
Research article

Fractional nonlinear wave dynamics for neural signal processing in bioengineering: Hamiltonian bifurcations and exact soliton solutions

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Abstract: The propagation of nonlinear electrical impulses in neuronal media can be modeled through nonlinear fractional differential equations that incorporate the memory and hereditary characteristics of biological tissues. The motivation of this study is to provide a comprehensive analytical and dynamical investigation of the time-fractional nonlinear neuron model (TFNNM), with the aim of obtaining explicit wave structures and gaining deeper insight into the qualitative behavior of the model. From an analytical perspective, the study seeks to derive exact traveling-wave solutions that can characterize different nonlinear wave patterns, including dark, periodic, and kink-type structures. From a dynamical perspective, the study aims to explore the underlying phase-space behavior of the reduced system by examining its Hamiltonian structure, equilibrium points, and their stability and classification. In particular, the identification of center, saddle, and degenerate cusp configurations under appropriate parameter conditions on A, B , and C provides a systematic understanding of the possible dynamical regimes of the model. Thus, the study is motivated by the need to combine exact solution techniques with dynamical-systems analysis to obtain a more complete understanding of the nonlinear behavior and wave propagation characteristics of the TFNNM. Moreover, the phase portraits demonstrate that the signs of dynamical parameters strongly control the number and type of real equilibrium points and consequently determine the topology of the phase space. Eight representative parameter regimes are examined, showing configurations containing a central equilibrium surrounded by saddle structures, two center-type equilibria separated by a saddle, and regimes in which the nonzero equilibria become imaginary. The obtained dynamical behavior provides a qualitative interpretation of the traveling-wave structures and establishes a direct connection between the parameter space, equilibrium configurations, and nonlinear wave dynamics. These results enhance the understanding of fractional neuron wave propagation and provide a useful analytical framework for studying nonlinear signal transmission in biological media.

Keywords: time-fractional nonlinear neuron model; soliton dynamics; pulse propagation; generalized projective Riccati equation method; tanh-function method; phase-portraits analysis

Mathematics Subject Classification: 37K40, 39B62, 33B10, 26A48, 26A51

1. Introduction

The propagation of nerve impulses is one of the fundamental processes governing information transmission in biological nervous systems. Mathematical models based on nonlinear partial differential equations have played an important role in describing the generation, propagation, and interaction of these electrical signals. Among them, the time-fractional nonlinear neuron model (TFNNM) has attracted considerable attention because it incorporates the memory and hereditary characteristics of biological tissues through fractional-order derivatives. Unlike classical integer-order models, the TFNNM provides a more realistic description of neuron dynamics by accounting for the nonlocal behavior of nerve impulse propagation, thereby offering deeper insight into the complex mechanisms underlying signal transmission along axons. Consequently, the model has found widespread applications in neuroscience, mathematical biology, nonlinear science, and other branches of applied mathematics [1, 2].

During the past few decades, fractional calculus has become an indispensable mathematical framework for modeling complex physical and biological systems exhibiting memory, non-locality, and anomalous diffusion. Its applications extend to engineering, fluid mechanics, hydrodynamics, plasma physics, optical fibers, and biomedical sciences, where conventional integer-order models often fail to capture the underlying physical processes accurately. Owing to these advantages, nonlinear fractional evolution equations have become an active area of research, and numerous analytical and numerical techniques have been developed to investigate their dynamical behavior and exact solutions [3]. A variety of analytical techniques have been developed for constructing exact solutions of nonlinear evolution equations. These include the modified (G'/G) -expansion method [4, 5], extended Fan sub-equation method [6], the modified auxiliary equation method [7], the Sardar sub-equation method [8–10], the modified Sardar sub-equation method [11], the Φ^6 -model expansion method [12], the functional variable and first integral methods [13], the new extended direct algebraic and (G'/G^2) methods [14, 15], the auxiliary equation approach [16], the Laplace residual power series and new iterative methods [17], the modified extended tanh-function method [18], the Riccati equation mapping technique [19], he's semi-inverse variational method and the ansatz method [20], the extended generalized Kudryashov method and the Riccati equation mapping approach [21, 22], the conformable sub-equation method [23], and the Kudryashov method [24]. These approaches have considerably advanced the understanding of nonlinear wave propagation in diverse physical models, although each possesses its own strengths and limitations regarding computational complexity and the diversity of obtainable solutions.

1.1. Literature review

Several researchers have recently employed various analytical techniques to investigate the TFNNM and its nonlinear wave dynamics. For example, Alam et al. [25] applied the Kudryashov

method to analyze the parametric effects of the TFNNM and obtained kink-type soliton solutions. Akash et al. [26] employed the modified $\left(\frac{G'}{G}\right)$ -expansion and $\left(M - \frac{G'}{G} - E\right)$ methods to examine the solitary wave behavior and bifurcation characteristics of the model. These studies demonstrate that analytical approaches are effective for revealing the nonlinear dynamics of fractional neuron models. However, the diversity of exact solutions obtainable from different analytical methods remains an active area of investigation, and further exploration is required to identify additional wave structures with potential physical significance. Amin et al. [27], recent studies have explored nonlinear fractional soliton neuron models using analytical, machine-learning, and computational approaches to derive diverse exact wave solutions and analyze their propagation and stability. Various structures, including lump, breather, periodic, bright-dark, singular, anti-kink, and kink waves, have been reported through graphical and numerical investigations. Moreover, chaos diagnostics such as bifurcation diagrams, strange attractors, phase portraits, recurrence plots, fractal dimensions, return maps, and multistability have been used to characterize complex dynamics and external forcing effects. Rahaman et al. [28], recent research has investigated nonlinear fractional soliton neuron models in various scientific fields, focusing on nonlinear wave propagation and neural signal transmission. The ϕ^6 -model expansion scheme has been employed to derive diverse soliton solutions, supported by two- and three-dimensional graphical analyses. Furthermore, sensitivity analysis based on planar dynamical systems has been used to examine the complex behavior of the model and demonstrate the effectiveness of the proposed analytical approach. Alzaidi et al. [29], studied a time-space fractional soliton neuron model using the modified Sardar sub-equation (MSSE) and improved F-expansion methods to construct explicit analytical solutions. The obtained soliton solutions, together with bifurcation and sensitivity analyses, revealed the effects of fractional-order parameters on neuronal wave dynamics and signal propagation.

1.2. Motivation and contribution

The propagation of nonlinear electrical impulses in neuronal media is an important problem in neuroscience, mathematical biology, and nonlinear science. Conventional integer-order models may not adequately describe the memory and hereditary characteristics of biological tissues, whereas fractional-order formulations provide a framework for incorporating nonlocal temporal effects into the description of nerve impulse propagation. In this context, the time-fractional nonlinear neuron model (TFNNM) provides a useful mathematical framework for investigating nonlinear wave propagation in neuronal media and for examining the influence of fractional-order memory effects on neural signal dynamics. Although several analytical techniques have previously been applied to the TFNNM, the systematic application of the generalized projective Riccati equation method (GPREM) and the extended tanh-function method (ETHFM), together with a detailed dynamical-systems investigation, remains insufficiently explored. In particular, the influence of the fractional-order parameter on different nonlinear wave structures and the relationship between the analytical wave solutions and the underlying phase-space dynamics require further investigation. Motivated by these considerations, this study develops a combined analytical and dynamical framework for investigating the TFNNM. The GPREM and ETHFM are employed to derive explicit exact traveling-wave solutions, producing several classes of nonlinear structures, including dark, periodic, kink, singular, and rational wave solutions. The effect of the fractional-order parameter on the propagation characteristics of the obtained waves is further illustrated through three-dimensional surface plots,

two-dimensional profiles, contour plots, and density visualizations. In addition to the analytical treatment, the reduced ordinary differential equation is transformed into a planar dynamical system and investigated within a Hamiltonian framework. The equilibrium points are identified and classified according to the signs and magnitudes of the reduced parameters A , B , and C . The analysis reveals center, saddle, and degenerate cusp configurations, as well as parameter regimes in which the nonzero equilibrium points become imaginary. Thus, the present work connects the exact wave structures with the qualitative phase-space behavior of the model and provides a broader description of nonlinear signal propagation in fractional neuronal media.

1.3. Aim of the study

The main aim of this study is to develop a comprehensive analytical and dynamical investigation of the time-fractional nonlinear neuron model (TFNNM) in order to obtain exact nonlinear wave solutions and clarify the qualitative behavior of the associated dynamical system. In particular, the study aims to combine the generalized projective Riccati equation method (GPREM) and the extended tanh-function method (ETHFM) for constructing explicit traveling-wave solutions, while employing Hamiltonian and bifurcation analysis to examine the equilibrium structures and phase-space dynamics of the reduced model. The study further aims to establish the relationship between the fractional-order parameter, nonlinear wave propagation, equilibrium configurations, and the qualitative topology of the phase space. Through this combined analytical and dynamical approach, the work seeks to provide a useful mathematical framework for understanding nonlinear signal transmission and wave propagation in biological media exhibiting memory and hereditary effects.

1.4. Summarization

The remainder of this manuscript is organized as follows. Section 2 presents the mathematical formulation of the TFNNM together with the fractional complex wave transformation used to reduce the governing equation. Section 3.1 outlines the theoretical framework and implementation procedures of the GPREM and the ETHFM. In Section 4.1, these analytical methods are employed to derive several exact solitary wave solutions of the TFNNM. Section 5 provides a comprehensive graphical analysis of the obtained solutions through 3D surface plots and corresponding 2D line profiles, illustrating the influence of the fractional-order parameter on wave propagation. Section 6 drives detailed bifurcation analysis of TFNNM through its dynamical system and stability of equilibrium points. Section 7 includes details discussion of the obtained results. Finally, Section 8 concludes the paper by summarizing the principal findings, highlighting the significance of the obtained results, and discussing potential directions for future research.

2. Time-fractional nonlinear neuron model

In this work, Jumarie's modified Riemann–Liouville fractional derivative [18] is employed because it provides a convenient analytical framework for studying nonlinear fractional evolution equations. Its property that the fractional derivative of a constant is zero, together with its compatibility with traveling-wave transformations and modified chain and product rules, significantly simplifies the derivation of exact solutions [30]. Physically, the fractional-order parameter represents the memory and hereditary characteristics of neuron tissues, enabling the model to describe nonlocal

temporal effects in nerve impulse propagation [31, 32]. Although Jumarie's derivative has been the subject of theoretical debate and is less universally accepted than the Caputo derivative, it remains a popular and effective operator for obtaining analytical solutions of nonlinear fractional models through symbolic methods such as the GPREM and the ETHFM.

Fractional calculus, soliton theory and classical neuron model contribute to form TFNNM that ensures how neural systems work in the presence of time and space. TFNNM provides an accurate and efficient way to study the behavior of neurons and their signals. The TFNNM can be expressed, as [25]

$$D_t^{2\alpha} U = \frac{\partial}{\partial x} \left[(P^2 + QU + RU^2) \frac{\partial U}{\partial x} \right] - S \frac{\partial^4 U}{\partial x^4}, \quad 0 < \alpha \leq 1, \quad (2.1)$$

where α denotes fractional order derivative, t denotes time, x denotes position along the nerve axon, P defines sound velocity (SV) of the nerve membrane (NM), Q, R denote the nature of the nonlinearity and the phase transition of the elastic constants of the NM respectively, S denotes frequency dependence of the SV of the membrane and U denotes change in membrane density (MD) influenced by the action potential (AP). The parameters P , Q , and R cannot be adjusted freely as they depict the thermodynamic properties of the NM. To realize its whole physical importance, the soliton solutions are really helpful. We provide the definition of fractional derivative (FD) described by Jumarie [18, 33], as

$$D_t^\alpha U(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} (U(t) - U(0)) dt, & \alpha \leq 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} (U(t) - U(0)) dt, & 0 < \alpha \leq 1, \\ \left[U^{(n)}(x) \right]^{\alpha-n}, & n \leq \alpha \leq n+1, n \geq 1. \end{cases} \quad (2.2)$$

Equation (2.2) holds some following properties.

$$\begin{aligned} D_t^\alpha c &= 0, & \alpha \geq 0, c &= \text{constant}, \\ D_t^\alpha [cU(x)] &= c[D_t^\alpha U(x)], & \alpha \geq 0, c &= \text{constant}, \\ D_x^\alpha x^\gamma &= \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, & \gamma \geq \alpha \geq 0, \\ D_x^\alpha [U(x)\lambda(x)] &= \lambda(x)[D_x^\alpha U(x)] + U(x)[D_x^\alpha \lambda(x)], \\ D_x^\alpha [U(x(t))] &= U_x(x)x^\alpha(t). \end{aligned} \quad (2.3)$$

A complex traveling wave transformation for Eq (2.1) is considered, as

$$U = U(x, t) = U(\xi), \quad \xi = Kx \pm \frac{Vt^\alpha}{\Gamma(\alpha+1)}. \quad (2.4)$$

Substitution of Eq (2.4) into Eq (2.1) and after two times integration yields the following ODE.

$$(V^2 - K^2 P^2)U - \frac{1}{2}K^2 QU^2 - \frac{1}{3}K^2 RU^3 + K^4 S U'' = 0. \quad (2.5)$$

By putting the constant of integration C equal to zero. Using homogeneous balancing principle for U^3 and U'' yields $N = 1$. Table 1 summarizes previous studies of the TFNNM, highlighting the methods used and the principal wave structures reported. This comparison provides context for the analytical methods developed in the following sections.

Table 1. Comparison of previous studies of the time-fractional nonlinear neuron model.

Reference	Methodology used	Main outcomes/solutions
Alam et al. [25]	Kudryashov method.	Kink-type soliton solutions.
Akash et al. [26]	Modified $\frac{G'}{G}$ -expansion method, and (M- $\frac{G'}{G}$ -E) method.	Diverse types of solitary wave behavior.
Amin et al. [27]	Machine-learning, chaos diagnostics, bifurcation diagrams, strange attractors, phase portraits, recurrence plots, fractal dimensions, return maps, and multistability.	Various structures, including lump, breather, periodic, bright-dark, singular, anti-kink, and kink waves.
Rahaman et al. [28]	ϕ^6 model expansion scheme, sensitivity analysis.	Derive diverse soliton solutions, supported by two- and three-dimensional graphical analyses.
Alzaidi et al. [29]	Modified Sardar sub-equation (MSSE) and improved F-expansion methods to construct explicit analytical solutions.	Dark, bright, dark-bright, periodic, singular and other kinds of solitons.

3. Analytical methods for constructing exact traveling-wave solutions

3.1. Generalized projective Ricatti equation method

Step 1. According to GPREM [34], the general solution is assumed in the form

$$U(\xi) = a_0 + \sum_{r=1}^N f(\xi)^{r-1} [a_r f(\xi) + b_r g(\xi)], \quad (3.1)$$

where, a_0 , a_r and b_r are unknown parameters need to be determined with $r = (1, 2, 3, \dots, N)$. $f(\xi)$ and $g(\xi)$ satisfy following auxiliary equations.

$$\begin{aligned} f'(\xi) &= \delta f(\xi)g(\xi), \\ g'(\xi) &= \lambda + \delta g^2(\xi) - mf(\xi), \end{aligned} \quad (3.2)$$

and,

$$g^2(\xi) = -\delta \left(\lambda - 2mf(\xi) + \frac{m^2 + i}{\lambda} f^2(\xi) \right), \quad (3.3)$$

where $i = \pm 1$ while m, λ are nonzero coefficients with $\delta = \pm 1$.

If $m = \lambda = 0$, then Eq (3.3) preserves following solution.

$$U(\xi) = \sum_{r=0}^N A_r g^r(\xi), \quad (3.4)$$

where, $g(\xi)$ satisfies following auxiliary equation.

$$g'(\xi) = g^2(\xi). \quad (3.5)$$

Depending on the values of these coefficients $f(\xi)$ and $g(\xi)$ satisfies following solutions.

Case 1: For $\delta = -1, \iota = -1$, and $\lambda > 0$, $f(\xi)$ and $g(\xi)$ have following solutions.

$$f_1(\xi) = \frac{\lambda \operatorname{sech}(\sqrt{\lambda}\xi)}{m \operatorname{sech}(\sqrt{\lambda}\xi) + 1}, \quad g_1(\xi) = \frac{\sqrt{\lambda} \tanh(\sqrt{\lambda}\xi)}{m \operatorname{sech}(\sqrt{\lambda}\xi) + 1}. \quad (3.6)$$

Case 2: For $\delta = -1, \iota = 1$, and $\lambda > 0$, $f(\xi)$ and $g(\xi)$ satisfy following solutions.

$$f_2(\xi) = \frac{\lambda \operatorname{csch}(\sqrt{\lambda}\xi)}{m \operatorname{csch}(\sqrt{\lambda}\xi) + 1}, \quad g_2(\xi) = \frac{\sqrt{\lambda} \coth(\sqrt{\lambda}\xi)}{m \operatorname{csch}(\sqrt{\lambda}\xi) + 1}. \quad (3.7)$$

Case 3: For $\delta = 1, \iota = -1$, and $\lambda > 0$, $f(\xi)$ and $g(\xi)$ possess following solutions.

$$f_3(\xi) = \frac{\lambda \sec(\sqrt{\lambda}\xi)}{m \sec(\sqrt{\lambda}\xi) + 1}, \quad g_3(\xi) = \frac{\sqrt{\lambda} \tan(\sqrt{\lambda}\xi)}{m \sec(\sqrt{\lambda}\xi) + 1}. \quad (3.8)$$

Case 4: For $\delta = 1, \iota = 1$ with $\lambda > 0$, $f(\xi)$ and $g(\xi)$ contain following solutions.

$$f_4(\xi) = \frac{\lambda \csc(\sqrt{\lambda}\xi)}{m \csc(\sqrt{\lambda}\xi) + 1}, \quad g_4(\xi) = \frac{\sqrt{\lambda} \cot(\sqrt{\lambda}\xi)}{m \csc(\sqrt{\lambda}\xi) + 1}. \quad (3.9)$$

Step 2. Inserting Eqs (3.1)–(3.3) into Eq (2.5), and equating the coefficient of powers of $f^i(\xi)g^j(\xi)$ ($i = 0, 1, 2, \dots, j = 0, 1, \dots$) to zero, we get a system of algebraic equations. After solving this system simultaneously, we evaluate the values of unknown coefficients.

Step 3. By putting values of these known coefficients in Step 3 and values of $f(\xi), g(\xi)$ in cases (1)–(5) into the Eq (3.1), solutions of Eq (2.5) are determined.

3.2. Extended tanh-function method

Step 1. According to ETHFM [35], the general solution is defined in the form

$$f(\xi) = \sum_{r=1}^N a_r \Psi^r(\xi), \quad (3.10)$$

where a_r are nonzero constants need to be determined for $r = (0, 1, 2, \dots, N)$. $\Psi(\xi)$ satisfies Ricatti equation.

$$\Psi'(\xi) = d + \Psi^2(\xi). \quad (3.11)$$

Depending on different values of d , $\Psi(\xi)$ satisfies following solutions.

Case 5: For $d < 0$, $\Psi(\xi)$ has following solution.

$$\Psi(\xi) = -\sqrt{-d} \tanh(\sqrt{-d}\xi) \quad \text{or} \quad \Psi(\xi) = -\sqrt{-d} \coth(\sqrt{-d}\xi). \quad (3.12)$$

Case 6: For $d = 0$, $\Psi(\xi)$ possesses following solution.

$$\Psi(\xi) = -\frac{1}{\xi}. \quad (3.13)$$

Case 7: For $d > 0$, $\Psi(\xi)$ satisfies following solution.

$$\Psi(\xi) = \sqrt{d}\tan(\sqrt{d}\xi) \quad \text{or} \quad \Psi(\xi) = -\sqrt{d}\cot(\sqrt{d}\xi). \quad (3.14)$$

Step 2. Inserting Eq (3.10) along with Eq (3.11) into Eq (2.5), we get a polynomial in $\Psi(\xi)$. Equating the coefficient of different powers of $\Psi(\xi)$ to zero, we get a system of algebraic equations that can be solved simultaneously, and values of required unknown constants are determined.

Step 3. By putting values of these unknown constants in **Step 2** and values of $\Psi(\xi)$ in Eqs (3.12)–(3.14) into Eq (3.10), solutions of Eq (2.5) are determined.

4. Application of analytical methods

4.1. Application of generalized projective Riccati equation method

As $N = 1$, so the solution can be assumed in the following form using Eq (3.1).

$$U(\xi) = a_0 + a_1 f(\xi) + b_1 g(\xi), \quad (4.1)$$

where a_0, a_1, b_1 are unknown coefficients need to be determined. Substituting Eq (4.1) in Eq (2.5), the following system of algebraic equations is obtained by setting coefficients of higher powers of $f^i(\xi)g^j(\xi)$ to zero.

$$\frac{-2K^4 S a_1 \delta^3 m^2}{\lambda} - \frac{2K^4 S a_1 \delta^3 \iota}{\lambda} + \frac{K^2 R a_1 b_1^2 \delta m^2}{\lambda} + \frac{K^2 R a_1 b_1^2 \delta \iota}{\lambda} - \frac{K^2 R a_1^3}{3} = 0, \quad (4.2)$$

$$\frac{K^2 R a_0 b_1^2 \delta m^2}{\lambda} + \frac{K^2 R a_0 b_1^2 \delta \iota}{\lambda} - \frac{K^2 Q a_1^2}{2} + \frac{K^2 Q b_1^2 \delta m^2}{2\lambda} + \frac{K^2 Q b_1^2 \delta \iota}{2\lambda} - 2K^2 R a_1 b_1^2 \delta m$$

$$+ 4K^4 S a_1 \delta^3 m - K^4 S a_1 \delta m - K^2 R a_0 a_1^2 = 0, \quad (4.3)$$

$$- 2K^4 S \delta^3 \lambda a_1 + K^4 S \delta \lambda a_1 + K^2 R \delta \lambda a_1 b_1^2 - 2K^2 R \delta m a_0 b_1^2 - K^2 Q \delta m b_1^2 - K^2 R a_0^2 a_1$$

$$- K^2 P^2 a_1 - K^2 Q a_0 a_1 + V^2 a_1 = 0, \quad (4.4)$$

$$- b_1 K^2 P^2 + b_1 V^2 + \frac{1}{3} K^2 R b_1^3 \delta \lambda - K^2 Q a_0 b_1 - K^2 R a_0^2 b_1 = 0, \quad (4.5)$$

$$2K^4 S b_1 \delta^3 m - K^4 S b_1 m \delta - \frac{2}{3} K^2 R b_1^3 \delta m - 2K^2 R a_0 a_1 b_1 - K^2 Q a_1 b_1 = 0, \quad (4.6)$$

$$\frac{K^2 R b_1^3 \delta \iota}{3\lambda} - \frac{2K^4 S b_1 \delta^3 \iota}{\lambda} + \frac{K^2 R b_1^3 \delta m^2}{3\lambda} - \frac{2K^4 S b_1 m^2}{\lambda} - K^2 R a_1^2 b_1 = 0, \quad (4.7)$$

$$K^2 R a_0 b_1^2 \delta \lambda + a_0 V^2 + \frac{1}{2} K^2 Q b_1^2 \delta \lambda - \frac{1}{3} K^2 R a_0^3 - \frac{1}{2} K^2 Q a_0^2 - a_0 K^2 P^2 = 0. \quad (4.8)$$

By solving above system simultaneously, we obtain the values of unknown coefficients as follows:

$$K = \sqrt{\frac{-1}{12RS\delta^3\lambda - 6RS\delta\lambda}} Q, \quad V = \sqrt{\frac{-(6P^2R - Q^2)}{72S\delta^3\lambda - 36S\delta\lambda}} \frac{Q}{R}, \quad m = \sqrt{\frac{-(-2\delta^3\iota - \delta\iota)}{2\delta^3 - \delta - 4}},$$

$$a_1 = 0, \quad a_0 = -\frac{Q}{2R}, \quad b_1 = \sqrt{\frac{-1}{4\lambda\delta}} \frac{Q}{R}.$$

The solutions are then evaluated using Eq (4.1) along with Eq (3.6) for the obtained values of coefficients as follows.

Case 1: For $\delta = -1, \iota = -1$ and $\lambda > 0$, the solution is obtained, as

$$U_1(x, t) = -\frac{Q}{2R} + \frac{\sqrt{-\frac{1}{\lambda\delta}} Q \lambda \tanh\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right)}{2R\left(m \operatorname{sech}\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right) + 1\right)}. \quad (4.9)$$

Case 2: For $\delta = -1, \iota = 1$ and $\lambda > 0$, the solution is obtained, as

$$U_2(x, t) = -\frac{Q}{2R} + \frac{\sqrt{-\frac{1}{\lambda\delta}} Q \lambda \coth\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right)}{2R\left(m \operatorname{csch}\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right) + 1\right)}. \quad (4.10)$$

Case 3: For $\delta = 1, \iota = -1$ and $\lambda > 0$, the following solution is obtained.

$$U_3(x, t) = -\frac{Q}{2R} + \frac{\sqrt{-\frac{1}{\lambda\delta}} Q \sqrt{\lambda} \tan\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right)}{2R\left(m \sec\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right) + 1\right)}. \quad (4.11)$$

Case 4: For $\delta = 1, \iota = 1$ and $\lambda > 0$, the following solution is obtained.

$$U_4(x, t) = -\frac{Q}{2R} + \frac{\sqrt{-\frac{1}{\lambda\delta}} Q \sqrt{\lambda} \cot\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right)}{2R\left(m \csc\left(\sqrt{\lambda}\left(Kx + \frac{Vt^\alpha}{\Gamma(\alpha+1)}\right)\right) + 1\right)}. \quad (4.12)$$

4.2. Solutions of extended tanh-function method

As $N = 1$, the general solution can be assumed in the following form using Eq (3.10).

$$U(\xi) = a_0 + a_1 \Psi(\xi). \quad (4.13)$$

Substituting Eq (4.13) along with Eq (3.11) into Eq (2.5), system of algebraic equations is obtained by setting coefficients of highest power of $\Psi^k(\xi)$ to zero, (i.e. $k = 0, 1, 2, \dots, N$).

$$-\frac{1}{3}K^2Ra_1^3 + 2K^4Sa_1 = 0, \quad (4.14)$$

$$-\frac{1}{2}K^2Qa_1^2 - K^2Ra_0a_1^2 = 0, \quad (4.15)$$

$$2K^4Sa_1d - K^2Ra_0^2a_1 - a_1K^2P^2 - K^2Qa_0a_1 + a_1V^2 = 0, \quad (4.16)$$

$$-a_0K^2P^2 + a_0V^2 - \frac{1}{2}K^2Qa_0^2 - \frac{1}{3}K^2Ra_0^3 = 0. \quad (4.17)$$

By solving above system simultaneously, we obtain the values of unknown coefficients as follows:

$$K = \sqrt{-\frac{1}{24SdR}} Q, \quad V = \sqrt{-\frac{6P^2R - Q^2}{144Sd}} \frac{Q}{R}, \quad a_0 = -\frac{Q}{2R}, \quad a_1 = \sqrt{-\frac{1}{4d}} \frac{Q}{R}.$$

The solutions are then determined using Eq (4.13) along with Eqs (3.12)–(3.14) for the obtained coefficients as follows.

Case 5: For $d < 0$, the obtained solution is expressed, as

$$\begin{aligned}
 U_6(x, t) &= -\frac{Q}{2R} - \frac{\sqrt{-\frac{1}{d}} Q \sqrt{-d} \tanh\left(\sqrt{-d}\left(\frac{\sqrt{-\frac{6}{SdR}} Q x}{12} + \frac{\sqrt{-\frac{6P^2R-Q^2}{Sd}} Q t^\alpha}{12R \Gamma(\alpha+1)}\right)\right)}{2R}, \quad \text{or} \\
 &= -\frac{Q}{2R} - \frac{\sqrt{-\frac{1}{d}} Q \sqrt{-d} \coth\left(\sqrt{-d}\left(\frac{\sqrt{-\frac{6}{SdR}} Q x}{12} + \frac{\sqrt{-\frac{6P^2R-Q^2}{Sd}} Q t^\alpha}{12R \Gamma(\alpha+1)}\right)\right)}{2R}.
 \end{aligned} \tag{4.18}$$

Case 6: For $d = 0$, the following solution is obtained.

$$U_7(x, t) = -\frac{Q}{2R} - \frac{\sqrt{-\frac{1}{d}} Q}{2R\left(\frac{\sqrt{-\frac{6}{SdR}} Q x}{12} + \frac{\sqrt{-\frac{6P^2R-Q^2}{Sd}} Q t^\alpha}{12R \Gamma(\alpha+1)}\right)}. \tag{4.19}$$

Case 7: For $d > 0$, the solution is expressed, as

$$\begin{aligned}
 U_8(x, t) &= -\frac{Q}{2R} + \frac{\sqrt{-\frac{1}{d}} Q \sqrt{d} \tanh\left(\sqrt{d}\left(\frac{\sqrt{-\frac{6}{SdR}} Q x}{12} + \frac{\sqrt{-\frac{6P^2R-Q^2}{Sd}} Q t^\alpha}{12R \Gamma(\alpha+1)}\right)\right)}{2R}, \quad \text{or} \\
 &= -\frac{Q}{2R} + \frac{-\sqrt{-\frac{1}{d}} Q \sqrt{d} \cot\left(\sqrt{d}\left(\frac{\sqrt{-\frac{6}{SdR}} Q x}{12} + \frac{\sqrt{-\frac{6P^2R-Q^2}{Sd}} Q t^\alpha}{12R \Gamma(\alpha+1)}\right)\right)}{2R}.
 \end{aligned} \tag{4.20}$$

5. Graphical analysis of the proposed model

This section presents surface, contour, and density plots of the TFNNM solutions obtained using the GPREM and ETHFM, illustrating the effect of the fractional-order parameter on the wave profiles.

Figure 1 presents the surface, contour, and density plots of $U_1(x, t)$ in Eq (4.9) for $\alpha = 0.77, 0.92$, and 1, illustrating the effect of the fractional order on the wave profile.

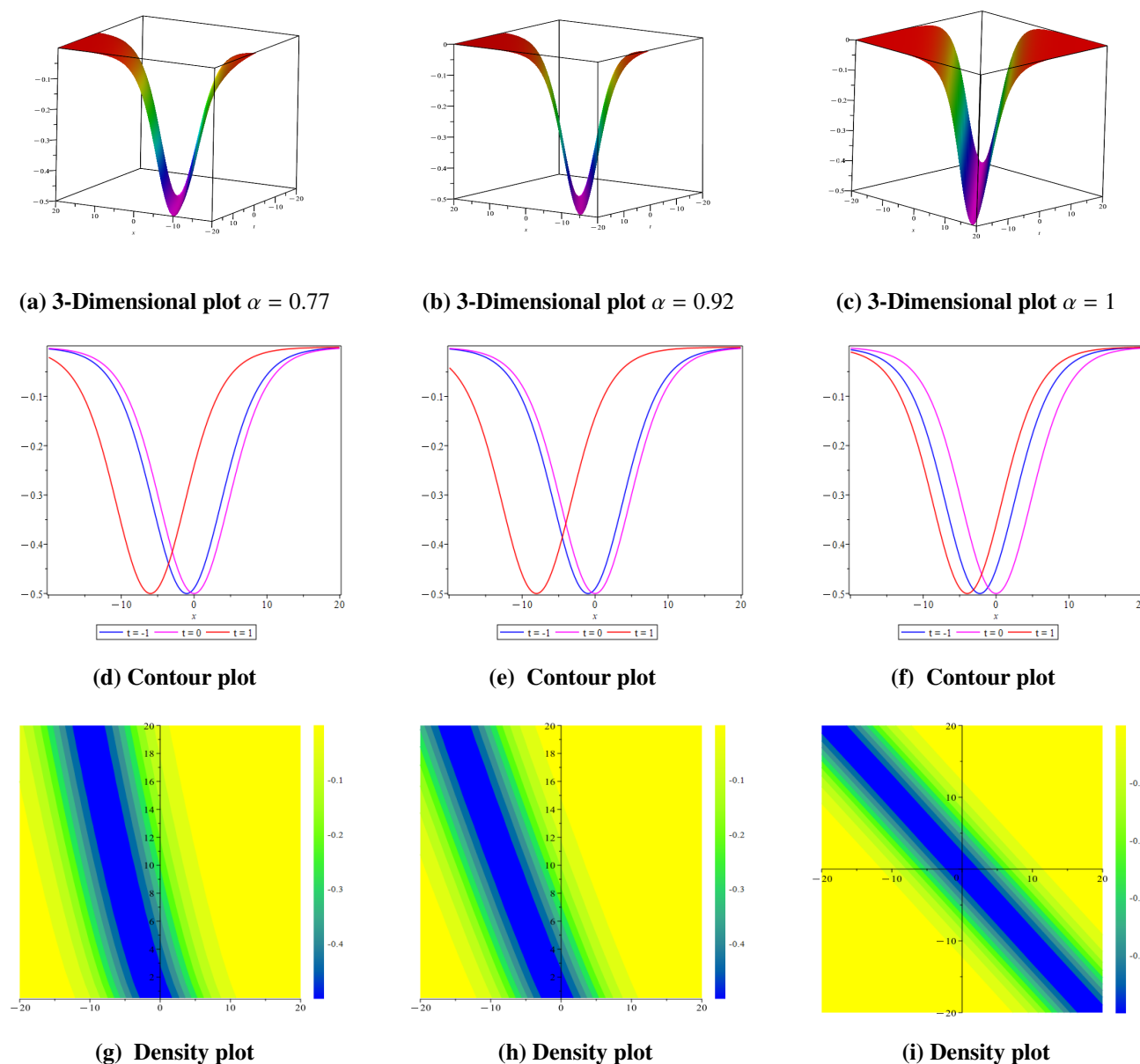


Figure 1. Three-dimensional surface, contour, and density plots illustrating the dark wave solution $U_1(x, t)$ obtained from Eq (4.9) of the time-fractional nonlinear neuron model for $\alpha = 0.77, 0.92$, and 1 . The parameters are $P = Q = R = S = \lambda = 1$ and $m = 2$. The dark wave represents a localized depression in the neuronal wave amplitude relative to the surrounding background state. The plots demonstrate how the wave propagates through the spatial-temporal domain while maintaining its localized dark-wave profile. Variations in the fractional order α modify the wave structure and propagation behavior, illustrating the influence of fractional dynamics on the evolution of the neuronal signal.

Figure 2 shows the surface, contour, and density plots of $U_2(x, t)$ in Eq (4.10) for $\alpha = 0.96, 0.99$, and 1 , allowing comparison of the wave patterns at different fractional orders.

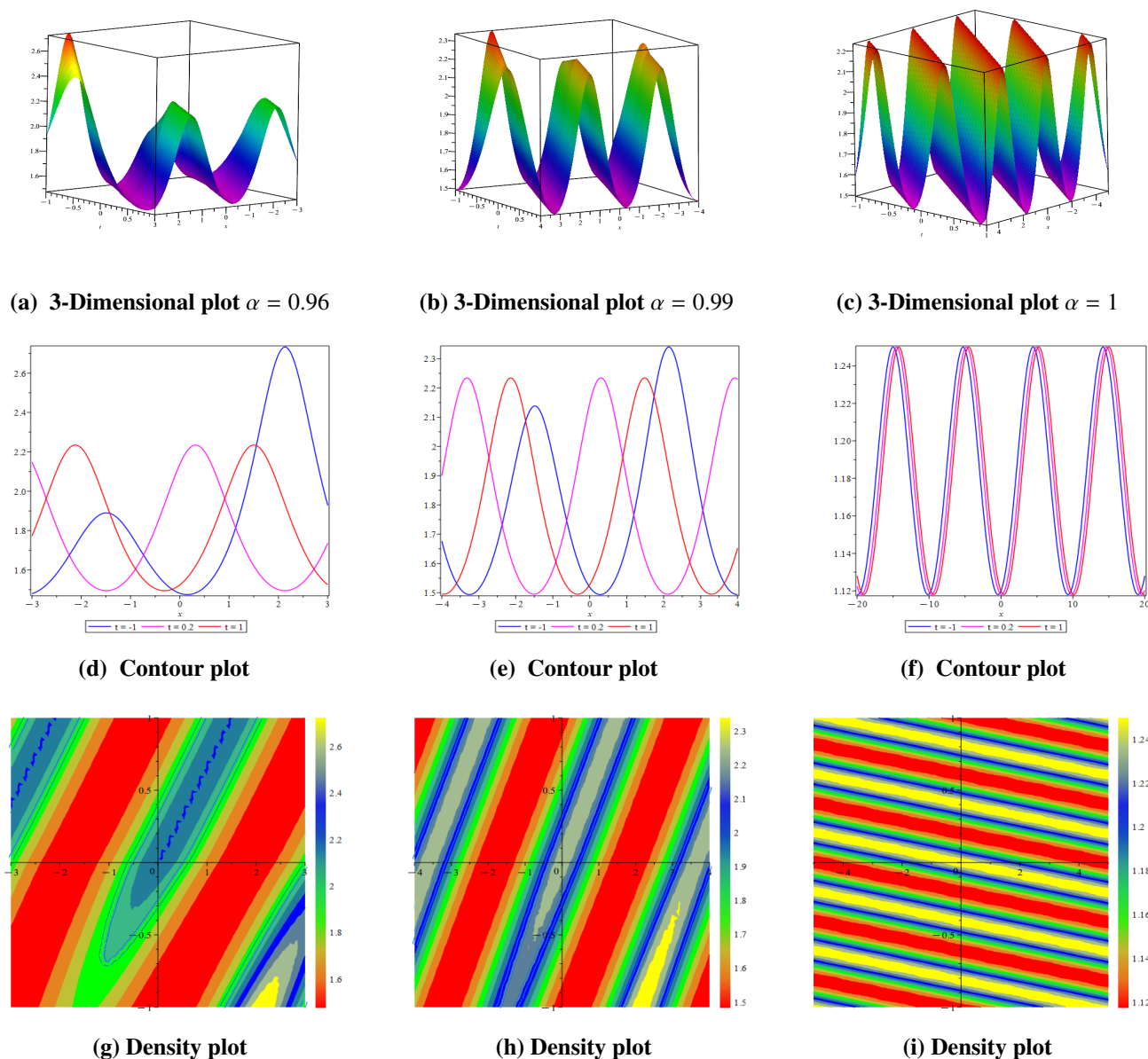


Figure 2. Three-dimensional surface, contour, and density plots illustrating the periodic wave solution $U_2(x, t)$ obtained from Eq (4.10) of the time-fractional nonlinear neuron model. The plots correspond to $\alpha = 0.96, 0.99$, and 1 , with $P = 1.2$, $Q = 2$, $R = -2$, $S = 0.99$, $\lambda = 0.54$, and $m = 0.9$. The periodic wave describes a regularly oscillating neuronal signal with recurrent spatial-temporal amplitude variations. Its periodic structure indicates sustained wave propagation with repeated oscillatory patterns, which can represent recurring or rhythmic signal activity in the neuronal medium. The changes associated with the fractional order illustrate the effect of memory-dependent fractional dynamics on the wave amplitude and propagation profile.

Figure 3 illustrates the surface, contour, and density plots of $U_6(x, t)$ in Eq (4.18) for $\alpha = 0.96, 0.99$, and 1 , showing how the fractional order affects the transition profile.

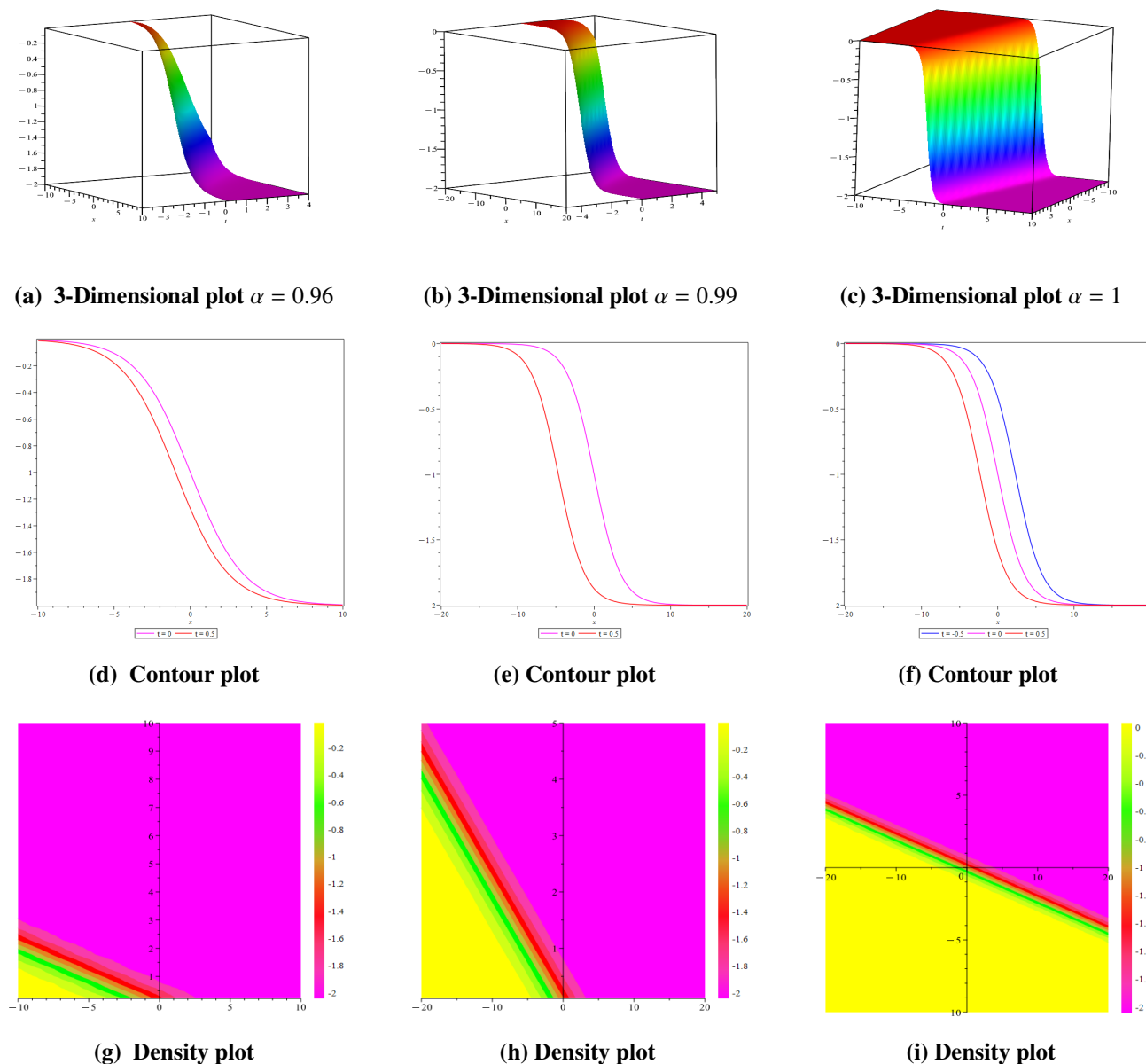


Figure 3. Three-dimensional surface, contour, and density plots illustrating the kink wave solution $U_6(x, t)$ obtained from Eq (4.18) of the time-fractional nonlinear neuron model. The plots correspond to $\alpha = 0.96, 0.99$, and 1 , with $P = 5$, $Q = 10$, $R = 5$, $S = 10$, and $d = -1$. The kink wave represents a smooth transition between two distinct asymptotic states of the neuronal field. Its monotonic transition profile describes the transfer of the system from one stable or quasi-stationary level to another as the wave propagates through the medium. The surface, contour, and density plots demonstrate the persistence of this transition structure and reveal how the fractional order influences the steepness, localization, and propagation characteristics of the neuronal wave.

Figure 4 displays the surface, contour, and density plots of $U_8(x, t)$ in Eq (4.20) for $\alpha = 0.8, 0.98$,

and 1, highlighting the variation of the wave profile with the fractional order.

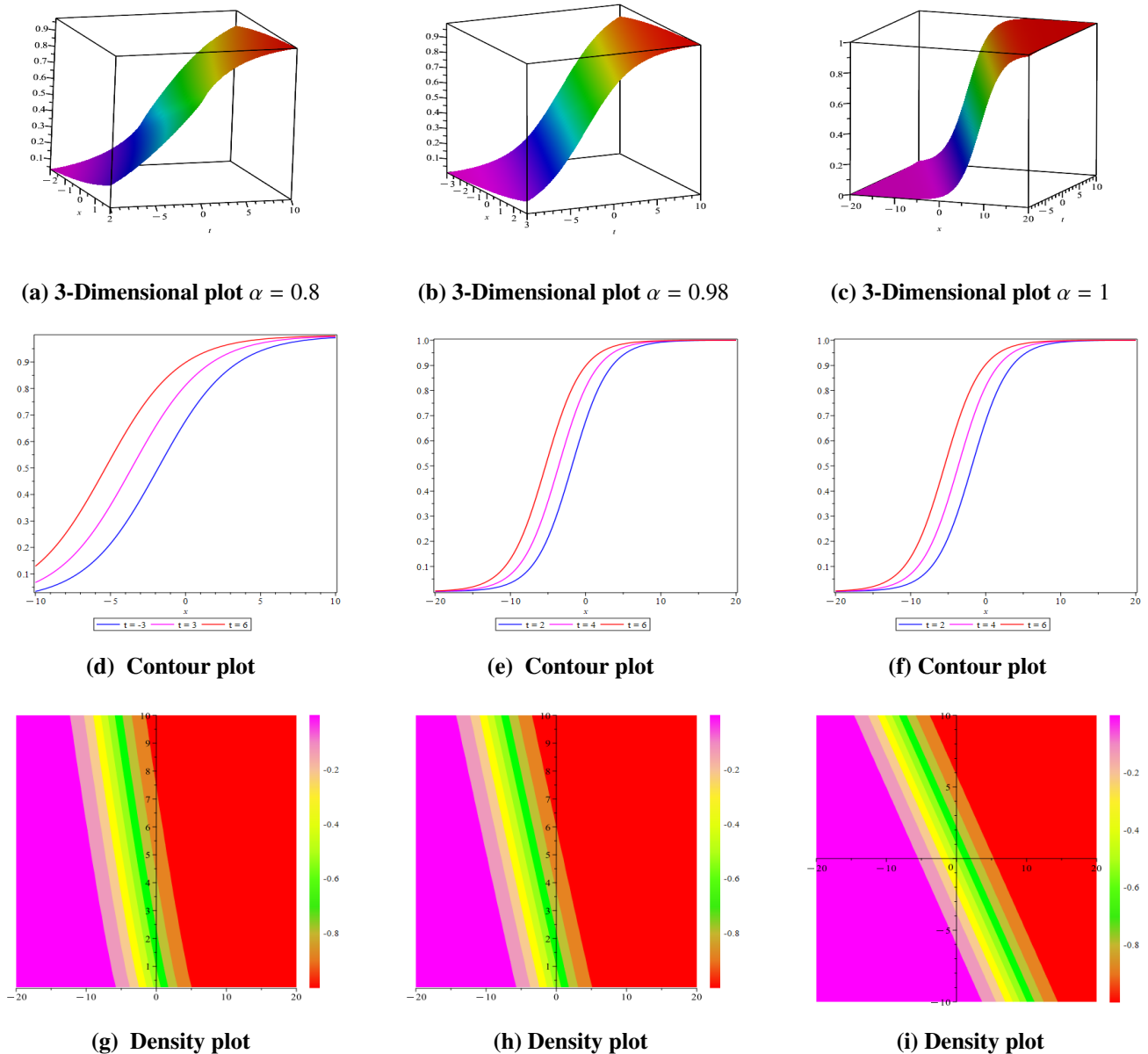


Figure 4. Three-dimensional surface, contour, and density plots illustrating the kink wave solution $U_8(x, t)$ obtained from Eq (4.20) of the time-fractional nonlinear neuron model. The plots correspond to $\alpha = 0.8, 0.98$, and 1, with $P = Q = R = S = d = 1$. The kink profile describes a localized transition of the neuronal wave field between two different asymptotic amplitude levels. Such a transition-type structure reflects the ability of the nonlinear medium to support persistent signal changes during propagation. The graphical results show that variation in the fractional order changes the shape and propagation characteristics of the kink, demonstrating the role of fractional effects in controlling the temporal evolution of neuronal signals.

6. Dynamical system

Bifurcation is a phenomenon taking place in the nonlinear system when the existing parameters arrive at a critical threshold. Bifurcation has a major function in nonlinear systems as it converts the boundedness of traveling wave solutions into unboundedness, and periodic solutions to non-periodic. This section investigates the qualitative behavior of Nonlinear ODE of the TFNNM (2.1) using the theory of planar dynamical systems [36–38] that helps us to construct Hamiltonian function of the dynamical system, and identify the equilibrium points. Equation (2.5) is defined, as

$$(V^2 - K^2 P^2)U - \frac{1}{2}K^2 Q U^2 - \frac{1}{3}K^2 R U^3 + K^4 S U'' = 0. \quad (6.1)$$

Equation (6.1) can also be written, as

$$U'' = -\frac{V^2 - K^2 P^2}{K^4 S}U + \frac{Q}{2K^2 S}U^2 + \frac{R}{3K^2 S}U^3. \quad (6.2)$$

Galilean transformation converted Eq (6.1) into second-order planar dynamical system, introducing new variables. Equation (6.1) can be rewritten, as

$$\begin{cases} U' = W, \\ W' = -AU + BU^2 + CU^3, \end{cases} \quad (6.3)$$

where

$$A = \frac{V^2 - K^2 P^2}{K^4 S}, \quad B = \frac{Q}{2K^2 S}, \quad C = \frac{R}{3K^2 S}. \quad (6.4)$$

6.1. Conservative Hamiltonian function

Multiplying Eq (6.1) by U' and integrating once gives the Hamiltonian (energy) function [37, 39]

$$H(U, W) = \frac{W^2}{2} + \frac{A}{2}U^2 - \frac{B}{3}U^3 - \frac{C}{4}U^4 = h, \quad (6.5)$$

where h is the integration constant. Equation (6.5) must satisfy the conservation condition

$$\frac{dH}{d\xi} = 0. \quad (6.6)$$

Indeed, using the chain rule, Eq (6.6) can be expanded, as

$$\frac{dH}{d\xi} = \frac{\partial H}{\partial U}U' + \frac{\partial H}{\partial W}W'. \quad (6.7)$$

Partial derivatives for the Hamiltonian function $H(U, W)$ are given by

$$\frac{\partial H}{\partial U} = AU - BU^2 - CU^3, \quad \frac{\partial H}{\partial W} = W. \quad (6.8)$$

Using the governing system (6.3) along with Eq (6.8), we obtained the following result

$$\frac{dH}{d\xi} = (AU - BU^2 - CU^3)W + W(-AU + BU^2 + CU^3) = 0.$$

Therefore,

$$H(U, W) = h = \text{constant} \quad (6.9)$$

along every trajectory of the system. Hence, the Hamiltonian function (6.5) is a conserved quantity, and the system represents a conservative Hamiltonian dynamical system.

6.2. Equilibrium points

The equilibrium points of the dynamical system (6.3) are obtained by setting

$$U' = 0, \quad W' = 0.$$

Dynamical system (6.3) yields,

$$W = 0, \quad \text{and} \quad -AU + BU^2 + CU^3 = 0.$$

Factoring the second term, we get

$$U(-A + BU + CU^2) = 0$$

that implies

$$U = 0, \quad (-A + BU + CU^2) = 0.$$

Hence, the equilibrium points are $E_0 = (0, 0)$, $E_1 = \left(\frac{-B + \sqrt{B^2 + 4AC}}{2C}, 0\right)$, and $E_2 = \left(\frac{-B - \sqrt{B^2 + 4AC}}{2C}, 0\right)$, provided that $B^2 + 4AC \geq 0$.

The Jacobian matrix of system (6.3) is expressed, as

$$J = \begin{pmatrix} 0 & 1 \\ -A + 2BU + 3CU^2 & 0 \end{pmatrix}, \quad (6.10)$$

$$J(U, W) = A - 2BU - 3CU^2. \quad (6.11)$$

6.3. Nature of equilibrium points

Nature of the equilibrium points is discussed by using Eq (6.11). For our purpose, theory of planar dynamical system [40–42] suggests following cases, regarding nature of equilibrium points.

- (1) If $J(U, W) < 0$, then (U, W) is a saddle point.
- (2) If $J(U, W) > 0$, then (U, W) is a center point.
- (3) If $J(U, W) = 0$, then (U, W) is a cusp point.

6.4. Stability of equilibrium points via eigenvalues

Eigenvalues of the dynamical system

$$J = \begin{pmatrix} 0 & 1 \\ -A + 2BU + 3CU^2 & 0 \end{pmatrix}. \quad (6.12)$$

Consequently, for the equilibrium point $E_0 = (0, 0)$, Eq (6.11) provides following result.

$$J(U, W) = \lambda^2 = -A. \quad (6.13)$$

Therefore, behavior of $E_0 = (0, 0)$ depends on value of A .

- If $A > 0$, then

$$J(U, W) = \lambda^2 = -A > 0, \implies \lambda = \pm i\sqrt{A},$$

and $(0, 0)$ is a center point.

- If $A < 0$, then

$$J(U, W) = \lambda^2 = A < 0, \implies \lambda = \pm \sqrt{-A},$$

and $(0, 0)$ is a saddle point.

- If $A = 0$, then

$$J(U, W) = \lambda^2 = 0,$$

and $(0, 0)$ is a cusp point. The stability of the origin changes at $A = 0$, and a critical condition of bifurcation is evaluated, as $V = \pm KP$.

At the equilibrium point $E_1 = \left(\frac{-B + \sqrt{B^2 + 4AC}}{2C}, 0 \right)$, Eq (6.11) provides following result.

$$J(U, W) = \lambda^2 = \frac{\sqrt{B^2 + 4AC}(-B + \sqrt{B^2 + 4AC})}{2C}. \quad (6.14)$$

Hence, behavior of $\left(\frac{-B + \sqrt{B^2 + 4AC}}{2C}, 0 \right)$ depends on the value of discriminant $(\sqrt{B^2 + 4AC})$.

- If $B^2 + 4AC > 0$, then

$$J(U, W) = \lambda^2 > 0,$$

and E_1 is a center, provided that $C > 0$.

- If $B^2 + 4AC < 0$, then

$$J(U, W) = \lambda^2 < 0,$$

and E_1 is saddle point, provided that $C < 0$.

- If $B^2 + 4AC = 0$, then

$$J(U, W) = \lambda^2 = 0,$$

and E_1 is cusp point, provided that $C \neq 0$.

Similarly, at equilibrium point $E_2 = \left(\frac{-B - \sqrt{B^2 + 4AC}}{2C}, 0 \right)$, Eq (6.11) extracts following outcomes.

$$J(U, W) = \lambda^2 = \frac{\sqrt{B^2 + 4AC}(B + \sqrt{B^2 + 4AC})}{2C}. \quad (6.15)$$

Depending on the value of discriminant $\sqrt{B^2 + 4AC}$, following results are obtained.

- If $B^2 + 4AC > 0$, then

$$J(U, W) = \lambda^2 > 0,$$

provided E_2 is a center point, provided that $C > 0$.

- If $B^2 + 4AC < 0$, then

$$J(U, W) = \lambda^2 < 0,$$

and E_2 is saddle point, provided that $C < 0$.

- If $B^2 + 4AC = 0$, then

$$J(U, W) = \lambda^2 = 0,$$

and E_2 is cusp point, provided that $C \neq 0$.

Table 2 summarizes the eight representative parameter combinations and the corresponding equilibrium-point behavior. The associated phase portraits are shown in Figure 5.

Table 2. Bifurcation analysis for the parameter values of A , B , and C used in Figure 5.

Figure	Parameters case	Parameter values	Behavior of points
5(a)	$A > 0, B > 0, C > 0$	$A = 1.1, B = 0.2, C = 0.5$	$J(E_0) > 0, J(E_1) < 0, J(E_2) < 0$
5(b)	$A > 0, B > 0, C < 0$	$A = 2.1, B = 1.5, C = -0.5$	$J(E_0) > 0, E_1, E_2$ are non real
5(c)	$A > 0, B < 0, C > 0$	$A = 1.1, B = -0.2, C = 0.5$	$J(E_0) > 0, J(E_1) < 0, J(E_2) < 0$
5(d)	$A > 0, B < 0, C < 0$	$A = 2.1, B = -1.5, C = -0.5$	$J(E_0) > 0, E_1, E_2$ are non real
5(e)	$A < 0, B > 0, C > 0$	$A = -2.1, B = 1.5, C = 0.5$	$J(E_0) < 0, E_1, E_2$ are non real
5(f)	$A < 0, B > 0, C < 0$	$A = -1.1, B = 0.2, C = -0.5$	$J(E_0) < 0, J(E_1) > 0, J(E_2) > 0$
5(g)	$A < 0, B < 0, C > 0$	$A = -2.1, B = -1.5, C = 0.5$	$J(E_0) < 0, E_1, E_2$ are non real
5(h)	$A < 0, B < 0, C < 0$	$A = -1.1, B = -0.2, C = -0.5$	$J(E_0) < 0, J(E_1) > 0, J(E_2) > 0$

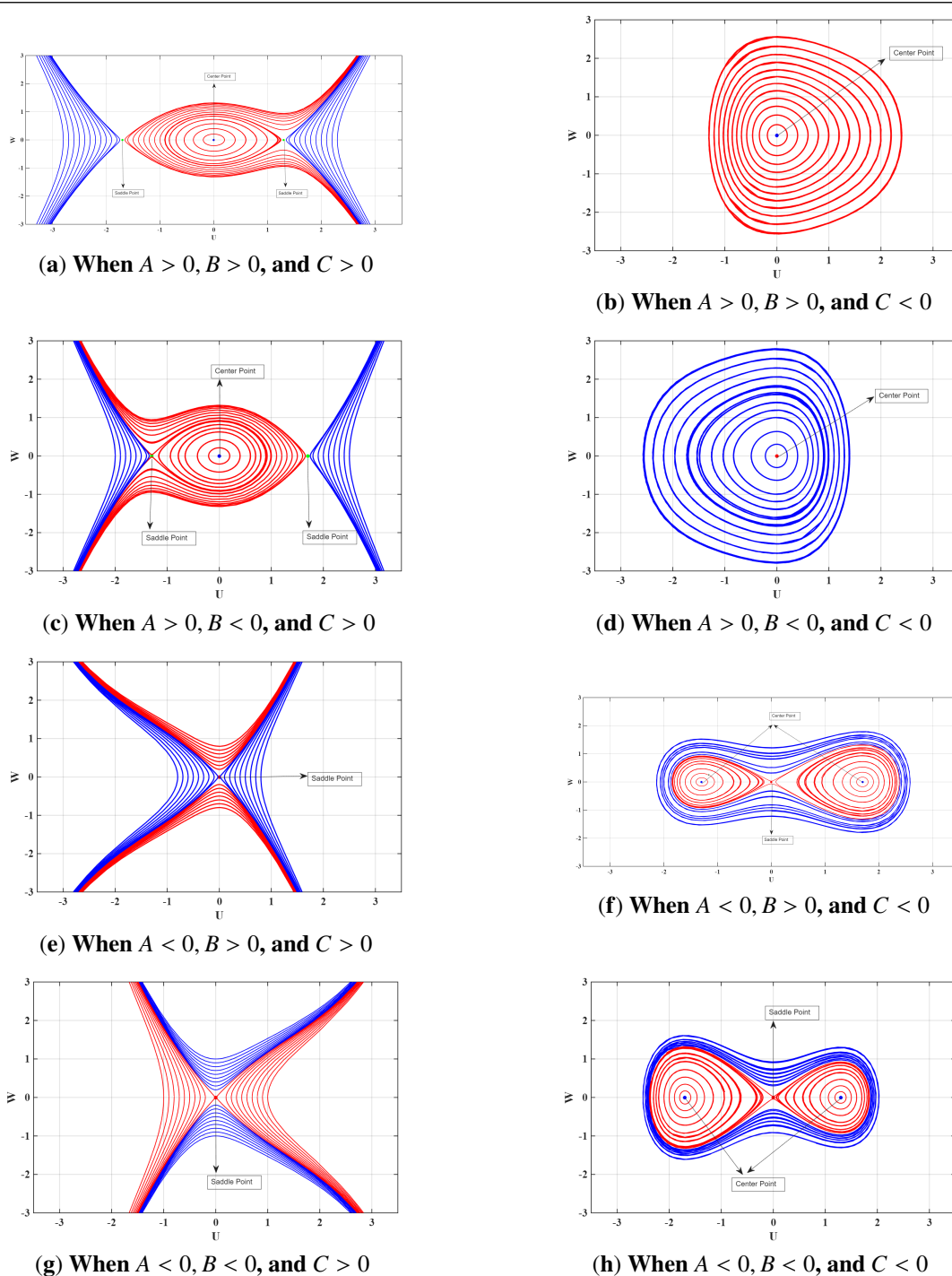


Figure 5. Phase portraits of the planar dynamical system associated with the time-fractional nonlinear neuron model under eight different combinations of the signs of A, B , and C . In particular, panels (a) and (c) exhibit a center at (E_0) surrounded by saddle-type nonzero equilibria, whereas panels (f) and (h) display two center-type nonzero equilibria separated by a saddle at the origin. For panels (b), (d) representation of the centre point and, (e), (g), depicts the saddle, only the origin is a real equilibrium, while the remaining equilibrium points are non real. These results demonstrate the strong dependence of the phase-space structure and equilibrium configuration on the signs and magnitudes of the reduced dynamical parameters A, B , and C .

Table 3 lists the parameter-sign cases, equilibrium coordinates, and initial conditions (ICs) used for the phase portraits in Figure 5. These data complement the equilibrium classification in Table 2.

Table 3. Parameter cases, equilibrium points, and initial conditions for the phase portraits in Figure 5.

Plot	Cases	Points	ICs
5(a)	$A > 0, B > 0,$ $C > 0$	(0,0), (1.29,0), (-1.69,0)	(0.2, 0.02), (0.4, 0.02), (0.6, 0.02), (0.8, 0.02), (1, 0.02), (1.1, 0.02), (1.25, 0.02), (-1, 0.02), (-1.1, 0.02), (-1.2, 0.02), (-1.3, 0.02), (-1.4, 0.02), (-1.5, 0.02), (-1.6, 0.02), (-1.7, 0.02), (1.35, 0.02), (1.4, 0.02) (1.5,0.02), (1.6, 0.02), (1.7, 0.02), (1.8, 0.02), (1.9, 0.02), (2, 0.02), (2.1, 0.02), (2.2, 0.02), (2.3, 0.02), (-1.75, 0.02), (-1.8, 0.02), (-1.9, 0.02), (-2, 0.02), (-2.1, 0.02), (-2.2, 0.02), (-2.3, 0.02), (-2.4, 0.02), (-2.5, 0.02), (-2.6, 0.02), (-2.8, 0.02)
5(b)	$A > 0, B > 0,$ $C < 0$	(0,0)	(0.2, 0.02), (0.4, 0.02), (0.6, 0.02), (0.8, 0.02), (1, 0.02), (1.1, 0.02), (1.4, 0.02), (1.6, 0.02), (1.8, 0.02), (2, 0.02), (2.2, 0.02), (2.4, 0.02), (2.6, 0.02)
5(c)	$A > 0, B < 0,$ $C > 0$	(0,0), (1.69,0), (-1.29,0)	(0.2, 0.02), (0.4, 0.02), (0.6, 0.02), (0.8, 0.02), (0.9, 0.02), (0.92, 0.02), (0.99, 0.02), (1, 0.02), (1.1, 0.02), (1.2, 0.02), (1.3, 0.02), (1.4, 0.02), (1.5, 0.02), (1.6, 0.02), (1.65, 0.02), (1.75, 0.02), (1.8, 0.02), (1.9, 0.02), (2, 0.02), (2.1, 0.02), (2.2, 0.02), (2.3, 0.02), (2.4, 0.02), (2.5, 0.02), (-1.3, 0.02), (-1.4, 0.02), (-1.5, 0.02), (-1.6, 0.02), (-1.65, 0.02), (-1.75, 0.02), (-1.8, 0.02), (-1.9, 0.02), (-2, 0.02)
5(d)	$A > 0, B < 0,$ $C < 0$	(0,0)	(0.2, 0.02), (0.4, 0.02), (0.6, 0.02), (0.8, 0.02), (0.9, 0.02), (0.92, 0.02), (0.99, 0.02), (1, 0.02), (1.1, 0.02), (1.2, 0.02), (1.3, 0.02), (1.4, 0.02), (1.5, 0.02), (1.6, 0.02), (1.65, 0.02)
5(e)	$A < 0, B > 0,$ $C > 0$	(0,0)	(0.1, 0.02), (0.2, 0.02), (0.3, 0.02), (0.4, 0.02), (0.5, 0.02), (0.6, 0.02), (0.7, 0.02), (0.8, 0.02), (0.9, 0.02), (1, 0.02), (1.03, 0.02), (-0.1, 0.02), (-0.2, 0.02), (-0.3, 0.02), (-0.4, 0.02), (-0.5, 0.02), (-0.6, 0.02), (-0.7, 0.02), (-0.8, 0.02), (-0.9, 0.02), (-1, 0.02), (-1.03, 0.02), (0.02, 0.1), (0.02, 0.2), (0.02, 0.3), (0.02, 0.4), (0.02, 0.5), (0.02, 0.6), (0.02, 0.7), (0.02, 0.8), (0.02, 0.9), (0.02, 1), (0.02, 1.1), (0.02, -0.1), (0.02, -0.2), (0.02, -0.3), (0.02, -0.4), (0.02, -0.5), (0.02, -0.6), (0.02, -0.7), (0.02, -0.8), (0.02, -0.9), (0.02, -1), (0.02, -1.1)
5(f)	$A < 0, B > 0,$ $C < 0$	(0,0), (-1.69,0), (1.29,0)	(-1.3, 0.02), (-1.4, 0.02), (-1.5, 0.02), (-1.6, 0.02), (-1.7, 0.02), (-1.8, 0.02), (-1.83, 0.02), (-1.85, 0.02), (1.8, 0.02), (1.9, 0.02), (2, 0.02), (2, 0.02), (2.1, 0.02), (2.2, 0.02), (2.25, 0.02), (2.35, 0.02), (2.3, 0.02), (2.38, 0.02), (0.02, 0.3), (0.02, 0.5), (0.02, 0.8), (0.02, 0.9), (0.02, 1), (0.02, 1.2)
5(g)	$A < 0, B < 0,$ $C > 0$	(0,0)	(0.1, 0.02), (0.2, 0.02), (0.3, 0.02), (0.4, 0.02), (0.5, 0.02), (0.6, 0.02), (0.7, 0.02), (0.8, 0.02), (0.9, 0.02), (1, 0.02), (1.03, 0.02), (-0.1, 0.02), (-0.2, 0.02), (-0.3, 0.02), (-0.4, 0.02), (-0.5, 0.02), (-0.6, 0.02), (-0.7, 0.02), (-0.8, 0.02), (-0.9, 0.02), (-1, 0.02), (-1.03, 0.02) (0.02, 0.1), (0.02, 0.2), (0.02, 0.3), (0.02, 0.4), (0.02, 0.5), (0.02, 0.6), (0.02, 0.7), (0.02, 0.8), (0.02, 0.9), (0.02, 1), (0.02, 1.1), (0.02, -0.1), (0.02, -0.2), (0.02, -0.3), (0.02, -0.4), (0.02, -0.5), (0.02, -0.6), (0.02, -0.7), (0.02, -0.8), (0.02, -0.9), (0.02, -1), (0.02, -1.1)
5(h)	$A < 0, B < 0,$ $C < 0$	(0,0), (-1.69,0), (1.29,0)	(-1.8, 0.02), (-1.9, 0.02), (-2, 0.02), (-2.1, 0.02), (-2.2, 0.02), (-2.3, 0.02), (-2.36, 0.02), (-2.37, 0.02), (1.4, 0.02), (1.5, 0.02), (1.6, 0.02), (1.7, 0.02), (1.75, 0.02), (1.8, 0.02), (1.83, 0.02), (1.845, 0.02), (1.847, 0.02), (0.02, 0.6), (0.02, 0.7), (0.02, 0.8), (0.02, 0.9)

7. Result and discussion

This section explains the graphical representation of the solutions to the TFNNM, showing different types of wave patterns. By choosing suitable numerical values for the unknown coefficients, various wave patterns are determined here. These graphs help us to understand how the fractional coefficients and time-space coefficients affect the wave patterns. 3D and 2D plots give a clearer picture of how the solutions to the TFNNM behave.

Figure 1 presents the three-dimensional, contour, and density representations of the dark-wave solution $U_1(x, t)$ given by Eq (4.9). The parameter values are fixed at $P = Q = R = S = \lambda = 1$ and $m = 2$, while the fractional-order parameter is varied as $\alpha = 0.77$, $\alpha = 0.92$, and $\alpha = 1$. The obtained structure exhibits a localized depression in the wave amplitude with respect to the surrounding background level. The three-dimensional surfaces demonstrate that the dark profile remains localized while propagating through the spatial-temporal domain. The corresponding contour and density plots further confirm the persistence and localization of the wave structure. An important feature of Figure 1 is the effect of the fractional order. When α is changed from 0.77 to 0.92 and subsequently to the classical value $\alpha = 1$, the spatial-temporal distribution of the dark wave changes progressively. Since the fractional-order parameter enters the traveling-wave variable through $t^\alpha/\Gamma(\alpha + 1)$, its variation modifies the effective temporal evolution of the wave. Consequently, the position and propagation characteristics of the localized depression are altered even though the remaining model parameters are unchanged. Thus, decreasing α from unity introduces a stronger fractional-memory contribution, whereas the limiting case $\alpha = 1$ recovers the corresponding integer-order temporal behavior. This demonstrates that the fractional parameter provides an effective mechanism for controlling the propagation characteristics of localized neuronal signals.

Figure 2 illustrates the periodic-wave solution $U_2(x, t)$ obtained from Eq (4.10). In this case, the parameter values are $P = 1.2$, $Q = 2$, $R = -2$, $S = 0.99$, $\lambda = 0.54$, and $m = 0.9$, while the fractional orders $\alpha = 0.96$, $\alpha = 0.99$, and $\alpha = 1$ are considered. Unlike the localized dark structure in Figure 1, the solution in Figure 2 exhibits recurrent oscillations distributed throughout the spatial-temporal domain. The contour and density representations clearly reveal the repeated wave bands associated with the periodic nature of the solution. The comparison of the three fractional orders shows that the temporal memory encoded by α modifies the phase and spatial-temporal arrangement of the periodic oscillations. As α approaches unity, the fractional temporal contribution becomes progressively weaker and the wave approaches its classical temporal propagation behavior. For smaller values of α , the memory effect modifies the temporal scaling and consequently changes the location and profile of the oscillatory structures. Therefore, the fractional order can be interpreted as a control parameter for the propagation and temporal organization of recurring neuronal signals. The persistence of the periodic pattern for all three values also indicates that the underlying nonlinear interaction continues to support sustained oscillatory propagation despite the variation in fractional order.

Figure 3 depicts the kink-type solution $U_6(x, t)$ obtained from Eq (4.18). The selected parameters are $P = 5$, $Q = 10$, $R = 5$, $S = 10$, and $d = -1$, with the fractional orders specified in the figure as $\alpha = 0.96$, $\alpha = 0.99$, and $\alpha = 1$. The solution possesses a monotonic transition between two distinct asymptotic states. This behavior is fundamentally different from the localized depression of Figure 1 and the recurrent oscillations of Figure 2. The three-dimensional surfaces demonstrate the propagation of the transition front, whereas the contour and density plots provide a clearer representation of its

spatial localization and transition region. The fractional-order parameter has a direct influence on the steepness and temporal displacement of the kink. Changes in α modify the fractional temporal scaling in the traveling coordinate and therefore alter the position and shape of the transition layer. In particular, the variation of α changes how rapidly the neuronal field evolves from one asymptotic level to the other. Hence, the fractional parameter provides a mathematical representation of memory-dependent effects on the propagation of transition-type neuronal signals. The persistence of the kink structure for different fractional orders confirms that the nonlinear model can support robust transition waves even when the temporal memory characteristics of the medium are changed.

Figure 4 shows another kink-type structure corresponding to $U_8(x, t)$ from Eq (4.20). Here, $P = Q = R = S = d = 1$, and the fractional orders are $\alpha = 0.8$, $\alpha = 0.98$, and $\alpha = 1$. The graphical results demonstrate a localized transition of the neuronal field between two different amplitude levels. The three-dimensional plots reveal the propagation of the transition front, while the contour and density plots highlight its localization and spatial-temporal evolution. A clear parametric effect is observed when α is varied. For $\alpha = 0.8$, the fractional temporal contribution is stronger, producing a different spatial-temporal positioning and profile of the kink compared with the near-classical case $\alpha = 0.98$. At $\alpha = 1$, the fractional model approaches the corresponding classical temporal evolution. Thus, increasing α toward unity progressively changes the temporal scaling of the kink and modifies its propagation characteristics. These observations demonstrate that the fractional order is not merely a mathematical parameter but an important mechanism for regulating the temporal evolution of nonlinear neuronal waves.

Figure 5 presents the phase portraits of the planar dynamical system obtained from the traveling-wave reduction of the time-fractional nonlinear neuron model. The phase-plane behavior is examined under different combinations of the signs of the parameters A , B , and C . These parameters determine the existence and nature of the equilibrium points and, consequently, the qualitative structure of the trajectories. The corresponding equilibrium points are obtained from $W = 0$, $-AU + BU^2 + CU^3 = 0$, which yields $E_0 = (0, 0)$ and, when $B^2 + 4AC \geq 0$, $E_1 = \left(\frac{-B + \sqrt{B^2 + 4AC}}{2C}, 0\right)$, $E_2 = \left(\frac{-B - \sqrt{B^2 + 4AC}}{2C}, 0\right)$. The Jacobian criterion used in the manuscript classifies these points as centers, saddles, or cusps according to the sign of the corresponding Jacobian quantity.

Figure 5(a): $A > 0$, $B > 0$, and $C > 0$.

Figure 5(a) represents the phase portrait for the parameter values $A = 1.1$, $B = 0.2$, $C = 0.5$. For this parameter configuration, three equilibrium points are present at $E_0 = (0, 0)$, $E_1 = (1.29, 0)$, $E_2 = (-1.69, 0)$. The origin E_0 behaves as a center, whereas E_1 and E_2 are saddle points. Accordingly, the phase portrait contains a family of closed trajectories surrounding the central equilibrium point, indicating bounded oscillatory motion in the neighborhood of E_0 . The saddle points located on either side of the center generate separatrix-type trajectories that organize the surrounding phase flow. The coexistence of a center and two saddle points demonstrates that the positive values of A , B , and C permit a rich bounded phase-space structure with distinct regions of oscillatory and saddle-type dynamics. The closed orbits surrounding the center indicate periodic behavior of the reduced traveling-wave dynamical system, while the trajectories approaching or departing from the saddle points represent critical wave configurations separating different dynamical regimes. The equilibrium classification and parameter values agree with Table 2 of the manuscript.

Figure 5(b): $A > 0$, $B > 0$, and $C < 0$.

In Figure 5(b), the parameter values are selected as $A = 2.1$, $B = 1.5$, $C = -0.5$. Under this

condition, the origin $E_0 = (0,0)$ remains the relevant real equilibrium point and is classified as a center, while the other two equilibrium solutions E_1 and E_2 become imaginary. Consequently, the phase portrait consists predominantly of nested closed trajectories surrounding the origin. In contrast to Figure 5(a), the saddle points situated away from the origin disappear from the real phase plane because the corresponding equilibrium coordinates are no longer real. The resulting family of closed orbits indicates bounded and periodic motion around the center. Thus, changing the sign of C from positive to negative substantially modifies the topology of the phase portrait: The two real saddle structures observed in Figure 5(a) are replaced by a single central oscillatory region. This result demonstrates the important role of the cubic nonlinear coefficient C in determining the existence of additional real equilibrium states and the global organization of the phase trajectories.

Figure 5(c): $A > 0$, $B < 0$, and $C > 0$.

Figure 5(c) corresponds to $A = 1.1$, $B = -0.2$, $C = 0.5$. The three real equilibrium points are $E_0 = (0,0)$, $E_1 = (1.69, 0)$, $E_2 = (-1.29, 0)$. Here, E_0 is a center, while E_1 and E_2 are saddle points. The phase portrait therefore possesses a central family of closed trajectories around the origin together with saddle structures on both sides of the center. A notable feature of this case is that changing the sign of B compared with Figure 5(a) reverses the locations of the two nonzero equilibrium points: The positive and negative equilibrium positions are shifted from approximately $(1.29, 0)$ and $(-1.69, 0)$ in Figure 5(a) to $(1.69, 0)$ and $(-1.29, 0)$ in Figure 5(c). Nevertheless, the qualitative topology remains characterized by a center surrounded by closed orbits and two external saddle points. This indicates that the quadratic nonlinear coefficient B strongly affects the spatial arrangement of the equilibrium states while the combination $A > 0$ and $C > 0$ preserves the center–saddle configuration.

Figure 5(d): $A > 0$, $B < 0$, and $C < 0$.

Figure 5(d) illustrates the phase portrait for $A = 2.1$, $B = -1.5$, $C = -0.5$. For these parameter values, the origin $E_0 = (0,0)$ is a center, whereas the two nonzero equilibrium solutions are imaginary. The phase plane consequently contains a single central family of closed trajectories surrounding the origin, with no real saddle points appearing in the plotted domain. The closed and approximately concentric trajectories demonstrate bounded oscillatory behavior of the dynamical system. Compared with Figure 5(c), where $C > 0$ produces two additional real saddle points, the negative value of C removes these nonzero real equilibria and simplifies the phase-space topology. Hence, the change in the signs of B and C modifies the global organization of the trajectories while maintaining the center character of the origin because A remains positive. This case confirms that the signs and magnitudes of the nonlinear coefficients play a decisive role in determining whether the reduced neuron model exhibits multiple equilibrium states or a single bounded oscillatory regime.

Figure 5(e): $A < 0$, $B > 0$, and $C > 0$.

Figure 5(e) considers the parameter combination $A = -2.1$, $B = 1.5$, $C = 0.5$. In this case, the origin $E_0 = (0,0)$ changes its nature and becomes a saddle point because $A < 0$. The other two equilibrium points E_1 and E_2 are imaginary for the selected parameter values. Consequently, the phase portrait is dominated by trajectories organized around the saddle point at the origin. Unlike Figures 5(a)–(d), no family of closed periodic orbits surrounds the origin. Instead, the trajectories form saddle-type branches that approach or depart from the equilibrium according to the direction of the phase flow. This transition from a center for $A > 0$ to a saddle for $A < 0$ clearly demonstrates the critical role of the parameter A in determining the local stability and qualitative behavior of the dynamical system. In particular, the change in the sign of A transforms the origin from a bounded oscillatory state into an

unstable saddle configuration, thereby producing a substantial topological change in the phase portrait.

Figure 5(f) corresponds to $A < 0$, $B > 0$, and $C < 0$, with $A = -1.1$, $B = 0.2$, and $C = -0.5$. In contrast to panel (e), the selected parameter combination produces three real equilibrium points: the origin E_0 and two nonzero equilibria E_1 and E_2 . The Jacobian satisfies $J(E_0) < 0$, whereas $J(E_1) > 0$ and $J(E_2) > 0$. Hence, the origin is a saddle, while the two nonzero equilibria are center-type points. The resulting phase portrait contains two families of closed trajectories surrounding the nonzero centers, with the saddle at the origin separating the two regions of bounded motion. This configuration indicates that the nonlinear coefficients (B) and (C), together with the negative value of (A), can generate two distinct center regions in the phase space.

Figure 5(g) represents the case $A < 0$, $B < 0$, and $C > 0$, with $A = -2.1$, $B = -1.5$, and $C = 0.5$. Here, the origin remains a saddle because $A < 0$, whereas the nonzero equilibria are non-real for the selected parameter values. Thus, similar to panel (e), only the origin contributes a real equilibrium structure to the phase portrait. However, the reversal of the sign of (B) changes the nonlinear balance in the reduced equation and consequently changes the orientation and geometric organization of the phase trajectories. This comparison between panels (e) and (g) demonstrates that the sign of (B) affects the nonlinear asymmetry of the phase-space structure even when $A < 0$ and $C > 0$ are maintained.

Figure 5(h) corresponds to $A < 0$, $B < 0$, and $C < 0$, with $A = -1.1$, $B = -0.2$, and $C = -0.5$. Three real equilibrium points are obtained in this case. The Jacobian satisfies $J(E_0) < 0$, while $J(E_1) > 0$ and $J(E_2) > 0$. Consequently, the origin is a saddle and the two nonzero equilibria are centers. The phase portrait therefore exhibits two center-type regions separated by the saddle at the origin. This structure is qualitatively similar to panel (f), but it is generated with the opposite sign of (B). The comparison between panels (f) and (h) indicates that reversing the sign of (B) does not necessarily change the center/saddle classification when $A < 0$ and $C < 0$; instead, it changes the location and arrangement of the nonzero equilibrium points in the phase plane.

Physical interpretation of present research

In Figure 1 we have pointed out that the dark wave is a decrease in the amplitude or electric activity, as compared to the background, of the wave at a point or in a region. Within the framework of the model of the nonlinear neuron, such a localized depression may be linked to an inhibition or reduction of the activity of the neuron, to a local energy redistribution, and to a nonlinear signal propagation through a biological membrane. The physical role of propagation, nonlinear, dispersive, localization and fractional-order parameters have also been explained. The physical interpretation of the periodic wave has been extended (Figure 2). A regular sequence of peaks and troughs is the sustained oscillatory behavior while propagating through the signal. These periodic phenomenon can be tied to rhythmic electrical activity, periodic firing of neurons, synchronized oscillations, and periodic signaling. It has also been demonstrated that the balance that is responsible for the periodic structure is given by the nonlinear and dispersive parameters, while the memory characteristics of the neuron medium are given by the fractional-order parameter. In the present paper, we have discussed the kink structure as a transition front between two different states of the neuronal medium as summarized in the sketch of Figure 3. On the physical side, this kind of wave may be a state transition from resting to excited membrane states in the propagation of a signal from a neuron. The discussion now focuses on its significance in the context of excitation fronts, switching between membrane states, nonlinear phase transitions and propagation of stable electrical signals in biological neural

tissue. The second kink solution is further emphasized in Figure 4 as being physical. Excellent propagation-front shape, almost unchanged, suggests a strong excitation structure due to the balance between nonlinear and dispersive effects. On a physical level this behaviour has been linked to the stable propagation of an excitation state along the membrane of a neuron and may be applicable to neural state switching and the propagation of an excitation front. Each of the phase portraits in Figure 5 has been expanded individually, to include the physical interpretation. The phase portraits convey information about the bounded oscillations, transition states, and the organization of the nonlinear neuronal-wave dynamics of the reduced traveling-wave system. Particularly, panels Figure 5(a)–(c) has saddle type equilibria encircling a center. The closed trajectories are bounded oscillatory regimes, and the saddle points with the corresponding structures of the separatrices are the boundaries of transition between different dynamical regimes. The physical realizations are associated with the coexistence of bound oscillatory dynamics and critical transition states in the reduced neuron dynamics. In Figure 5(a), the nonzero equilibrium points are not real and so the origin is the only real equilibrium point of interest. The outcome are closed trajectories, which represent a mostly bounded oscillatory regime. The physical state of this configuration corresponds to a simplified dynamical state where the extra real equilibrium states that exist with alternative nonlinear membrane configurations are not present. The phase space organizations of the center-saddle type are unchanged when the sign of the quadratic nonlinear coefficient B is reversed in Figure 5(c). This demonstrates physically that the quadratic nonlinear response can modify the location of alternative neuronal states without necessarily changing the overall type of bounded oscillatory dynamics. The nonzero equilibria are now again non-real in Figure 5(d) and the origin is still a center. The resulting closed trajectories will be associated with bounded oscillatory dynamics. The case shows how different values of the nonlinear coefficients can result in the stability of other real equilibria points while retaining an oscillatory regime near the origin. In Figure 5(e) the change of A from positive to negative changes the origin from a center to a saddle. From a physical point of view, this corresponds to a qualitative change in the traveling-wave dynamics by moving from a bounded oscillatory state to a saddle-type configuration, that is, in the local organization of the traveling wave dynamics. This outcome shows that the parameter A has a crucial role in the stability and the qualitative behavior of the neuronal-wave system, especially when it is reduced. In the phase portraits of Figures 5(f) and Figure 5(h), there are two nonzero equilibria, both of which are center, with a saddle at the origin. There are two families of bounded oscillatory states represented by the closed orbits about the nonzero centers and a saddle at the origin which divides them. Physically this structure means that the different bounded wave states coexist and it shows the potential for multiple dynamically distinct regimes of the reduced neuron model. In the case of Figure 5(g), only the origin is a real equilibrium and is saddle-type. The phase portrait is thus an unbounded or transition dominated dynamical organization and not a family of closed oscillatory states around the origin. This case also illustrates the role of the joint signs of the reduced nonlinear parameters in controlling the available dynamical states.

8. Conclusions

This study presents a comprehensive analytical and dynamical investigation of the time-fractional nonlinear neuron model (TFNNM), which provides a mathematical framework for describing the

propagation of nonlinear electrical impulses through neuronal media while accounting for the memory and hereditary characteristics of biological tissues. The generalized projective Riccati equation method (GPREM) and the extended tanh-function method (ETHFM) were successfully employed to construct several families of exact traveling-wave solutions, including dark, periodic, kink-type, rational, and singular wave structures. The fractional-order formulation enables the model to incorporate nonlocal temporal effects that are not represented in conventional integer-order descriptions of neuronal signal propagation. The physical interpretation of the obtained solutions provides several possible connections with neuronal signal dynamics. The dark-wave solutions represent localized reductions in the amplitude of neuronal activity relative to the surrounding background and may therefore be associated with localized inhibition or suppression of electrical activity in a neuronal medium. The periodic-wave solutions describe regularly repeating oscillatory patterns and may be related to rhythmic neuronal activity, repetitive firing, synchronized oscillations, and periodic signal transmission. In contrast, the kink-type solutions represent transition fronts connecting two different states of the neuronal medium and can be interpreted as mathematical representations of transitions between resting and excited membrane states. Such wave structures are particularly relevant to the description of excitation fronts and state switching during the propagation of electrical signals along neuronal membranes. These interpretations are consistent with the physical discussion of the wave profiles presented in the manuscript. The obtained results also demonstrate that the nonlinear and dispersive parameters play an important role in controlling the shape, localization, and propagation characteristics of the neuronal waves, whereas the fractional-order parameter provides a mechanism for incorporating memory effects into the dynamics. Consequently, variation of the fractional order can modify the propagation characteristics of the nonlinear waves and provides a mathematical means of investigating neuronal media with different memory responses. This feature may be useful in theoretical studies of biological tissues in which the present state of the system depends not only on its instantaneous state but also on its previous activity. From the dynamical systems perspective, the reduced ordinary differential equation was transformed into a planar Hamiltonian system with a conserved energy function. The equilibrium analysis revealed that the qualitative behavior of the reduced neuronal-wave system is strongly controlled by the parameters A , B , and C . Eight representative parameter regimes demonstrated several distinct phase-space configurations, including a center surrounded by saddle points, two center-type equilibria separated by a saddle at the origin, and regimes in which the nonzero equilibria become non-real. The closed trajectories around center-type equilibria represent bounded oscillatory wave regimes, whereas saddle points and their associated separatrix structures identify critical boundaries between different dynamical states. Thus, the bifurcation analysis provides additional information about the possible transitions between qualitatively different neuronal-wave behaviors. From an application-oriented viewpoint, these results provide a theoretical framework for interpreting different modes of nonlinear signal transmission in biological neural media. The dark and periodic structures can be used as mathematical representations of localized suppression and repetitive neuronal activity, respectively, while kink-type waves provide a useful description of propagating excitation or switching fronts. The coexistence of centers and saddles in the phase plane further indicates the possibility of multiple bounded and transition-type dynamical states within the reduced neuronal system. Therefore, the analytical solutions and bifurcation structures obtained in this work can serve as mathematical tools for studying how nonlinearities, dispersion, propagation characteristics, and memory effects influence

neuronal signal transmission.

Author contributions

Faisal Hayat Shaheen: Software, formal analysis, methodology, editing, writing original manuscript; Amal Alabdullatif: Investigation, validation, resources, funding acquisition, review, editing; Khizar Farooq: Software, review, and validation. All authors have read and agreed to publish the manuscript.

Use of Generative–AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest to report regarding the present study.

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