchy polynomial
z_0	Root of $Q_n^{[m]}(z)$ determining peak positions
n	RW order
n_p	Number of fundamental RWs in the asymptotic decomposition

2. Higher-order RW solutions in the CH–KP equation

The KP-reduction method begins with the KP hierarchy, which encompasses a broad class of integrable equations described via a τ -function, often expressed as a Wronskian or Grammian determinant. By imposing specific reduction constraints, such as identifying certain variables, applying symmetry conditions, or restricting parameter relations, the hierarchy is systematically reduced to lower-dimensional integrable equations, like the NLS or Boussinesq equations. The reduced τ -function then generates exact solutions, including RWs, solitons, and breathers, through operations such as logarithmic differentiation. This approach allows the construction of complex wave patterns by adjusting the τ -function's form and parameters while maintaining integrability.

We begin by rewriting the gCH–KP Eq (1.3) in its bilinear form, which allows us to use Hirota's method to construct RW solutions. These solutions are then derived using the KP-reduction technique.

Theorem 1. *By introducing the symmetry reduction*

$$\kappa = x + \sigma t, \quad \mathscr{Y} = y + \varsigma z, \quad (2.1)$$

together with the logarithmic transformation

$$u = \frac{12 \sigma \delta}{\beta} (\ln \tau)_{\kappa\kappa}, \quad (2.2)$$

the (3 + 1)-dimensional gCH–KP Eq (1.3) is reduced to the following bilinear equation:

$$\left(\delta \sigma D_{\kappa}^4 + (\alpha + \sigma) D_{\kappa}^2 + (\lambda + \mu \varsigma^2) D_{\mathscr{Y}}^2 \right) \tau \cdot \tau = 0, \quad (2.3)$$

where $\tau = \tau(\kappa, \mathscr{Y})$ is an analytic function of the reduced variables κ and $\mathscr{Y}$, and σ and ς are real parameters associated with the symmetry reduction.

Proof. Under the reduction (2.1), the chain rule gives

$$u_x = v_{\kappa}, \quad u_t = \sigma v_{\kappa}, \quad u_{xt} = \sigma v_{\kappa\kappa}, \quad u_{yy} = v_{\mathcal{Y}\mathcal{Y}}, \quad u_{zz} = \varsigma^2 v_{\mathcal{Y}\mathcal{Y}}. \quad (2.4)$$

Hence, Eq (1.3) becomes

$$(\alpha + \sigma)v_{\kappa\kappa} + \beta(v_{\kappa}^2 + vv_{\kappa\kappa}) + \delta \sigma v_{\kappa\kappa\kappa} + (\lambda + \mu \varsigma^2)v_{\mathcal{Y}\mathcal{Y}} = 0, \quad (2.5)$$

where $v = v(\kappa, \mathcal{Y})$. Thus the reduction is an exact invariant ansatz for the solution class $u(x, y, z, t) = v(x + \sigma t, y + \varsigma z)$.

Next, setting

$$v = \frac{6\delta\sigma}{\beta} q_{\kappa\kappa}, \quad (2.6)$$

in (2.5) and integrating the resulting equation twice with respect to κ , with the integration constants absorbed into the admissible background normalization, gives

$$\delta \sigma (q_{\kappa\kappa\kappa\kappa} + 3q_{\kappa\kappa}^2) + (\alpha + \sigma) q_{\kappa\kappa} + (\lambda + \varsigma^2 \mu) q_{\mathcal{Y}\mathcal{Y}} = 0, \quad (2.7)$$

where $q = q(\kappa, \mathcal{Y})$.

Following the P -polynomial formalism introduced in [48], we recall the following representative examples:

$$P_{2\kappa}(q) = q_{\kappa\kappa}, \quad P_{2\mathcal{Y}}(q) = q_{\mathcal{Y}\mathcal{Y}}, \quad P_{4\kappa}(q) = q_{\kappa\kappa\kappa\kappa} + 3q_{\kappa\kappa}^2. \quad (2.8)$$

Consequently, Eq (2.7) can be expressed compactly in terms of P -polynomials as

$$\delta \sigma P_{4\kappa}(q) + (\alpha + \sigma) P_{2\kappa}(q) + (\lambda + \varsigma^2 \mu) P_{2\mathcal{Y}}(q) = 0. \quad (2.9)$$

Introducing a change of dependent variable

$$q = 2 \ln \tau \Leftrightarrow v = \frac{6\sigma\delta}{\beta} q_{\kappa\kappa} = \frac{12\sigma\delta}{\beta} (\ln \tau)_{\kappa\kappa}, \quad (2.10)$$

and by invoking the relationship between the Bell polynomials and the Hirota D -operator as established in [48], and applying the results from Propositions 1 and 2 in [49], we obtain the bilinear representation (2.3). Thus, the theorem is proved. $\square$

We note that the transformation in Eq (2.2) plays a role analogous to the classical Cole–Hopf transformation. To derive this transformation, we substitute the seed solution

$$v = \exp(k_i \kappa + w_i \mathcal{Y}), \quad (2.11)$$

into the linear part of the nonlinear Eq (2.5) with $\beta = 0$. Here, k_i is an arbitrary constant, while w_i denotes the dispersion parameter. This substitution yields an algebraic relation; from this relation, the corresponding dispersion relation follows

$$w_i = \frac{k_i \sqrt{-(\mu \varsigma^2 + \lambda) (\delta \sigma k_i^2 + \alpha + \sigma)}}{\mu \varsigma^2 + \lambda}. \quad (2.12)$$

Furthermore, motivated by the leading-order analysis of the Painlevé test, we introduce the transformation

$$v = R(\ln \tau)_{\kappa\kappa}. \quad (2.13)$$

Substituting this expression, together with

$$\tau = 1 + \exp\left(k_i \kappa + \frac{\sqrt{-(\mu \zeta^2 + \lambda)(\delta \sigma k^2 + \alpha + \sigma)} k}{\mu \zeta^2 + \lambda} \mathscr{Y}\right), \quad (2.14)$$

into the full Eq (2.5), we obtain

$$R = 0, \quad \frac{12\delta\sigma}{\beta}, \quad (2.15)$$

which exactly reproduces the transformation given in Eq (2.2).

Remark 1. We emphasize that the integrability statement in this work concerns the reduced bilinear Eq (2.3) or, equivalently, the corresponding nonlinear Eq (2.5). We have also performed the Painlevé analysis for the full $(3 + 1)$ -dimensional model. Although the leading-order behavior and the resonances, $j = -1, 4, 5, 6$ coincide with those of the reduced bilinear equation, the compatibility condition at the resonance $j = 4$ is not satisfied for the full model. Thus, the Painlevé analysis provides supporting evidence for the integrability of the reduced bilinear equation, but it does not by itself establish integrability. In particular, the failure of the compatibility condition at $j = 4$ shows that the full $(3 + 1)$ -dimensional model does not pass the Painlevé test and therefore does not provide evidence of integrability through this criterion.

The reduction (2.1) does not represent the most general four-variable RW solution; rather, it provides a reduced form through which a class of higher-order RW solutions can be constructed. Equation (2.3) represents the reduced form of the KP hierarchy in the variables $(\kappa, \mathscr{Y})$. According to the KP-hierarchy reduction framework, the reduced bilinear equation admits rational solutions generated by a degenerate KP τ -function. Under the reduction constraints used in Theorem 2, the KP τ -function is restricted to depend only on the reduced variables $(\kappa, \mathscr{Y})$. The rational limit of the KP Gram-type solutions then leads to a determinant representation whose matrix elements are expressed through Schur polynomials. The determinant representation requires the explicit form of these matrix elements, which are given in terms of Schur polynomials defined through the generating function

$$\sum_{n=0}^{\infty} \mathcal{S}_n(V) \lambda^n = \exp\left(\sum_{k=1}^{\infty} v_k \lambda^k\right), \quad (2.16)$$

which generates a sequence of symmetric polynomials $\mathcal{S}_n(V)$ in the infinite variable set $V = (v_1, v_2, v_3, \dots)$. These functions arise naturally in the study of integrable systems and symmetric functions. Expanding the exponential leads to an explicit formula

$$\mathcal{S}_n(V) = \sum_{\substack{l_1, l_2, \dots \geq 0 \\ l_1 + 2l_2 + \dots = n}} \prod_{j=1}^n \frac{v_j^{l_j}}{l_j!}, \quad (2.17)$$

where the sum runs over all non-negative integer sequences $(l_1, l_2, \dots)$ such that the weighted sum equals n . This formula resembles the exponential generating function for complete symmetric functions. For example, we have

$$\mathcal{S}_0(y) = 1, \quad \mathcal{S}_1(y) = v_1, \quad \mathcal{S}_2(y) = \frac{1}{2}v_1^2 + v_2, \quad \mathcal{S}_3(y) = \frac{1}{6}v_1^3 + v_1v_2 + v_3, \quad \dots$$

In particular, the given exponential generating function corresponds to the vacuum τ -function of the KP hierarchy. Through the KP-reduction method, more general exact solutions can be obtained by modifying or composing such generating functions, making $\mathcal{S}_n(V)$ a fundamental tool in constructing solutions to the KP equations.

With the Schur polynomials and the bilinear form of the gCH–KP equation established, we are now equipped to construct explicit RW solutions. The following theorem summarizes this construction and provides a general formula for the RW solutions.

Theorem 2. For $\kappa = x + \sigma t$ and $\mathcal{Y} = y + \varsigma z$, symmetry-reduced higher-order RW solutions of the gCH–KP Eq (1.3) are

$$u_n(\kappa, \mathcal{Y}) = \frac{12\delta\sigma}{\beta} (\ln \tau_n)_{\kappa\kappa}, \quad (2.18)$$

where

$$\tau_n(\kappa, \mathcal{Y}) = \det_{1 \leq i, j \leq n} (\phi_{2i-1, 2j-1}), \quad (2.19)$$

and

$$\phi_{i,j} = \sum_{v=0}^{\min(i,j)} \left(-\frac{1}{12}\right)^v \mathcal{S}_{i-v}(X^+ + v\mathbf{h}) \mathcal{S}_{j-v}(X^- + v\mathbf{h}). \quad (2.20)$$

The vectors $X^\pm = (X_1^\pm, X_2^\pm, \dots)$ and $\mathbf{h} = (h_1, h_2, \dots)$ are defined as

$$X_{2r+1}^+ = \frac{3^{-2r}}{6(2r+1)!} \left(i \sqrt{\frac{-(\alpha+\sigma)}{\delta\sigma}} \kappa - 4^r \sqrt{\frac{-(\alpha+\sigma)^2}{\delta\sigma(\varsigma^2\mu+\lambda)}} \mathcal{Y} \right) + a_{2r+1}, \quad r = 0, 1, 2, \dots \quad (2.21)$$

$$X_{2r+1}^- = \frac{3^{-2r}}{6(2r+1)!} \left(i \sqrt{\frac{-(\alpha+\sigma)}{\delta\sigma}} \kappa + 4^r \sqrt{\frac{-(\alpha+\sigma)^2}{\delta\sigma(\varsigma^2\mu+\lambda)}} \mathcal{Y} \right) - a_{2r+1}^*, \quad r = 0, 1, 2, \dots \quad (2.22)$$

$$X_{2r}^\pm = 0, \quad r = 1, 2, 3, \dots \quad (2.23)$$

The first few h_k values are

$$\left[h_1 = \frac{2i}{9} \sqrt{3} \quad h_2 = -\frac{5}{108} \quad h_3 = -\frac{5i}{729} \sqrt{3} \quad h_4 = \frac{287}{116640} \quad h_5 = \frac{17i}{43740} \sqrt{3} \right]. \quad (2.24)$$

These coefficients are obtained from the following generating function:

$$\sum_{r=1}^{\infty} h_r \lambda^r = \ln \left[\frac{2i\sqrt{3}}{\lambda} \tanh\left(\frac{\lambda}{6}\right) \tanh\left(\frac{\lambda}{6} + \frac{2\pi i}{3}\right) \right]. \quad (2.25)$$

Here, a_i indicates arbitrary complex parameters, a_i^* denotes the complex conjugate of a_i , and $i = \sqrt{-1}$. Moreover, for the solutions to be real, the following conditions on σ and ς must be satisfied:

$$\delta\sigma(\alpha+\sigma) < 0, \quad \delta\sigma(\varsigma^2\mu+\lambda) < 0. \quad (2.26)$$

Proof. The proof of the theorem is presented in Appendix. $\square$

We emphasize the distinction between the full $(3 + 1)$ -dimensional gCH–KP equation and the symmetry-reduced solution class considered in this work. The reduction $u(x, y, z, t) = v(\kappa, \mathcal{Y})$ is an exact substitution into Eq (1.3), rather than an approximation. Indeed, according to Eq (2.4), the main Eq (1.3) is transformed exactly into Eq (2.5). Therefore, any solution $v(\kappa, \mathcal{Y})$ of Eq (2.5) generates, through the coordinate transformation above, an exact solution $u(x, y, z, t) = v(x + \sigma t, y + \zeta z)$ of the $(3 + 1)$ -dimensional equation. The subsequent transformation $v = (12\sigma\delta/\beta)(\ln \tau)_{\kappa\kappa}$ is likewise an exact transformation between Eq (2.5) and its bilinear form (2.3). Consequently, the rational solutions constructed from the KP-hierarchy τ -functions are exact solutions of the original gCH–KP equation within this symmetry-reduced class.

Dynamics of higher-order RWs

We analyze the dynamical behavior of the higher-order rational solution τ_n by utilizing the matrix entries given in (2.20). The simplest RW, known as the Peregrine-type RW, can be obtained by setting $n = 1$ in Theorem 2. By doing so, we can derive

$$u_1(\kappa, \mathcal{Y}) = \frac{12\delta\sigma}{\beta}(\ln \tau_1)_{\kappa\kappa}, \quad (2.27)$$

in which

$$\tau_1 = \frac{\alpha + \sigma}{36\delta\sigma}\kappa^2 - \frac{\Im(a_1)}{3}\sqrt{\frac{-(\alpha + \sigma)}{\delta\sigma}}\kappa + \frac{\alpha + \sigma^2}{36\delta\sigma(\zeta^2\mu + \lambda)}\mathcal{Y}^2 + \frac{\Re(a_1)}{3}\sqrt{\frac{(\alpha + \sigma)^2}{\delta\sigma(\zeta^2\mu + \lambda)}}\mathcal{Y} - |a_1|^2 - \frac{1}{12}, \quad (2.28)$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number z , respectively, and κ and $\mathcal{Y}$ are given in (2.1). The solution u_1 generates two kinds of waveforms in different coordinate planes: a RW in the (x, y) , (x, z) , (t, y) , and (t, z) planes, and a line soliton in the (x, t) and (y, z) planes.

Solution (2.27) has three critical points which are listed below.

$$(\kappa, \mathcal{Y}) = \left(\frac{6\sigma\delta\Im(a_1)}{\alpha + \sigma}\sqrt{\frac{-(\alpha + \sigma)}{\delta\sigma}}, -\frac{6\delta\sigma\Re(a_1)(\zeta^2\mu + \lambda)}{(\alpha + \sigma)^2}\sqrt{\frac{-(\alpha + \sigma)^2}{\delta\sigma(\zeta^2\mu + \lambda)}} \right), \quad (2.29)$$

$$(\kappa, \mathcal{Y}) = \left(\frac{3}{\alpha + \sigma} \left(2\Im(a_1)\sigma\sqrt{\frac{-(\alpha + \sigma)}{\delta\sigma}}\delta \pm \sqrt{-\delta\sigma(\alpha + \sigma)} \right), -\frac{6\delta\sigma\Re(a_1)(\zeta^2\mu + \lambda)}{(\alpha + \sigma)^2}\sqrt{\frac{-(\alpha + \sigma)^2}{\delta\sigma(\zeta^2\mu + \lambda)}} \right). \quad (2.30)$$

According to these critical points and the above mentioned coordinates involving the RW, the amplitudes of the peak and the holes are given by

$$A_P = -8\frac{\alpha + \sigma}{\beta}, \quad A_H = \frac{\alpha + \sigma}{\beta}. \quad (2.31)$$

It is clear from the amplitude formulas that the amplitudes are independent of the free parameter a_1 , whereas the location of the peak depends directly on a_1 . Moreover, with respect to the amplitudes

and under Condition (2.26), it is worth noting that for $A_P > 0$, bright RWs are obtained, whereas for $A_P < 0$, dark RWs result.

To obtain a centered RW (at $z = t = 0$), we can simply set $a_1 = 0$ (see Formula (2.29)). This scenario is illustrated in detail in Figure 1 through three-dimensional (3D), two-dimensional (2D), and density plots.

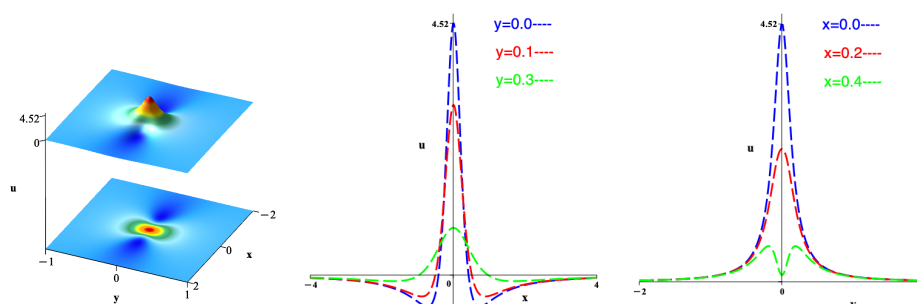


Figure 1. The first-order centered RW (2.27) for the parameters $\alpha = -6\sqrt{2} - 1$, $\beta = 15$, $\delta = \frac{1}{2}$, $\lambda = -2$, $\mu = 1$, $a_1 = 0$, and $z = t = 0$.

The RW solutions for different values of a_1 are presented in the first row of Figure 2, which includes both bright and dark RWs. As shown, the amplitude of the peak remains constant at 4.52, while the position of the peak is indicated in the figure.

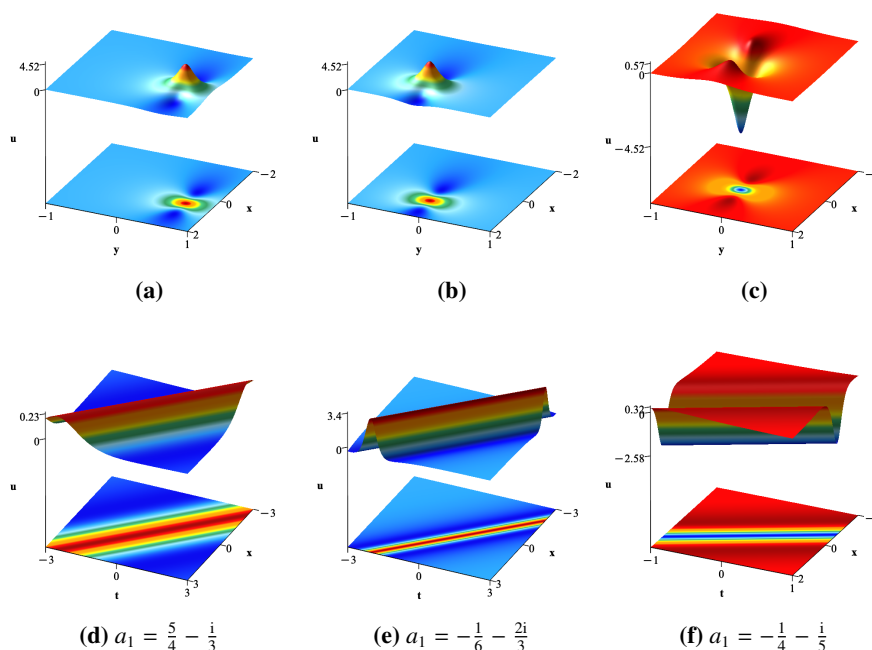


Figure 2. First-order rational solution (2.27) with $\alpha = -6\sqrt{2} - 1$, $\beta = -15$, $\delta = \frac{1}{2}$, $\lambda = -2$, and $\mu = 1$. The first row shows RWs in the (x, y) plane ($z = t = 0$), all with amplitude 4.52 but different peak locations; (c) shows a dark RW. The second row shows soliton lines in the (x, t) plane ($z = y = 0$), with different amplitudes; (f) shows the corresponding dark soliton.

We showed that the parameter a_1 does not affect the amplitude of the RW; however, when considering other coordinates, for example the (x, t) -plane, the scenario changes. In this case, for $y = z = 0$, the amplitude of the soliton peak is obtained through the following formula:

$$A_s = -8 \frac{\alpha + \sigma}{\beta(12 \Re(a_1)^2 + 1)}. \quad (2.32)$$

Unlike the RW, the amplitude of the soliton depends on the parameter a_1 . In the asymptotic limits, as $\lim_{a_1 \rightarrow 0}$, the soliton's amplitude coincides with the RW amplitude, while as $\lim_{a_1 \rightarrow \infty}$, the soliton's amplitude vanishes. The second row of Figure 2 illustrates these behaviors.

To obtain a second-order RW, we set $n = 2$ in the general solution (2.19). In this case, there are two free complex parameters, a_1 and a_3 . For instance, when $a_1 = a_3 = 0$, and $\alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2, \mu = 1$, the τ -function reduces to the following expression:

$$\begin{aligned} \tau_2 = & \frac{\sqrt{2}}{1458} \kappa^6 + \frac{2^{5/4}}{729} \kappa^5 + \left(\frac{2\mathcal{Y}^2}{81} + \frac{35}{5832} \right) \kappa^4 + \frac{2^{3/4}(96\mathcal{Y}^2 + 11)}{2916} \kappa^3 + \frac{\sqrt{2}(6912\mathcal{Y}^4 + 2976\mathcal{Y}^2 + 107)}{46656} \kappa^2 \\ & + \frac{2^{1/4}(384\mathcal{Y}^4 + 120\mathcal{Y}^2 + 1)}{1944} \kappa + \frac{73}{186624} + \frac{131}{3888} \mathcal{Y}^2 + \frac{11}{81} \mathcal{Y}^4 + \frac{16}{27} \mathcal{Y}^6. \end{aligned} \quad (2.33)$$

Accordingly, the second-order rational solution can be obtained using (2.18). This solution is illustrated in Figure 3, Panels (a) and (d). As shown in the figure, in the (x, y) -plane, there are two principal RW peaks located along the y -axis. The larger peak has an amplitude of 5.8798 and is located at $(x, y) = (0.0405, 0)$, while the smaller peak has an amplitude of 4.9559 and is located at $(x, y) = (-1.098, 0)$. In the (x, t) -plane, two parallel soliton lines are observed; again, the larger one has an amplitude of 5.8798, whereas the smaller one has an amplitude of 4.9559.

By increasing the absolute values of a_1 and a_3 , the dynamics of the solution change significantly. In particular, the two RW peaks split into three distinct peaks. For $a_1 = 0$ and $a_3 = 0.1$, the solution produces three separated peaks with different amplitudes in the (x, y) -plane, as shown in Panel (b), while in the (x, t) -plane, two parallel soliton lines are observed. For sufficiently large parameter values, for example $a_1 = 2$ and $a_3 = 3$, the three RW peaks become completely separated and attain the same amplitude, equal to that of the fundamental RW, namely 4.52 (see Panel (c)). In this case, only a single soliton line appears in the (x, t) -plane, as shown in Panel (f).

The second-order RW can be regarded as the “three sister RWs”. This term refers to a well-known phenomenon in oceanography where three large, consecutive RWs appear within a short span of time and distance, often causing severe damage to ships and offshore structures. The name “three sisters” originates from historical sailors’ reports and scientific observations describing these waves as a trio occurring one after another [50]. This phenomenon is clearly illustrated in the first row of Figure 3.

For $n = 3$ in Theorem 2, the third-order RW solution can be derived. This solution involves three free complex parameters, a_j for $j = 1, 3, 5$. In the specific case where $a_1 = a_3 = a_5 = 0$, and $\alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2, \mu = 1$, the corresponding τ -function can be expressed as Eq (A.19) in Appendix.

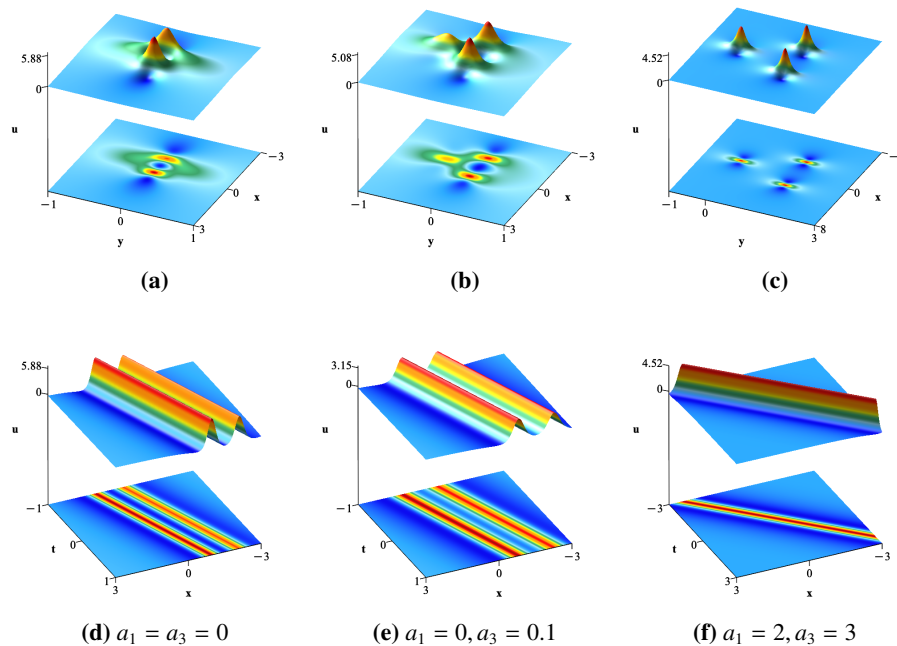


Figure 3. The second-order rational solution τ_2 with parameters $\alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2, \mu = 1$. The first row illustrates the RWs in the (x, y) plane with $z = t = 0$. The second row illustrates the soliton lines in the (x, t) plane with $z = y = 0$.

Analogous to the second-order case, this configuration yields three interacting RW peaks located along the x -axis, as illustrated in Figure 4(a). By increasing the magnitudes of the free parameters a_j , these waves separate into six distinct peaks. Figure 4(b)–(d) display different RW structures corresponding to various parameter selections. For relatively large values of a_3 (with the remaining parameters being small), a triangular RW pattern emerges, whereas a large a_5 gives rise to a pentagonal pattern. In both cases, the peak amplitudes are identical and equal to that of the fundamental Peregrine-type RW. Similar to the first- and second-order solutions, depending on the choice of coordinates, soliton lines can be generated from the third-order solution.

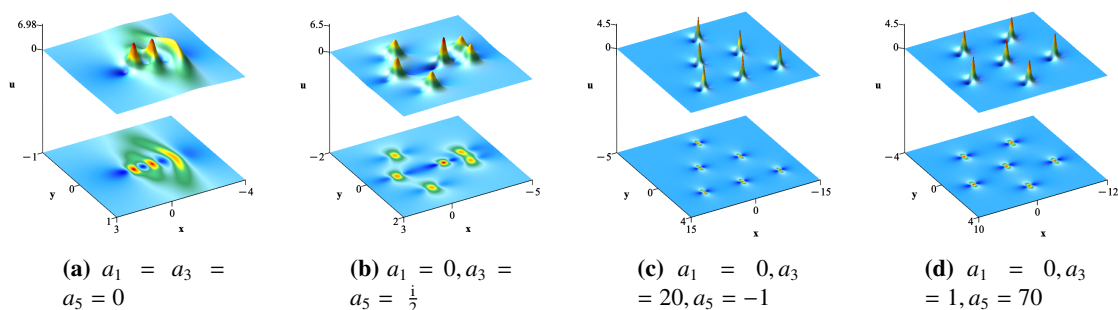


Figure 4. The third-order RW solution τ_3 with the parameters $\alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2, \mu = 1$, and $z = t = 0$.

Proposition 1. *The internal parameters a_{2r+1} enter the general RW solution in Theorem 2 as constant shifts in the odd components of the vectors $X^\pm$, thereby controlling the spatial structure of the higher-order RWs. For small $|a_{2r+1}|$, the peaks may merge and exhibit different amplitudes, whereas as $|a_{2r+1}|$ increases, they separate into well-separated fundamental Peregrine-type RWs with amplitudes approaching the fundamental value. In the first-order case, a_1 only shifts the peak and hole positions and does not affect their amplitudes as given by Eq (2.31).*

The orientations of these patterns (triangular and pentagonal) vary depending on the arguments of a_3 and a_5 . This effect is illustrated in Figure 5 for different arguments of a_3 . We will revisit this issue in the next section, where we explore the relationship between the RW patterns and the root structure of the YV polynomials.

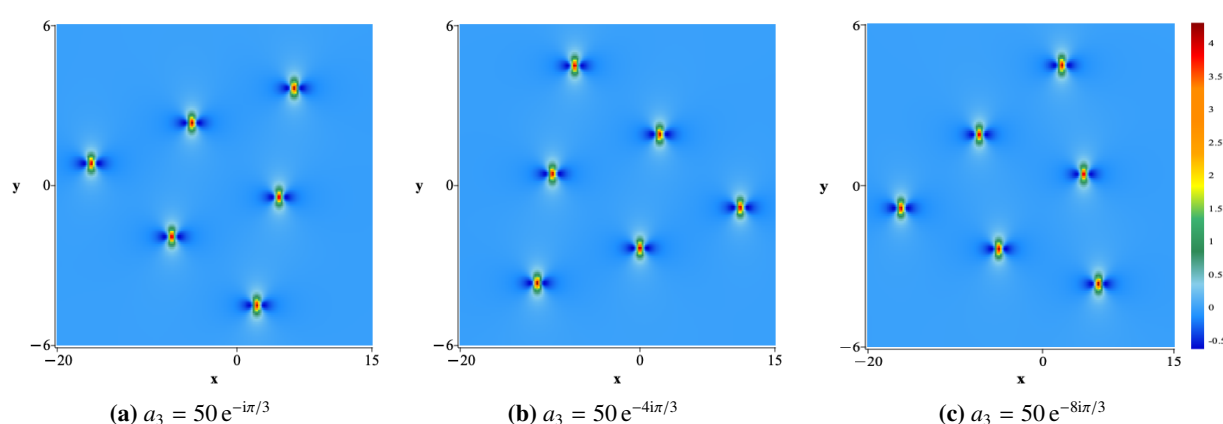


Figure 5. The third-order RW with the parameters $a_1 = i, a_5 = 1, \alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2, \mu = 1$, and $z = t = 0$.

Remark 2. *The reduction introduced in Theorem 1 also clarifies the relation between the generalized (3+1)-dimensional gCH–KP Eq (1.3) and its (2+1)-dimensional counterpart (1.2). Indeed, by considering solutions depending on the single effective transverse variable*

$$\mathcal{Y} = y + \zeta z,$$

namely $u(x, y, z, t) \rightarrow u(x, \mathcal{Y}, t)$, the transverse derivative terms in Eq (1.3) satisfy

$$u_{yy} = u_{\mathcal{Y}\mathcal{Y}}, \quad u_{zz} = \zeta^2 u_{\mathcal{Y}\mathcal{Y}}.$$

Consequently, the transverse part of Eq (1.3) becomes $(\lambda + \mu\zeta^2)u_{\mathcal{Y}\mathcal{Y}}$. Therefore, after a suitable rescaling of the transverse variable $\mathcal{Y}$, Eq (1.3) reduces to the (2 + 1)-dimensional CH–KP equation. In particular, the choice $\mu = 0$ gives the reduction to Eq (1.2) directly.

This observation shows that the RW solutions derived here for the gCH–KP equation also provide the corresponding higher-order RW solutions for the (2 + 1)-dimensional CH–KP equation as a special case.

Furthermore, another extension of the CH–KP equation introduced in [42] is given by

$$(u_t + \alpha u_x + \beta u u_x + \delta u_{xxt})_x + \lambda u_{yy} + \mu u_{tt} = 0. \quad (2.34)$$

It is straightforward to observe that under the identification $z \rightarrow t$, Eq (1.3) reduces to this form. Consequently, the solutions obtained in this work can also be mapped to solutions of this equation via the same transformation. This implies that the constructed τ -function solutions, together with the associated higher-order RW structures, remain valid for this extended model as well.

The main contribution of this work, however, is not the additional transverse variable itself, but the construction of the general higher-order τ -function solutions and the analytical characterization of their RW patterns through the YV hierarchy polynomials. These results provide a unified description of polygonal RW structures and their predicted locations, which, to the best of our knowledge, have not been established previously for Eqs (1.2) and (2.34).

Remark 3. For $n = 1, \dots, 4$, it is possible to recast the τ -function (2.19) into a symmetric polynomial form by an appropriate choice of the parameters a_j . Specifically, it can be expressed as

$$\tilde{\tau}_n(\kappa, \mathcal{Y}) = \sum_{j=0}^{n(n+1)/2} \sum_{m=0}^j \omega_{m,j} \kappa^{2m} \mathcal{Y}^{2(j-m)}, \quad (2.35)$$

where the coefficients $\omega_{m,j}$ depend on $\alpha, \beta, \delta, \lambda, \mu$, and the parameters σ, ς . This representation contains only even powers of the variables κ and $\mathcal{Y}$, and $\tilde{\tau}_n$ is a nonzero polynomial of total degree $n(n+1)$.

The structure of $\tilde{\tau}_n$ in (2.35) mirrors the τ -functions used in symbolic-algebra constructions of multi-RW solutions for the KP and Boussinesq equations [24]. This correspondence suggests that the symbolic solutions presented in Ref. [41] for Eq (1.2), in Ref. [44] for the gCH–KP Eq (1.3), and in Ref. [42] for the extended $(2+1)$ -dimensional Eq (2.34) can all be understood within a unified τ -function framework.

In the general higher-order case, the τ_n -function involves n free parameters, $a_1, a_3, a_5, \dots, a_{2n-1}$, which provide considerable freedom in shaping the RW pattern, leading to configurations such as polygons and circular arrays. By contrast, the reduced form $\tilde{\tau}_n$ produces exactly n localized peaks positioned along the x -axis (see Figure 6). In this setting, the parameters no longer control the peak locations and instead affect only their amplitudes.

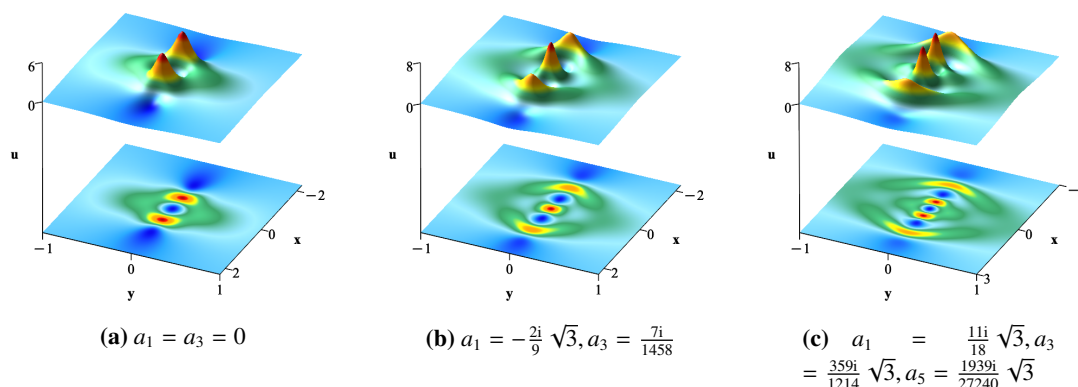


Figure 6. Second-, third-, and fourth-order reduced RWs corresponding to even-power polynomials (2.35) with the parameters $\alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2, \mu = 1$, and $z = t = 0$.

Conjecture 1. *Based on our observations of the second-, third-, and fourth-order RWs of the gCH–KP equation (not shown here due to length), which involve n free parameters a_i , ($i = 1, 3, 5, \dots, 2n + 1$), we conjecture that when one of the free parameters is significantly larger than the others, the resulting RW forms a corresponding polygonal pattern. In other words, the dominant free parameter dictates the geometric arrangement of the peaks, while the amplitudes of all peaks remain equal to that of the fundamental Peregrine-type RW.*

From a physical viewpoint, these polygonal patterns represent the spatial organization of multiple extreme-wave events. Each vertex corresponds to a localized RW peak, while the polygonal arrangement describes how these extreme events are distributed around the center. The increasing number of sides, from triangular to pentagonal and higher-order patterns, reflects the richer spatial structures that can arise in multidimensional nonlinear dispersive systems.

By the way, calculating the determinant in (2.19) for large values of n is very costly and may even be impossible, so we need to replace this calculation with an alternative approach. In Section 3, we combine these observations with the properties of the YV polynomials' roots to show that, without performing the extensive calculations required to obtain higher-order RWs, one can simply use the root structure of the YV polynomials to determine the locations of the separated fundamental RW components in the large- $|a_{2m+1}|$ regime.

3. Polygonal RW patterns in the gCH–KP equation

In Section 2, we constructed general RW solutions to the gCH–KP Eq (1.3) using the KP hierarchy reduction method. We demonstrated that these solutions depend on internal parameters a_{2j+1} , and when one such parameter, say a_{2m+1} , is taken to be large while the others remain of order $O(1)$, (i.e., finite and not growing with respect to a_{2m+1}), the resulting RW solution exhibits a distinct geometric pattern shaped like a $(2m + 1)$ -gon. This behavior was further analyzed by connecting it to the root structures of the YV polynomials in [30]. In that work, Clarkson and Mansfield showed that the roots of these polynomials, which generate rational solutions of the second Painlevé equation and its hierarchy, form highly regular and symmetric configurations, most notably triangular lattices.

Such regularity in the YV root patterns suggests a deep correspondence between the algebraic geometry underlying the Painlevé hierarchies and the polygonal RW structures arising in the gCH–KP equation. This analogy provides further insight into how internal parameters in integrable systems can encode hidden geometric symmetries in their rational or RW solutions.

Building on this insight, we proposed an analytical asymptotic description of these RW patterns by expressing the solutions as a superposition of fundamental Peregrine-type RWs positioned according to the roots of these polynomials. To illustrate the effectiveness of this approach, we considered the gCH–KP Eq (1.3), constructed its general RW solutions using Theorem 2, and derived the predicted RW patterns accordingly.

3.1. YV hierarchy polynomials

We recall from [51] that the YV hierarchy polynomials $Q_n^{[m]}(z)$ can be expressed in determinant form as

$$Q_n^{[m]}(z) = c_n \begin{vmatrix} p_n^{[m]}(z) & p_{n+1}^{[m]}(z) & \cdots & p_{2n-1}^{[m]}(z) \\ p_{n-2}^{[m]}(z) & p_{n-1}^{[m]}(z) & \cdots & p_{2n-3}^{[m]}(z) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-n+2}^{[m]}(z) & p_{-n+3}^{[m]}(z) & \cdots & p_1^{[m]}(z) \end{vmatrix}, \quad (3.1)$$

where the functions $p_k^{[m]}(z)$ are defined through the generating function

$$\sum_{k=0}^{\infty} p_k^{[m]}(z) \epsilon^k = \exp(z\epsilon - \frac{2^{2m}}{2m+1} \epsilon^{2m+1}). \quad (3.2)$$

Here, m is a positive integer, and we take $p_k^{[m]}(z) = 0$ for $k < 0$. The constant c_n is given by $c_n = \prod_{j=1}^n (2j-1)!!$. Moreover, the polynomials $p_k^{[m]}(z)$ satisfy the recurrence relation

$$p_k^{[m]}(z) = [p_{k+1}^{[m]}(z)]', \quad (3.3)$$

which provides a useful tool for recursive construction.

Figure 7 displays the root plots of the YV polynomial hierarchy $Q_n^{[m]}(z)$ in the complex z -plane for selected values of n and m .

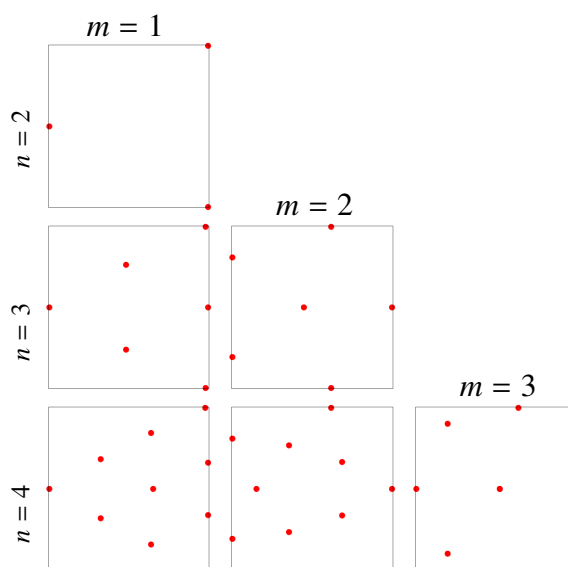


Figure 7. Plots of the root structures of the YV polynomial hierarchy $Q_n^{[m]}(z)$ for $1 \leq n \leq 5$ and $1 \leq m \leq 4$. The horizontal axis represents $\Re(z)$, and the vertical axis represents $\Im(z)$.

Theorem 3. [51] The general YV polynomial $Q_n^{[m]}(z)$ has degree $\frac{n(n+1)}{2}$ and can be expressed as

$$Q_n^{[m]}(z) = z^{n_0(n_0+1)/2} q_n^{[m]}(\zeta), \quad \zeta = z^{2m+1}, \quad (3.4)$$

where $q_n^{[m]}(\zeta)$ is a monic polynomial with real coefficients and a nonzero constant term. The integer n_0 is defined by

$$n_0 = \begin{cases} n \bmod (2m+1), & \text{if } 0 \leq n \bmod (2m+1) \leq m, \\ 2m - (n \bmod (2m+1)), & \text{if } n \bmod (2m+1) > m. \end{cases} \quad (3.5)$$

This expression ensures $0 \leq n_0 \leq m$. The multiplicity of the root at zero is $\frac{n_0(n_0+1)}{2}$; if $n_0 = 0$, then zero is not a root. Due to this restriction on n_0 , the resulting n_0 value is unique.

Conjecture 2. It was proposed in [30] that, within the generalized YV polynomial hierarchy $Q_n^{[m]}(z)$, all roots different from zero are simple. When combined with Theorem 3, this assumption leads to the conclusion that the total number of nonzero simple zeros of the polynomials is given by

$$n_p = \frac{1}{2}(n(n+1) - n_0(n_0+1)), \quad (3.6)$$

where n_0 is defined as in Eq (3.5).

The distribution of the zeros of the YV polynomials is crucial for analyzing RW patterns. These roots act as geometric templates that determine the spatial arrangement and symmetry of the corresponding RW solutions in integrable systems. Although the precise correspondence between polynomial root configurations and RW patterns was first established for the NLS equation [51], a similar mechanism also arises in the gCH–KP system. When the internal parameter a_{2m+1} becomes large, the resulting RW pattern is governed by the root distribution of $Q_n^{[m]}(z)$: each simple root corresponds to a fundamental Peregrine-type RW, while a root at the origin with the multiplicity $n_0(n_0+1)/2$ produces an n_0 -th order central RW. In what follows, we demonstrate that this algebraic–geometric correspondence extends naturally to the gCH–KP equation, giving rise to a rich family of polygonal RW structures.

3.2. Analytical RW patterns in the gCH–KP equation

In Section 2, we observed that RWs form distinct geometric patterns composed of multiple Peregrine waves: triangular patterns for large a_3 , pentagonal for a_5 , heptagonal for a_7 , nonagonal for a_9 , hendecagonal for a_{11} , and tridecagonal for a_{13} , and so on. The RW solutions constructed in Theorem 2 also contain free internal complex parameters $\{a_1, a_3, \dots, a_{2n-1}\}$. We normalize $a_1 = 0$ by an appropriate shift of the $(\varkappa, \mathscr{Y})$ coordinates. However, we retain a_1 in the solution expressions for clarity, as its role will be clarified in the following. We demonstrate that when any of these parameters becomes large, the resulting RWs display clear and predictable geometric patterns, fully determined by the root structures of the YV polynomial hierarchy.

Consider the case where one internal parameter a_{2m+1} is large, with $1 \leq m \leq n-1$, where the index m corresponds to the position of a_{2m+1} within the set of parameters $\{a_3, a_5, \dots, a_{2n-1}\}$ associated with the RW solution $u_n(x, t)$. Assume also that all remaining parameters, namely $\{a_3, a_5, \dots, a_{2m-1}, a_{2m+3}, \dots, a_{2n-1}\}$, remain of order $\mathcal{O}(1)$. To analyze this regime, we define the rescaled parameter

$$\Lambda \equiv \left(-\frac{2m+1}{2^{2m}} a_{2m+1} \right)^{\frac{1}{2m+1}}. \quad (3.7)$$

To analyze the asymptotic structure of higher-order RWs and establish their connection with the YV polynomial roots, we use the scaling freedoms of the gCH–KP equation to remove inessential constants. Therefore, without loss of generality, we choose

$$\alpha = -1 - 12\sqrt{\delta}, \quad \lambda = -2, \quad \mu = \sigma = \varsigma = 1. \quad (3.8)$$

This choice does not restrict the generality of the solutions obtained in Theorem 2; it only simplifies the asymptotic analysis. In particular, $\sigma = \varsigma = 1$ gives the normalized variables $\kappa = x + t$ and $\mathscr{Y} = y + z$, while the choices of α and λ simplify the fundamental RW expression and the resulting asymptotic formulas. Under this normalization, the large a_{2m+1} asymptotics of RWs in the Eq (1.3) can be described by the following theorems.

Theorem 4. Let $u_n(\kappa, \mathscr{Y})$, with $\kappa = x + t$, $\mathscr{Y} = y + z$, be the n th-order RW solution of the gCH–KP Eq (1.3), depending on the internal parameters $\{a_3, a_5, \dots, a_{2n-1}\}$. Suppose that a single parameter a_{2m+1} is large. In the outer region where

$$\sqrt{\kappa^2 + \mathscr{Y}^2} = O(|a_{2m+1}|^{1/(2m+1)}),$$

the RW $u_n(\kappa, \mathscr{Y})$ asymptotically decomposes into n_p fundamental RWs, where n_p is given by (3.6).

Each of these fundamental waves is asymptotically given by a shifted version of the first-order solution

$$u_1(\kappa, \mathscr{Y}) = \frac{12\delta}{\beta} \ln \tau_{1,\kappa\kappa}, \quad (3.9)$$

with

$$\tau_1 = -\frac{1}{3\sqrt{\delta}}\kappa^2 - 4\mathscr{Y}^2 - \frac{1}{12}, \quad (3.10)$$

centered at locations $(\hat{\kappa}_0, \hat{\mathscr{Y}}_0)$, determined by the corresponding simple root z_0 of $\mathcal{Q}_n^{[m]}(z)$, through the formulas

$$\hat{\kappa}_0 = \delta^{\frac{1}{4}} \Re[-i\sqrt{3}\Lambda z_0] + \psi, \quad (3.11)$$

$$\hat{\mathscr{Y}}_0 = \frac{\sqrt{3}}{6} \Im[-i\sqrt{3}\Lambda z_0], \quad (3.12)$$

where Λ is given in Eq (3.7) and

$$\psi = -\frac{4}{3}(n-1). \quad (3.13)$$

The error of this fundamental RW approximation is $O(|a_{2m+1}|^{-1/(2m+1)})$. Moreover, in the vicinity of each such point $(\hat{\kappa}_0, \hat{\mathscr{Y}}_0)$, the full solution admits the asymptotic expansion

$$u_n(\kappa, \mathscr{Y}; a_3, a_5, \dots, a_{2n-1}) = u_1(\kappa - \hat{\kappa}_0, \mathscr{Y} - \hat{\mathscr{Y}}_0) + O(|a_{2m+1}|^{-1/(2m+1)}), \quad (3.14)$$

when $|a_{2m+1}| \gg 1$ and $[(\kappa - \hat{\kappa}_0)^2 + (\mathscr{Y} - \hat{\mathscr{Y}}_0)^2]^{1/2} = O(1)$.

Outside these neighborhoods, specifically when $(\kappa, \mathscr{Y})$ is not close to any of the fundamental wave centers or when

$$\sqrt{\kappa^2 + \mathscr{Y}^2} \gg |a_{2m+1}|^{1/(2m+1)},$$

the solution $u_n(\kappa, \mathscr{Y})$ decays to the background value ($u = 0$) as $|a_{2m+1}| \rightarrow \infty$.

Proof. When a_{2m+1} is large and the other parameters are $O(1)$ in the RW solution (2.18), at $(\kappa, \mathcal{Y})$ where $\sqrt{\kappa^2 + \mathcal{Y}^2} = O(|a_{2m+1}|^{\frac{1}{2m+1}})$, by denoting

$$\lambda = a_{2m+1}^{-\frac{1}{2m+1}}, \quad (3.15)$$

and recalling the expression of Schur polynomials (2.17), we have

$$\begin{aligned} \mathcal{S}_j(\mathbf{X}^+ + \nu \mathbf{h}) &= \mathcal{S}_j(X_1^+, \nu h_2, X_3^+, \nu h_4, \dots) \\ &= \lambda^{-j} \mathcal{S}_j(X_1^+ \lambda, \nu h_2 \lambda^2, X_3^+ \lambda^3, \nu h_4 \lambda^4, \dots) \\ &\sim \lambda^{-j} \mathcal{S}_j[X_1^+ \lambda, 0, \dots, 0, 1, 0, \dots] \\ &= \mathcal{S}_j(X_1^+, 0, \dots, 0, a_{2m+1}, 0, \dots). \end{aligned}$$

In summary, we have

$$\mathcal{S}_j(\mathbf{X}^+ + \nu \mathbf{h}) = \mathcal{S}_j(\mathbf{v}) \left[1 + O\left(a_{2m+1}^{-1/(2m+1)}\right) \right], \quad |a_{2m+1}| \gg 1, \quad (3.16)$$

with vector

$$\mathbf{v} = (X_1^+, 0, \dots, 0, a_{2m+1}, 0, \dots), \quad (3.17)$$

where X_1^+ comes from Eq (2.21) using (3.8), that is

$$X_1^+ = \frac{i}{\sqrt{3}} (\delta^{-\frac{1}{4}} \kappa + 2\sqrt{3} i \mathcal{Y}).$$

We omit the νh_1 component of $X_1^+ + \nu h_1$ in $\mathbf{v}$, even though $h_1 \neq 0$ is given in (2.24), because the term νh_1 is significantly smaller than X_1^+ and therefore is negligible in leading order.

From the definition of Schur polynomials in Eq (2.17), we observe that $\mathcal{S}_j(\mathbf{v})$ is connected to the polynomial $p_j^{[m]}(z)$ defined in Eq (3.2) via the relation

$$\mathcal{S}_j(\mathbf{v}) = \Lambda^j p_j^{[m]}(z).$$

Therefore,

$$\mathcal{S}_j(\mathbf{X}^+ + \nu s) \sim \Lambda^j p_j^{[m]}(z), \quad |a_{2m+1}| \gg 1. \quad (3.18)$$

Similarly, using the identity $X_1^- = -(X_1^+)^*$ from (2.22), we obtain

$$\mathcal{S}_j(\mathbf{X}^- + \nu \mathbf{h}) \sim (-\Lambda^*)^j p_j^{[m]}(z^*), \quad |a_{2m+1}| \gg 1. \quad (3.19)$$

By applying these expressions and following a similar procedure to that in [51], we deduce that the function τ in Eq (2.19) exhibits the following asymptotic behavior

$$\tau \sim \kappa |a_{2m+1}|^{\frac{n(n+1)}{2m+1}} \left| Q_n^{[m]}(z) \right|^2, \quad |a_{2m+1}| \gg 1, \quad (3.20)$$

where κ is a constant that depends on (n, m) . This asymptotic result indicates that the solution $u_n(\kappa, \mathcal{Y})$ in Eq (2.18) remains approximately zero, except in the vicinity of certain points $(\tilde{\kappa}_0, \tilde{\mathcal{Y}}_0)$, where $\Lambda^{-1} X_1^+$ coincides with a root of the polynomial $Q_n^{[m]}(z)$; that is

$$z_0 = \Lambda^{-1} \frac{i}{\sqrt{3}} (\delta^{-\frac{1}{4}} \tilde{\kappa}_0 + 2\sqrt{3} i \tilde{\mathcal{Y}}_0), \quad (3.21)$$

matches a root of the YV polynomial $Q_n^{[m]}(z)$. As a result, the positions $(\tilde{\kappa}_0, \tilde{\mathcal{Y}}_0)$ can be obtained as

$$\tilde{\kappa}_0 = \delta^{\frac{1}{4}} \Re[-i \sqrt{3} \Lambda z_0], \quad \tilde{\mathcal{Y}}_0 = \frac{\sqrt{3}}{6} \Im[-i \sqrt{3} \Lambda z_0]. \quad (3.22)$$

We show that in a small neighborhood of each point $(\tilde{\kappa}_0, \tilde{\mathcal{Y}}_0)$, specifically when $\sqrt{(\kappa - \tilde{\kappa}_0)^2 + (\mathcal{Y} - \tilde{\mathcal{Y}}_0)^2} = O(1)$, the RW solution $u_n(\kappa, \mathcal{Y})$ from Eq (2.18) with a large parameter a_{2m+1} closely resembles a fundamental RW. Although the precise peak of this fundamental wave does not coincide exactly with $(\tilde{\kappa}_0, \tilde{\mathcal{Y}}_0)$, it remains in close proximity. To establish this, we use a more detailed asymptotic expansion for $\mathcal{S}_j(\mathbf{X}^+ + \nu \mathbf{s})$ valid near $(\tilde{\kappa}_0, \tilde{\mathcal{Y}}_0)$, where the vector $\hat{\mathbf{v}}$ is redefined as follows [51]:

$$\hat{\mathbf{v}} = (\mathcal{X}_1^+ + \nu h_1, 0, \dots, 0, a_{2m+1}, 0, \dots). \quad (3.23)$$

The vector $\hat{\mathbf{v}}$ now includes an additional term, νh_1 . This new νh_1 component contributes an extra $O(1)$ correction to the fundamental RW's position $(\hat{\kappa}_0, \hat{\mathcal{Y}}_0)$ within the RW patterns, as will be demonstrated below.

The polynomial $\mathcal{S}_j(\hat{\mathbf{v}})$ is connected to $p_j^{[m]}(z)$ from Eq (3.2). Exploiting this relationship, we can express the asymptotic form of $\mathcal{S}_j(\mathbf{X}^+ + \nu \mathbf{h})$ as follows:

$$\mathcal{S}_j(\mathbf{X}^+ + \nu \mathbf{h}) = \Lambda^j p_j^{[m]}(z + \nu h_1 \Lambda^{-1}) \left[1 + O\left(a_{2m+1}^{-2/(2m+1)}\right) \right], \quad (3.24)$$

where $z = \Lambda^{-1} \mathcal{X}_1^+$. Similarly, we obtain

$$\mathcal{S}_j(\mathbf{X}^- + \nu \mathbf{h}) = (-\Lambda^*)^j [p_j^{[m]}(z + \nu h_1 \Lambda^{-1})]^* \left[1 + O\left(a_{2m+1}^{-2/(2m+1)}\right) \right], \quad (3.25)$$

since h_1 in Eq (2.24) is purely imaginary. Applying these refined asymptotics for $\mathcal{S}_j(\mathbf{X}^\pm + \nu \mathbf{s})$, we can now evaluate the leading-order behavior of $\tau(\kappa, \mathcal{Y})$ in Eq (2.19) near the point $(\tilde{\kappa}_0, \tilde{\mathcal{Y}}_0)$. To do this, we use determinant identities along with the Laplace expansion to express $\tau(\kappa, \mathcal{Y})$ as in [26]

$$\begin{aligned} \tau(\kappa, \mathcal{Y}) = & \sum_{0 \leq \nu_1 < \dots < \nu_n \leq 2n-1} \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^+ + \nu_j \mathbf{h}) \right] \\ & \times \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^- + \nu_j \mathbf{h}) \right], \end{aligned} \quad (3.26)$$

where $\mathcal{S}_j \equiv 0$ when $j < 0$. When a_{2m+1} becomes large, the parameter Λ also grows correspondingly. Under this condition, the leading-order contributions to $\tau(\kappa, \mathcal{Y})$ arise primarily from two specific index sets: $\nu = (0, 1, \dots, n-1)$ and $\nu = (0, 1, \dots, n-2, n)$. For the first index choice of $\nu = (0, 1, \dots, n-1)$, i.e., $\nu_j = j-1$, we have

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^+ + \nu_j \mathbf{h}) \right] \\ &= \left(\frac{\sqrt{3}i}{6} \right)^{\frac{n(n-1)}{2}} \Lambda^{\frac{n(n+1)}{2}} \det_{1 \leq i, j \leq n} \left[p_{2i-j}^{[m]}(z + (j-1)h_1 \Lambda^{-1}) \right] \left[1 + O\left(a_{2m+1}^{-2/(2m+1)}\right) \right]. \end{aligned} \quad (3.27)$$

Since $z = \Lambda^{-1} \mathcal{X}_1^+$ is a linear function of $\varkappa$ and $\mathcal{Y}$, and from Eq (3.21), we have $z_0 = \Lambda^{-1} \mathcal{X}_1^+(\tilde{\varkappa}_0, \tilde{\mathcal{Y}}_0)$, it follows that

$$z + (j-1)h_1\Lambda^{-1} = z_0 + \left[\mathcal{X}_1^+(\varkappa - \tilde{\varkappa}_0, \mathcal{Y} - \tilde{\mathcal{Y}}_0) + (j-1)h_1 \right] \Lambda^{-1}. \quad (3.28)$$

Expanding $p_{2i-j}^{[m]}(z + (j-1)h_1\Lambda^{-1})$ around z_0 , we then get

$$p_{2i-j}^{[m]}(z + (j-1)h_1\Lambda^{-1}) = p_{2i-j}^{[m]}(z_0) + \left(p_{2i-j}^{[m]} \right)'(z_0) \left[\mathcal{X}_1^+(\varkappa - \tilde{\varkappa}_0, \mathcal{Y} - \tilde{\mathcal{Y}}_0) + (j-1)h_1 \right] \Lambda^{-1} + O(\Lambda^{-2}). \quad (3.29)$$

By substituting this expansion into the determinant in Eq (3.27), the leading-order $O(1)$ term of the determinant is $Q_n^{[m]}(z_0)$, which vanishes because z_0 is a root of the YV polynomial. To derive the remaining terms, we recall the property (3.3) of $p_j^{[m]}(z)$ polynomials, which yields

$$\left(p_{2i-j}^{[m]} \right)'(z_0) = p_{2i-j-1}^{[m]}(z_0).$$

Applying this relation causes most terms in the determinant expansion of Eq (3.27) to cancel out, so that only one leading term remains significant.

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[p_{2i-j}^{[m]}(z + (j-1)h_1\Lambda^{-1}) \right] \\ &= \det_{1 \leq i, j \leq n} \left[p_{2i-1}^{[m]}(z_0), p_{2i-2}^{[m]}(z_0), \dots, p_{2i-N-1}^{[m]}(z_0) [\mathcal{X}_1^+(\varkappa - \tilde{\varkappa}, \mathcal{Y} - \tilde{\mathcal{Y}}) + (n-1)h_1] \right] \Lambda^{-1} + O(\Lambda^{-2}) \\ &= c_n^{-1} \left[Q_n^{[m]} \right]'(z_0) [\mathcal{X}_1^+(\varkappa - \tilde{\varkappa}, \mathcal{Y} - \tilde{\mathcal{Y}}) + (n-1)h_1] \Lambda^{-1} + O(\Lambda^{-2}), \end{aligned}$$

where c_n is defined in Eq (3.1). In the final step, we applied the Wronskian property of the determinant representing $Q_n^{[m]}(z)$ from Eq (3.1). Since h_1 in Eq (2.24) is purely imaginary, we incorporate the $(N-1)h_1$ term into the shift of $\tilde{\varkappa}_0$, yielding

$$\det_{1 \leq i, j \leq n} \left[p_{2i-j}^{[m]}(z + (j-1)h_1\Lambda^{-1}) \right] = c_n^{-1} \left[Q_n^{[m]} \right]'(z_0) \mathcal{X}_1^+(\varkappa - \hat{\varkappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) \Lambda^{-1} + O(\Lambda^{-2}), \quad (3.30)$$

where $\hat{\varkappa}_0$ and $\hat{\mathcal{Y}}_0$ are given in Eqs (3.11) and (3.12). Inserting this into Eq (3.27), and noting that $\Lambda^{-1} = O(a_{2m+1}^{-1/(2m+1)})$, we get

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^+ + \nu_j \mathbf{h}) \right] \\ &= c_n^{-1} \left[Q_n^{[m]} \right]'(z_0) \left(\frac{\sqrt{3}i}{6} \right)^{\frac{n(n-1)}{2}} \Lambda^{\frac{n(n+1)-2}{2}} \mathcal{X}_1^+(\varkappa - \hat{\varkappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) \left[1 + O(a_{2m+1}^{-1/(2m+1)}) \right]. \end{aligned} \quad (3.31)$$

Similarly, we have

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^- + \nu_j \mathbf{h}) \right] \\ &= c_n^{-1} \left[Q_n^{[m]} \right]'(z_0^*) \left(\frac{\sqrt{3}i}{6} \right)^{\frac{N(N-1)}{2}} (-\Lambda^*)^{\frac{n(n+1)-2}{2}} \mathcal{X}_1^-(\varkappa - \hat{\varkappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) \left[1 + O(a_{2m+1}^{-1/(2m+1)}) \right]. \end{aligned} \quad (3.32)$$

Hence, the contribution to Eq (3.26) arising from the first index set $\nu = (0, 1, \dots, n-1)$ is given by

$$\hat{\kappa} \left| \left[\mathcal{Q}_n^{[m]} \right]' (z_0) \right|^2 |a_{2m+1}|^{\frac{n(n+1)-2}{2m+1}} \mathcal{X}_1^+(\kappa - \hat{\kappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) \mathcal{X}_1^-(\kappa - \hat{\kappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) \left[1 + \mathcal{O}\left(a_{2m+1}^{-1/(2m+1)}\right) \right], \quad (3.33)$$

where

$$\hat{\kappa} = c_n^{-2} (-1)^{n^2-1} \left(\frac{1}{12} \right)^{\frac{n(n-1)}{2}} \left(\frac{2m+1}{2^{2m}} \right)^{\frac{n(n+1)-2}{2m+1}}. \quad (3.34)$$

Next, we compute the contribution to $\tau(\kappa, \mathcal{Y})$ in Eq (3.26) from the second index choice $\nu = (0, 1, \dots, n-2, n)$. Using Eq (3.24), we obtain

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^+ + \nu_j \mathbf{h}) \right] \\ &= \left(\frac{\sqrt{3}i}{6} \right)^{\frac{n(n-1)+2}{2}} \Lambda^{\frac{n(n+1)-2}{2}} \det_{1 \leq i, j \leq N} \left[p_{2i-1-\nu_j}^{[m]}(z + \nu_j h_1 \Lambda^{-1}) \right] \left[1 + \mathcal{O}\left(a_{2m+1}^{-2/(2m+1)}\right) \right]. \end{aligned} \quad (3.35)$$

The leading-order asymptotics of the determinant are obtained by ignoring the terms $\nu_j h_1 \Lambda^{-1}$ and substituting z with z_0 , yielding

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^+ + \nu_j \mathbf{h}) \right] \\ &= \left(\frac{\sqrt{3}i}{6} \right)^{\frac{n(n-1)+2}{2}} \Lambda^{\frac{n(n+1)-2}{2}} \det_{1 \leq i, j \leq n} \left[p_{2i-1-\nu_j}^{[m]}(z_0) \right] \left[1 + \mathcal{O}\left(a_{2m+1}^{-1/(2m+1)}\right) \right] \\ &= \left(\frac{\sqrt{3}i}{6} \right)^{\frac{n(n-1)+2}{2}} \Lambda^{\frac{n(n+1)-2}{2}} \left[\mathcal{Q}_n^{[m]} \right]' (z_0) \left[1 + \mathcal{O}\left(a_{2m+1}^{-1/(2m+1)}\right) \right]. \end{aligned} \quad (3.36)$$

Similarly, we have

$$\begin{aligned} & \det_{1 \leq i, j \leq n} \left[\left(\frac{\sqrt{3}i}{6} \right)^{\nu_j} \mathcal{S}_{2i-1-\nu_j}(\mathbf{X}^- + \nu_j \mathbf{h}) \right] \\ &= \left(\frac{\sqrt{3}i}{6} \right)^{\frac{n(n-1)+2}{2}} (-\Lambda^*)^{\frac{n(n+1)-2}{2}} \left[\mathcal{Q}_n^{[m]} \right]' (z_0^*) \left[1 + \mathcal{O}\left(a_{2m+1}^{-1/(2m+1)}\right) \right]. \end{aligned} \quad (3.37)$$

The contribution to Eq (3.26) from this second index choice is

$$\hat{\kappa} \left| \left[\mathcal{Q}_n^{[m]} \right]' (z_0) \right|^2 |a_{2m+1}|^{\frac{n(n+1)-2}{2m+1}} \left[-\frac{1}{12} + \mathcal{O}\left(a_{2m+1}^{-1/(2m+1)}\right) \right], \quad (3.38)$$

where $\hat{\kappa}$ is given in Eq (3.34). Combining both contributions to $\tau(\kappa, \mathcal{Y})$ in Eq (3.26), we obtain

$$\begin{aligned}
\tau(\kappa, \mathcal{Y}) &= \hat{\kappa} \left| \left[Q_n^{[m]} \right]' (z_0) \right|^2 |a_{2m+1}|^{\frac{n(n+1)-2}{2m+1}} \left[X_1^+(\kappa - \hat{\kappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) X_1^-(\kappa - \hat{\kappa}_0, \mathcal{Y} - \hat{\mathcal{Y}}_0) - \frac{1}{12} \right] \\
&\quad \times \left[1 + O\left(a_{2m+1}^{-1/(2m+1)}\right) \right] \\
&= -\frac{1}{12} \hat{\kappa} \left| \left[Q_n^{[m]} \right]' (z_0) \right|^2 |a_{2m+1}|^{\frac{n(n+1)-2}{2m+1}} \left[\frac{4}{\sqrt{\delta}} (\kappa - \hat{\kappa}_0)^2 + 48(\mathcal{Y} - \hat{\mathcal{Y}}_0)^2 + 1 \right] \left[1 + O\left(a_{2m+1}^{-1/(2m+1)}\right) \right].
\end{aligned} \tag{3.39}$$

Based on the assumption in Conjecture 2 that all nonzero roots of the YV polynomials are simple, the leading-order term of $\tau(\kappa, \mathcal{Y})$ is guaranteed to be nonzero. Hence, when a_{2m+1} is large and $(\kappa, \mathcal{Y})$ lies near $(\hat{\kappa}_0, \hat{\mathcal{Y}}_0)$, the asymptotic expression above, together with Eq (2.18), ensures that

$$u_n(\kappa, \mathcal{Y}) = \frac{12\delta}{\beta} \ln \left[-\frac{1}{3\sqrt{\delta}} (\kappa - \hat{\kappa}_0)^2 - 4(\mathcal{Y} - \hat{\mathcal{Y}}_0)^2 - \frac{1}{12} \right]_{\kappa\kappa} + O\left(a_{2m+1}^{-1/(2m+1)}\right), \tag{3.40}$$

which proves Eq (3.14). $\square$

So far, we have studied the asymptotic behavior of the RW solution in the outer region. Next, we present its asymptotic form in the vicinity of the origin.

Theorem 5. Consider the RW solution $u_n(\kappa, \mathcal{Y})$ with $\kappa = x + t$, $\mathcal{Y} = y + z$, and a large parameter a_{2m+1} . In the vicinity of the origin, specifically where $\sqrt{\kappa^2 + \mathcal{Y}^2} = O(1)$, the solution $u_n(\kappa, \mathcal{Y})$ can be well approximated by a lower-order RW $u_{n_0}(\kappa, \mathcal{Y})$, where the order n_0 is determined by n and m through the relation (3.5), satisfying $0 \leq n_0 \leq n - 2$.

This lower-order RW u_{n_0} has the internal parameters $(\hat{a}_1, \hat{a}_3, \dots, \hat{a}_{2n_0-1})$, which are related to the original parameters $(a_1, a_3, \dots, a_{2n-1})$ of u_n via the formula

$$\hat{a}_{2r-1} = a_{2r-1} + (n - n_0)h_{2r-1}, \quad r = 1, 2, \dots, n_0, \tag{3.41}$$

where h_{2r-1} are defined constants from the problem setup (2.25).

The approximation error satisfies

$$u_n(\kappa, \mathcal{Y}; a_1, a_3, a_5, \dots, a_{2n-1}) = u_{n_0}(\kappa, \mathcal{Y}; \hat{a}_1, \hat{a}_3, \hat{a}_5, \dots, \hat{a}_{2n_0-1}) + O\left(|a_{2m+1}|^{-1}\right), \tag{3.42}$$

valid for large $|a_{2m+1}|$ and $(\kappa, \mathcal{Y})$ near the origin.

Importantly, although the original solution u_n has $a_1 = 0$, the parameter $\hat{a}_1$ of the reduced RW u_{n_0} is generally nonzero. If $n_0 = 0$, no such lower-order RW emerges near the origin, the solution $u_n(\kappa, \mathcal{Y})$ decays to the background value ($u = 0$) as $|a_{2m+1}| \rightarrow \infty$.

Proof. The determinant $\tau(\kappa, \mathcal{Y})$ in Eq (2.19) can be represented as a $3n \times 3n$ determinant [26]

$$\tau(\kappa, \mathcal{Y}) = \begin{vmatrix} \mathbf{0}_{n \times n} & \Phi_{n \times 2n} \\ -\Psi_{2n \times n} & \mathbf{I}_{2n \times 2n} \end{vmatrix}, \tag{3.43}$$

where the matrix elements are given by

$$\Phi_{i,j} = \left(\frac{\sqrt{3}i}{6} \right)^{j-1} \mathcal{S}_{2i-j} [\mathbf{X}^+ + (j-1)\mathbf{h}], \quad \Psi_{i,j} = \left(\frac{\sqrt{3}i}{6} \right)^{i-1} \mathcal{S}_{2j-i} [\mathbf{X}^- + (i-1)\mathbf{h}], \tag{3.44}$$

and $\mathbf{0}$ denotes the zero matrix, while i represents the identity matrix and $\mathcal{S}_j \equiv 0$ when $j < 0$.

To simplify $\tau(\kappa, \mathcal{Y})$ in the large- a_{2m+1} limit, we expand the elements of the matrices Φ and Ψ in terms of powers of a_{2m+1} and its complex conjugate a_{2m+1}^* , using a method analogous to that of Eq (58) in [51]. By systematically applying row and column transformations, as done in [51], we discard higher-order terms while retaining only the dominant contributions in each row or column. As a result, the determinant $\tau(\kappa, \mathcal{Y})$ asymptotically reduces to the structure given in Eq (64) of [51].

$$\tau(\kappa, \mathcal{Y}) = \hat{\beta} \begin{vmatrix} \mathbf{0}_{n_0 \times n_0} & \hat{\Phi}_{n_0 \times 2n_0} \\ -\hat{\Psi}_{2n_0 \times n_0} & \mathbf{I}_{2n_0 \times 2n_0} \end{vmatrix} \left[1 + O(a_{2m+1}^{-1}) \right], \quad (3.45)$$

where $\hat{\beta}$ is a constant determined by the pair (m, n) , which also incorporates a specific power of $|a_{2m+1}|$. Under this scaling, the matrix entries adopt the simplified structure

$$\hat{\Phi}_{i,j} = \left(\frac{\sqrt{3}i}{6} \right)^{j-1} \mathcal{S}_{2i-j} [\mathbf{Y}^+ + (j-1 + \nu_0)\mathbf{h}], \quad \hat{\Psi}_{i,j} = \left(\frac{\sqrt{3}i}{6} \right)^{i-1} \mathcal{S}_{2j-i} [\mathbf{Y}^- + (i-1 + \nu_0)\mathbf{h}], \quad (3.46)$$

where the vectors $\mathbf{Y}^\pm$ are defined by removing the a_{2m+1} terms from $\mathbf{X}^\pm$

$$\mathbf{Y}^+ = \mathbf{X}^+ - (0, \dots, 0, a_{2m+1}, 0, \dots), \quad \mathbf{Y}^- = \mathbf{X}^- + (0, \dots, 0, a_{2m+1}^*, 0, \dots), \quad (3.47)$$

and $\nu_0 = n - n_0$, with n_0 defined in (3.5).

In the case of the NLS equation, all odd-indexed h_j terms vanish, i.e., $h_1 = h_3 = \dots = 0$. As a result, the determinant in Eq (3.45) can be further simplified by choosing $\nu_0 = 0$ in Eq (3.46), consistent with the relation (65) in [51]. This leads to an asymptotic reduction of $\tau(\kappa, \mathcal{Y})$ into a lower-order RW $u_{n_0}(\kappa, \mathcal{Y})$, whose internal parameters remain unchanged from those of the original $u_n(\kappa, \mathcal{Y})$. In contrast, for the gCH-KP equation, the odd-indexed terms h_{odd} are nonzero. Therefore, we decompose the vector $\mathbf{h}$ into its even and odd components as follows:

$$\mathbf{h} = \mathbf{h}_{\text{odd}} + \mathbf{h}_{\text{even}}, \quad \mathbf{h}_{\text{odd}} = (h_1, 0, h_3, 0, \dots), \quad \mathbf{h}_{\text{even}} = (0, h_2, 0, h_4, \dots). \quad (3.48)$$

By applying a modified version of the identity analogous to Eq (65) in [51], we find that the determinant in Eq (3.45) simplifies to the following form:

$$\begin{aligned} \hat{\Phi}_{i,j} &= \left(\frac{\sqrt{3}i}{6} \right)^{j-1} \mathcal{S}_{2i-j} [\mathbf{Y}^+ + (j-1)\mathbf{h} + \nu_0 \mathbf{h}_{\text{odd}}], \\ \hat{\Psi}_{i,j} &= \left(\frac{\sqrt{3}i}{6} \right)^{i-1} \mathcal{S}_{2j-i} [\mathbf{Y}^- + (i-1)\mathbf{h} + \nu_0 \mathbf{h}_{\text{odd}}]. \end{aligned} \quad (3.49)$$

Finally, we incorporate the $\nu_0 \mathbf{h}_{\text{odd}} = (n - n_0) \mathbf{h}_{\text{odd}}$ contribution into $\mathbf{y}^\pm$ by redefining the internal parameters accordingly

$$\hat{a}_{2r-1} = a_{2r-1} + (n - n_0)h_{2r-1}, \quad r = 1, 2, \dots, n_0. \quad (3.50)$$

This redefinition is justified because all $\mathbf{h}_{\text{odd}}$ terms are purely imaginary in the case of the gCH-KP equation, which preserves the consistency between $\mathbf{Y}^+$ and $\mathbf{Y}^-$.

Consequently, the determinant representation of $\tau(\kappa, \mathcal{Y})$ in Eq (3.45), with the matrix elements defined in Eq (3.49), yields a RW of order n_0 , where the internal parameters $\hat{a}_{2r-1}$ are related to the original parameters a_{2r-1} of the n th-order RW $u_n(\kappa, \mathcal{Y})$ via Eq (3.50), which corresponds to Eq (3.41). The approximation error is $O(|a_{2m+1}|^{-1})$, consistent with the asymptotics in Eq (3.45). This completes the proof of Theorem 5. $\square$

We emphasize that the term “prediction” used in this work refers specifically to an asymptotic prediction in the regime $|a_{2m+1}| \rightarrow \infty$, with the remaining internal parameters kept $O(1)$. The peak locations are not inferred solely from numerical agreement. Rather, they follow analytically from the asymptotic reduction of the exact τ -function. In particular, Eq (3.20) shows that the leading-order structure of the τ -function is governed by $|Q_n^{[m]}(z)|^2$, so that each nonzero simple root z_0 of the corresponding YV polynomial generates a localized fundamental RW component. The explicit mapping from z_0 to the physical coordinates is given by Eqs (3.11) and (3.12). The local expansion in Eq (3.14) further shows that the resulting fundamental RW approximation has an error of $O(|a_{2m+1}|^{-1/(2m+1)})$. Therefore, the YV roots determine the leading-order locations analytically, while the numerical results in Section 4 serve as an independent verification of the asymptotic prediction and its error scaling. This statement is not intended to imply an exact peak-location formula for arbitrary finite values of the internal parameters.

3.3. Dynamical analysis of predicted RWs

In this section, we analyze the dynamical behavior of the predicted RWs and provide a brief evaluation of the error between these approximate solutions and the corresponding exact ones. From the theorems above, we understand that under the large-parameter limit, RW patterns share significant common features across the gCH–KP Eq (1.3) studied. At the heart of each pattern lies a lower-order RW, whose order n_0 is determined exclusively by the original wave’s order n and the index m of the dominant internal parameter. The detailed shape of this central wave, however, depends on its internal parameters as specified in Eq (3.42). These parameters dictate whether the central RW forms a single peak or multiple humps.

Beyond the central region, the overall RW pattern decomposes into multiple fundamental RWs. The number of these fundamental structures corresponds to the number of nonzero simple roots of the YV polynomial $Q_n^{[m]}(z)$. The spatial and temporal locations $(\hat{x}_0, \hat{\mathcal{T}}_0)$ of these individual waves are governed by a linear mapping from the real and imaginary parts of each nonzero simple root z_0 . Thus, the arrangement of the outer RWs is directly shaped by the complex root structure of the underlying polynomial. The linear map is

$$\begin{bmatrix} \hat{x}_0 \\ \hat{\mathcal{T}}_0 \end{bmatrix} = \mathbf{D}_\delta \mathbf{M}_\Lambda \begin{bmatrix} \Re(z_0) \\ \Im(z_0) \end{bmatrix} + \psi, \quad (3.51)$$

where

$$\mathbf{D}_\delta = \begin{bmatrix} \sqrt{3} \delta^{1/4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{M}_\Lambda = \begin{bmatrix} \Im(\Lambda) & \Re(\Lambda) \\ -\Re(\Lambda) & \Im(\Lambda) \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi \\ 0 \end{bmatrix}, \quad (3.52)$$

and Λ and ψ are defined in Eqs (3.7) and (3.13), respectively.

The matrix $\mathbf{M}_\Lambda$ represents the rotation and dilation associated with the complex factor Λ . In fact, we have

$$\mathbf{M}_\Lambda^T \mathbf{M}_\Lambda = |\Lambda|^2 I,$$

and hence $\mathbf{M}_\Lambda/|\Lambda|$ is an orthogonal rotation matrix. The coordinate-scaling matrix $\mathbf{D}_\delta$ accounts for the fixed scaling between the complex-plane coordinates and the physical coordinates $(\varkappa, \mathscr{Y})$, while the translation vector $\boldsymbol{\psi}$ produces the uniform translation in the $\varkappa$ -direction. Thus, the root configuration of $Q_n^{[m]}(z)$ is mapped to the locations of the outer fundamental RWs through a rotation–dilation process, followed by coordinate scaling and a uniform translation. In Figure 8, we apply a rotation to the root locations of the YV polynomial using the transformation $z_0 \mapsto (-i\sqrt{3})\Lambda z_0$ in order to align their pattern with that of our analytical predictions.

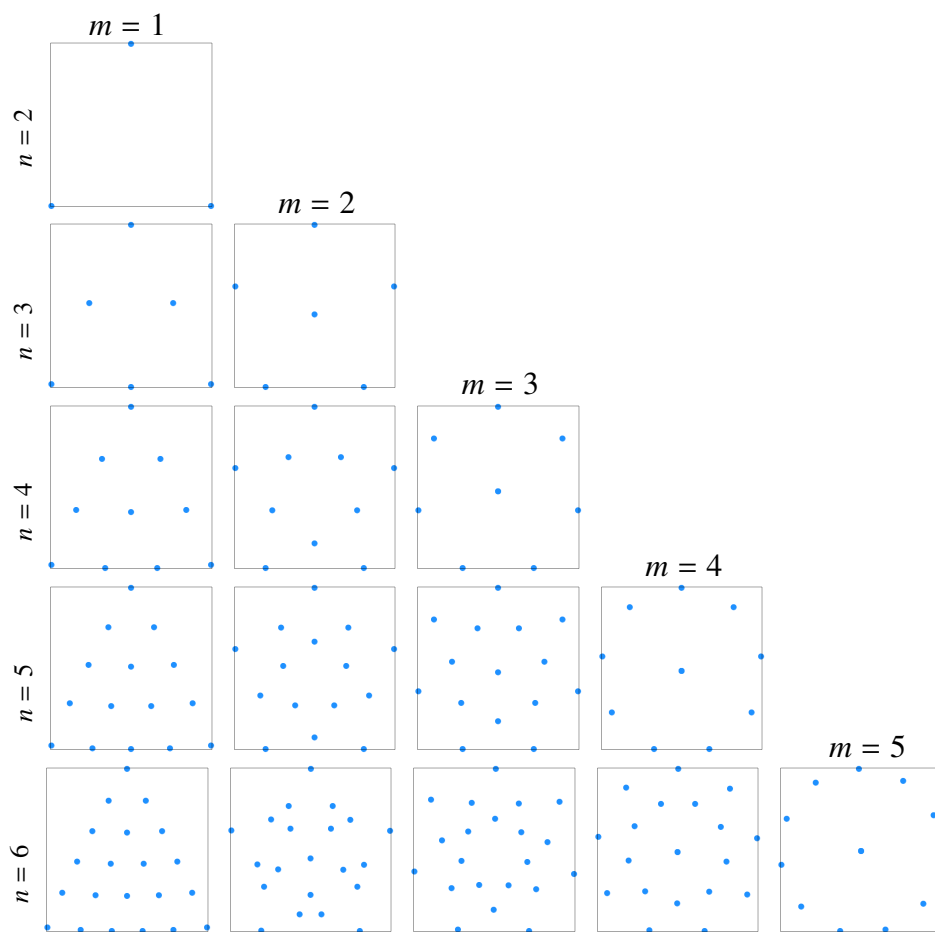


Figure 8. Orientations of the locations of the roots of the YV polynomial $Q_n^{[m]}(z)$ under the transformation $z_0 \mapsto z_0(-i\sqrt{3})\Lambda$, where z_0 denotes any root of the polynomial and Λ is defined in Eq (3.7), and $a_{2m+1} = -10i$. The horizontal axis represents $\Re(z)$, and the vertical axis represents $\Im(z)$.

We now turn to the prediction and validation of RW patterns. In this context, Theorems 4 and 5 provide an analytical approximation of the RW solution under the regime of a large internal parameter

a_{2m+1} , expressed as

$$u_n(\kappa, \mathcal{Y}) \approx u_{n_0}(\kappa, \mathcal{Y}) + \sum_{k=1}^{n_p} u_1(\kappa - \hat{\kappa}_0^{(k)}, \mathcal{Y} - \hat{\mathcal{Y}}_0^{(k)}), \quad (3.53)$$

where n_0 is defined by Eq (3.5), and $u_{n_0}(\kappa, \mathcal{Y})$ represents a lower-order RW located at the center. The internal parameters $(\hat{a}_1, \hat{a}_3, \dots, \hat{a}_{2n_0-1})$ of this central wave are determined by Eq (3.41). The function $u_1(\kappa, \mathcal{Y})$ corresponds to the fundamental RW as given in Eq (3.9), with its position $(\hat{\kappa}_0^{(k)}, \hat{\mathcal{Y}}_0^{(k)})$ determined from Eqs (3.11) and (3.12) using each nonzero simple root $z_0^{(k)}$ of the YV polynomial $Q_n^{[m]}(z)$. The total number of such fundamental RWs, denoted by n_p , is given by Eq (3.6).

When the dominant internal parameter a_{2m+1} is selected according to the values listed in Table 2 for $2 \leq n \leq 6$ and $1 \leq m \leq n-1$, with all other internal parameters set to zero, the resulting predicted RW patterns in the (x, y) -plane, as described by Eq (3.53), are illustrated in Figure 9. As shown in Figure 9, the asymptotic prediction based on the roots of the YV hierarchy accurately reproduces the spatial arrangement of the higher-order RWs. For instance, when $n = 4$ and the parameter a_3 dominates, the solution decomposes into 10 fundamental RWs arranged in a triangular configuration. This agrees with the three nonzero simple roots of the corresponding $Q_4^{[1]}(z)$ polynomial. Similarly, for a large a_5 , 10 separated fundamental RWs appear in a pentagonal configuration, consistent with the root distribution of $Q_4^{[2]}(z)$. The agreement between the predicted positions and the numerically obtained peaks confirms the validity of the YV-polynomial-based prediction.

Table 2. Values of the large parameters a_{2m+1} of RWs in Figure 9.

n	a_3	a_5	a_7	a_9	a_{11}
2	15				
3	-10	100			
4	-25	50	-100		
5	-5	20	-80	200	
6	7	15	55	335	-7700

The associated linear transformation matrix $\mathbf{M}_\Lambda$ is given in Eq (3.52), and it satisfies $\mathbf{M}_\Lambda/(\sqrt{3}|\Lambda|)$, which is a rotation matrix. Consequently, the predicted RW structures in Figure 9 can be directly derived from the root configurations of the YV polynomials by applying a combination of rotation, dilation, and translation, without shear or stretching. This implies that, without performing lengthy determinant calculations, one can readily predict the approximate locations of the higher-order RW peaks simply by using the root positions of the YV polynomials and the corresponding lower-order RW solutions. This correspondence is visually evident when comparing Figure 9 with the root plots in Figure 8.

The patterns shown in the bottom row of Figure 9 (for $n = 6$), where the internal parameter values match those used in Figure 8, represent a fully developed realization of the linear transformation described in Eq (3.51), applied to the YV root geometry. These transformed root structures were previously displayed in Figure 8. A noteworthy feature of this transformation is the inclusion of a constant shift in the x -direction, given by $\psi = -\frac{4(n-1)}{3}$, which results in an overall horizontal displacement of the entire RW pattern. This effect becomes increasingly pronounced for larger values

of n , as evident in the figure. However, because the gCH–KP equation is invariant under spatial and temporal translations, this shift does not alter the actual structure or shape of the RW pattern.

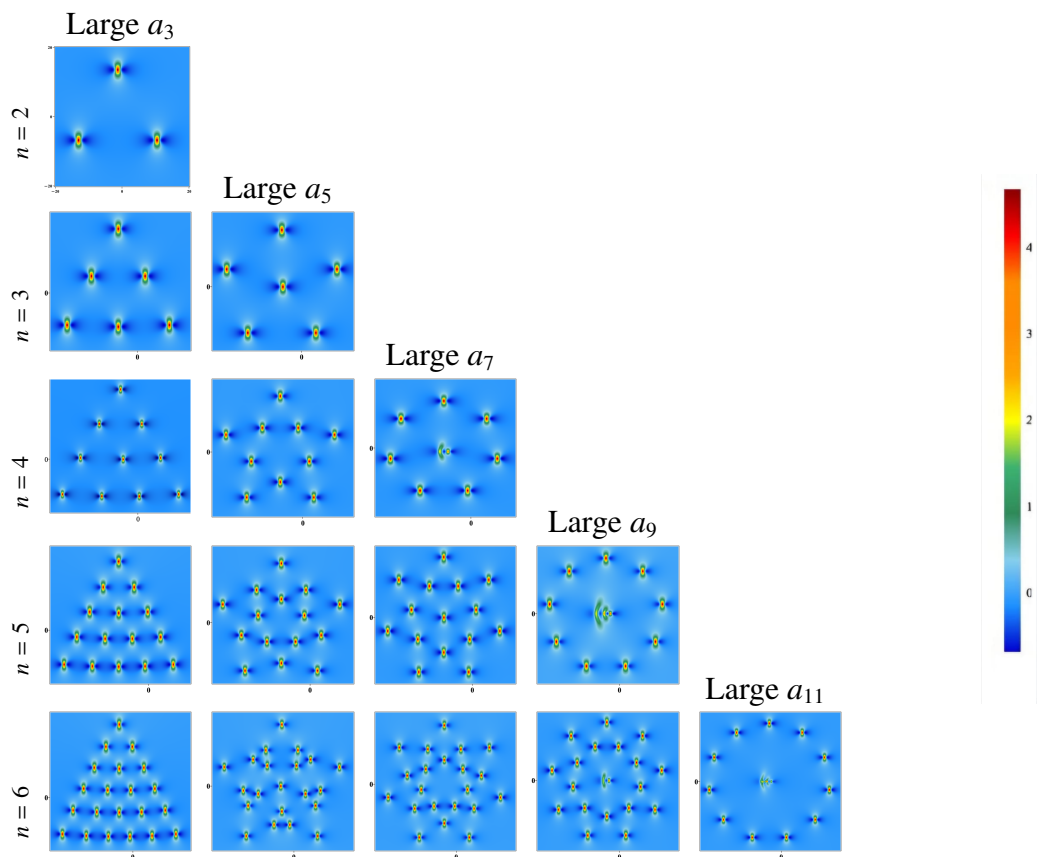


Figure 9. Predicted RW patterns $u_n(x, y, z, t)$ (3.53) for $2 \leq n \leq 6$ with the dominant parameter a_{2m+1} , $m = 1, \dots, n-1$. Here, $\delta = \frac{1}{2}$, $\beta = 15$, and $z = t = 0$; the values of a_{2m+1} are given in Table 2, with all other internal parameters set to zero. The horizontal axis represents x , and the vertical axis represents y .

We next present a quantitative assessment of the asymptotic predictions derived in Theorem 4. Our analysis addresses two key aspects. First, we study how the positioning error of RWs evolves as the odd-order parameter a_{2m+1} increases. Specifically, we explore the asymptotic regime governing the spatial and temporal locations of Peregrine waves at large distances from the origin. Our objective is to verify the theoretical error rate $\mathcal{O}(|a_{2m+1}|^{-1/(2m+1)})$. Second, we evaluate the deviation between the exact and asymptotically predicted solutions, with particular emphasis on the third-order case for large values of $|a_3|$.

First: We present the predicted RW patterns for $n = 2$ to $n = 6$ in Figure 9. For a specific case, we consider only two representative fourth-order RW configurations. The first is a triangular structure induced by a large a_3 with a fixed argument $\arg(a_3) = -\pi/4$; the second is a pentagonal structure arising from a large, positive real a_5 . In both cases, all remaining parameters are set to zero, isolating the effect of the dominant term. These two configurations are schematically illustrated in Panels (a) and (c) of Figure 10, respectively. Within each pattern, we isolate a single Peregrine wave, highlighted for clarity, and compare its actual peak position (x, y) with the predicted location $(\hat{x}_0, \hat{y}_0)$ derived from

Eqs (3.11) and (3.12) for $z = t = 0$ as $|a_3|$ or $|a_5|$ increases. The discrepancy is measured using the Euclidean distance

$$\text{Error of peak location} = \sqrt{(\hat{x}_0 - x)^2 + (\hat{y}_0 - y)^2}.$$

These errors are plotted as solid curves in Figure 10 (b) and (d), corresponding to the triangular and pentagonal cases. The plots were generated on a log–log scale using the `loglogplot` command in Maple. For reference, the asymptotic scaling laws $|a_3|^{-1/3}$ and $|a_5|^{-1/5}$ are shown as dashed lines. The numerical results confirm that the error decreases at the predicted rate of $O(|a_{2m+1}|^{-1/(2m+1)})$, thereby validating the theoretical estimates for the Peregrine wave location derived in Theorem 4.

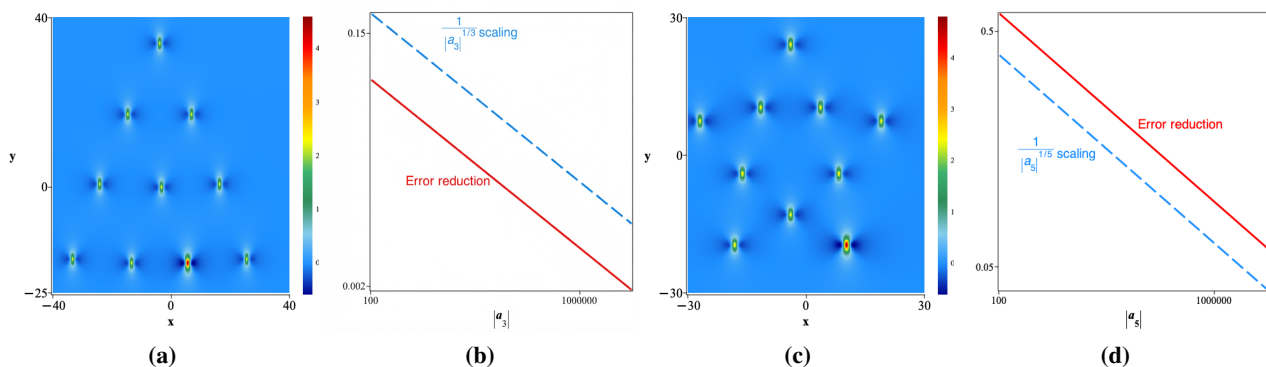


Figure 10. Reduction of prediction errors in Eqs (3.11) and (3.12) for Peregrine wave locations as $|a_3|$ or $|a_5|$ increases. **Panel a:** Triangular pattern of fourth-order RWs for a large $|a_3|$ with $\arg(a_3) = -\pi/4$. **Panel b:** Error versus $|a_3|$ for the Peregrine wave identified as the highlighted peak. **Panel c:** Pentagonal pattern of fourth-order RWs for a large $|a_5|$ with $\arg(a_5) = -\pi/4$. **Panel d:** Error versus $|a_5|$ for the Peregrine wave identified as the highlighted peak.

Second: To assess the accuracy of the predicted RW solutions constructed using YV polynomials, we compare them with the exact solutions obtained via the τ -functions for the gCH–KP equation. Although this analysis applies to RWs of any order, we focus on the third-order case where $|a_3|$ is large as a representative example.

Figure 11 displays the exact third-order RW patterns for several values of $|a_3|$, showing the spatial arrangement and intensity of the six peaks. To quantify the difference between the predicted and exact solutions, we compute the subtraction of the predicted solution from the exact solution, as shown in Figure 12. This difference field highlights two key aspects:

- (i) The background, where both solutions closely agree, resulting in near-zero values.
- (ii) The localized peak regions, where discrepancies may occur.

As $|a_3|$ increases, the error around the peaks becomes weaker and more localized, indicating that the predicted solution increasingly matches the exact one.

To further quantify this agreement, we measured the amplitudes of each peak in the exact solution, subtracted the corresponding amplitudes in the predicted solution, and calculated the average of these differences. This procedure was repeated for five increasing values of $|a_3|$, providing a representative error for each case. Finally, we plotted this error on a logarithmic scale for both the parameter and the error $\log(|a_3|)$ vs. $\log(\text{error})$ ($\log(\varepsilon)$) and applied spline interpolation to visualize the trend. The

resulting plot (Panel f) shows that as $|a_3|$ increases, the average discrepancy decreases, confirming the convergence of the predicted solution toward the exact solution in the large- $|a_3|$ regime.

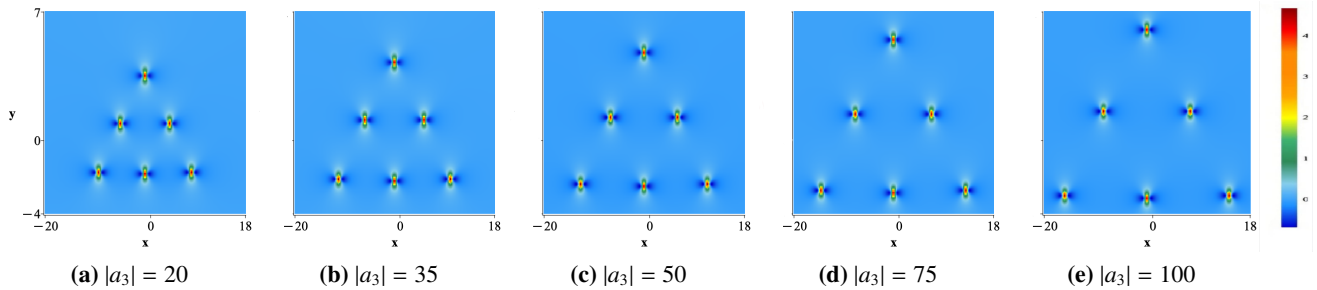


Figure 11. The third-order RW solution (exact solution) for the parameters $\delta = \frac{1}{2}, \beta = 15, a_0 = a_5 = z = t = 0$.

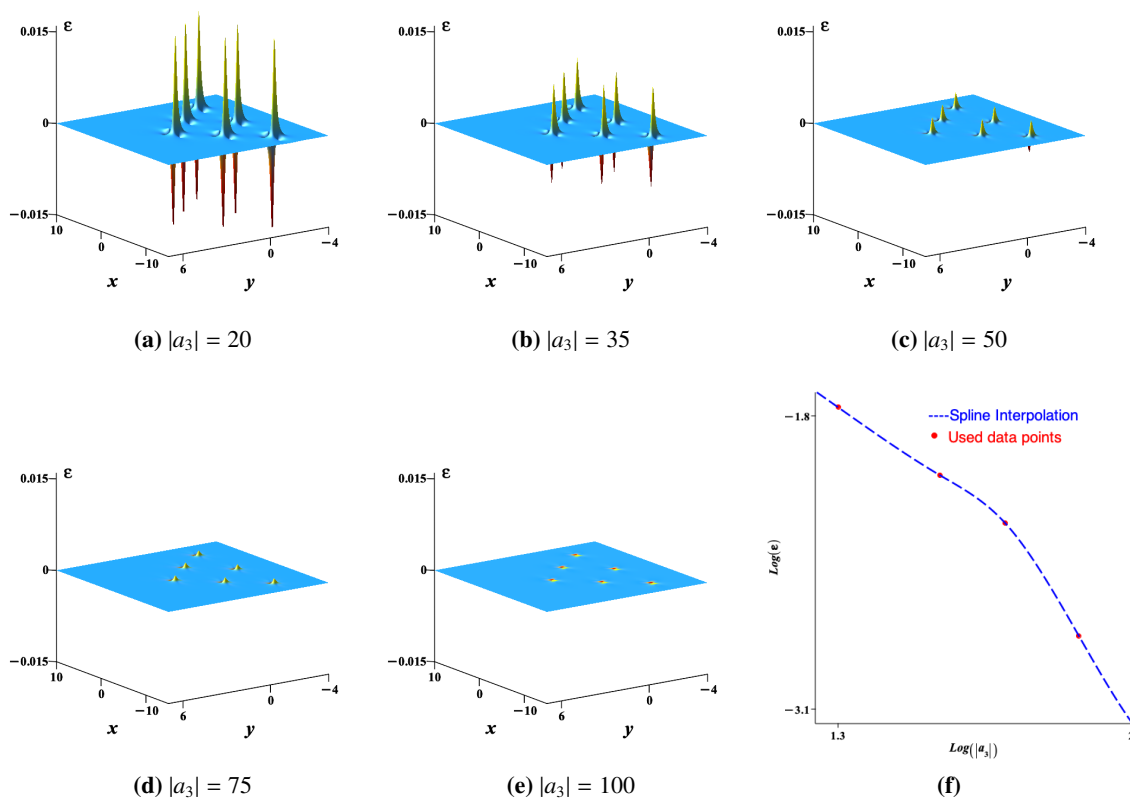


Figure 12. Error ε between the exact and predicted third-order RW solutions for $\delta = \frac{1}{2}, \beta = 15$, and $a_0 = a_5 = 0$. Panels (a)–(e) show the 3D difference fields, highlighting their agreement in the background and localized discrepancies near the six peaks. Panel (f) shows $\log(a_3)$ versus $\log(\varepsilon)$ with spline interpolation; the decreasing trend indicates convergence of the predicted solution to the exact one as $|a_3|$ increases.

4. Discussion

In summary, we analytically investigated RW patterns in the $(3 + 1)$ -dimensional gCH–KP Eq (1.3), demonstrating that large internal parameters lead to geometric arrangements, such as triangles, pentagons, and heptagons, composed of Peregrine waves, often featuring a lower-order RW at the center. These patterns correspond to the roots of the YV polynomial hierarchy, with their orientation determined by the phase of the large parameter. Our findings generalize to other integrable systems, revealing universal RW structures connected to rational solutions of the second Painlevé equation. This link clarifies previously observed numerical patterns and highlights the key role of Painlevé equations in nonlinear wave dynamics.

To conclude, we present the following observations.

- Theorem 2 provides a unified determinant formulation for higher-order rational solutions of the equation via the KP-hierarchy reduction method. Through a single τ -function representation, it generates RWs of arbitrary order, with the internal parameters controlling the spatial structure of the solutions.

A key feature of these solutions is their coordinate-dependent behavior. In planes such as (x, y) , (x, z) , (t, y) , and (t, z) , the solutions appear as fully localized RWs, whereas in the (x, t) and (y, z) planes, they reduce to line solitons or parallel soliton structures. Hence, the theorem naturally unifies RWs and soliton lines within a single analytical framework.

Moreover, by specifying the free parameters a_j appropriately, the general τ -function reduces to even-power rational polynomials, yielding symmetric RW solutions aligned along a single spatial direction. These reduced solutions coincide with those previously obtained for the $(2 + 1)$ - and $(3 + 1)$ -dimensional CH–KP equations using ad hoc test-function and ansatz-based methods. In this sense, Theorem 2 not only generalizes but also unifies earlier constructions within a single integrable KP-reduction framework, which then serves as the exact foundation for the subsequent algebraic–geometric analysis based on YV polynomials.

- Theorem 4 indicates that when the internal parameter a_{2m+1} is sufficiently large, the n th-order RW described by Eq (2.18) breaks apart into n_p well-separated fundamental RWs of Peregrine type. The spatial arrangement of these waves replicates the root structure of the YV polynomial $Q_n^{[m]}(z)$, exhibiting the same rotational symmetry of angle $2\pi/(2m + 1)$. While the overall shape of the RW pattern matches the polynomial's root distribution, a global scaling and rotation distinguish the two. These transformations arise from the multiplicative factor in Eqs (3.11) and (3.12): The rotation angle corresponds to the argument of $-a_{2m+1}$ divided by $2m + 1$, and the scaling factor is $\left[(2m + 1)^{2-2m}|a_{2m+1}|\right]^{1/(2m+1)}$.
- Theorem 4 characterizes the multiplicity of the zero root in the generalized YV polynomials $Q_n^{[m]}(z)$, and shows that their root configurations possess rotational symmetry with an angle $2\pi/(2m + 1)$ in the complex z -plane. In the classical case where $m = 1$, corresponding to the original YV polynomials $Q_n(z)$, this result implies that zero is either not a root or a simple root, i.e., $n_0 = 0$ or 1 , consistent with earlier studies [52, 53]. Owing to this rotational symmetry, any of the $(2m + 1)$ -th roots of $-(2m + 1)^{2-2m}a_{2m+1}$ can be used on the right-hand side of Eqs (3.11) and (3.12), as the choice corresponds to an equivalent orientation of the symmetric root structure.

- As an illustrative application of Theorems 4 and 5, we revisit the numerical observations reported in Ref. [31]. Specifically, under the bilinear RW solution (2.18), an n th-order RW exhibits a prominent ring structure when the parameter a_{2n-1} becomes large. In this scenario, $m = n - 1$, leading to $n_0 = n - 2$ by the second equation in (3.5). According to our theoretical framework, the central region of this RW corresponds to an $(n - 2)$ th-order RW, encircled by $n_p = 2n - 1$ fundamental (Peregrine) RWs. These outer waves are symmetrically arranged along a ring with $2\pi/(2n - 1)$ -rotational symmetry, precisely matching the structure observed in [31].
- Another noteworthy implication of our theorems is their connection to a heuristic result discussed in Ref. [54], which states that an n th order RW is composed of $n(n + 1)/2$ elementary RWs. While this count includes overlapping or merging structures and is thus qualitative, our theory offers a framework to interpret it. When a_{2m+1} is large, the theorems indicate that the solution decomposes into two parts: a lower-order RW of order n_0 at the center, and $n_p = n(n + 1)/2 - n_0(n_0 + 1)/2$ Peregrine waves away from the center. By induction, the central RW of order n_0 contributes exactly $n_0(n_0 + 1)/2$ elementary RWs. Summing both contributions, the total number of elementary RWs in the structure is $n(n + 1)/2$, aligning with the claim in Ref. [54].
- The RWs are composed of multiple Peregrine waves arranged in polygonal formations that correspond to the dominant large internal parameter: Triangular shapes appear for large values of a_3 , pentagonal for large values of a_5 , heptagonal for large values of a_7 , nonagonal for large values of a_9 , hendecagonal (11-sided) for large values of a_{11} , and tridecagonal (13-sided) for large values of a_{13} . Such polygonal configurations along the diagonal in Figure 9 are often referred to in the literature as single-shell ring structures.

Beyond these peripheral Peregrine waves, some RWs also feature a lower-order RW at their core. For example, the sixth-order RWs shown in the bottom row of Figure 9 display varied central structures depending on the large parameter: Panels one and four (large a_3 and a_9 , respectively) exhibit a fundamental Peregrine wave at the center of the patterns; the second panel (large a_5) hosts a second-order RW; the fifth panel (large a_{11}) features a third-order RW. For the parameter selections used in the figure, these central lower-order RWs correspond to super-RWs, characterized by the highest peak amplitudes within their respective orders.

- The analysis of the geometric structures of higher-order RWs through the roots of YV polynomials has been investigated previously for several integrable nonlinear wave equations, including the Boussinesq equation [55]

$$u_{tt} + u_{xx} - (u^2)_{xx} - \frac{1}{3}u_{xxxx} = 0, \quad (4.1)$$

the Benjamin-Ono (BO) equation [56]

$$u_{tt} + (u^2)_{xx} - u_{xxxx} = 0, \quad (4.2)$$

and the NLS equation [51],

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u = 0, \quad (4.3)$$

and other KP-type systems. In these studies, higher-order RWs can be interpreted as nonlinear superpositions of fundamental RWs, and their asymptotic peak distributions are governed by the root structures of the associated YV polynomials. In particular, when an internal parameter of a higher-order solution becomes sufficiently large, the solution decomposes into a set of fundamental RWs whose positions are determined by the nonzero roots of the corresponding YV polynomial.

Although this algebraic mechanism is common among these integrable models, the explicit form of the fundamental RW and the transformation from the YV polynomial roots to the physical coordinates depend on the underlying equation. Consequently, different equations lead to different expressions for the fundamental solution and different formulas for the positions of the RW peaks, while the governing role of the YV polynomial roots remains unchanged.

In the present work, we establish this YV-polynomial-controlled RW structure for Eq (1.3). The obtained rational solutions have the same universal decomposition property as those found in previous integrable systems; however, the corresponding fundamental RW and the mapping between the YV roots and the physical peak locations are specific to the gCH–KP equation. To emphasize the similarities and differences between the present results and previous studies, we summarize in Table 3 the relationship among the NLS, Boussinesq, BO, and the present gCH–KP equations, including their fundamental RWs and the corresponding peak-position relations.

Table 3. Comparison of the YV polynomial-based RW structures for the NLS, Boussinesq, BO, and the present gCH–KP equations.

Eqs	Fundamental RWs	Positions of fundamental RWs	Relation with existing models
Eq (1.3)	$u(\kappa, \mathcal{Y}) = \frac{12\delta}{\beta} \ln(\tau_1)_{\kappa\kappa},$ $\tau_1 = -\left(\frac{1}{3\sqrt{\delta}}\kappa^2 + 4\mathcal{Y}^2 + \frac{1}{12}\right)$	$(\hat{\kappa}_0, \hat{\mathcal{Y}}_0) =$ $(\delta^{\frac{1}{4}}\Re[-i\sqrt{3}z_0\Lambda] - \frac{4}{3}(n-1), \frac{\sqrt{3}}{6}\Im[-i\sqrt{3}z_0\Lambda])$	$\kappa = x + t, \mathcal{Y} = y + z$
Eq (4.1)	$u(x, t) = 2\ln(\tau_B)_{xx},$ $\tau_B = x^2 + t^2 + 1$	$(\hat{x}_0, \hat{t}_0) =$ $(\Re[-2i\sqrt{3}z_0\Lambda] - \frac{4}{3}(n-1), \Im[-2i\sqrt{3}z_0\Lambda])$	$\tau_1(\kappa, \mathcal{Y}) =$ $-\frac{1}{12}\tau_B(2\delta^{-\frac{1}{4}}\kappa, 4\sqrt{3}\mathcal{Y})$
Eq (4.2)	$u(x, t) = 1 - 6\ln(\tau_{BO})_{xx},$ $\tau_{BO} = x^2 + 2t^2 + \frac{3}{2}$	$(\hat{x}_0, \hat{t}_0) =$ $(\Re[3z_0\Lambda] + \frac{2\sqrt{6}}{3}(n-1), -\sqrt{2}\Im[3z_0\Lambda])$	$\tau_1(\kappa, \mathcal{Y}) =$ $-\frac{1}{18}\tau_{BO}(\sqrt{6}\delta^{-\frac{1}{4}}\kappa, 6\mathcal{Y})$
Eq (4.3)	$u(x, t) = \left(1 - \frac{g}{f}\right)e^{it},$ $g = 4(1 + 2it), f = 1 + 4x^2 + 4t^2$	$(\hat{x}_0, \hat{t}_0) = (\Re[z_0\Lambda], \Im[z_0\Lambda])$	$\tau_1(\kappa, \mathcal{Y}) =$ $-\frac{1}{12}f(\delta^{-\frac{1}{4}}\kappa, 2\sqrt{3}\mathcal{Y})$

The fundamental RW solutions listed in Table 3 can be understood as different realizations of a unified KP-hierarchy's rational structure expressed through their corresponding τ -functions. The relationships between these solutions are established via dimensional reductions, variable transformations, and rescalings of the KP variables, which preserve the quadratic form of the underlying rational τ -functions.

For the Boussinesq equation, the fundamental solution of the present gCH–KP system reduces to the standard Boussinesq RW structure by eliminating the transverse KP variable and introducing the Boussinesq variables

$$x = 2\delta^{-1/4}\kappa, \quad t = 4\sqrt{3}\mathcal{Y},$$

which corresponds to the reduction $\mathcal{Y} = \frac{t}{4\sqrt{3}}$ and $\kappa = \frac{\delta^{1/4}}{2}x$. Together with the scaling relation between the τ -functions given in the table, the quadratic form of the gCH–KP τ -function is

mapped exactly into the standard Boussinesq rational τ -function. Consequently, the fundamental RWs of both systems coincide up to simple variable rescaling and normalization.

For the BO equation, the correspondence is obtained through a KP-hierarchy reduction in which the dependence on the transverse KP variable is removed by imposing the appropriate reduction constraint on the phase variables. Introducing the BO variables through

$$x = \sqrt{6}\delta^{-1/4}\kappa, \quad t = 6\mathcal{Y},$$

together with the associated reduction in the KP phase functions, the gCH–KP τ -function admits the scaling relation in the table. Under this transformation, the quadratic form of the gCH–KP τ -function is mapped into the standard BO rational τ -function, showing that the two fundamental RW structures differ only by rescaling and normalization, while sharing the same KP-hierarchy origin.

Finally, for the NLS equation, the fundamental RW is constructed from a complex function g and a real function f . Although this solution can be obtained through the KP reduction method [26, 51], it can also be derived by the Darboux transformation [21]. In contrast to these results, in the present work, we establish the relation between the τ -function τ_1 and the YV polynomial structure, as summarized in Table 3.

Moreover, the position formulas in Table 3 demonstrate both the similarities and differences among these integrable systems. A common feature is that the locations of fundamental RWs are determined by the roots of the corresponding YV polynomials through the KP reduction framework. However, the explicit position formulas are different because of the different reductions and coordinate transformations connecting the KP variables with the physical variables. For the Boussinesq, BO, and gCH–KP equations, the KP variables are connected with the physical coordinates through a nontrivial transformation, resulting in a shift in the x/κ -direction of the fundamental RW positions. Therefore, the roots of the corresponding YV polynomials determine the locations after applying this coordinate mapping. In contrast, for the NLS equation, the reduction directly gives the physical variables (x, t) , and no additional spatial shift appears. Consequently, the positions of the fundamental RWs are determined directly by the YV roots without the extra coordinate transformation.

5. Conclusions

In this work, we explored higher-order RW solutions of the $(3 + 1)$ -dimensional gCH–KP equation. Using the KP-reduction method together with Hirota's bilinear formalism and the properties of Schur polynomials, we derived fully algebraic, rational solutions that reveal rich and intricate dynamical behaviors, all governed by the internal free parameters a_{2j+1} .

We observed that the solution u_n generates two kinds of waveforms in different coordinate planes: RWs and soliton lines. In the case of RWs, although the parameters a_{2j+1} do not affect the amplitudes of the main peaks, they play a key role in shaping where and when the peaks appear. Interestingly, when a single parameter dominates, the corresponding n th order RW splits into fundamental Peregrine waves arranged in elegant polygonal patterns, such as triangular, pentagonal, or higher $(2m + 1)$ -gons, while all peaks retain the same amplitude as the fundamental RW.

A particularly striking insight is the link between these geometric patterns and the roots of YV polynomials. This connection provides a simple, efficient way to predict the positions and arrangement of peaks in high-order RWs, without needing to calculate the full, computationally heavy τ -functions.

For all numerical computations in Sections 3 and 4, we fixed $\delta = \frac{1}{2}$, so that the normalization (3.8) gives $\alpha = -6\sqrt{2} - 1$. We also take $\beta = 15$, $\lambda = -2$, $\mu = 1$, $\sigma = \varsigma = 1$. Thus, the parameter set used in the numerical validation is identical to that adopted in the examples of Section 2.

Beyond their mathematical significance, these results may have practical implications for extreme-wave prediction and control. In ocean-wave modeling, the YV-root-based formulas could provide a rapid way to estimate the locations and spatial arrangement of extreme-wave peaks. In nonlinear optics, the same framework may help characterize and design high-intensity pulse structures. More generally, the underlying algebraic–geometric mechanism may provide a useful mathematical framework for studying localized extreme events in other nonlinear systems.

Future work may focus on extending the present analysis to RW patterns generated via τ -function representations whose entries are governed by Adler–Moser and Okamoto polynomial hierarchies [57–59]. Since these hierarchies arise from Painlevé-type structures and admit additional free parameters, they may produce more general root configurations than the YV case. In particular, it would be interesting to investigate how these root structures determine the asymptotic spatial–temporal locations and geometric arrangements of fundamental RWs under large-parameter limits. Another direction is to examine whether the associated mapping remains asymptotically linear at leading order or develops higher-order nonlinear corrections.

Author contributions

Conceptualization, methodology, investigation, formal analysis, visualization, validation, and writing–review and editing, M. M., K. H., E. H. and S. S.; software, data curation, and writing–original draft preparation, M. M., K. H. and S. S.; resources, supervision, and project administration, K. H. and S. S.; funding acquisition, S. S. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

We have used generative AI tools solely to improve the English language of the manuscript, including grammar, wording, and typographical errors. No AI tools were used to generate the scientific content, analysis, results, or conclusions.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix

Proof of Theorem 2. We follow the KP-reduction approach developed in [26, 60, 61]. We first recall two standard facts concerning Gram-determinant solutions of the KP hierarchy. If $\mathcal{M}_{i,j}^{(N)}$, $\varphi_i^{(N)}$, and $\psi_j^{(N)}$ are functions of x_1 , x_2 , and x_3 that satisfy

$$\begin{aligned}\partial_{x_1} \mathcal{M}_{i,j}^{(N)} &= \varphi_i^{(N)} \varpi_j^{(N)}, \\ \partial_{x_2} \mathcal{M}_{i,j}^{(N)} &= \varphi_i^{(N+1)} \varpi_j^{(N)} + \varphi_i^{(N)} \varpi_j^{(N-1)}, \\ \partial_{x_3} \mathcal{M}_{i,j}^{(N)} &= \varphi_i^{(N+2)} \varpi_j^{(N)} + \varphi_i^{(N+1)} \varpi_j^{(N-1)} + \varphi_i^{(N)} \varpi_j^{(N-2)}, \\ \mathcal{M}_{i,j}^{(N+1)} &= \mathcal{M}_{i,j}^{(N)} + \varphi_i^{(N)} \varpi_j^{(N+1)}, \\ \partial_{x_k} \varphi_i^{(N)} &= \varphi_i^{(N+k)}, \quad \partial_{x_k} \varpi_j^{(N)} = -\varpi_j^{(N-k)}, \quad (k = 1, 2, 3),\end{aligned}\tag{A.1}$$

then the determinant

$$\tau_N = \det_{1 \leq i, j \leq n} (\mathcal{M}_{i,j}^{(N)}),\tag{A.2}$$

formed from any admissible sequence of rows and columns satisfies the KP-hierarchy's bilinear equation

$$(D_{x_1}^4 - 4D_{x_1} D_{x_3} + 3D_{x_2}^2) \tau_N \cdot \tau_N = 0.\tag{A.3}$$

In particular, define

$$\mathcal{M}_{i,j}^{(N)} = \mathcal{A}_i \mathcal{B}_j \mathcal{M}^{(N)}, \quad \mathcal{M}^{(N)} = \frac{1}{p+q} \left(-\frac{p}{q} \right)^N e^{\theta_1 + \theta_2},\tag{A.4}$$

with

$$\theta_1 = px_1 + p^2x_2 + p^3x_3, \quad \theta_2 = qx_1 - q^2x_2 + q^3x_3,\tag{A.5}$$

and

$$\mathcal{A}_i = \sum_{k=0}^i \frac{a_k}{(i-k)!} (f_1(p) \partial_p)^{i-k}, \quad \mathcal{B}_j = \sum_{l=0}^j \frac{b_l}{(j-l)!} (f_2(q) \partial_q)^{j-l}.\tag{A.6}$$

Then

$$\varphi_i^{(N)} = \mathcal{A}_i p^N e^{\theta_1}, \quad \psi_j^{(N)} = \mathcal{B}_j (-q)^{-N} e^{\theta_2},\tag{A.7}$$

satisfy (A.1). Consequently, (A.2) provides a solution of (A.3).

We now impose the dimension-reduction condition

$$(\partial_{x_3} - 3\partial_{x_1})\tau_N = C\tau_N. \quad (\text{A.8})$$

For the kernel (A.4),

$$(\partial_{x_3} - 3\partial_{x_1})\mathcal{M}^{(N)} = (p^3 - 3p + q^3 - 3q)\mathcal{M}^{(N)}.$$

Introduce $\mathcal{K}_1$ and $\mathcal{K}_2$ through

$$p^3 - 3p = \mathcal{K}_1 + \mathcal{K}_1^{-1}, \quad q^3 - 3q = \mathcal{K}_2 + \mathcal{K}_2^{-1}, \quad (\text{A.9})$$

and choose

$$f_1(p) = \frac{\mathcal{K}_1}{\mathcal{K}_1'}, \quad f_2(q) = \frac{\mathcal{K}_2}{\mathcal{K}_2'}.$$

Therefore, $f_1(p)\partial_p = \mathcal{K}_1\partial_{\mathcal{K}_1}$ and $f_2(q)\partial_q = \mathcal{K}_2\partial_{\mathcal{K}_2}$. The reduction point is determined by $\mathcal{K}_1(p_0) = \mathcal{K}_2(q_0) = 1$. Since $\mathcal{K} + \mathcal{K}^{-1} = 2$ at $\mathcal{K} = 1$, together with the regularity condition $\mathcal{F}'(p_0) = 0$, we obtain

$$p_0^3 - 3p_0 = 2, \quad 3p_0^2 - 3 = 0,$$

and hence $p_0 = q_0 = -1$.

At $p = q = -1$, the action of the reduction operator on the matrix elements becomes

$$(\partial_{x_3} - 3\partial_{x_1})\mathcal{M}_{i,j}^{(N)} = 2 \sum_{k=0}^i \sum_{k \text{ even}} \frac{1}{k!} \mathcal{M}_{i-k,j}^{(N)} + 2 \sum_{l=0}^j \sum_{l \text{ even}} \frac{1}{l!} \mathcal{M}_{i,j-l}^{(N)}. \quad (\text{A.10})$$

We now choose the odd indices

$$(i_1, \dots, i_n) = (1, 3, \dots, 2n-1), \quad (j_1, \dots, j_n) = (1, 3, \dots, 2n-1).$$

When the derivative of the determinant is evaluated by cofactors, every term in the first sum of (A.10) with $k > 0$ produces two identical rows and therefore vanishes. Similarly, all terms with $l > 0$ in the second sum produce two identical columns. Hence

$$(\partial_{x_3} - 3\partial_{x_1})\tau_N = 4C\tau_N, \quad (\text{A.11})$$

and therefore (A.8) holds with $C \rightarrow 4C$. Consequently, putting $N = 0$ and $x_3 = 0$ in (A.3), we obtain

$$(D_{x_1}^4 - 12D_{x_1}^2 + 3D_{x_2}^2)\tau \cdot \tau = 0. \quad (\text{A.12})$$

It remains to determine the explicit form of the reduced determinant. Using the generator [60]

$$\mathcal{G} = \exp(\gamma_1 \partial_{\ln \mathcal{K}_1} + \gamma_2 \partial_{\ln \mathcal{K}_2})$$

and the branch

$$p(\mathcal{K}_1) = e^{2\pi i/3} \mathcal{K}_1^{1/3} + e^{-2\pi i/3} \mathcal{K}_1^{-1/3},$$

with the analogous expression for $q(\mathcal{K}_2)$, the expansion of $\mathcal{G}\mathcal{M}^{(0)}/\mathcal{M}^{(0)}$ at $\mathcal{K}_1 = \mathcal{K}_2 = 1$ gives

$$\frac{\mathcal{G}\mathcal{M}^{(0)}}{\mathcal{M}^{(0)}} = \sum_{\nu=0}^{\infty} \left(-\frac{\gamma_1\gamma_2}{12}\right)^{\nu} \times \exp \left[\sum_{k \geq 1} (X_k^+ + \nu h_k) \gamma_1^k + \sum_{k \geq 1} (X_k^- + \nu h_k) \gamma_2^k \right]. \quad (\text{A.13})$$

Here, $\mathcal{X}_k^\pm$ are specified as

$$\mathcal{X}_k^+ = \frac{\exp(2i\pi/3) + (-1)^r \exp(-2i\pi/3)}{3^r \cdot r!} [x_1 + (-2)^r x_2] + a_r, \quad (\text{A.14})$$

$$\mathcal{X}_k^- = \frac{\exp(2i\pi/3) + (-1)^r \exp(-2i\pi/3)}{3^r \cdot r!} [x_1 - (-2)^r x_2] + (-1)^r a_r^*, \quad (\text{A.15})$$

while the common coefficients h_k are determined by

$$\sum_{k=1}^{\infty} h_k \lambda^k = \ln \left[\frac{2i\sqrt{3}}{\lambda} \tanh\left(\frac{\lambda}{6}\right) \tanh\left(\frac{\lambda}{6} + \frac{2\pi i}{3}\right) \right].$$

Comparing the coefficients in (A.13) and using

$$\exp\left(\sum_{k=1}^{\infty} X_k \lambda^k\right) = \sum_{m=0}^{\infty} \mathcal{S}_m(\mathbf{X}) \lambda^m,$$

we obtain

$$\phi_{i,j} = \sum_{v=0}^{\min(i,j)} \left(-\frac{1}{12}\right)^v \mathcal{S}_{i-v}(\mathbf{X}^+ + v\mathbf{h}) \mathcal{S}_{j-v}(\mathbf{X}^- + v\mathbf{h}).$$

Therefore, after taking the odd-index rows and columns,

$$\tau_n = \det_{1 \leq i,j \leq n} (\phi_{2i-1,2j-1}), \quad (\text{A.16})$$

satisfies (A.12).

Finally, under

$$x_1 = \frac{\sqrt{3}}{6 \sqrt{-\frac{\delta\sigma}{\alpha+\sigma}}} \varkappa, \quad x_2 = \frac{-\sqrt{3}i}{12 \sqrt{-\frac{\delta\sigma(\varsigma^2\mu+\lambda)}{(\alpha+\sigma)^2}}} \mathscr{Y}, \quad (\text{A.17})$$

together with $\varkappa = x + \sigma t$ and $\mathscr{Y} = y + \varsigma z$, Eq (A.12) becomes

$$\left(\delta\sigma D_{\varkappa}^4 + (\alpha + \sigma) D_{\varkappa}^2 + (\lambda + \mu\varsigma^2) D_{\mathscr{Y}}^2\right) \tau \cdot \tau = 0,$$

which is (2.3). The dependent-variable transformation

$$u_n = \frac{12\delta\sigma}{\beta} (\ln \tau_n)_{\varkappa\varkappa}, \quad (\text{A.18})$$

then yields the stated higher-order RW solutions of the gCH–KP Eq (1.3) along with (A.16).

For the reality condition, $f_1(-1)$ and $f_2(-1)$ are purely imaginary. Choosing $b_k = (-1)^k a_k^*$ gives

$$(\mathcal{M}_{i,j}^{(0)})^* = (-1)^{i+j} \mathcal{M}_{j,i}^{(0)},$$

so that the odd-index determinant is real. The variable transformations are real under

$$\delta\sigma(\alpha + \sigma) < 0, \quad \delta\sigma(\varsigma^2\mu + \lambda) < 0.$$

Therefore (2.18)–(2.24) give the required general higher-order RW solutions. $\square$

The corresponding τ -function for the third-order RW with the parameters $a_1 = a_3 = a_5 = 0$, and $\alpha = -6\sqrt{2} - 1, \beta = 15, \delta = \frac{1}{2}, \lambda = -2$, and $\mu = 1$ is as follows.

$$\begin{aligned} \tau_3 = & -\frac{\kappa^{12}}{318864600} - \frac{\sqrt[4]{2^3}}{39858075}\kappa^{11} - \frac{\sqrt{2}}{1275458400}(144\mathscr{Y}^2 + 251)\kappa^{10} - \frac{\sqrt[4]{2}}{127545840}(192\mathscr{Y}^2 + 125)\kappa^9 \\ & - \left(\frac{1}{295245}\mathscr{Y}^4 + \frac{43}{4251528}\mathscr{Y}^2 + \frac{6907}{20407340}\right)\kappa^8 - \frac{\sqrt[4]{2^3}}{10203720}(18432\mathscr{Y}^4 + 22272\mathscr{Y}^2 + 4283)\kappa^7 \\ & - \frac{\sqrt{2}}{40814680}(110592\mathscr{Y}^6 + 403200\mathscr{Y}^4 + 267792\mathscr{Y}^2 + 30889)\kappa^6 - \frac{\sqrt[4]{2}}{4081466880}(884736\mathscr{Y}^6 \\ & + 1405440\mathscr{Y}^4 + 578352\mathscr{Y}^2 + 40415)\kappa^5 + \left(-\frac{8}{32805}\mathscr{Y}^8 - \frac{262}{295245}\mathscr{Y}^6 - \frac{2309}{2834352}\mathscr{Y}^4 \right. \\ & \left. - \frac{5033}{22674816}\mathscr{Y}^2 - \frac{40061}{4353564672}\right)\kappa^4 - \frac{\sqrt[4]{2^3}}{32651735040}(21233664\mathscr{Y}^8 + 36274176\mathscr{Y}^6 \\ & + 21346560\mathscr{Y}^4 + 4111200\mathscr{Y}^2 + 93125)\kappa^3 - \frac{\sqrt{2}}{13064010}(764411904\mathscr{Y}^{10} + 2202992640\mathscr{Y}^8 \\ & + 2356715520\mathscr{Y}^6 + 880473600\mathscr{Y}^4 + 131893200\mathscr{Y}^2 + 1439075)\kappa^2 - \frac{\sqrt[4]{2}}{2727920}(42467328\mathscr{Y}^{10} \\ & + 63037440\mathscr{Y}^8 + 49282560\mathscr{Y}^6 + 108500\mathscr{Y}^4 + 1400850\mathscr{Y}^2 + 8825)\kappa - \frac{64}{5475}\mathscr{Y}^{12} - \frac{248}{1625}\mathscr{Y}^{10} \\ & - \frac{587}{3960}\mathscr{Y}^8 - \frac{4393}{4720}\mathscr{Y}^6 - \frac{21671}{181328}\mathscr{Y}^4 - \frac{60983}{435672}\mathscr{Y}^2 - \frac{47569}{4179422012}. \end{aligned} \tag{A.19}$$



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