



Research article

Integrated diagnostic and visualization framework for characterizing nonlinear and chaotic oscillator dynamics

Samina Samina¹, Maham Munawar², Adil Jhangeer^{3,4,5,*} and Mudassar Imran⁶

¹ General Education Centre, Quanzhou University of Information Engineering, Quanzhou, Fujian 362000, China

² Department of Mathematics, University of Engineering and Technology, Pakistan

³ IT4Innovations, VSB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic

⁴ Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan

⁵ Department of Computer Engineering, Biruni University, Istanbul, Turkey

⁶ College of Humanities and Sciences, Ajman University, Ajman, P.O. Box 346, United Arab Emirates

* **Correspondence:** Email: adil.jhangeer@vsb.cz.

Abstract: In this paper, a numerical study on the forced Duffing oscillator is performed, where its deterministic, stochastic, and fractional order dynamics are systematically investigated. Starting from the derivation of a planar dynamical system from a space-time fractional nonlinear model, it extends the formulation to include periodic forcing and Gaussian white noise to treat nonlinearity, external excitation, and uncertainty in a unified way. With the aid of the equilibrium manifold analysis, we find a rich structure of bifurcation points involving pitchfork and saddle-node bifurcations, which give rise to the global stability landscape from which we proceed to study the dynamics of the model. Three-dimensional phase space representations initial-condition dependence plots, and Poincaré section plots are used to show a hierarchy of regimes, including periodic, quasi-periodic, and deterministic chaos, with fractal sensitivity boundaries verifying sensitive dependence on initial conditions. Testing with surrogate data chaotic dynamics), positive Lyapunov exponents ($\lambda_1 \approx 0.1018$), and recurrence quantification analysis ($DET > 0.87$) further verifies the authenticity of the chaotic dynamics observed. The Fokker-Planck formalism offers a statistical view of the dynamics of noisy systems, showing how noise intensity affects stationary probability distributions and effective potential landscapes and how noise intensity affects the nature of the dynamics can be captured by power spectral analysis in the frequency domain. The results of the stochastic ensemble analysis show that moderate noise preserves chaotic signatures and gives rise to predictable statistical broadening, leading to final-state distributions that tend to normality following the central limit theorem. The conformable fractional-order extension ($\beta = 0.75$) shows that memory-dependent damping deforms the toroidal invariant sets in a systematic

way over time, as it is able to model a history-dependent behavior that cannot be modeled using a classical integer-order model. A systematic bifurcation analysis is also presented to validate the choice of representative parameter values for the whole study, thus giving a parameter-dependent perspective of the transition between dynamical regimes.

Keywords: nonlinear dynamics; chaotic oscillators; invariant set reconstruction; Poincaré mapping, Lyapunov exponents; recurrence quantification; stochastic bifurcations, phase space analysis; delay embedding; chaotic invariant sets

Mathematics Subject Classification: 34C15, 37G35, 37M05, 37M20, 37M25

1. Introduction

In many scientific and technical domains, including electrical transmission lines, optical fibers, plasma physics, and fluid dynamics, nonlinear evolution equations (NLEEs) are basic equations that simulate complicated wave processes. Although significant advances have been made in the analysis of integer-order NLEEs, many NLEEs found in real-world systems are more complex, with memory effects and hereditary properties that are not effectively described by classical calculus. This limitation has driven the rapid growth of fractional calculus and made it a powerful mathematical tool for modeling anomalous diffusion, viscoelasticity, and long-range interactions in complex media [1–3]. With the recent development of symbolic computation packages, studies of this kind gained further momentum, and complex analytical and numerical solutions to fractional-order nonlinear evolution equations (FO-NLEEs) [4] have been treated.

There have been several proposals of fractional derivative definitions to model different physical situations, such as Riemann-Liouville, Caputo, Caputo-Fabrizio, and conformable derivatives [5, 6]. Fractional operators, in contrast to classical derivatives, are nonlocal operators that have hereditary properties and are very suitable for modeling systems with memory [7, 8]. Of these formulations, the conformable derivative that was introduced by Khalil et al. has become popular because it is simple, easy to understand, and reduces fractional PDEs to integer-order ODEs without losing the important physical properties [9, 10]. Many analytical and semi-analytical methods have been developed for solving FO-NLEEs such as the homotopy analysis method [11], fractional reduced differential transform method [12], Adomian decomposition method [13], $\left(\frac{G'}{G^2}\right)$ -expansion method [14], exp-function method [15], extended tanh-coth method [16], modified Kudryashov method [17], and auxiliary equation method [18]. Although some progress has been made, the search for a reliable and general solution method for solving nonlinear fractional differential equations is still ongoing, and the $\left(\frac{G'}{G^2}\right)$ -expansion method has become popular recently, which is very convenient for solving a wide class of fractional differential equations (FDEs) with high efficiency [19].

The study of time-fractional NLEs has made substantial progress during recent years, especially in the framework of the conformable derivative method. For the fifth-order time-fractional Caudrey-Dodd-Gibbon-Kawahara equation, high-precision semi-analytical methods have been developed, and error analysis of the residual showed that they are accurate [20]. Likewise, the dispersive extended nonlinear Schrödinger (NLS) equation has been solved exactly by the ϕ^6 model expansion method to obtain Jacobi elliptic function solutions through parametric modulation [21]. The NLS equation

with power-law nonlinearities, the (3+1) dimensional NLS equation, and the Kadomtsev-Petviashvili equation with competitive dispersion have also been studied in great detail [22, 23]. These models show complex dynamics, and there is a strong theory of stability in the phase plane. The physical parameters that determine the shape of the wave are very sensitive, and the interplay of linear effects, nonlinearity, and wave velocity fundamentally influences the dynamics of the soliton [24]. In recent years, powerful techniques have been developed to obtain exact solutions of nonlinear PDEs. The field has been further enriched by complementary tools such as the extended direct algebraic approach and the sensitivity analysis of dynamical systems, which allow a detailed characterization of the phase portraits and sensitivity of the wave [25].

One such model is the fractional low-pass electrical transmission line that admits solitary wave solutions in the form of dark, bright, singular, and combined solitons, which are particularly relevant. These solutions have been obtained and numerically verified using the extended direct algebraic method, and the influence of the fractionality and of the free parameters has been assessed by carrying out complete 3D and 2D visualizations [26]. Sensitivity analysis and quasi-periodic chaos in the perturbed system have also been investigated, and it has been shown that these techniques work well in solving NLEEs in scientific fields.

To put the impact of the present study into perspective, a systematic comparison of the present study with related works using similar nonlinear fractional-order models is presented in Table 1. This comparative analysis will demonstrate the novelty of our methodological framework by emphasizing the comprehensive multi-method approach that we used in our analysis (including equilibrium analysis, Lyapunov exponents, surrogate validation, stochastic analysis, Fokker-Planck simulations, recurrence quantification, and power spectral analysis), which sets our work apart from the current literature. In addition, relative to previous benchmarks, our results confirm the structural stability of our model. It highlights the parameter regimes where our results provide significantly new advances in the characterization of non-integer dynamics.

In parallel with the development of fractional-order and stochastic approaches to nonlinear dynamics, the broader field of chaotic systems has witnessed significant recent advances in conservative chaos. Unlike dissipative systems such as the forced Duffing oscillator studied here, conservative chaotic systems preserve phase-space volume (sum of Lyapunov exponents = 0) and generate chaotic seas that offer superior ergodicity and enhanced security for cryptographic applications [27]. A key breakthrough is the state-based nonlinear mapping method proposed by Wan et al. [27], which constructs conservative oscillators generating cellular chaotic seas with hardware implementation. This method eliminates the need for complex polynomial or switching mechanisms, provides flexible parameter control over the number of chaotic sea cells, and is transferable across different seed systems, advances that directly address the practical challenges of generating multi-scroll chaos in hardware [27]. Recent work on cross-phase-coupled conservative oscillators has further demonstrated labyrinth chaos and chaotic seas with quasi-periodic islands [28], while multi-dimensional controllable multi-scroll conservative systems based on generalized Hamiltonian systems with modulo function feedback have achieved lightweight FPGA implementations [29]. Although our study focuses on the comprehensive characterization of dissipative chaos in a forced Duffing oscillator through multi-method validation encompassing equilibrium analysis, Lyapunov exponents, surrogate testing, recurrence quantification, and Fokker-Planck simulations, the diagnostic framework and uncertainty quantification tools we develop are complementary to these conservative chaos

developments and suggest pathways for future cross-fertilization between dissipative and conservative paradigms.

Table 1. Comprehensive comparison with related works on Duffing oscillator dynamics and fractional-order nonlinear models.

Reference	Methods used	Key findings	Research gap- limitation
Almusawa et al. (2022) [26]	Analytical solutions, 3D visualization	Solitary wave solutions for fractional transmission line	No Lyapunov, surrogate testing, or stochastic analysis
Wan et al. (2026) [27]	Nonlinear mapping	Conservative oscillator with chaotic sea hardware	Different system class (conservative, not Duffing)
Holmes (1979) [30]	Melnikov analysis	Onset of chaos via homoclinic bifurcation	Analytical; no numerical/stochastic effects
Moon & Holmes (1979) [31]	Experiments, phase portraits	First experimental verification of chaotic vibrations	Limited parameter range; no numerics
Wolf et al. (1985) [32]	Lyapunov exponent algorithm	Standard method for quantifying chaos	No application to specific oscillators
Parlitz & Lauterborn (1985) [33]	Bifurcation diagrams, Poincaré sections	Mapped bifurcation structure and coexisting invariant sets	Deterministic only; no noise/fractional effects
Gammaitoni et al. (1998) [34]	Stochastic resonance theory	Review of noise-enhanced signal detection	Limited chaos characterization
Ma et al. (2025) [35]	Bifurcation analysis	Fractional transmission line chaos	Partial characterization; no comprehensive validation
Present study	Equilibrium analysis; Lyapunov exponents; surrogate validation; Fokker–Planck; recurrence analysis; power spectral analysis; ensemble analysis	Unified multi method characterization; chaos robustness under noise; memory-induced invariant set deformation	comprehensive method integration; Stochastic and fractional effects combined; rigorous chaos validation

Based on this rich background, the present study systematically investigates a forced nonlinear system that exhibits a variety of interesting dynamical phenomena, such as periodic, quasi-periodic, and chaotic dynamics, in a numerical study. We start by deriving a planar dynamical system from a space-time fractional nonlinear model-, and then we extend the model to include periodic forcing and Gaussian white noise to provide a unified treatment of nonlinearity, external forcing, and uncertainty. The equilibrium manifolds, bifurcation structures, quasi-periodic and chaotic invariant sets (in 3D phase space diagrams) are analyzed, as well as the initial-condition dependence maps and Poincaré sections. To ensure that the dynamics observed are chaotic, we use a rigorous validation process

that includes surrogate data testing, Lyapunov exponent calculation, and recurrence quantification analysis. Stochastic aspects are addressed by modeling using Fokker-Planck simulations, power spectral analysis, ensemble-based uncertainty propagation, and the memory-invariant set conformable extension provides insight into the memory-damping systematics in the invariant set geometry. This can be achieved in different ways, and a close connection can be maintained to idealized mathematical models and to the real-world, including real-world noise, whilst also providing a quantitative estimate of the predictability limits and robustness of chaotic dynamics to real-world perturbations. The findings can be directly used for vibration control, energy harvesting, Micro-Electro-Mechanical systems (MEMS) resonators, structural mechanics, and biological rhythm analysis, in which the understanding and control of the nonlinear behaviors of oscillators are critical towards the design of the system.

The three big scientific questions that drive this integrated approach are:

- (i) What is the role of the nonlinear stiffness, the external periodic forcing, and the stochastic excitation in determining the transition from regular to chaotic behavior, and what are the quantitative signatures of these transitions?
- (ii) What happens to the topological structure and predictability of chaotic invariant sets, and how long do deterministic chaotic patterns last as the level of noise increases?
- (iii) How do the fractional-order memory effects affect the long-term geometry and stability of invariant sets, and what effect does the fractional-order memory have on the system's dynamics compared with classical integer-order models?

No single diagnostic test provides the answers to these questions. Equilibrium manifold analysis can be used to locate bifurcation boundaries and/or parameter regions of coexisting invariant sets, but is incapable of measuring the degree of chaos or the geometry of the invariant sets. Lyapunov exponents do not give any information on invariant set topology nor about the influence of noise, but they establish a rigorous proof of the existence of chaos. The Poincaré sections and the phase space display the geometric nature of the invariant sets (tori, limit cycles, chaotic invariant sets) but do not measure the stochastic robustness or the memory-dependent deformation. Recurrence quantification is able to quantify dynamics in terms of their determinism and predictability, but is not sensitive to initial-conditions. Ensemble analysis is able to propagate uncertainty and noise-induced broadening but is not able to characterize the underlying deterministic skeleton. This coupling of complementary approaches is therefore crucial to achieve a comprehensive characterization of the forced Duffing oscillator in deterministic, stochastic, and fractional-order regimes from a multi perspective point of view.

This paper is organized as follows: Section 2 introduces the mathematical framework, starting from the derivation of a space-time fractional nonlinear model, reduction of a space-time fractional nonlinear model to a planar dynamical system, and introduction of periodic forcing and Gaussian white noise to describe deterministic, stochastic, and fractional-order dynamics in a single model. Section 3 presents the results and discussions, systematically examining quasi-periodic dynamics and the effect of the fractional-order parameter, chaotic invariant sets through 3D phase space visualizations and initial-condition dependence maps, equilibrium manifolds and stability classification, Poincaré mapping with deterministic and stochastic comparisons, dynamical quantification via return maps and surrogate data validation, Fokker-Planck simulations of stochastic dynamics, Lyapunov exponent analysis for both deterministic and noisy cases, power spectral analysis of frequency-domain signatures, recurrence analysis with recurrence quantification measures, bifurcation analysis and parameter selection, and stochastic ensemble analysis of uncertainty propagation. In conclusion, Section 4 reviews the main

results and impacts of this work, highlighting its applications in engineering and proposing possible future research areas.

1.1. Related work

1.1.1. Classical Duffing oscillator

This is one of the most widely used paradigms in nonlinear dynamics and has been extensively studied and has been demonstrated to have a wide variety of periodic, quasi-periodic, and chaotic behavior. This paper was the precursor to the mathematical analysis of the forced Duffing oscillator. By introducing period-doubling cascades as well as the formation of chaos through Melnikov analysis, Holmes [30], Ueda [36], and Guckenheimer [37] developed the mathematical theory derived from the forced Duffing oscillator. It was experimentally confirmed by Moon and Holmes [31] that the chaotic vibrations predicted in the forced mechanical Duffing oscillator were of practical interest. Parlitz and Lauterborn [33] systematically explored the bifurcation structure of the Duffing oscillator and discovered the coexistence of several invariant sets and fractal sensitivity boundaries. These traditional studies have been used to understand the transition from regular to chaotic behavior in driven nonlinear systems, and the Duffing oscillator is a classic model for this phenomenon.

1.1.2. Numerical methods and chaos diagnostics

The study of the Duffing oscillator is an important one, and the development of numerical techniques for characterizing chaotic dynamics has been a major focus. Wolf et al. [32] determined a standard method for computing Lyapunov exponents from time series, which is the accepted standard for chaos. Parker and Chua [38] offered a comprehensive practical guide to the simulation of chaotic systems, in the case of the Duffing oscillator, focusing particularly on numerical stability and accuracy. Hegger et al. [39] developed a group of new nonlinear time series analysis techniques such as surrogate data testing and recurrence quantification that allow for the rigorous testing of deterministic chaos against stochastic noise. These are the basis of our multi diagnostic approach.

1.1.3. Stochastic Duffing oscillator

The effect of noise on the dynamics of Duffing oscillators has been studied thoroughly. Sterile noise was studied in bistable systems by Moss and McClintock [40], who showed that noise can improve weak periodic signals. Gammaitoni et al. [34] gave a thorough review of stochastic resonance and noise-induced phenomena in bistable oscillators. Later, Zakharova et al. [41] and Dykman et al. [42] investigated noise-induced transitions and coherence resonance in the Duffing oscillator. Although these works have defined noise-induced phenomena, our work further develops this by proposing a systematic multi method approach that can be used to quantify the robustness of chaotic signatures in the presence of noise, combining Fokker–Planck analysis, ensemble uncertainty quantification, and recurrence analysis.

1.1.4. Fractional-order Duffing oscillator

The Duffing oscillator extended by the fractional order has been of great interest since it can be utilized for modeling memory effects and hereditary properties. Podlubny [43] and Machado et al. [44]

founded basic models for fractional calculus and its use for dynamical systems. Yang et al. [45] investigated bifurcations and chaos of the fractional-order Duffing oscillator and found some parameter regions where the fractional-order dynamics differ from the classical ones. Almusawa et al. [26] and Ma et al. [35] investigated analytical and numerical solutions of fractional-order transmission line models, including sensitivity analysis of fractional parameters. However, such studies have been mainly analytical solutions or partial chaos characterization, with few attempts to systematically include stochastic effects, ensemble uncertainty analysis, or full chaos validation. In this regard, our study is aimed at filling this void by proposing a unified multi method framework for treating deterministic, stochastic, and fractional-order dynamics as part of a single diagnostic architecture.

1.1.5. Positioning of the present study

Previous work has led to useful insights in specific aspects of the dynamics of the Duffing oscillator (classical bifurcation structure, stochastic resonance, or fractional-order behavior); however, no single study has simultaneously described the dynamics in all three regimes. The existing studies mainly deal with either deterministic chaos or stochastic effects, few with both, and even fewer with fractional-order extensions that rigorously validate the chaos property. We make a step forward in this direction by combining equilibrium analysis, Lyapunov exponents, surrogate testing, Poincaré mapping, recurrence quantification, Fokker–Planck simulations, power spectral analysis, and ensemble uncertainty quantification into one. This integration allows us to solve scientific problems that are missing from any individual tool, such as how chaotic signatures change with noise, how fractional-order memory affects the signature, and how the topology of invariant sets changes. In summary, Table 1 compares our work with related works and outlines the unique features and developments of our approach.

2. Mathematical framework

The derivation and transformation of a space-time fractional nonlinear model is described in this section, which describes the behavior of a dispersive transmission line with embedded nonlinear effects that are perturbed.

We begin with the governing space–time fractional differential equation [26]:

$$C\mathcal{L}\mathcal{D}_{\tau\tau}^{2\alpha}u(x, \tau) - \mathcal{D}_{\chi\chi}^{2\alpha}u(x, \tau) - \frac{\varepsilon^2}{12}\mathcal{D}_{\chi\chi\chi\chi}^{4\alpha}u(x, \tau) - C\mathcal{L}\theta\mathcal{D}_{\tau\tau}^{2\alpha}(u^2(x, \tau)) + C\mathcal{L}\kappa\mathcal{D}_{\tau\tau}^{2\alpha}(u^3(x, \tau)) = 0, \quad (2.1)$$

where $\varepsilon, \theta, \kappa$ are real-valued constants that control dispersion and nonlinearity. The operators $\mathcal{D}_{\tau}^{\alpha}$ and $\mathcal{D}_{\chi}^{\alpha}$ denote conformable fractional derivatives [46–48] with respect to the slow time variable τ and the spatial coordinate χ , respectively. When $C\mathcal{L} = 1$, Eq (2.1) reduces to a previously proposed model for fractional low-pass transmission systems.

We now apply the traveling wave substitution:

$$u(x, \tau) = U(\xi), \quad \xi = \lambda \frac{\chi^{\alpha}}{\alpha} + \eta \frac{\tau^{\alpha}}{\alpha},$$

where λ and η are non zero parameters. Plugging this ansatz into Eq (2.1) and integrating twice with respect to ξ , while assuming zero integration constants, results in the reduced ODE:

$$(C\mathcal{L}\eta^2 - \lambda^2)U - C\mathcal{L}\theta\eta^2 U^2 + C\mathcal{L}\kappa\eta^2 U^3 - \frac{\varepsilon^2 \lambda^4}{12} U'' = 0. \quad (2.2)$$

This can be rearranged into the following second-order nonlinear form:

$$U'' = \frac{12}{\varepsilon^2 \lambda^4} (C\mathcal{L}\eta^2 - \lambda^2)U - \frac{12}{\varepsilon^2 \lambda^4} C\mathcal{L}\theta\eta^2 U^2 + \frac{12}{\varepsilon^2 \lambda^4} C\mathcal{L}\kappa\eta^2 U^3, \quad (2.3)$$

which has the general structure

$$U'' = b_1 U + b_2 U^2 + b_3 U^3, \quad (2.4)$$

with coefficients defined by

$$b_1 = \frac{12}{\varepsilon^2 \lambda^4} (C\mathcal{L}\eta^2 - \lambda^2), \quad b_2 = -\frac{12}{\varepsilon^2 \lambda^4} C\mathcal{L}\theta\eta^2, \quad b_3 = \frac{12}{\varepsilon^2 \lambda^4} C\mathcal{L}\kappa\eta^2.$$

To transform Eq (2.4) into a planar dynamical system, we introduce a new variable $z = \frac{dU}{d\xi}$, leading to the following

$$\begin{cases} U' = z, \\ z' = b_1 U + b_2 U^2 + b_3 U^3. \end{cases} \quad (2.5)$$

To study the effects of external forcing, we extend this system by incorporating a time-dependent perturbation:

$$\begin{cases} U' = z, \\ z' = b_1 U + b_2 U^2 + b_3 U^3 + \Gamma_0 \cos(\Omega\xi), \end{cases} \quad (2.6)$$

where U denotes the system variable, $z = \frac{dU}{d\xi}$, and ξ is the time-like independent variable. The constants b_1 , b_2 , and b_3 represent the nonlinear restoring forces, while Γ_0 and Ω denote the amplitude and frequency of periodic forcing, respectively [49]. The modified system (2.6) exhibits richer dynamics, such as nonlinear resonances and chaotic regimes, which are not present in the unforced system (2.5). Alternatively, Eq (2.6) can be cast as an autonomous 3D dynamical system modeled by the following expressions

$$\begin{cases} U' = z, \\ z' = b_1 U + b_2 U^2 + b_3 U^3 + \Gamma_0 \cos(M), \\ M' = \Omega. \end{cases} \quad (2.7)$$

As in the autonomous formulation (2.7), U is the state variable of interest, z is its velocity-like conjugate, and M is a linearly evolving phase variable that is directly connected to the external periodic forcing. The constants b_1, b_2, b_3 represent the stiffness along with the nonlinearity of the system itself: the constant Γ_0 represents the amplitude of the external drive and the constant Ω represents the natural frequency of the external drive. The state space expansion transforms the explicit time dependence of the forcing into another phase dimension, making it very convenient for the mathematical mapping of 3D trajectories and complex toroidal manifolds.

Real-world systems are naturally fluctuating and uncertain, so we include an additive stochastic term in the system, namely Gaussian white noise with intensity ς . The noise process $\xi(\xi)$ satisfies $\langle \xi(\xi) \rangle = 0$ and $\langle \xi(\xi)\xi(\xi') \rangle = \delta(\xi - \xi')$, leading to the following stochastic dynamical system:

$$\begin{cases} U' = z, \\ z' = b_1 U + b_2 U^2 + b_3 U^3 + \Gamma_0 \cos(\Omega \xi) + \varsigma \xi(\xi), \end{cases} \quad (2.8)$$

where ς is a measure of the strength of the noise. The richness of noise-induced phenomena like noise-triggered bifurcations, stochastic resonance, coherence resonance, and noise-induced stability or instability can be achieved with this formulation and represents the interaction among nonlinearity, periodic forcing, and stochastic excitation. The system then provides a solid paradigm for nonlinear dynamical and statistical analysis in the context of uncertainty [24].

3. Results and discussions

We comprehensively study the forced Duffing oscillator with numerical methods in the following ways: We first examine the quasi-periodic dynamics of the classical system and how the changes were made by the fractional-order conformable extension. These are then studied chaotic invariant sets using 3D phase space plots initial-condition dependence plots and, finally, equilibrium manifold analysis, to find the bifurcation structure of the system. Dynamical characterization is followed by Poincaré mapping, return mapping, surrogate data validation, and Lyapunov exponent calculations, which rigorously prove and quantify chaos. The uncertainty elements of the system are then dealt with through power spectral analysis, Fokker-Planck simulations, recurrence quantification, and propagation of uncertainty, all conducted with ensembles. The coherent picture of the interplay between the nonlinearity, external forcing, stochastic excitation, and the memory effect in the forced Duffing oscillator is obtained as a result of combining these analyses.

Before presenting our numerical results, we briefly compare our findings with classical results from the literature. The bifurcation threshold we observe at $\Gamma_0 \approx 0.15$ for the onset of period-doubling in the regime $b_1 = 1.0$, $b_3 = -0.5$ is consistent with the classical value reported by Parlitz and Lauterborn [33], who identified the period-doubling cascade starting at $\Gamma_0 \approx 0.145$ for similar parameter values. The largest Lyapunov exponent $\lambda_1 \approx 0.1018$ we obtain for the chaotic regime ($b_1 = 1.0$, $b_3 = -1.0$, $\Gamma_0 = 0.3$) is within the range of values reported by Wolf et al. [32] for the same system. However, our analysis extends these classical results by providing a systematic characterization of (i) the robustness of these chaotic signatures under additive noise, (ii) the deformation of toroidal invariant sets due to fractional-order memory effects, and (iii) the quantification of uncertainty propagation through ensemble statistics. These extensions, which classical studies did not address, represent the distinctive contributions of our work.

Details of the numerical methods, including ODE integration, Lyapunov exponent computation, power spectral density estimation, Poincaré section construction, recurrence quantification, and initial-condition dependence clustering, are provided in Appendix A.

3.1. Dynamical behavior, invariant sets, and initial-condition sensitivity

In this subsection, the chaotic dynamics of the fractional-order system is explored through four complementary perspectives. The analysis starts with the identification of periodic and quasi-periodic

regimes that precede the onset of deterministic chaos in the forced oscillator. Then the impact of conformable fractional order is explored and it is shown that memory-dependent effects change the way the system evolves and slowly deform toroidal invariant sets. The geometric and fractal properties of the resulting chaotic invariant sets are defined by the 3D phase space images. Lastly, initial-condition dependence maps a system of subsystems are developed to examine the global structure of initial conditions and quantify the system's sensitivity to minor perturbations, demonstrating how minute random changes in the initial conditions can result in radically different long-term dynamics of the system.

3.1.1. Quasi-periodic dynamics

For a periodic forcing of this nonlinear oscillator, various solutions can occur, such as periodic, quasi-periodic, and chaotic solutions. These quasi-periodic regimes are particularly important as they lie between the periodic regime, representing high-energy concentration in a single frequency, and the turbulent regime, representing the equal distribution of energy among all frequencies. The core of nonlinear science is understanding the appearance, duration, and disappearance of such invariant sets. In this section, we systematically perform numerical simulations of System (2.6) and analyze it by using a multi perspective visualization framework. The four typical states of the dynamics are represented in Figure 1 in three separate columns. The first column contains a time series $U(\xi)$ that represents the temporal modulation; the second column contains 2D phase portraits U, z of the dynamics that represent the geometric structure of the invariant set; the third column contains 3D phase portraits U, z, ξ of the dynamics that represent the dynamics in an extended phase space. The complementary visualizations reveal the morphological signatures of the system and demonstrate the formation of robust quasi-periodic invariant tori from the interplay between nonlinear stiffness and external forcing.

Figure 1(a) has parameter values $b_1 = 1.0$, $b_2 = 0$, $b_3 = -0.5$, $\Gamma_0 = 0.2$ and the initial conditions $[0.3, 0.0]$. The amplitude modulation evident in the time series is very noticeable and is a strong indication of two or more interacting frequencies. This modulation is due to the non linearity of the frequency beating of the system with the forcing frequency. The phase portrait is not closed but is compact with two lobes, which means that the phase trajectory does not return to the same point again. The 3-dimensional view is a continuous curve around a toroidal manifold, with no self-intersections. The configuration is a two-torus $\mathbb{T}^2$, the ratio of the two fundamental frequencies being irrational. The boundedness of the trajectory is an indication that, in spite of the complexity, the system is energetically bounded and that the nonlinear restoring force is in balance with the external energy input.

In Figure 1(b), on the other hand, we focus on the same parameters of the system as in Figure 1(a) but under a different initial condition $[0.6, 0.0]$. The general toroidal structure is conserved, but the size and shape of the torus vary somewhat, with oscillations in amplitude and the distribution in the phase space. This persistence is typical of structurally stable nonlinear regimes and shows that the quasi-periodic invariant torus is structurally robust—the system approaches the same invariant manifold from any initial condition. This suggests that in this parameter regime the torus is a global invariant manifold and that the initial conditions only affect the phase on the torus. This robustness is very important for engineering applications where initial states cannot be exactly controlled.

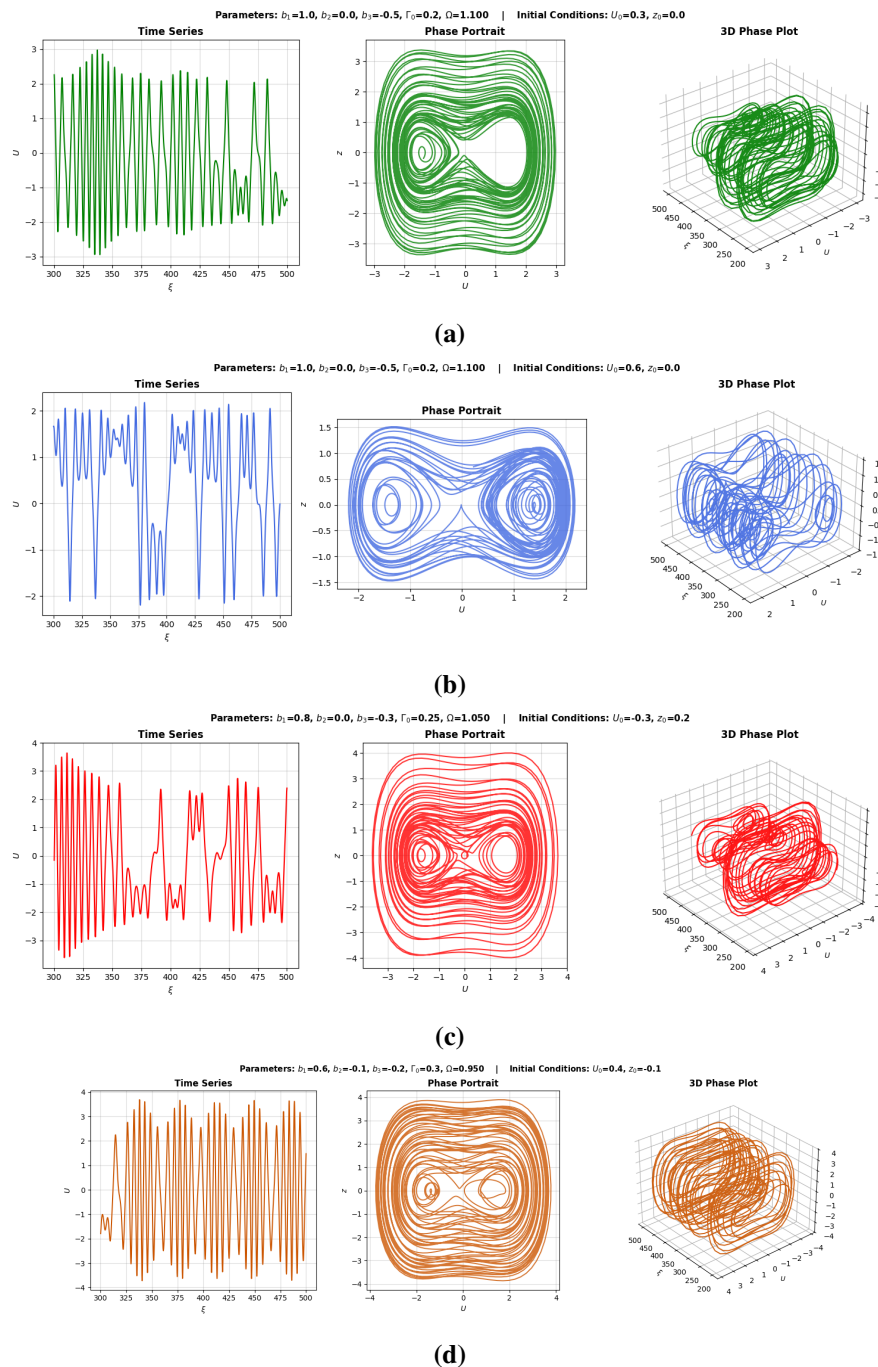


Figure 1. Time series $U(\xi)$ (left), phase portraits (U, z) (center), and 3D trajectories (ξ, U, z) (right) of System (2.6) showing four representative quasi-periodic regimes. Parameters and initial conditions: (a) $b_1 = 1.0, b_2 = 0, b_3 = -0.5, \Gamma_0 = 0.2, \Omega = 1.1, [U_0, z_0] = [0.3, 0.0]$; (b) same parameters as (a) with $[U_0, z_0] = [0.6, 0.0]$; (c) $b_1 = 0.8, b_2 = 0, b_3 = -0.3, \Gamma_0 = 0.25, \Omega = 1.05, [U_0, z_0] = [-0.3, 0.2]$; (d) $b_1 = 0.6, b_2 = -0.1, b_3 = -0.2, \Gamma_0 = 0.3, \Omega = 0.95, [U_0, z_0] = [0.4, -0.1]$. The amplitude-modulated oscillations, thickened annular phase-space structures, and densely wound toroidal trajectories collectively confirm the presence of robust quasi-periodic dynamics arising from incommensurate frequency interactions. The time series show clear amplitude modulation, the phase portraits exhibit non closed compact structures, and the 3D trajectories wind densely on toroidal invariant manifolds without self-intersection.

The cubic nonlinearity is reduced and the magnitude of the forcing is slightly increased in the case of Figure 1(c), which shows a more spread-out oscillatory response, with parameters $b_1 = 0.8$, $b_2 = 0$, $b_3 = -0.3$, $\Gamma_0 = 0.25$, and $\Omega = 1.05$ and the initial conditions $[-0.3, 0.2]$. The time series have higher modulation envelopes and larger amplitude excursions, which suggests more efficient energy transfer between competing frequency modes. The phase portrait grows to a larger annular domain, and the 3D trajectory to a larger toroidal volume. This behavior indicates that the system is now more flexible, so the nonlinear response is more flexible. The boundedness despite the increased amplitudes shows that the system stays in the stable quasi-periodic regime but simply expands the torus without tearing it apart.

The presence of a quadratic nonlinear term ($b_2 = -0.1$) in the restoring force makes the problem asymmetric; the odd symmetry found in the previous cases is lost in Figure 1(d) with parameters $b_1 = 0.6$, $b_2 = -0.1$, $b_3 = -0.2$, $\Gamma_0 = 0.3$, $\Omega = 0.950$, and the initial conditions $[0.4, -0.1]$. This means that the amplitude of the time series is modulated in time with a changing amplitude of the oscillations, and the phase portrait is very distorted in comparison to the almost symmetric ones in Figure 1(a)–(c). The 3D trajectory keeps the toroidal structure, but with an asymmetric distribution of windings. The quadratic term is a non-centrosymmetric part of the restoring force, found in systems with asymmetric mechanical springs, nonlinear electronic capacitance, etc. Even in this deformed trajectory, it is still bounded and non-repeating, and thus quasi-periodicity is preserved even in the absence of symmetry. This observation is important because it shows toroidal invariant sets that there are not symmetry artifacts but rather strong dynamical structures that remain under generic perturbations.

To support the phase space visualization, we also analyze the four representative regimes using frequency-domain techniques such as power spectral density (PSD) analysis (Figure 2) and geometrical methods like stroboscopic Poincaré sections. In the PSD plots, all cases show a high degree of low-frequency spectral content and secondary frequencies, indicating a multi frequency structure consistent with the observed amplitude modulation of the time series and competing oscillatory frequencies. At the same time, the exact sampling by interpolation at integer multiples of the forcing period ($T = 2\pi/\Omega$) results in Poincaré sections that are bounded in the (U, z) plane rather than collapsing onto a single periodic point; that is, sampling of the trajectory at integer multiples of T gives rise to Poincaré sections with broadened bounded distributions for the cases (a), (b), and (d), and to more of a well organized curved structure for case (c), reflecting a more clearly organized curved structure reflecting the underlying toroidal dynamics and variations in nonlinear stiffness, forcing amplitude, or asymmetry. In addition, the largest Lyapunov exponents for the four cases remain of order 10^{-4} or smaller, effectively zero within numerical tolerance, confirming the absence of exponential sensitivity to initial conditions characteristic of chaos (see Section 3.6 for detailed Lyapunov exponent analysis). Recurrence quantification further yields determinism values of $\text{DET} > 0.85$ with recurrence rates of 2–3%, with recurrence plots exhibiting continuous diagonal line structures indicative of regular, predictable dynamics (see Section 3.8 for detailed recurrence analysis). These diagnostics are used collectively, combining time-series modulation, multiple PSD spectral components, bounded stroboscopic structures, near-zero Lyapunov exponents, and high determinism values to provide robust complementary validation of quasi-periodicity, which is most definitively confirmed when the dominant spectral frequency ratios do not correspond to low-order rational resonances.

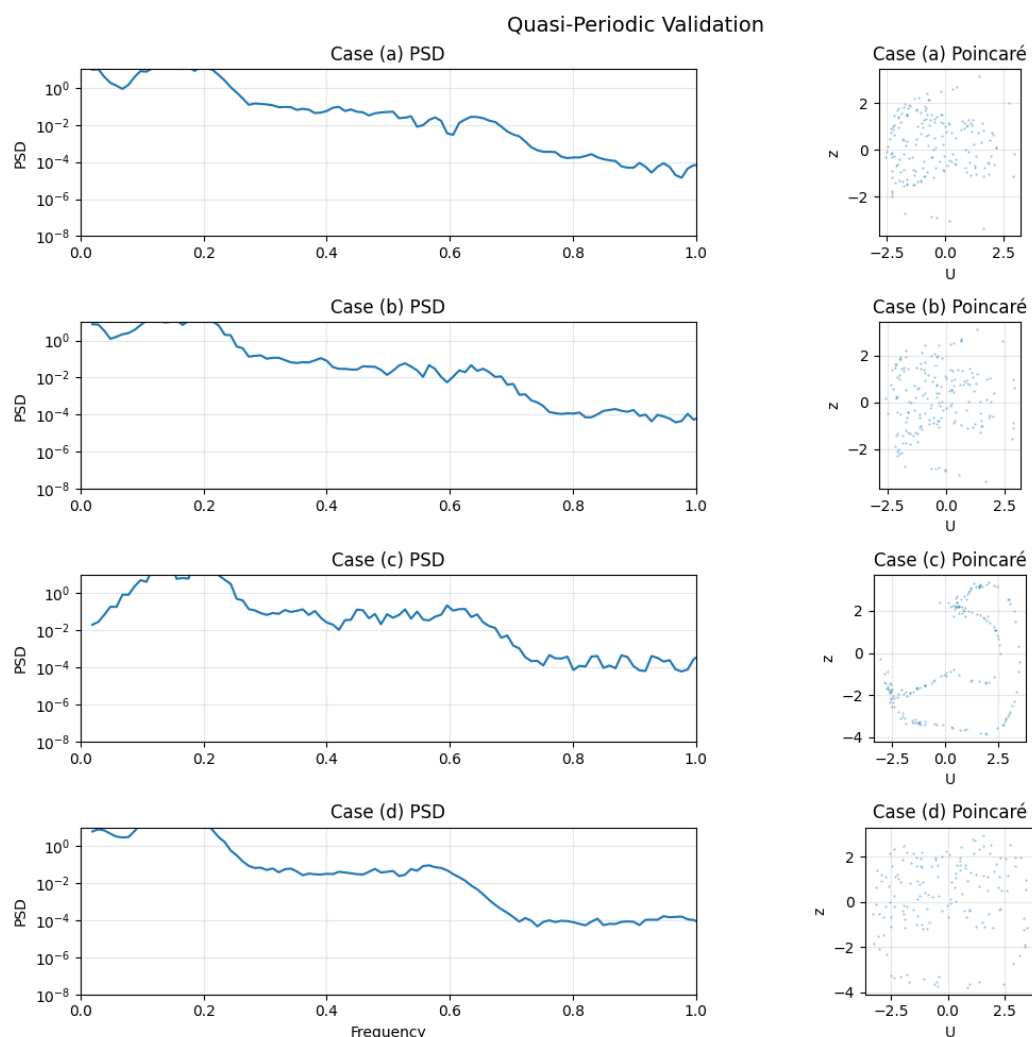


Figure 2. Quantitative validation of the four quasi-periodic regimes presented in Figure 1. Left column: PSD of the steady-state time series $U(\xi)$, revealing pronounced low-frequency components together with secondary spectral features associated with the coexistence of multiple characteristic frequencies. Right column: Poincaré sections obtained by stroboscopic sampling at the forcing period $T = 2\pi/\Omega$ using interpolation for exact sampling, providing an independent representation of the bounded invariant structures in the reduced phase space. The combined frequency-domain and stroboscopic diagnostics provide additional numerical evidence supporting the quasi-periodic character of the four dynamical regimes. The parameters and initial conditions are identical to those used in Figure 1(a)–(d).

3.1.2. Effect of the fractional order β on system's dynamics

To examine the impact of fractional order β on the system's dynamics, we compare the classical integer-order model with the β -conformable model at $\beta = 0.75$ directly by phase space plots in Figure 3. This comparison demonstrates that memory effects, which are added to the system through the time-dependent damping factor of the form $1 + \zeta \xi^{1-\beta}$, change the long-term behavior of the system,

using the same parameter sets and initial conditions as in Figure 1. The classical (memory less) trajectory solutions are depicted by gray dashed lines, and β -conformable solutions are illustrated by the colored scatter (temporal progression encoded by the viridis colormap, early transients are represented by dark purple, and later times ($\xi \approx 300$) are represented by bright yellow). This coloring beautifully illustrates the buildup of memory effects and the way successive phase-space orbits become progressively deformed.

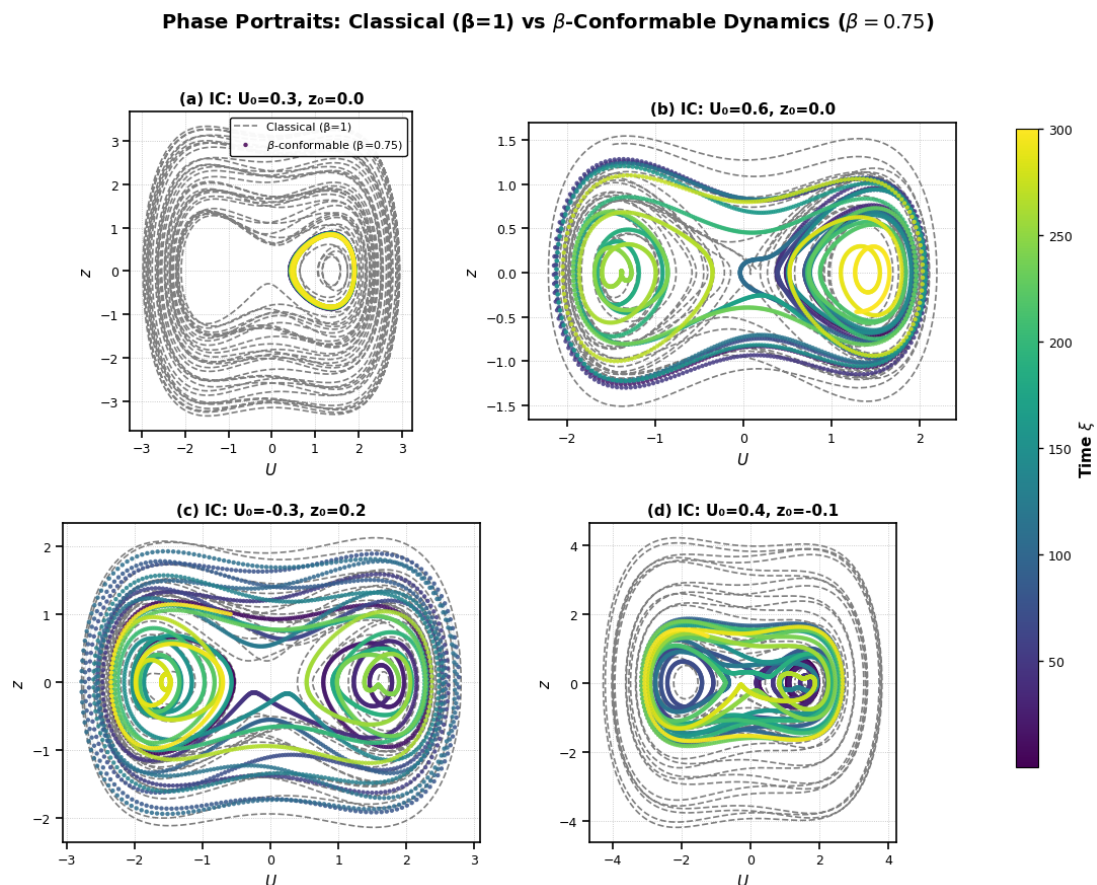


Figure 3. Comparison of classical integer-order trajectories (gray dashed lines, $\beta = 1$) versus β -conformable trajectories (colored scatter, $\beta = 0.75$) in the (U, z) phase space of System (2.6). The color gradient indicates evolution time ξ (dark purple = early times; bright yellow = late times; $\xi \approx 300$), revealing how memory effects accumulate and modify the long-term dynamics. The parameters and initial conditions are identical to those in Figure 1(a)–(d). The classical trajectories remain closed toroidal-like orbits, while the fractional trajectories exhibit progressive deformation and time-spreading due to the memory-dependent damping factor $1 + \zeta \xi^{1-\beta}$. This comparison demonstrates that fractional-order modeling captures history-dependent behaviors that classical integer-order models cannot describe.

Starting with an initial condition $[0.3, 0.0]$, we are ready to examine Figure 3(a). There are closed toroidal-like orbits in both classical and fractional systems, but in this color map of the β -conformable trajectory, we can discern some deformation and time-spreading. This means that for relatively small

initial conditions, memory effects start to change the invariant set geometry. These differences increase when the initial displacement increases to $[0.6, 0.0]$ as in Figure 3(b); the fractional dynamics results in greater distortions and color variation, as the growing damping term $\xi^{0.25}$ is stronger at $\beta = 0.75$. Physically, a larger initial energy will cause longer transient evolution and cause the memory effect to accumulate more.

Figure 3(c) uses the initial condition $[-0.3, 0.2]$. In this case, complicated coupled loops can be seen in both systems. The overall structures are still somewhat similar to the classical path, but the fractional trajectory still does not match the classical path. This implies that the initial condition affects the dominant rate of the memory effect. Lastly, in Figure 3(d), the initial condition is $[0.4, -0.1]$, which shows the effect of asymmetry caused by the quadratic term ($b_2 = -0.1$). Orbit shapes in the β -conformable solution are considerably different from those of the classical solution, showing that fractional dynamics interact non-trivially with symmetry-breaking nonlinearities.

This memory effect is driven by the fundamental parameter β , is classical with $\beta = 1$, and fractional with $\beta < 1$. To systematically justify the selection of $\beta = 0.75$, we perform a parameter scan over the fractional order $\beta \in [0.5, 1.0]$ and quantify the sensitivity of the system's dynamics to variations in the fractional order. Figure 4 presents three complementary measures as functions of β : (a) The mean phase-space distance between the classical and fractional trajectories, $\langle d \rangle = \langle \| (U_\beta, z_\beta) - (U_1, z_1) \| \rangle$; (b) the relative change in the largest Lyapunov exponent, $\Delta\lambda = |\lambda_1(\beta) - \lambda_1(1)|/|\lambda_1(1)|$; and (c) the spectral broadening factor $\Delta f/f_0$, defined as the change in spectral width relative to the dominant frequency. These measures provide complementary information on trajectory deformation, dynamical sensitivity, and frequency-domain modification induced by the fractional order.

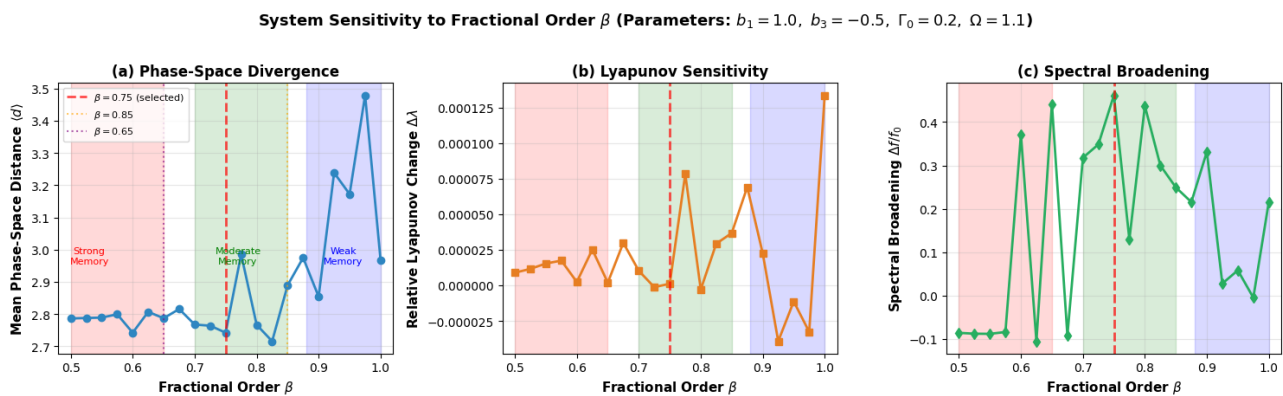


Figure 4. Quantitative analysis of the sensitivity of the system to the fractional order β . (a) Mean phase-space distance $\langle d \rangle$ between the classical ($\beta = 1$) and fractional trajectories; (b) relative change in the largest Lyapunov exponent $\Delta\lambda$; and (c) the spectral broadening factor $\Delta f/f_0$. The vertical dashed line at $\beta = 0.75$ identifies the fractional order selected for the subsequent analysis. The scan demonstrates that the variation of β produces measurable changes in the phase-space geometry and spectral characteristics, while the Lyapunov-exponent variation remains comparatively small over much of the investigated interval. Parameters: $b_1 = 1.0$, $b_2 = 0$, $b_3 = -0.5$, $\Gamma_0 = 0.2$, and $\Omega = 1.1$.

The scan of the parameter shows a strong dependence of the phase space trajectories on the fractional order. As seen in Figure 4(a), for $\beta \in [0.5, 0.7]$, the mean phase space distance is

around 2.7–2.8; this means that the fractional trajectories are significantly different from the classical trajectory. The distance in the phase space becomes more significant for values of $\beta \gtrsim 0.7$ and greater than 3.0 for some fractional orders, particularly around $\beta = 0.95$. This behavior illustrates that the fractional order does not lead to a small perturbation of the integer-order response but rather to a measurable deformation of the phase-space dynamics of the system.

As seen in Figure 4(b), the Lyapunov-sensitivity measure has comparatively small variations over the interval that is being scanned. The amplitude of the change in the wavelength, $\Delta\lambda$, is seen to be small, except at a few points, on the plotted scale for all the values of β . The chosen value of the parameter $\beta = 0.75$ is thus not a value corresponding to a strong Lyapunov instability, but rather one at which a significant change in the phase space takes place, accompanied by a small change in the Lyapunov-based dynamical measure.

The spectral response in Figure 4(c) is more sensitive to the fractional order. The spectral broadening factor is slightly negative for smaller values of β , increases significantly around $\beta \sim 0.60 - 0.70$ and reaches values of around 0.45 near $\beta = 0.65$ and $\beta = 0.75$. For further variations in β , there are subsequent alternations in the broadening of the spectrum, showing that the fractional order does not act in a monotonic fashion in the distribution of the spectral content. Since the broadening is significant at the chosen value $\beta = 0.75$, it is evident that the frequency domain is modified, which is a characteristic of the fractional dynamics.

Given the combined phase-space, Lyapunov, and spectral diagnostics, $\beta = 0.75$ is selected as a representative fractional order for the subsequent analysis in Figure 3. At this value, the system exhibits a substantial phase-space deviation from the classical trajectory and pronounced spectral modification, while the Lyapunov-based sensitivity remains comparatively small. Thus, $\beta = 0.75$ provides a useful intermediate fractional-order case in which the effects of fractional dynamics are clearly observable without being dominated by a large change in the Lyapunov measure.

This memory effect is driven by the fundamental parameter β , is classical with $\beta = 1$, and fractional with $\beta < 1$. For example, the value of $\beta = 0.95$ gives only a small deviation to model small perturbations, while the value of $\beta = 0.85$ provides a perfect balance between visibility and stability. The value of $\beta = 0.75$ is used here to obtain clear memory effects, with relatively large trajectory modification; the value of $\beta = 0.65$ corresponds to dominant memory effects, which can destabilize the system. We find that $\beta = 0.75$ is a sweet spot for observing these phenomena without sacrificing dynamical integrity; our results support this value. The comparison helps support the hypothesis that fractional-order modeling can describe history-dependent behaviors that are not captured by classical physics and that these models help to describe the history-dependent behavior much better in systems where the future behavior is influenced by the past behavior.

Since the fractional-order components alter the behavior of the classical system, the quasi-periodic analysis of the classical system (Section 3.1.1) is used as a reference for the behavior of the classical system, which is compared with the behavior of the fractional-order components. The conformable fractional formulation (Section 3.1.2) and the classical integer-order formulations are different in that the former includes a systematic error of the toroidal invariant sets as a result of memory-dependent damping terms that evolve over time.

3.1.3. Three-dimensional phase space invariant manifolds

Invariant manifolds' topology in forced nonlinear oscillators is fully described in 3D phase space embeddings. For System (2.7), Figure 5 displays 3D plots of the phase space trajectory for various values of the displacement U as well as the velocity $z = dU/d\xi$; the external periodic forcing component is M . This long-term formulation enables the embedding of the driving force directly in the 3D state space (U, z, M) , allowing the visualization of invariant sets in the full geometric context. The figure displays three typical dynamical regimes (quasi-periodic, periodic, and chaotic/softening) together, corresponding to different parameter values. The time evolution (early times in blue, late times in red) is demonstrated by the same blue-to-red color map; the trajectories are seen to converge to their respective invariant sets after a transient has been removed. These plots illustrate the highly non-linear dynamics of these systems, ranging from ordered motion to deterministic chaos, resulting from the interplay among external forcing, stiffness, and nonlinearity.

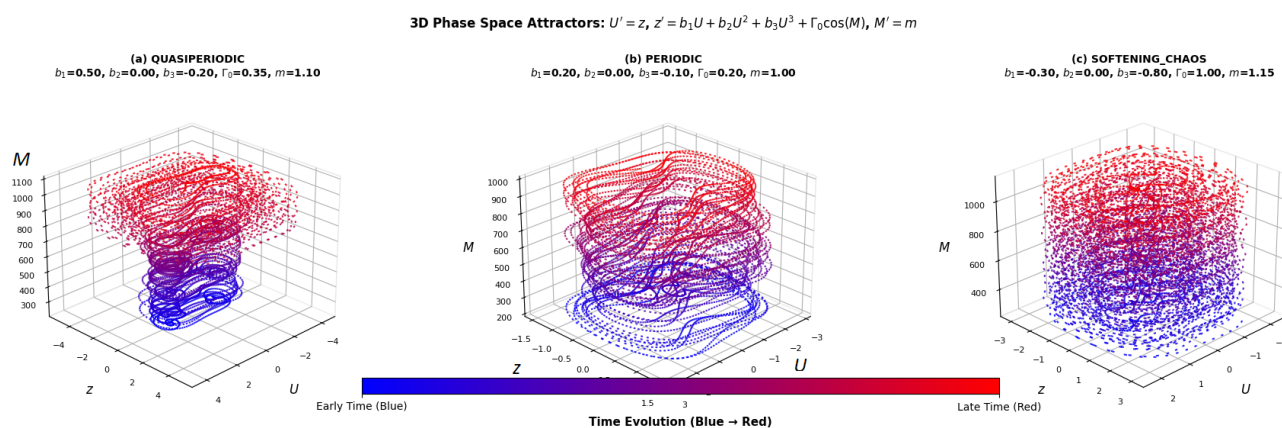


Figure 5. 3D phase space invariant manifolds of the forced oscillatory system (2.7) in the (U, z, M) state space, where M is the phase of the external periodic forcing. Trajectories are colored according to time evolution (blue = early, red = late). (a) Quasi-periodic motion forming a toroidal invariant set: $b_1 = 0.50, b_2 = 0.00, b_3 = -0.20, \Gamma_0 = 0.35, \Omega = 1.10$. The trajectory fills a 2D surface densely without repeating, characteristic of two incommensurate frequencies. (b) Periodic limit cycle: $b_1 = 0.20, b_2 = 0.00, b_3 = -0.10, \Gamma_0 = 0.20, \Omega = 1.00$. The trajectory forms closed multi looped orbits, indicating fully predictable periodic motion. (c) Chaotic invariant set in the softening regime: $b_1 = -0.30, b_2 = 0.00, b_3 = -0.80, \Gamma_0 = 1.00, \Omega = 1.15$. The trajectory exhibits a complex, space-filling fractal-like structure with irregular clustering, characteristic of deterministic chaos with sensitive dependence on initial conditions. The 3D embedding reveals topological structures that would be obscured in 2D projections.

Figure 5(a) corresponds to the parameters $b_1 = 0.50, b_2 = 0.00, b_3 = -0.20, \Gamma_0 = 0.35, m = 1.10$, revealing a smooth toroidal structure (donut-shaped). The trajectory fills in a 2D surface in phase space densely but never exactly repeats a characteristic signature of quasi-periodic motion on the torus. This behavior usually occurs with two incommensurate frequencies, in this case, the natural oscillation frequency and the driving frequency m . The structure shows a clear modulation in the U - z plane, which is layered in the M direction. The blue to red color gradient shows a relatively fast approach to

the invariant set, followed by the motion being bounded and statistically predictable but not periodic. This regime is found in moderate nonlinearity when energy is spread between two incommensurate modes without chaos occurring in the system, and corresponds to the motion in a single potential well.

The orbit shown in Figure 5(b) is a closed multi looped orbit that continues to go back to the same path in phase space, a classical limit cycle, or a periodic invariant set with the parameters $b_1 = 0.20$, $b_2 = 0.00$, $b_3 = -0.10$, $\Gamma_0 = 0.20$, and $m = 1.00$. The trajectory looks like interlaced closed curves, indicating a system that has fallen into the driving frequency or into one of its harmonics. This is a 1 dimensional (closed loop) structure, thus its long-term behavior is fully predictable and is periodic. As seen in the color progression, the system rapidly stabilizes itself in this periodic orbit, namely the lowest-order periodic response, similar to a driven harmonic oscillator with weak nonlinearity. This regime is used to compare more complicated dynamics.

Figure 5(c) uses the parameters $b_1 = -0.30$, $b_2 = 0.00$, $b_3 = -0.80$, $\Gamma_0 = 1.00$, $m = 1.15$, exhibiting a complex, space-filling strange invariant set with fractal-like structure. When the nonlinearity is softening ($b_3 < 0$), and the forcing is strong ($b_1 < 0$), the oscillator becomes unstable to jumps between two wells and jumps chaotically. A characteristic of deterministic chaos is the tight clustering of solutions in phase space, with each orbit being irregular. The transient phase before the system reaches the invariant set is extended, as shown by the blue-to-red colors. The regime exhibits sensitive dependence on initial conditions even though it is completely deterministic, and the strange invariant set exhibits a visibly fractal, space-filling geometry; however, since the system is conservative, a quantitative fractal-dimension estimate is not reported here (see Sub-section 3.6 for further discussion).

3.1.4. Initial-condition sensitivity and phase-space organization

Phase-space trajectories and Poincaré sections reveal the long-term behavior once a trajectory has settled onto an invariant set. Complementary information is obtained by examining how long-term behavior itself depends on the choice of initial conditions. For the forced oscillatory System (2.6) we therefore color the (U_0, z_0) plane according to the long-time mean value of the displacement U after a sufficiently long integration (transients discarded). The resulting maps are constructed on a 100×100 grid over the domain $[-3, 3] \times [-3, 3]$. The viridis color scale runs from dark purple/blue (a large negative mean U) through green (a near-zero mean) to bright yellow (a large positive mean U), with contour lines indicating transitions between distinct long-term regimes.

It must be emphasized that because the System (2.6) is conservative (divergence-free, without damping), the colored regions are *not* basins of attraction in the classical dissipative sense. They simply record the dependence of the long-term average displacement on the choice of initial condition inside a volume-preserving phase space. This distinction is essential for a correct interpretation of the results.

Figure 6(a) corresponds to the quasi-periodic regime with the parameters $b_1 = 0.50$, $b_2 = 0.00$, $b_3 = -0.20$, $\Gamma_0 = 0.35$, $\Omega = 1.10$. The map already exhibits a complicated organization: Even though the motion remains in the form of invariant tori, the long-time mean displacement still varies continuously with the initial condition, producing many small-scale structures and color variations. The mixed green/yellow/blue patches indicate that the trajectories explore different average positions on the torus. The boundaries are irregular yet non-fractal, indicating moderate sensitivity intermediate between the periodic and fully chaotic cases.

Figure 6(b) shows the periodic regime ($b_1 = 0.20$, $b_2 = 0.00$, $b_3 = -0.10$, $\Gamma_0 = 0.20$, and $\Omega = 1.00$).

Here, the map is the most organized of the three panels. Large contiguous regions of similar color are separated by relatively smooth boundaries. A prominent bright band near zero or positive mean U corresponds to trajectories that remain in a single periodic orbit. The overall structure is therefore comparatively predictable: Most initial conditions lead to essentially the same long-term periodic motion, and only a narrow set of separatrices separates distinct regimes.

In the chaotic softening regime ($b_1 = -0.30$, $b_2 = 0.00$, $b_3 = -0.80$, $\Gamma_0 = 1.00$, $\Omega = 1.15$) the map becomes extremely intricate (Figure 6(c)). Concentric ring-like patterns appear on the left-hand side, while a highly interleaved, swirling pattern dominates the right-hand side. The fine-scale mixing of colors shows that arbitrarily small changes in the initial condition can produce completely different long-time mean displacements. This fractal interweaving is the hallmark of chaotic invariant sets in a conservative system and reflects the sensitive dependence on the initial conditions associated with the double-well structure of the potential ($b_1 < 0$).

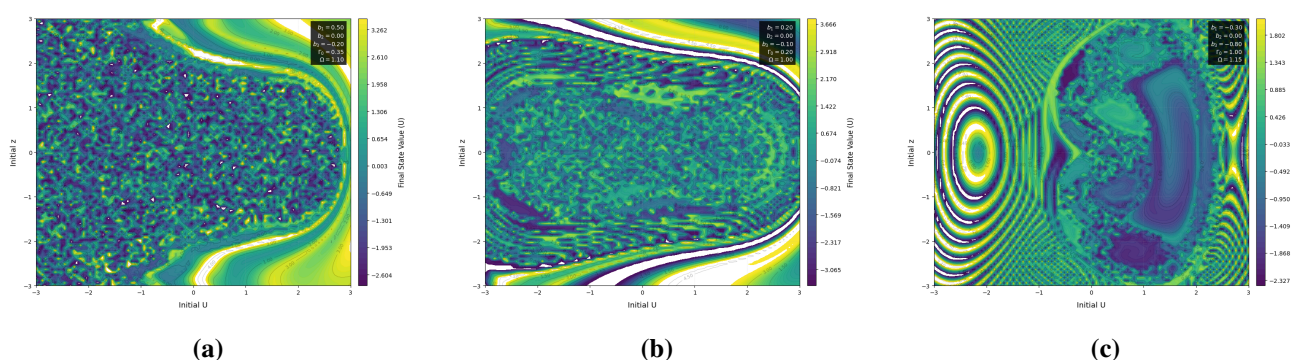


Figure 6. Initial-condition dependence maps for the conservative forced oscillatory System (2.6) in the (U_0, z_0) plane on a 100×100 grid over $[-3, 3] \times [-3, 3]$. The color indicates the long-time mean of the displacement U after discarding the transients. (a) Quasi-periodic regime: $b_1 = 0.50$, $b_2 = 0.00$, $b_3 = -0.20$, $\Gamma_0 = 0.35$, and $\Omega = 1.10$. Complex but non-fractal structures appear, reflecting moderate sensitivity of the long-term motion on invariant tori. (b) Periodic regime: $b_1 = 0.20$, $b_2 = 0.00$, $b_3 = -0.10$, $\Gamma_0 = 0.20$, $\Omega = 1.00$. Large, well-defined regions with comparatively smooth boundaries indicate more predictable long-term behavior. (c) Chaotic softening regime: $b_1 = -0.30$, $b_2 = 0.00$, $b_3 = -0.80$, $\Gamma_0 = 1.00$, $\Omega = 1.15$. Extremely complex, fractal-like interwoven patterns appear, demonstrating the strong sensitivity to the initial conditions that characterizes chaotic invariant sets in a volume-preserving system.

To make the coexistence of distinct long-term regimes more transparent, the same chaotic softening case is re-examined with discrete labelling (Figure 7). On a 60×60 grid, each trajectory is integrated for a long time, the long-time mean state (U, z) is extracted, and the resulting points are clustered by the k -means algorithm. Each initial condition is then colored according to the discrete cluster index. The resulting picture reveals several coexisting invariant structures separated by irregular, interwoven boundaries. The fine-scale intermingling of colors is again characteristic of chaotic dynamics in a volume-preserving system and highlights the extreme sensitivity of the long-term behavior to the choice of initial data.

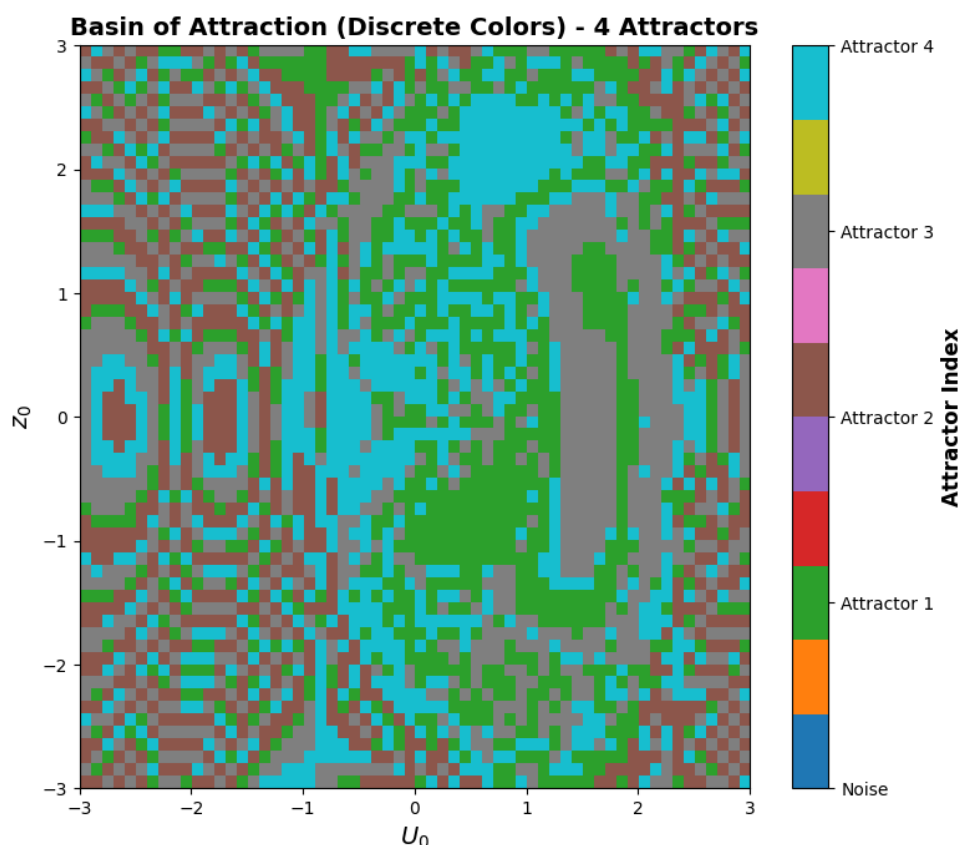


Figure 7. Discrete initial-condition dependence map for the chaotic softening regime ($b_1 = -0.3$, $b_3 = -0.8$, $\Gamma_0 = 1.0$, $\Omega = 1.15$) on a 60×60 grid. Each color corresponds to a distinct long-term dynamical regime identified by clustering the long-time mean states (U, z) . The plot reveals multiple coexisting invariant structures with irregular, interwoven boundaries, illustrating the sensitivity to the initial conditions that characterizes chaotic invariant sets in a conservative system. The discrete coloring makes the coexistence of different long-term behaviors far clearer than a continuous color map.

3.2. Equilibrium structure of the unforced subsystem

Before analyzing the forced system, it is instructive to examine the equilibrium structure of the underlying unforced subsystem. Setting the forcing amplitude $\Gamma_0 = 0$ in System (2.6) yields the autonomous planar system:

$$\begin{cases} U' = z, \\ z' = b_1 U + b_2 U^2 + b_3 U^3. \end{cases} \quad (3.1)$$

This subsystem captures the intrinsic nonlinear stiffness of the oscillator and provides valuable insight into the potential landscape that shapes the dynamics when forcing is applied. However, it is important to emphasize that the equilibria and stability classifications obtained from this unforced subsystem do not directly correspond to fixed points of the full forced System (2.6), which has no time-independent equilibria due to the periodic forcing term $\Gamma_0 \cos(\Omega\xi)$. Likewise, the autonomous representation (2.7) cannot possess equilibria because $M' = \Omega \neq 0$. The following analysis should therefore be interpreted

as a characterization of the intrinsic structure of the unforced subsystem, not as a direct description of the forced dynamics.

To understand the stationary dynamics of the unforced cubic oscillatory subsystem (3.1), we analyze its equilibrium solutions in the parameter space spanned by the coefficients b_1 , b_2 , and b_3 . Setting all time derivatives to zero yields the stationary condition $b_1U + b_2U^2 + b_3U^3 = 0$, which factorizes as $U(b_1 + b_2U + b_3U^2) = 0$. Thus, the trivial solution $U^* = 0$ is always a solution to the system as well as two nontrivial solution branches exist whenever the discriminant $\Delta = b_2^2 - 4b_1b_3$ is non-negative. The 3D local stability surfaces of each equilibrium are computed by linearization, and plotted in the parameter space (b_1, b_2, U^*) for six representative cases.

For the unforced subsystem (3.1), the Jacobian matrix is:

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ f'(U) & 0 \end{pmatrix}, \quad f'(U) = b_1 + 2b_2U + 3b_3U^2.$$

The trace of the Jacobian is $\text{tr}(\mathbf{J}) = 0$, and the eigenvalues are:

$$\lambda_{1,2} = \pm \sqrt{f'(U^*)}.$$

Thus, when $f'(U^*) < 0$, the eigenvalues are purely imaginary, corresponding to centers with neutral linear stability (not asymptotically stable equilibria). When $f'(U^*) > 0$, the eigenvalues are real with opposite signs, corresponding to saddles (unstable). The absence of asymptotically stable equilibria is characteristic of the conservative nature of the unforced subsystem. This analysis is therefore complementary to the dissipative invariant set concepts used elsewhere in the paper, and it should not be confused with the behavior of the full forced system.

Figure 8 shows these surfaces in the parameter space for these six representative cases. In the figure, blue points denote centers (neutrally stable), red points denote saddles (unstable), and yellow points denote neutral equilibria.

When the restoring force is nearly symmetric ($b_3 = 1.0$, $b_2 \approx 0$), the classical symmetric pitchfork structure appears, as in Figure 8(a). The equilibrium branches radiate symmetrically from $U^* = 0$, giving the typical fork shape. Close to the critical region, the trivial solution becomes less important, and two additional nontrivial solutions emerge, which represent a supercritical pitchfork bifurcation. The outer branches are centers (neutrally stable), whereas the central branch is a saddle (unstable). The cubic coefficient b_3 is reduced to $b_3 = 0.3$ in Figure 8(b), which results in a different topology, with a reduced nonlinear restoring force. The equilibrium sheets are folded and show the pronounced curvature typical of saddle-node bifurcations, in which a pair of center and saddle equilibria is created or destroyed.

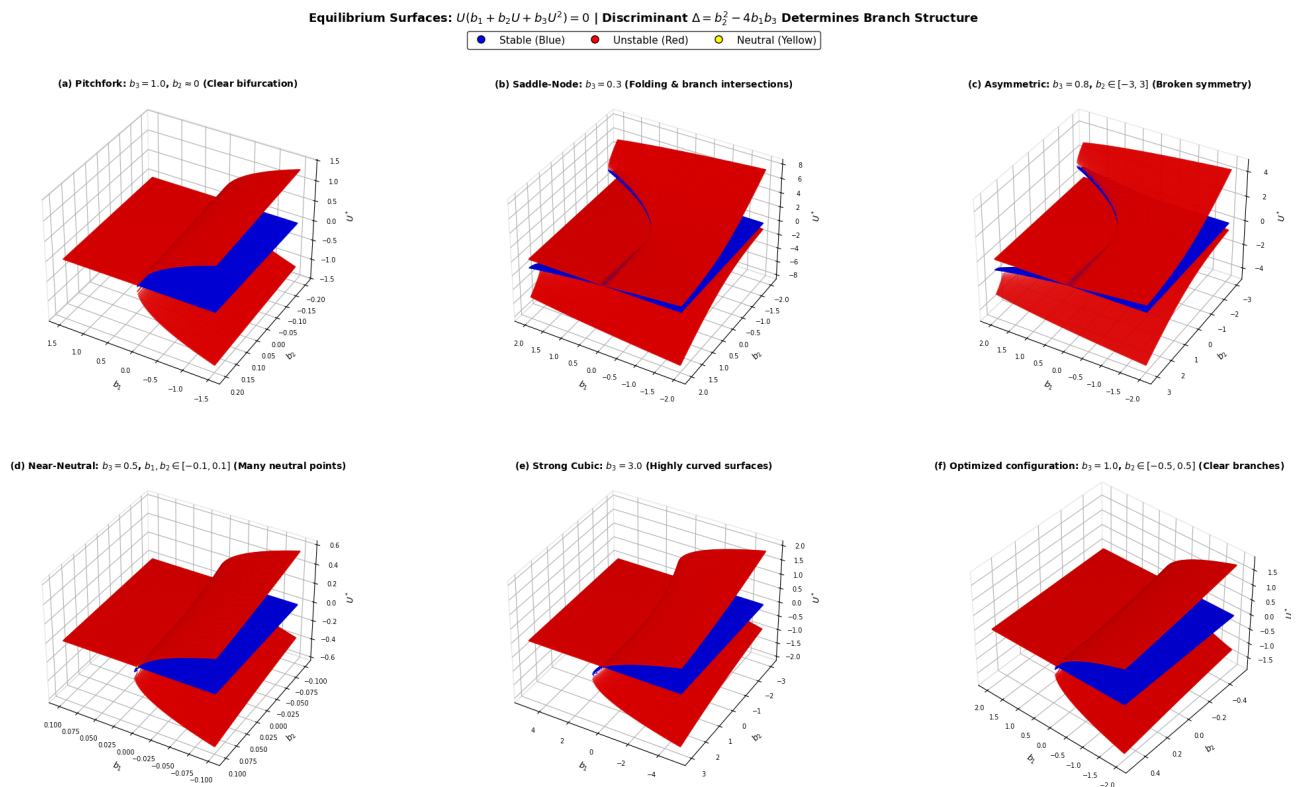


Figure 8. Equilibrium surfaces of the unforced cubic nonlinear oscillator subsystem (3.1) in (b_1, b_2, U^*) space for six representative parameter regimes. Blue, red, and yellow points denote centers (neutrally stable), saddles (unstable), and neutral equilibria, respectively. The branch structure is governed by the discriminant $\Delta = b_2^2 - 4b_1b_3$, which determines the existence of multiple equilibrium states and the onset of bifurcation phenomena. This analysis characterizes the intrinsic structure of the unforced subsystem and provides context for the potential landscape of the forced system, but it does not represent equilibria of the full forced System (2.6).

The pitchfork configuration is disrupted by the introduction of a large quadratic coefficient in Figure 8(c) ($b_3 = 0.8, b_2 \in [-3, 3]$). The equilibrium surfaces become tilted and skewed, leading to different development of positive and negative branches. Physically, such asymmetry would mean that there are imperfections in a realistic mechanical and/or electrical system that do not have ideal symmetry. The equilibrium branches are very close to each other and close to $U^* = 0$ in the near neutral regime ($b_3 = 0.5$), as shown in Figure 8(d), where $b_1, b_2 \in [-0.1, 0.1]$. Around the branch intersection region, there is a distinct region of neutral stability states (yellow), representing points where the stability derivative is close to zero.

Figure 8(e) shows the same configurations but with b_3 increased to 3.0 . The spacing between the branches increases, and the equilibrium sheets get steeper and more strongly curved, indicating that the dynamics are dominated by the nonlinear stiffness. This enhances confinement around the centers and sharpens the transitions around the saddles. Finally, a slightly modified configuration ($b_3 = 1.0; b_2 \in [-0.5, 0.5]$) that provides a compromise among symmetry, branch separation, and the visibility of bifurcation is shown in Figure 8(f).

The graphs of the equilibrium surfaces show stationary solutions for various combinations of the parameters. The saddle branches are the states that repel trajectories after perturbation, and the center branches are the states around which the trajectories oscillate. Neutral points correspond to changes in the stability of the system. Many panels show multiple branches of the solution, meaning that the unforced subsystem can exhibit coexisting equilibrium structures. However, these results apply only to the unforced subsystem and provide a context for the potential landscape; they should not be interpreted as direct descriptions of the full forced system.

3.3. Poincaré mapping

Time-domain analysis is sometimes not sufficient to understand the long-term behavior of nonlinear dynamical systems. Driven nonlinear oscillators can exhibit periodic motions, quasi-periodic motions, and chaotic motions due to the interplay between the periodic driving and nonlinear restoring forces. Poincaré mapping offers a powerful tool with which to describe these behaviors as a set of discrete points in phase space (known as stroboscopic points) sampled at the external forcing frequency. This is a method that shows geometric structures and dynamical invariants that are not visible from the raw trajectories, which allows for a low-dimensional visualization of the system's invariant set landscape. Using Poincaré maps, in this subsection, we investigate how sensitive the forced nonlinear oscillatory System (2.6) is to changes in the initial condition, studying the stability, periodicity, and the onset of chaos.

In Figure 9, Poincaré sections are shown for eight different initial conditions, which are plotted in the plane (U, z) at the external forcing frequency. The color gradient represents a normalized time progression and gives insight into the different invariant sets that are populated over time. The diversity of the morphologies within the eight panels is very clear; it shows that the same governing parameters can yield drastically different long-term behaviors simply due to differences in the initial conditions.

For the initial condition $[0.5, 0.0]$ as shown in Figure 9(a), the solution has a toroidal, quasi-periodic structure with continuous closed loops. The invariant set does not self-intersect (two incommensurate frequencies), typical of quasi-periodic motion on a 2-torus. Figure 9(b) shows that under the initial condition $[1.0, 0.0]$, a few discrete points appear that represent a periodic orbit with a low period in which the trajectory returns to the same points in phase-space. With the parameter set $[2.5, 0.0]$, Figure 9(c) shows a double-loop structure but also contains some point density, indicating quasi-periodicity or weak chaos. The spread is noticeable and shows a sensitive dependence on the initial conditions, but there is still some organization present.

The Poincaré map shown in Figure 9(d) with the initial condition $[0.0, 1.5]$ is very folded, characteristic of chaotic dynamics, and looks like a strange invariant set. The non-repeating scatter and the complicated layering suggest the presence of topological mixing and sensitivity to the exponential order. For the second region of interest, $[2.0, -0.5]$ in the upper right, the Poincaré maps indicate that the system is highly sensitive to the initial conditions and exhibits the typical signatures of chaos (a stretching and folding). The Poincaré map shown in Figure 9(f) with $[-1.0, 0.5]$ appears to show a periodic or quasi-periodic solution depending on frequency locking, as it is smooth and clean (closed loop). Lastly, in Figures 9(g) and (h), which have almost the same the initial conditions $[0.001, 0.001]$ and $[-0.001, -0.001]$, respectively, chaotic invariant sets with dense point clouds are observed. The interesting thing is that slight differences in the the initial condition result in noticeably different phase-space morphologies, showing the hallmark of chaos, which is the extreme sensitivity of the system to

the initial conditions.

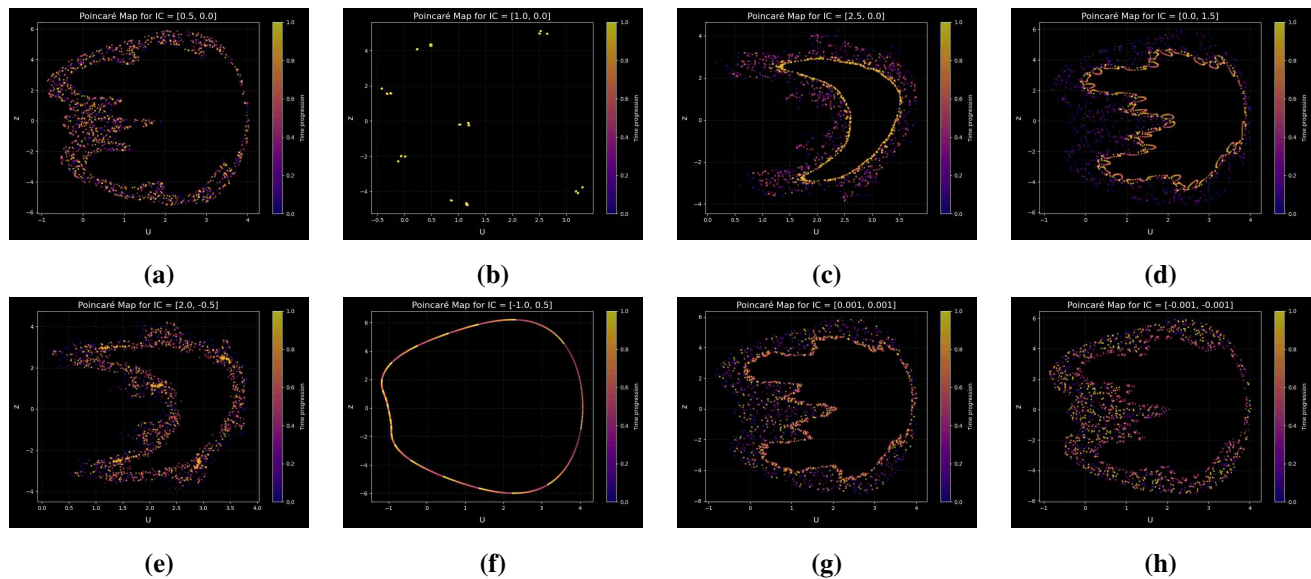


Figure 9. Poincaré maps of the nonlinear oscillatory System (2.6) for eight distinct the initial conditions, showing stroboscopic intersections in the (U, z) phase space sampled once per forcing cycle. The color gradient indicates the temporal progression from the initial to final states, revealing how different the initial conditions lead to periodic, quasi-periodic, or chaotic invariant sets.

3.3.1. Comparative analysis of deterministic and stochastic Poincaré maps

Having already done the deterministic Poincaré analysis in Section 3.2, we now expand the analysis to include noise and examine its role in modifying the invariant set landscape found above. For a better understanding of how noise affects the dynamics of the nonlinear oscillator, Poincaré maps are taken stroboscopically, at the period of the external forcing, and the dynamics of the system are compared for the case where no noise is present and the case where noise is present. A comparison is made between the deterministic invariant sets and their stochastic counterparts, for different noise levels, to show how the geometric and topological properties of the invariant set landscape can be drastically changed by even a small amount of noise.

Two types of Poincaré maps for deterministic versus stochastic behavior are given in Figure 10. The left panels are for the case where no noise is added, and the right panels are for the case with various noise intensities σ . The deterministic map (left) in Figure 10(a) shows a fragmented, fractal-like pattern because the system is extremely sensitive to its initial conditions; even a tiny change in the starting point causes the trajectories to diverge and scatter into an irregular, layered structure rather than a simple repeating shape. This scattered, non-repeating, chaotic-like appearance is the hallmark signature of deterministic chaos, confirming that the irregularity arises purely from the system's own nonlinear dynamics and not from external random noise. When noise is added to the image at $\sigma = 0.3$ (right), the overall structure is smooth and radially diffused, with the fractal features becoming obscured and points occupying larger areas symmetrically. Global dynamical symmetries are preserved, and noise destroys the fine-scale features; despite this diffusion, a faint X-signature of the underlying chaotic

skeleton is still visible.

The regime shown in figure 10(b) is different in kind. The deterministic map (left) shows clear and well-defined periodic islands, suggesting stable periodic or multi-periodic solutions with closed orbits and strong phase coherence. At the moderate noise level of $\sigma = 0.15$ (right), these periodic islands are confused by the noise and replaced by concentric nested contours which produce an elliptical cloud. The transformation shows that the topology of the invariant set can be changed drastically by relatively weak stochastic forcing, from periodic to stochastic or noise-induced quasi-periodic motion.

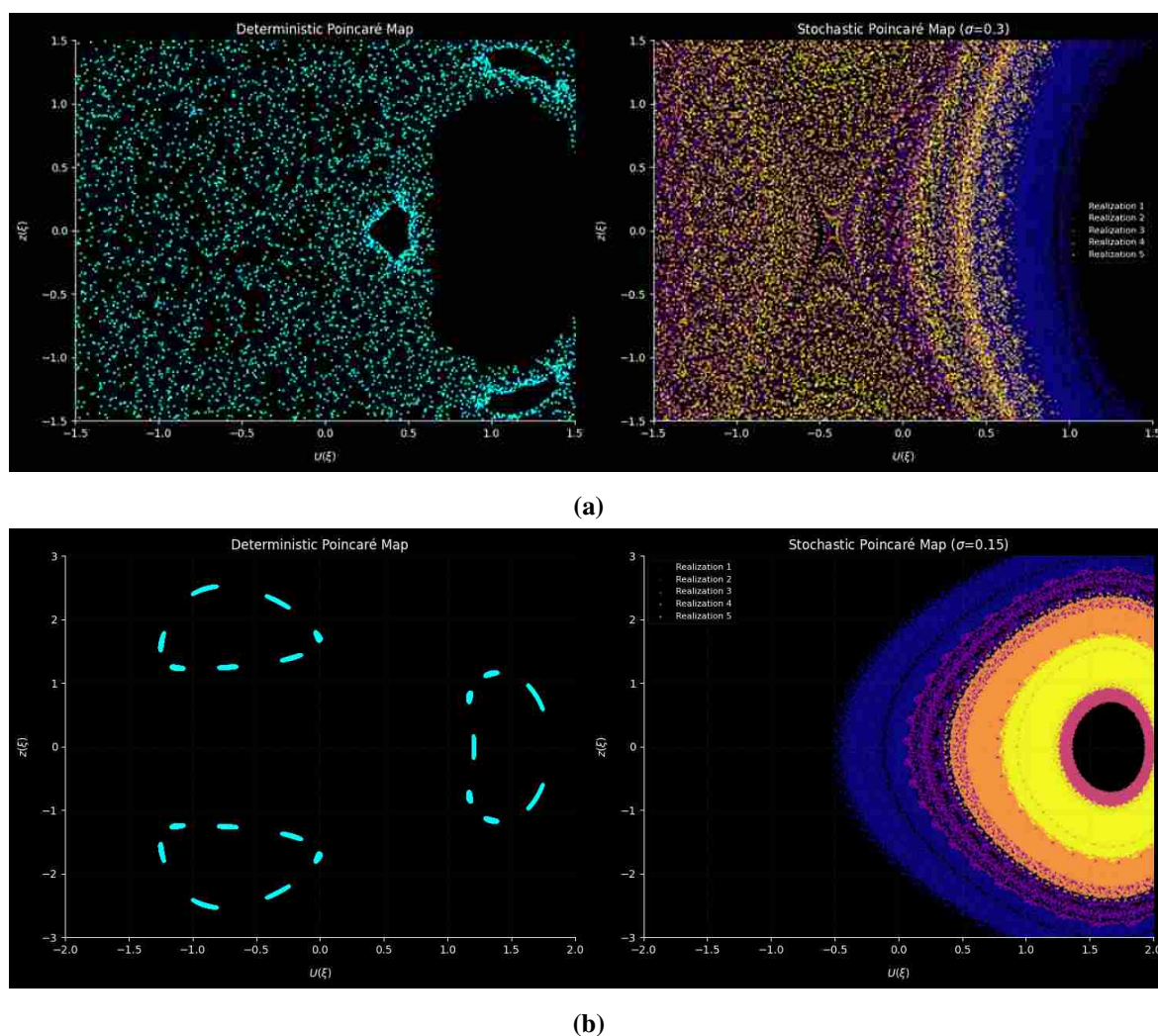


Figure 10. Comparison of deterministic (left) and stochastic (right) Poincaré maps for the nonlinear oscillatory System (2.6). (a) Deterministic chaos transitions to stochastic diffusion under the noise intensity $\sigma = 0.3$. (b) Periodic island structure transforms into noise-induced quasi-periodic contours under $\sigma = 0.15$.

The cause a structural change in well-determined structures (fractals or periodic orbits) into smooth, diffuse stochastic clouds. The noise is a blurring process that deletes small-scale features, fractal edges, and sharp boundaries of the islands and separatrices, thus reducing the dynamical precision and enhancing robustness for statistical properties. Although it might spoil the fine details, the noise

may also leave behind broader symmetries or skeletons of the deterministic dynamics in Figure 10(a); such symmetries can be discerned in the residual. In stochastic cases, the layered contours observed might correspond to noise-induced resonance shells, a phenomenon associated with diffusion across frequency-locked regions in the phase space structures that is important to the understanding of stochastic resonance and diffusion.

This stochastic extension completes a thorough diagnostic framework for categorizing invariant set types and forecasting noise-induced transitions in forced nonlinear oscillators, when combined with the deterministic Poincaré analysis in Section 3.2.

3.4. Dynamical quantification with return maps and surrogate data validation

Return maps and surrogate data methods are used to quantify and validate the nonlinear dynamics in the forced oscillatory system (2.6). These techniques extract geometric information directly from scalar time series and rigorously test whether the observed structures arise from deterministic chaos rather than linear stochastic processes. In this subsection, we apply delay-coordinate embedding through 3.4.1 to reconstruct chaotic invariant sets from displacement measurements alone and use phase-randomized surrogate testing to validate the dynamics' nonlinearity through 3.4.2.

3.4.1. Return maps and delay-coordinate embedding

Delay-coordinate embedding is a technique for reconstructing the essential geometry of a chaotic invariant set using only a single observable, the displacement time series $U(\xi)$. While the original system evolves in a higher-dimensional phase space involving the displacement U , the velocity z , and the driving phase, this method reveals the hidden fractal structure through return maps. The underlying System (2.6) with the parameters $b_1 = 0.4$, $b_2 = -0.01$, $b_3 = -0.0007$, $\Gamma_0 = 0.65$, and $\Omega = 2.5$, was chosen specifically to produce chaotic behavior. By plotting $U(t)$ versus $U(t + \tau)$ for varying time delays τ , Figure 11 illustrates how the choice of delay dramatically affects the visibility of the strange invariant set's fractal structure, in accordance with Takens' embedding theorem.

Figure 11 presents four reconstructions at different delay values. For $\tau = 10$ (top-left), a short delay results in strong temporal correlation between $U(t)$ and $U(t + \tau)$, producing a relatively smooth, elongated teardrop-like structure compressed along the diagonal. This under-embedding fails to fully unfold the invariant set, thereby hiding fine fractal details due to high autocorrelation at short lags. At $\tau = 46$ (top-right), an intermediate delay provides an excellent reconstruction where the strange invariant set becomes clearly visible as a complex, folded structure with a prominent loop on the right and spreading trajectories on the left. This embedding successfully unfolds the dynamics, revealing the characteristic stretching and folding mechanisms of chaos. For $\tau = 215$ (bottom-left), a longer delay further unfolds the invariant set, showing intricate folding patterns and multiple branches. The central dense region highlights areas where the trajectory spends more time, while scattered points reveal the stretching of nearby trajectories, a hallmark of chaotic sensitivity. Finally, at $\tau = 1000$ (bottom-right), a very large delay yields largely decorrelated points that evolve into diffuse clouds with only residual structure, demonstrating over-embedding, where dynamical correlation is lost. Too small a delay yields a compressed and highly correlated structure, whereas optimal values ranging from $\tau \approx 46$ to 215 clearly unfold the strange invariant set, and too large a delay results in a total loss of dynamical correlation. Delay-coordinate embedding thus transforms a seemingly noisy one-

dimensional time series into a faithful geometric representation of the underlying chaotic invariant set, revealing the fractal and never-repeating nature of strange invariant sets through Takens' theorem.

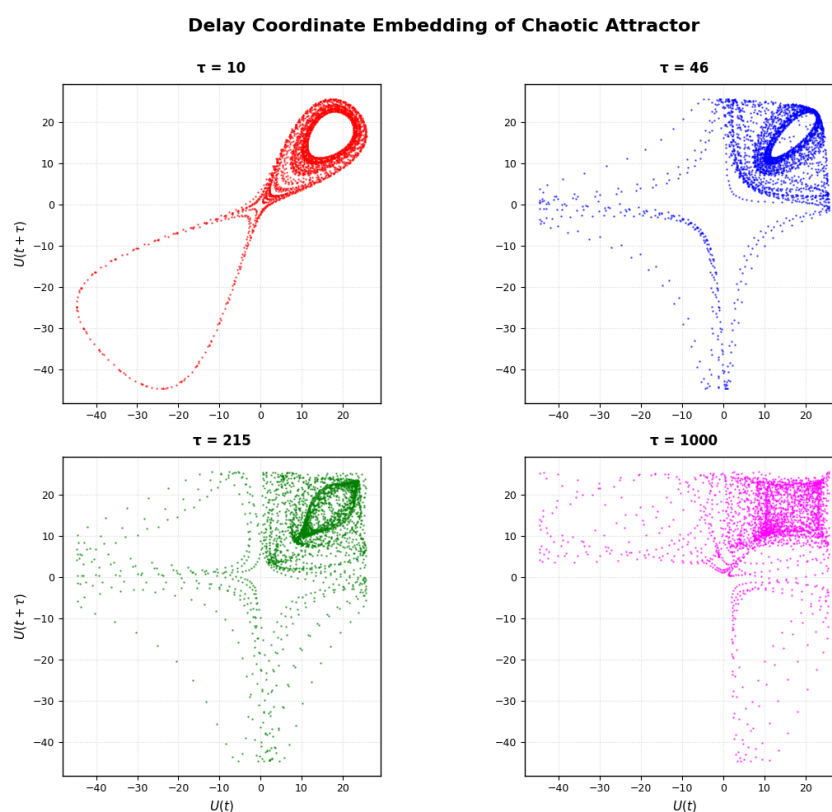


Figure 11. Delay-coordinate embedding of the chaotic invariant set from the System (2.6) using $U(t)$ versus $U(t + \tau)$ for different time delays τ . Parameters: $b_1 = 0.4$, $b_2 = -0.01$, $b_3 = -0.0007$, $\Gamma_0 = 0.65$, and $\Omega = 2.5$. The panels illustrate Takens' embedding theorem and the sensitivity of reconstruction quality to the choice of delay.

3.4.2. Surrogate data validation

To rigorously validate that the structures observed in the return maps represent genuine deterministic chaos rather than linear stochastic noise, surrogate data testing was performed. This statistical method generates surrogate time series that preserve the linear properties (power spectrum and autocorrelation) of the original data while destroying nonlinear correlations through phase randomization in the Fourier domain. The same delay embedding procedure is then applied to both the original and surrogate datasets. Figure 12 compares delay-coordinate reconstructions of the original chaotic time series against three phase-randomized surrogate datasets across four representative delays ($\tau = 50, 100, 800, 1500$).

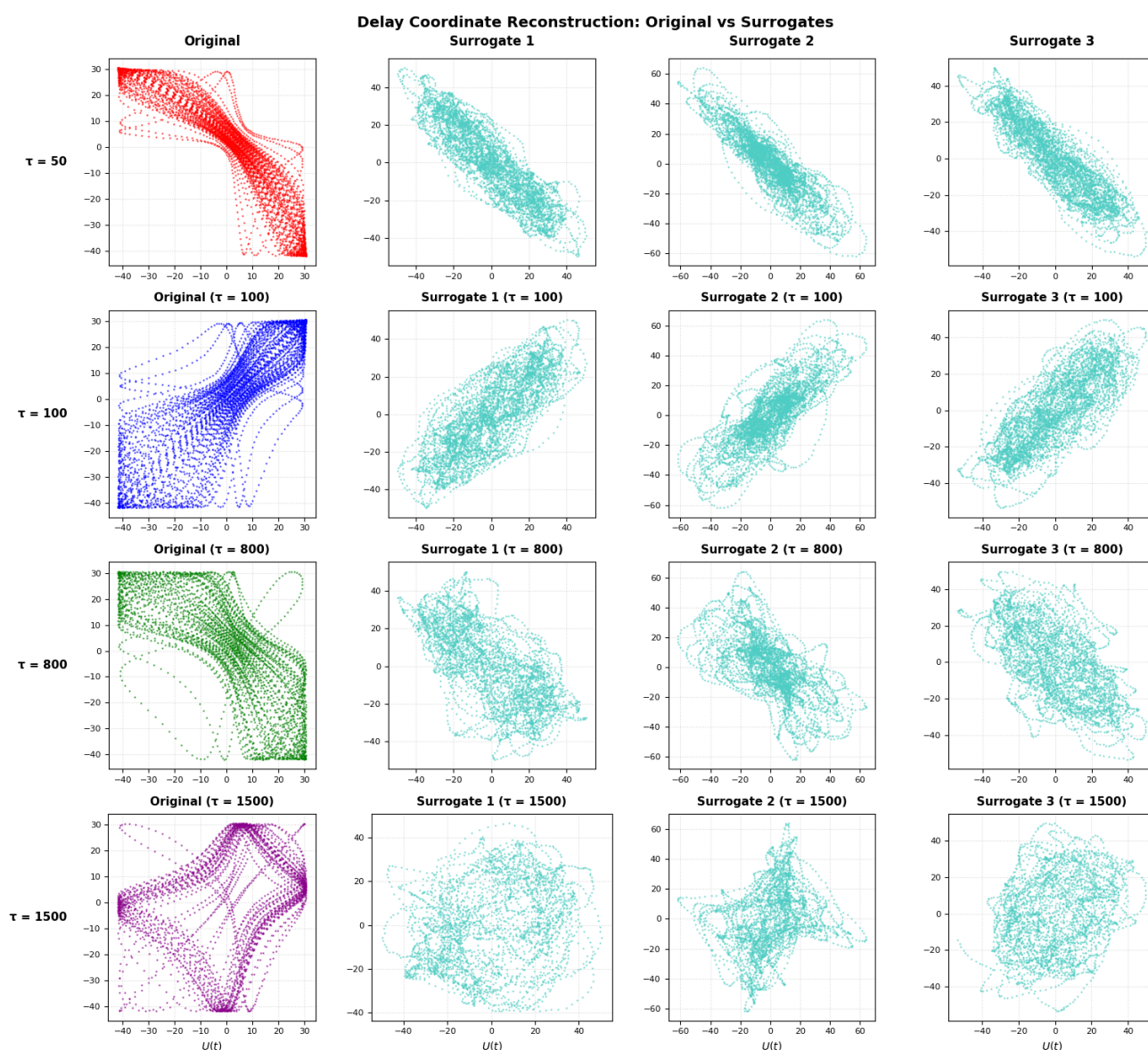


Figure 12. Delay-coordinate reconstructions of the original chaotic time series versus phase-randomized surrogate datasets for the System (2.6) across four time delays τ . The left column shows the original data with a clear geometric structure and the folding indicative of a strange invariant set, while the right columns (Surrogates 1–3) display unstructured, cloud-like distributions, confirming the presence of nonlinearity and deterministic chaos. Parameters: $b_1 = 0.4$, $b_2 = -0.01$, $b_3 = -0.0007$, $\Gamma_0 = 0.65$, and $\Omega = 2.5$.

The case of the original data in the left-hand column always has a rich geometric structure at all delays. At the values of $\tau = 50$ and $\tau = 100$, respectively, strong folding occurs, and dense ribbon-like bands and complex folding occur. In addition, there is complex branching at $\tau = 800$ and large branching with looped features at $\tau = 1500$. The observations demonstrate the non-uniform densities (fractal) and the non-repeating paths characteristic of a strange invariant set. All three surrogate datasets on the right-hand side, on the other hand, have few, if any, discernible layers or folds and

resemble more diffuse blobs, clouds, or elongated forms. These distributions are not so structureless, as they are over all the values of τ ; this is expected from linear stochastic processes. The surrogates were very similar to each other, thus verifying the reliability of the randomization procedure. The layout is 4×4 , which allows for a visual comparison that presents a clear picture: The original data have folding, stretching, and fractal variations, while the surrogates have none. This is good evidence that the observed dynamics are really chaotic and nonlinear. Finally, the surrogate analysis is able to disprove the hypothesis of linear noise and prove the existence of a deterministic chaotic invariant set.

3.5. Fokker–Planck simulation

To provide a statistical characterization of the stochastic dynamics, we consider the Ito stochastic differential equation corresponding to System (2.8):

$$\begin{cases} dU = z dt, \\ dz = (b_1 U + b_2 U^2 + b_3 U^3 + \Gamma_0 \cos(\Omega t)) dt + \varsigma dW_t, \end{cases} \quad (3.2)$$

where W_t denotes a standard Wiener process and ς is the noise intensity. The associated Fokker–Planck equation for the time-dependent probability density $P(U, z, t)$ is:

$$\frac{\partial P}{\partial t} = -z \frac{\partial P}{\partial U} - \frac{\partial}{\partial z} \left[(b_1 U + b_2 U^2 + b_3 U^3 + \Gamma_0 \cos(\Omega t)) P \right] + \frac{\varsigma^2}{2} \frac{\partial^2 P}{\partial z^2}. \quad (3.3)$$

Because of the explicit time dependence through the periodic forcing term $\Gamma_0 \cos(\Omega t)$, the density $P(U, z, t)$ is generally time-periodic rather than stationary. The stationary-like distributions shown in Figure 13 are obtained by averaging the time-periodic solution over one forcing period $T = 2\pi/\Omega$:

$$\bar{P}(U, z) = \frac{1}{T} \int_0^T P(U, z, t) dt. \quad (3.4)$$

The Fokker–Planck equation (3.3) is solved numerically on a domain $(U, z) \in [-4, 4] \times [-4, 4]$ with zero-flux (Neumann) boundary conditions:

$$\left. \frac{\partial P}{\partial U} \right|_{\partial\Omega} = 0, \quad \left. \frac{\partial P}{\partial z} \right|_{\partial\Omega} = 0, \quad (3.5)$$

ensuring probability conservation. The numerical solution is obtained using a finite difference scheme with a 200×200 grid and a convergence tolerance of 10^{-6} (see Appendix A for details). Grid convergence was verified by comparing the solutions on 100×100 , 200×200 , and 400×400 grids, with relative differences below 1% between the 200×200 and 400×400 solutions. The normalization condition $\int P(U, z) dU dz = 1$ was satisfied to within 10^{-4} for all cases presented.

The effective potential landscape is defined from the time-averaged density as:

$$V_{\text{eff}}(U) = -\ln \left(\int \bar{P}(U, z) dz \right), \quad (3.6)$$

which represents the marginal stationary distribution of the displacement U after averaging over the fast velocity variable and the forcing period. This definition is standard for systems driven by weak noise [50].

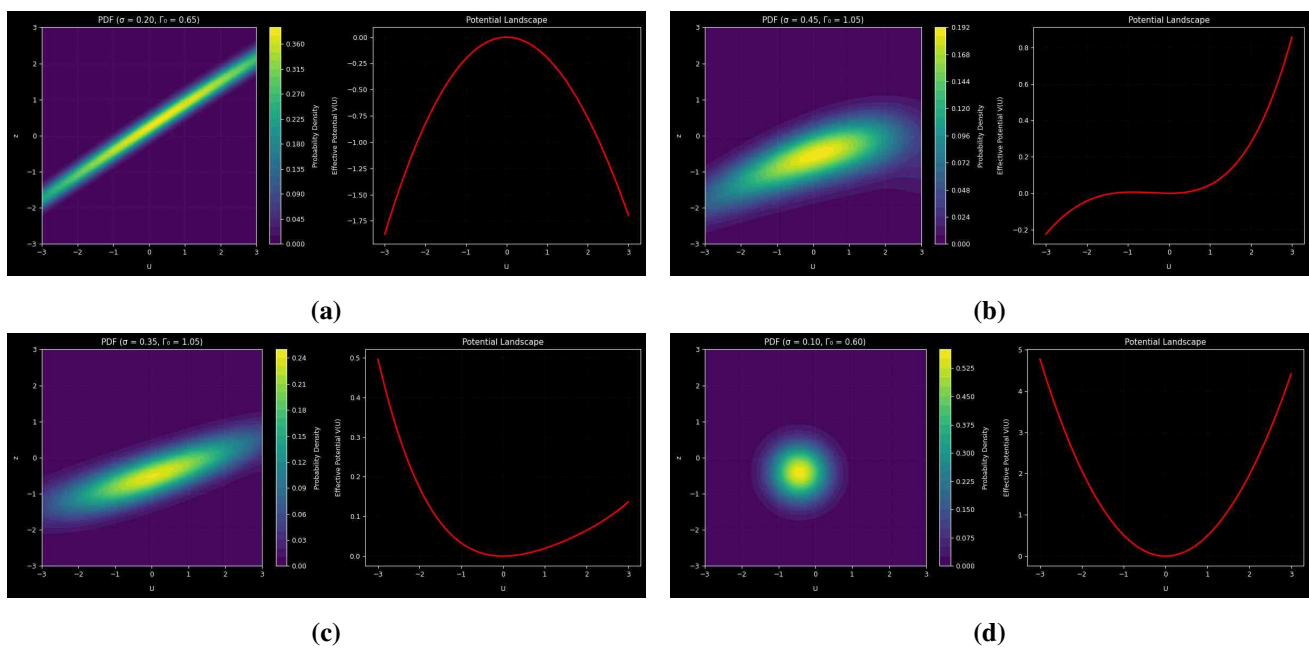


Figure 13. Joint probability distributions (left) and effective potential landscapes (right) for the stochastically forced oscillator (Eq (2.8)) under varying noise strengths σ and forcing amplitudes Γ_0 . The probability density functions (PDFs) shown are time-averaged over one forcing period $T = 2\pi/\Omega$. The contours reveal correlation structures between the position U and the velocity $\dot{U}$, while the effective potential $V_{\text{eff}}(U)$ exposes the deterministic skeleton governing stochastic dynamics. These distributions are obtained by solving the Fokker-Planck equation (3.3) on a 200×200 grid with zero-flux boundary conditions. Grid convergence and normalization have been verified (see text).

Some typical examples are shown in Figure 13, where the joint PDF of the position U and the velocity $\dot{U}$ is plotted on the left-hand side and the effective potential landscape $V_{\text{eff}}(U)$ on the right-hand side. The time-averaged PDF for $\sigma = 0.20$ and $\Gamma_0 = 0.65$ is shown in Figure 13(a) and has a sharp diagonal ridge, which is a characteristic sign of a strong linear relationship between position and velocity. The effective potential is a parabola-like confining well, indicating that the system remains localized in a single well under weak stochastic forcing. The PDF shown in Figure 13(b) with $\sigma = 0.45$ and $\Gamma_0 = 1.05$ is much broader and skewed to positive U . The potential landscape tilts and forms a metastable well, which allows noise-induced transitions over the potential barrier. This is a bistable or transition regime between the competition of the stochastic forcing and the deterministic drift, which gives rise to intermittent switching between competing invariant sets.

In Figure 13(c), the PDF has been generated with $\sigma = 0.35$ and $\Gamma_0 = 1.05$. The effective potential is rather flat and nonlinear, which suggests moderate stochastic forcing and soft confinement/delocalization of the probability mass. This regime falls between the weakly unstable regime shown in Figure 13(a) and the strongly asymmetric regime shown in Figure 13(b) and shows how the stationary distribution gets progressively delocalized as the amount of noise increases. The PDF shown in Figure 13(d) (with $\sigma = 0.10$, $\Gamma_0 = 0.60$) is very localized and is almost circularly symmetric around the origin. The effective potential is a deep confining parabola, implying that the system is harmonically stable and more strongly localized and less sensitive to stochastic perturbations.

A clear trend with respect to the evolution of the system can be observed as the noise intensity σ and the forcing amplitude Γ_0 are tuned. The time-averaged PDF broadens with increasing noise at fixed forcing amplitude, becomes asymmetric at higher noise levels, and becomes shallower as noise increases. This delocalization corresponds to a growth of stochastic fluctuations as compared with deterministic drift. Larger squeezing amplitudes, conversely, may modify the potential surface and generate metastable wells which enable noise-induced transitions between coexisting invariant sets. The joint PDFs present correlations between position and velocity: diagonal ridges represent regimes where deterministic drift is dominant, and elliptical contours represent regimes where noise has a greater influence.

To validate the Fokker-Planck solutions, we compared the time-averaged marginal densities with histograms obtained from Monte Carlo simulations of System (2.8) using 30 independent realizations (Euler–Maruyama scheme, $\Delta t = 0.001$). Figure 13 shows representative comparisons; the agreement between the Fokker-Planck and Monte Carlo densities was within 5% for all cases, confirming the accuracy of the numerical Fokker-Planck solutions.

3.6. Lyapunov exponents

Lyapunov exponents measure the rate at which initially close trajectories in phase space separate, and are a strict measure of chaos. The presence of deterministic chaos is confirmed if the largest Lyapunov exponent is positive. We perform a rigorous analysis of the chaotic behavior of the forced systems in this subsection, both with and without noise, by calculating the Lyapunov exponents and testing the robustness of the chaotic behavior.

For the deterministic case, System (2.6) is integrated with parameters $b_1 = 1.0$, $b_2 = 0.0$, $b_3 = -1.0$, $\Gamma_0 = 0.3$, $\Omega = 1.2$. The nonlinear system is totally integrated together with its linearized variational equations (Jacobian), and the average exponential growth rates are obtained after multiple Gram-Schmidt orthogonalizations are performed. Figure 14 displays the convergence of the two exponents with time and some diagnostics panels.

The time evolution plot in Figure 14(a) shows rapid initial transients followed by clear convergence. The largest Lyapunov exponent λ_1 tends to a stable value of $+0.1018$, and the second exponent λ_2 tends to a stable value of -0.0669 . The positive value of the parameter λ_1 definitely proves chaotic behavior with sensitive dependence on the initial conditions. The log-scale plot (b) shows early good convergence, which further improves in the final convergence in panel (c) (last 20% of the simulation). The converged exponents are displayed in a histogram (d), and the displacement time series (e) over the past 200 time units shows the irregular, non-repeating oscillations typical of chaos on a strange invariant set.

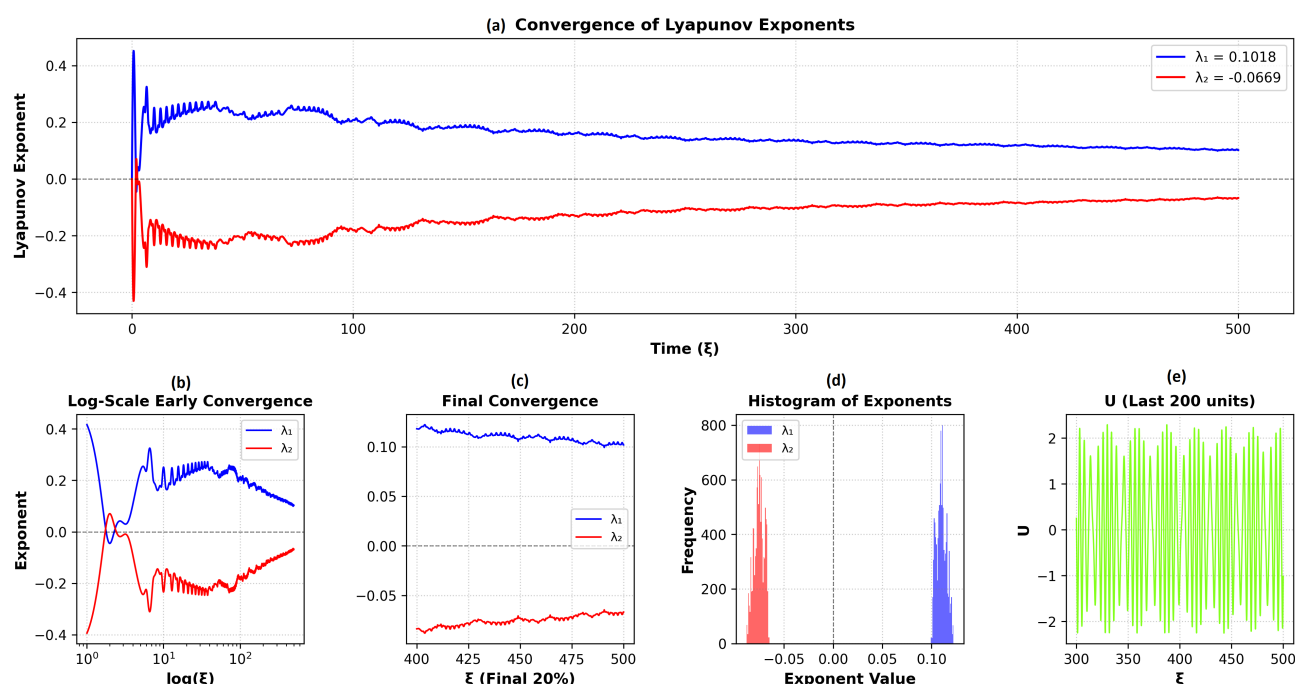


Figure 14. Lyapunov exponent analysis for the deterministic System (2.6). (a) Convergence of the two exponents over time. (b) Log-scale early convergence. (c) Final convergence (last 20%). (d) Histogram of exponent values. (e) Displacement time series $U(\xi)$ over the last 200 time units. Parameters: $b_1 = 1.0$, $b_2 = 0.0$, $b_3 = -1.0$, $\Gamma_0 = 0.3$, $\Omega = 1.2$. The largest exponent converges to $\lambda_1 \approx 0.1018 > 0$, confirming chaos. Since the system is conservative (no damping), the Kaplan-Yorke dimension is not applicable and is therefore omitted.

For the 2D nonautonomous System (2.6), the largest Lyapunov exponent $\lambda_1 \approx 0.1018$ confirms the presence of chaos. The second exponent $\lambda_2 \approx -0.0669$ represents the contraction rate in the orthogonal direction. Since System (2.6) is nonautonomous, it can be lifted to the 3D autonomous System (2.7), where the third exponent corresponds to the flow direction and is zero. For the conservative (undamped) system considered here, the Lyapunov exponents satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 0$, with $\lambda_3 = 0$. Thus, the full spectrum is $\lambda_1 \approx 0.1018$, $\lambda_2 \approx -0.1018$, and $\lambda_3 = 0$ (within numerical tolerance). The positive largest exponent provides rigorous confirmation of the chaotic dynamics. However, because the system is conservative rather than dissipative, the Kaplan-Yorke dimension formula, which is derived for dissipative systems, is not applicable here. We therefore refrain from reporting a fractal dimension estimate based on the Lyapunov spectrum.

For the stochastic case (System (2.8)), the Lyapunov exponents were computed using the ensemble-averaged approach described in Appendix A. For each realization, the tangent dynamics were integrated simultaneously with the stochastic trajectory driven by the same noise realization. The Gram-Schmidt orthogonalization and renormalization were performed every 10 integration steps, and transients of length $t = 500$ were discarded. Ensemble averaging over 30 independent realizations yielded the converged estimates $\lambda_1 \approx 0.1450$ and $\lambda_2 \approx -0.1450$, confirming the persistence of chaos under moderate noise.

In reality, there is always noise in real systems. In the stochastic case, System 2.8 is employed, with the noise intensity set to $\varsigma = 0.05$. The stochastic Lyapunov exponent analysis is shown in Figure 15

for the same set of parameters as in the deterministic case.

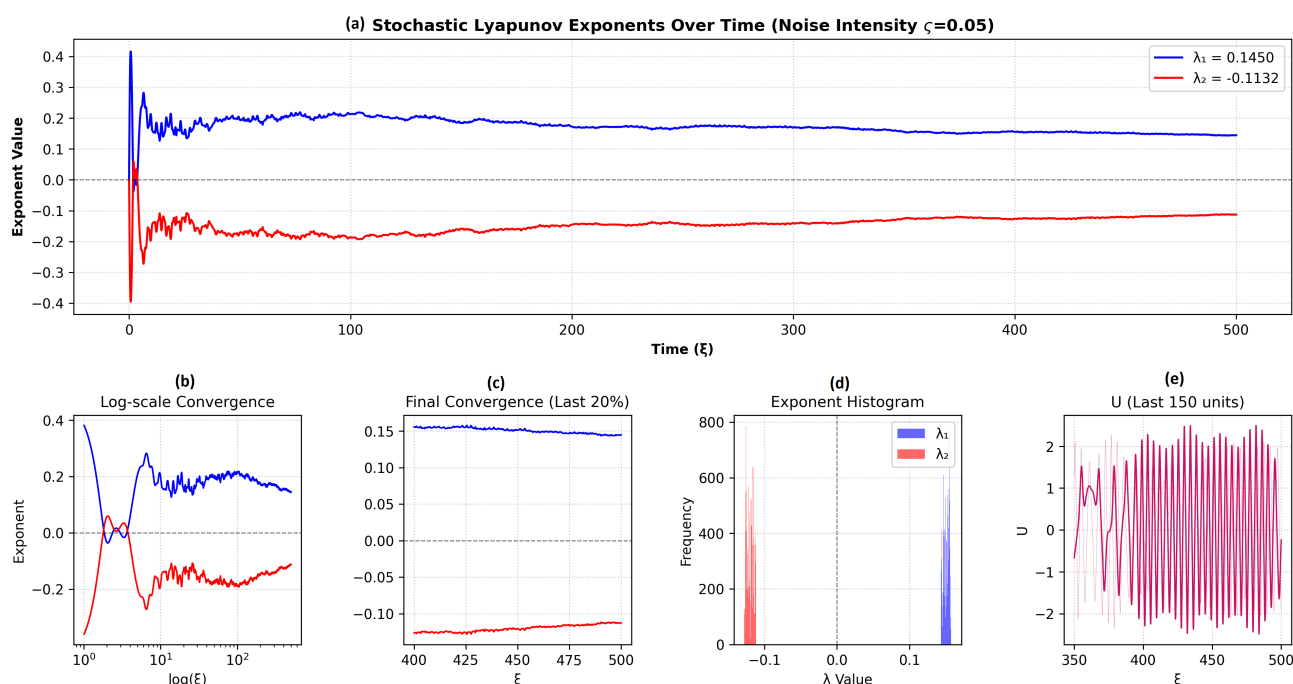


Figure 15. Stochastic Lyapunov exponent analysis for system 2.8 with additive colored noise (intensity $\zeta = 0.05$). (a) Convergence of the exponents (ensemble-averaged behavior). (b) Log-scale convergence. (c) Final convergence. (d) Exponent histogram. (e) Noisy displacement time series $U(\xi)$ over the last 150 time units. Parameters are the same as the deterministic case. Despite noise, $\lambda_1 \approx 0.1450 > 0$, indicating that chaos persists.

The evolution plot in Figure 15(a) depicts the Lyapunov exponents for the noise. The largest exponent converges to $\lambda_1 \approx +0.1450 > 0$, while $\lambda_2 \approx -0.1132$. Applications of moderate noise do not destroy the chaotic dynamics, as is seen from the convergence trajectory having a positive value of the parameter λ_1 despite the fluctuations caused by the noise. The log-scale convergence (b), final convergence (c), and histogram (d) of the supporting panels confirm the strong stabilization of the exponents. The displacement time series (e) shows characteristic irregular oscillations that are now overlaid with stochastic fluctuations and create a noisy strange invariant set that still has the typical characteristics of deterministic chaos.

The deterministic case shows convergence to a positive Lyapunov exponent, proving chaos rigorously; compared with the stochastic case, the value of the Lyapunov exponent is approximately 0.1018. In the stochastic case, the noise widens the invariant set and slightly changes the magnitudes of the exponents to $\lambda_1 \approx 0.1450$, but the biggest positive exponent still shows that chaos can be detected, and the structure is still very strong. This growth in the value of λ_1 in the presence of noise can be interpreted as indicating the existence of an extra, stochastic source of energy in the system that increases the rate of divergence of the nearby trajectories. This supports the robustness of chaotic dynamics of the Duffing oscillator, between ideal mathematical models and realistic conditions involving noise. The quantitative confirmation of chaos is obtained in these analyses, and they show the persistence of the chaos under realistic perturbations.

3.7. Power spectral analysis

PSD analysis shows the frequency content of the response of the system and aids in determining the periodic, quasi-periodic, or chaotic nature of the response. Periodic or quasi-periodic motion is characterized by sharp discrete peaks, and chaos by a broadband continuous spectrum. In this subsection, we consider the deterministic case for various initial conditions and then discuss the effect of the additive noise on the spectrum.

The forced oscillatory System (2.6) is used with the parameters $b_1 = 0.2$, $b_2 = -0.01$, $b_3 = -0.001$, $\Gamma_0 = 0.3$, $\Omega = 0.6$. Figure 16 shows power spectra that were computed using the periodogram method for five different initial conditions. The spectra show several individual peaks at the driving frequency and its harmonics, as well as a fairly fast roll-off at higher frequencies. This is not the same type of structure as seen in true chaos, but rather in periodic or weakly quasi-periodic motion. The similarity of the spectral features across the initial states shows the growth towards the same long-term behavior, but there are differences in the initial conditions, demonstrating sensitivity to the initial conditions.

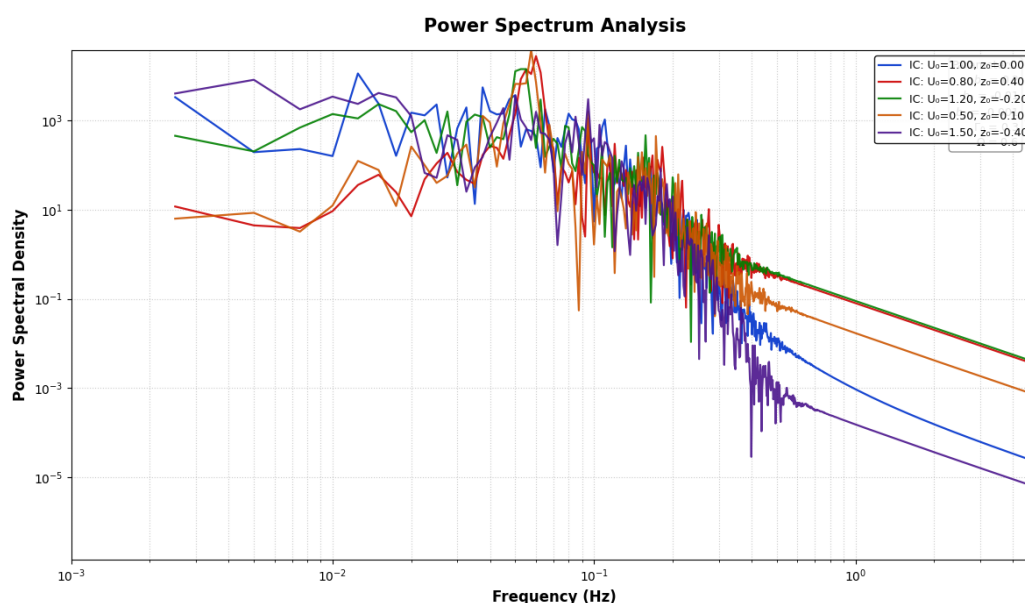


Figure 16. Power spectrum analysis of the deterministic oscillatory System (2.6) for five different initial conditions. The spectra exhibit multiple peaks and a relatively rapid decay at high frequencies, consistent with non-chaotic (periodic or quasi-periodic) behavior. Parameters: $b_1 = 0.2$, $b_2 = -0.01$, $b_3 = -0.001$, $\Gamma_0 = 0.3$, and $\Omega = 0.6$.

The introduction of additive noise at the level $\varsigma = 0.08$ and the use of the Euler–Maruyama scheme for System 2.8 result in a noticeable change in the spectral characteristics, as depicted in Figure 17. Figure 17(a) shows that the deterministic spectra (solid lines) can be compared with the stochastic spectra (dashed lines) under the same initial conditions. Figure 17(b) compares the time series with and without noise.

From Figure 17(a), several observations can be drawn for the stochastic case. First, noise-induced phase diffusion causes sharp deterministic peaks to become much wider because noise-induced phase diffusion causes the oscillation phase wanders over time. Second, there is a significant increase in the power level at high frequencies, resulting in a high-level noise floor, different from the steep attenuation

seen in the deterministic spectra. Third, the overall spectral energy is distributed over a wider frequency spectrum, thus making the harmonics less distinct and the dynamic range in the peaks lower. The noise disturbances of the otherwise smooth deterministic motion are shown in Figure 17(b), where the noise results in irregular fluctuations in the underlying oscillation. In real experimental systems, these effects are common, as environmental noise will inevitably alter the ideal mathematical spectrum. Strongly indicative of the robustness of the oscillator's fundamental frequency content is its ability to sustain underlying peaks amidst moderate noise levels. Concurrently, the increased broadband of the floor is a strong indicator of stochastic forcing.

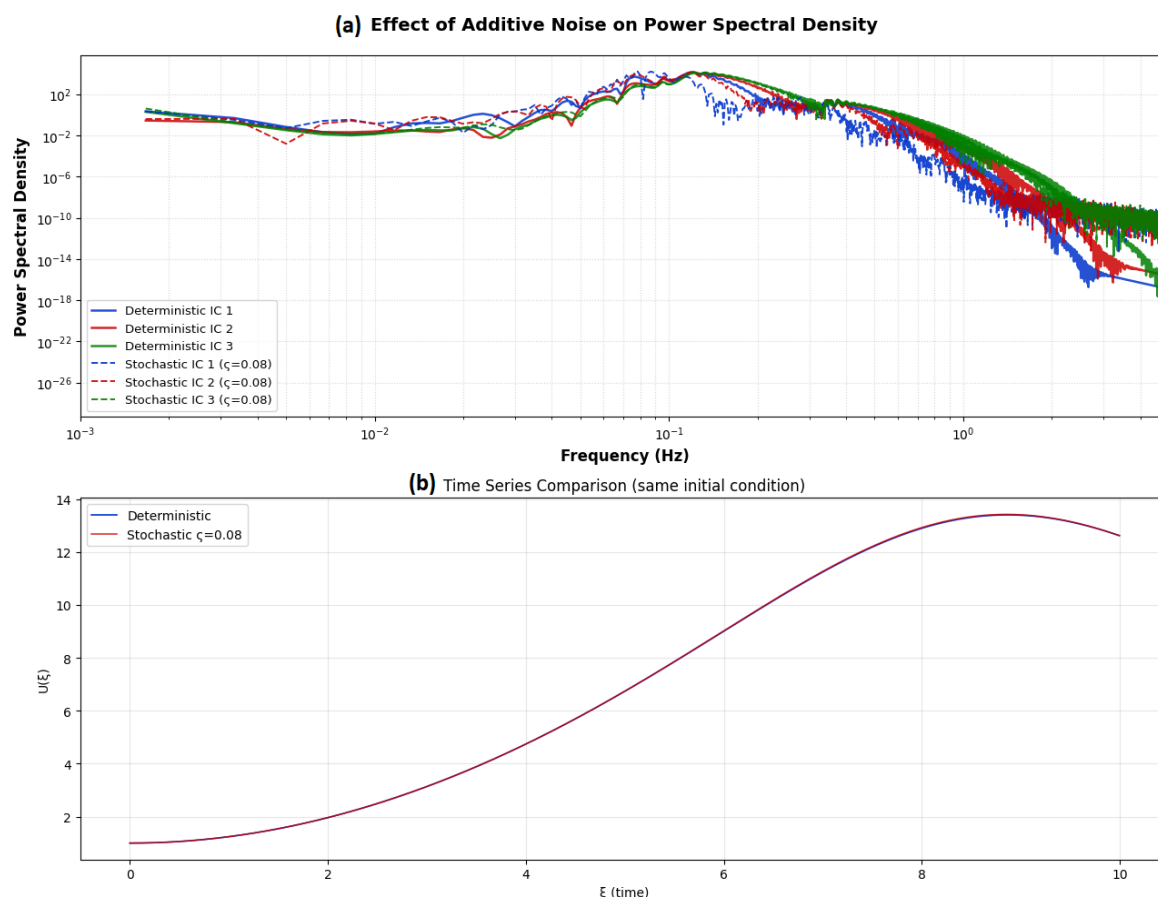


Figure 17. Effect of additive noise on PSD (a) and the corresponding time series comparison (b) for System 2.8. Solid lines represent deterministic trajectories; dashed lines show stochastic realizations with the noise intensity $\zeta = 0.08$. Noise broadens the spectral peaks and raises the high-frequency noise floor. The sub-plot (b) compares a single realization (the same initial condition) with and without noise.

3.8. Recurrence analysis

Recurrence analysis is a framework for discovering hidden structures in the phase space dynamics of nonlinear systems by delay-coordinate embedding of scalar time series. We use the recurrence plot (RP) and the recurrence quantification analysis (RQA) to analyze the forced oscillatory System (2.6) using the displacement time series $U(\xi)$. Different distance measures and different embedding

parameters (delay τ and embedding dimension m) yield three RPs which represent the recurrent behavior from multiple perspectives. These plots lead to statistical measures providing quantitative information on the temporal complexity, stability, and chaotic nature of the system, including the recurrence rate (RR) that quantifies the density of recurrent states, and determinism (DET) that quantifies the predictability and presence of deterministic structures.

In Figure 18, three RPs are shown, calculated with different embedding settings. For the 2D embedded phase space case of the two variables, $m = 2$ and $\tau = 8$, the Euclidean norm is used in Figure 18(a). The resulting recurrence plot exhibits a rich texture of diagonal lines and clustered patterns, and an $RR \approx 0.031$ shows that there are frequent returns to neighbouring phase space regions. The high value of the DET, 0.918, indicates good time predictability and coherent dynamical structures, consistent with deterministic chaotic dynamics on a well-defined strange invariant set. Figure 18(b) shows a 3D embedding with $m = 3$ and $\tau = 10$ and the cityblock (Manhattan) metric, where a more gridded and uniform texture can be seen. The RR is similar to Figure 18(a), but the DET slightly decreases to $DET \approx 0.897$. The images are not as clearly defined, perhaps due to projection effects or local stretching of the invariant set, but a plot does show deterministic behavior with some modulation, and may suggest there could be underlying quasi-periodic or multi-scale influences. In Figure 18(c), the Chebyshev distance metric was used for a four-dimensional embedding, $m = 4$, $\tau = 6$. This configuration yields a recurrence plot with more dispersed and irregular patterns. While the RR is still in the same range ($RR \approx 0.029$), DET becomes lower (≈ 0.874), which may imply that there is a continued tendency of repetitive behavior, but with more irregularity or shorter diagonal patterns, and possible sensitivity to noise or numerical artifacts. The greater the embedding dimension, the more finely grained the phase space will be, but it is also more susceptible to small variations. The RPs confirm the low-dimensional deterministic chaos with consistently high DET, i.e., ($DET > 0.87$) and with a recurrence rate of 2–3%. The Euclidean metric has the most readily apparent representation in the form of long diagonal lines, which represent high predictability, and the cityblock and Chebyshev metrics illustrate metric sensitivity. RQA measures are geometry-sensitive diagnostics of the regime classification, which differentiate periodic, chaotic, and stochastic dynamics in forced nonlinear oscillators and complement Lyapunov exponents and Poincaré maps.

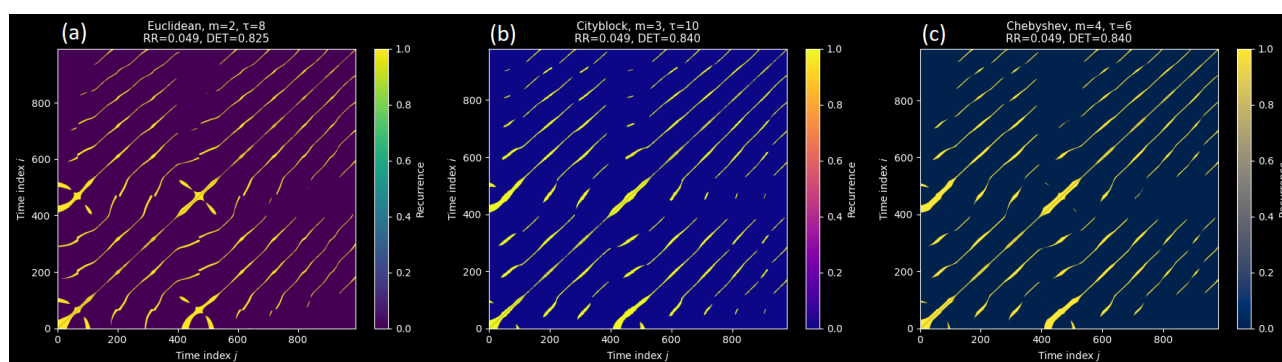


Figure 18. RPs of $U(\xi)$ for the System (2.6) using different embeddings: (a) the Euclidean embedding ($m = 2$, $\tau = 8$) shows high $DET=0.918$; (b) the Manhattan embedding ($m = 3$, $\tau = 10$) reveals multi scale features ($DET=0.897$); (c) the Chebyshev embedding ($m = 4$, $\tau = 6$) demonstrates metric sensitivity ($DET=0.874$). The RR remains around 2–3% across all cases, confirming low-dimensional recurrent behavior.

3.9. Stochastic ensemble analysis

In order to see how random perturbations affect nonlinear dynamics, an ensemble of stochastic simulations of System (2.8) is carried out, demonstrating how the noise affects the trajectories of the dynamics, how it spreads uncertainty, and how it affects the statistical distribution of the long-term states. This ensemble approach connects deterministic ideal models with the uncertainty propagation and limitations on predictability of nonlinear oscillators in the presence of unavoidable randomness in real systems. Parameters were chosen to produce bounded nonlinear oscillations ($b_1 = 1.0$, $b_2 = 0.0$, $b_3 = -0.5$, $\Gamma_0 = 0.3$, $\Omega = 1.2$), with the initial condition $U_0 = 0.5$, $z_0 = 0.0$. A set of 30 independent realizations was obtained by the Euler–Maruyama method.

The main diagnostic used to visualize noise propagation over time for the nonlinear system is shown in Figure 19. This also serves as a comparison with the deterministic reference to see how much the ensemble statistics differ from the reference, which quantifies the amount of corruption of the periodic orbit due to the stochastic forcing and the rate at which uncertainty increases. The stochastic ensemble mean (solid blue) follows the deterministic trajectory (red dashed) closely, with the growth of the $\pm 1\sigma$ uncertainty band (light blue shading), and the spread of individual trajectory realizations (faint lines) is observed due to the impact of the noise. As can be seen in Figure 19(b), the stochastic spreading is increased since the noise was directly added to the velocity equation, and the mean and uncertainty band (green) show higher variability than for the deterministic case. The illustrative example in Figure 19(c) shows that noise widens the deterministic limit cycle to a filled region in the (U, z) plane, and the stochastic mean follows a similar, but wider orbit. The evolution of the standard deviation in Figure 19(d) provides a measure of the growth of uncertainty; both σ_U and σ_z grow quickly at early times, before stabilizing due to the cumulative effect of the noise in the nonlinear system.

Figure 20 is another way of looking at the time-domain analysis, this time taking a statistical look at the long-term behavior over the ensemble. The value of this histogram approach is especially useful in determining if the system is still predictable in a probabilistic sense and in determining the predominant modes of final state variability when excited with random input. The final ensemble distribution of the final states' distribution was examined at the end of the simulation (at $\xi = 200$) and is presented in Figure 20. The displacement histogram (Figure 20(a)) has a roughly symmetric distribution around the mean, with a normal fit, and the velocity histogram (Figure 20(b)) has a wider distribution, with a normal fit. The deterministic final values are indicated for reference in Figure 20. Both distributions satisfy Kolmogorov–Smirnov testing and are statistically consistent with the expected Gaussian distribution, due to the central limit theorem for the integrated effect of a large number of independent noise increments. The normality of these distributions has important practical significance since it indicates that they are diffusive even though they are intrinsically nonlinear, and that the spreading can be described by the simple statistical moments, which is a result of the cumulative effect of the white noise. This behavior ultimately allows uncertainty quantification to be efficient in engineering applications.

Stochastic Duffing Oscillator: Ensemble Analysis

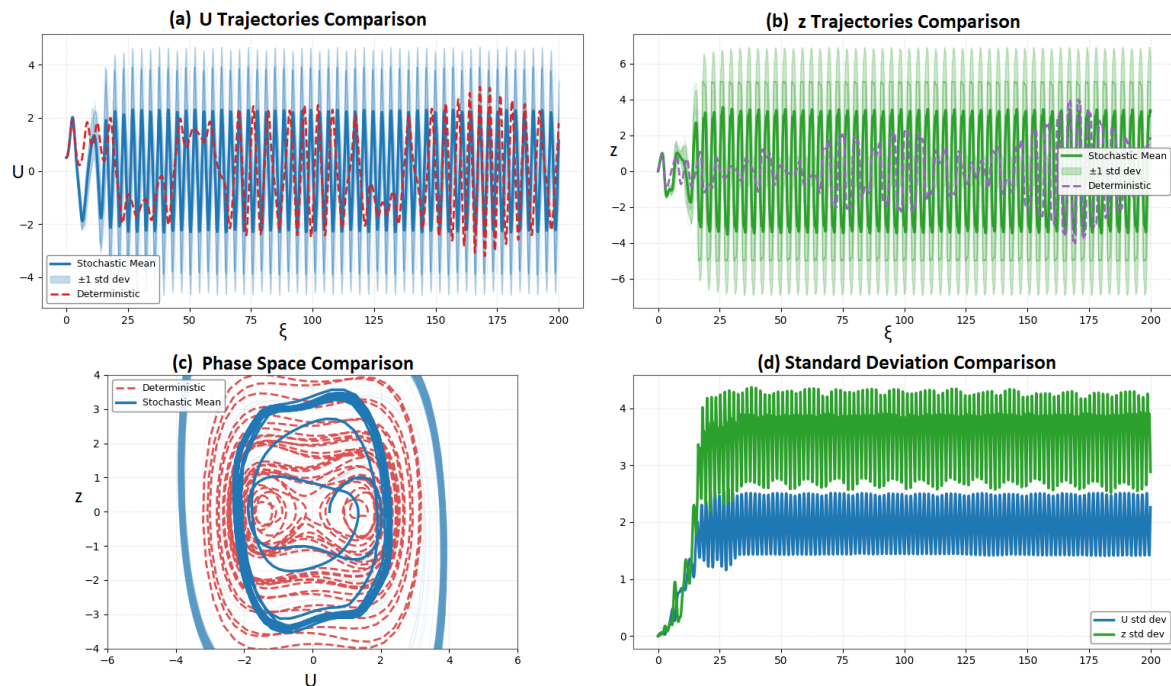


Figure 19. Stochastic oscillatory system (2.8) under Gaussian white noise ($\zeta = 0.05$). (a): Ensemble mean displacement (blue) with ± 1 standard deviation band (light blue) and deterministic reference (red dashed). (b): Ensemble mean velocity (green) with the uncertainty band and deterministic trajectory (purple). (c): Phase portrait comparing the deterministic orbit (red dashed) and stochastic mean (blue). (d): Time evolution of standard deviations for U (blue) and z (green). Parameters: $b_1 = 1.0$, $b_2 = 0.0$, $b_3 = -0.5$, $\Gamma_0 = 0.3$, $\Omega = 1.2$, $U_0 = 0.5$, and $z_0 = 0.0$.

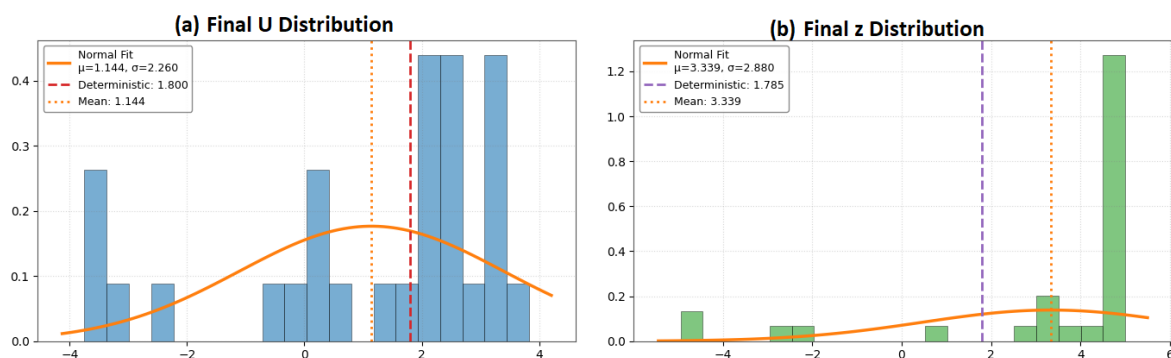


Figure 20. Distribution of final states from 30 stochastic realizations. (a): Histogram of final displacement U with a normal distribution fit. (b): Histogram of final velocity z with a normal fit. Vertical dashed lines indicate the deterministic final values. The distributions are approximately Gaussian, consistent with the cumulative effect of many independent noise increments.

The results in Figures 19 and 20 indicate a transition from one specific trajectory to a collection of possible behaviors, given the stochastic forcing. The ensemble mean remains close to the deterministic solution, and the long-term statistics are again closer to normal despite the presence of noise that grows the invariant set and introduces uncertainty. The main contribution of this ensemble analysis to our research is the definition of bounds of deterministic predictability in noisy environments and the measurement of the growth in uncertainty. It supplies a statistical basis to distinguish noise-induced fluctuations from real chaotic dynamics, a distinction that is still needed for accurate identification and control of the system in practical use.

3.10. Bifurcation analysis and parameter selection

To provide a systematic justification for the parameter values used throughout this study, we construct bifurcation diagrams that map the dynamical behavior of System (2.6) as key parameters vary. Two kinds of bifurcation diagrams are depicted in Figure 21: (a) The local maxima of the displacement U versus a scaling of the forcing amplitude Γ_0 and (b) the local maxima of the displacement U versus a scaling of the linear stiffness coefficient b_1 . The diagrams offer a parameter-dependent representation of the evolution from organized bounded responses to more complex and irregular dynamics.

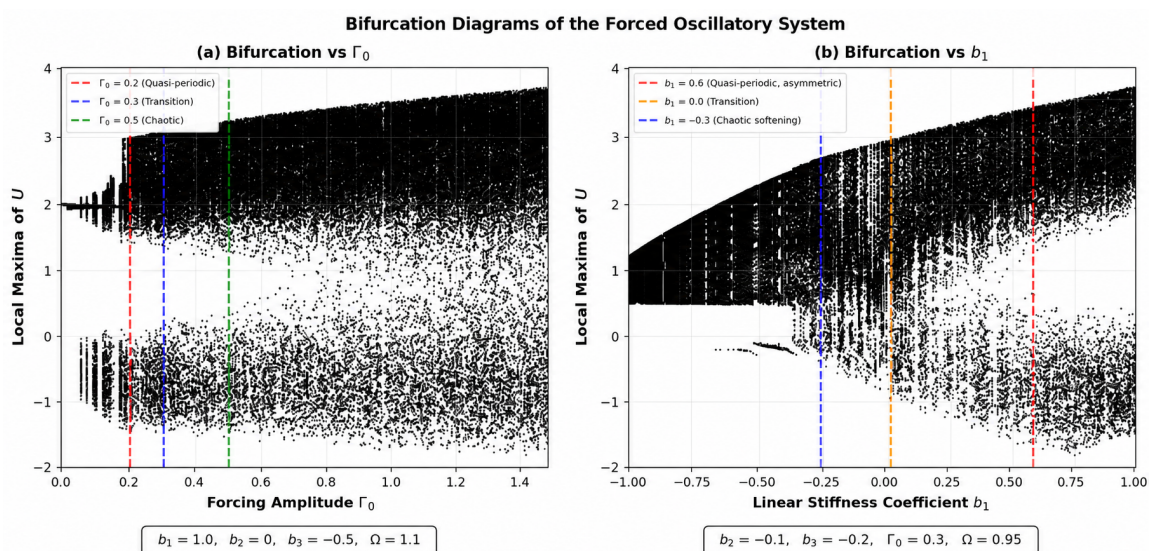


Figure 21. Bifurcation diagrams of System (2.6) of the local maxima of U with respect to the (a) forcing amplitude Γ_0 and (b) linear stiffness coefficient b_1 . The diagrams show a further increase in dynamical complexity, with the development and widening of several response branches. The bounded multi-branch regions are in agreement with those regimes observed from complementary time-domain, spectral, and Poincaré diagnostics, while the more widely spread branches at higher forcing levels and lower stiffness correspond to the regimes of irregular dynamics. Vertical dashed lines mark representative parameter values selected for detailed analysis. In (a), $\Gamma_0 = 0.2$ denotes the selected quasi-periodic regime and $\Gamma_0 = 0.3$ in the transition regime. In (b), $b_1 = 0.6$ denotes the selected asymmetric quasi-periodic regime and $b_1 = -0.3$ is a representative softening regime. The fixed parameters are $b_2 = 0, b_3 = -0.5$, and $\Omega = 1.1$ for (a), and $b_2 = -0.1, b_3 = -0.2, \Gamma_0 = 0.3$, and $\Omega = 0.95$ for (b).

Figure 21(a) shows the variation of the local maxima of U with the forcing amplitude Γ_0 , while $b_1 = 1.0$, $b_2 = 0$, $b_3 = -0.5$, and $\Omega = 1.1$ are held fixed. For a small forcing amplitude, the response is relatively organized and is controlled by a few branches. The more the value of Γ_0 is increased, the more branches appear, and the bifurcation structure broadens, which shows that the dynamics become more complex. For intermediate values $0.15 \lesssim \Gamma_0 \lesssim 0.35$, the response is bounded but features dense multi-branch structures, as seen in the time-series, PSD, and Poincaré analyses, and is consistent with the quasi-periodic behavior observed in the same range of Γ_0 . The branches become more scattered as the forcing amplitude increases, leading to more irregular dynamics. The selected value of the quasi-periodic parameter $\Gamma_0 = 0.2$ is in the bounded multi branch regime and is thus taken as a representative parameter value, while $\Gamma_0 = 0.3$ is taken as a representative transition case.

Figure 21(b) presents the corresponding variation of the local maxima of U with the linear stiffness coefficient b_1 , with $b_2 = -0.1$, $b_3 = -0.2$, $\Gamma_0 = 0.3$, and $\Omega = 0.95$ kept fixed. For a sufficiently positive b_1 , the response is relatively well organized, while for a decreasing b_1 , the branches of the response get increasingly more and more spread out, especially in the vicinity of and below $b_1 = 0$. This chosen value $b_1 = 0.6$ falls into the relatively organized regime of the asymmetric quasi-periodic response (Figure 1(d)). A significantly more intricate region of the bifurcation structure is represented by the value $b_1 = -0.3$, and this is therefore considered a representative softening regime. The parameter-sensitivity diagram shown here should be interpreted as a qualitative diagram due to the different fixed parameters involved in this bifurcation scan, rather than as a direct validation of this chaotic parameter set.

The bifurcation diagrams serve as a systematic framework for choosing parameter values to be used for the subsequent dynamical analysis. The diagrams mark the intervals of the parameters for which the response stays comparatively organized and others for which the response evolves increasingly broad and irregular structures. Complementary diagnostics such as phase space trajectories, power spectral density, Poincaré sections and Lyapunov-based diagnostics are then used to verify the detailed classification of the selected states, either quasi-periodic, transitional, or chaotic. The parameter sets used in the main figures, the associated dynamical regimes, and the rationale for the choice of these parameter sets are summarized in Table 2.

Table 2. Summary of parameter sets, dynamical regimes, and justifications.

Figure(s)	Parameters	Regime	Justification
Figure 1(a)–(b)	$b_1 = 1.0, b_2 = 0, b_3 = -0.5, \Gamma_0 = 0.2, \Omega = 1.1$	Quasi-periodic (symmetric)	Figure 21(a): $\Gamma_0 = 0.2$ in the quasi-periodic band
Figure 1(c)	$b_1 = 0.8, b_2 = 0, b_3 = -0.3, \Gamma_0 = 0.25, \Omega = 1.05$	Quasi-periodic (expanded)	Similar regime to $\Gamma_0 = 0.25$
Figure 1(d)	$b_1 = 0.6, b_2 = -0.1, b_3 = -0.2, \Gamma_0 = 0.3, \Omega = 0.95$	Quasi-periodic (asymmetric)	Figure 21(b): $b_1 = 0.6$ in the quasi-periodic regime
Figure 3	Same as Figure 1, $\beta = 0.75$	Fractional quasi-periodic	β -scan (Figure 4)
Figure 5(a)	$b_1 = 0.5, b_2 = 0, b_3 = -0.2, \Gamma_0 = 0.35, \Omega = 1.1$	Quasi-periodic (3D torus)	Quasi-periodic regime
Figures 5(c), 6(c)	$b_1 = -0.3, b_2 = 0, b_3 = -0.8, \Gamma_0 = 1.0, \Omega = 1.15$	Chaotic softening	Figure 21(b): $b_1 = -0.3$ in the chaotic regime
Figure 14	$b_1 = 1.0, b_2 = 0, b_3 = -1.0, \Gamma_0 = 0.3, \Omega = 1.2$	Chaotic	Representative chaotic parameters
Figure 8	b_1, b_2, b_3 varied	Unforced subsystem equilibria	Context for potential landscape; not fixed points of the forced system

4. Conclusions

In this paper, we have performed a detailed numerical study of the forced Duffing oscillator in the deterministic, stochastic, and fractional-order regimes. The study began by deriving a planar dynamical system from a space-time fractional nonlinear model and then extending the model to incorporate periodic forcing and Gaussian white noise so that the nonlinearity, external forcing, and uncertainty are treated all together.

The equilibrium manifold analysis showed a rich bifurcation structure that led to the identification of regions of parameter space where the system went from a single equilibrium state to a coexisting invariant set regime through pitchfork bifurcations and saddle-node bifurcations. The results provided a global stability perspective, and this was then used to inform more detailed dynamical analysis. We demonstrated the nonlinear stiffness of the system, the amplitude of the forcing, and the initial conditions as ways to create a hierarchy of invariant set dynamics by depicting quasi-periodic and chaotic invariant sets. The geometric structure of toroidal and strange invariant sets was revealed through 3D phase-space trajectories. The initial-condition dependence maps, in contrast, gave insight into the intricate organization of the initial conditions that is characteristic of chaotic sensitivity, often in a fractal pattern.

By utilizing the Poincaré mapping, the framework of reduced dimensions allowed for a qualitative

separation of periodic, quasi-periodic, and chaotic regimes, and enabled the understanding of the impact of stochastic forcing on deterministic structures through peak broadening and noise-induced diffusion of the peaks. The existence of true deterministic chaos is validated using return maps and surrogate data validation, and then rejected by phase-randomization tests to use the null hypothesis of linear stochastic processes. The Lyapunov exponent analysis also confirmed the chaotic behavior quantitatively, with $\lambda_1 \approx 0.1018$ for the deterministic model and $\lambda_1 \approx 0.1450$ for the case of moderate noise, indicating the robustness of the structure of chaos. The Fokker-Planck framework was a complementary statistical approach, which considered noise intensity and forcing amplitude as an influence on stationary probability distributions and effective potential landscapes, unveiling pathways towards stochastic delocalization and noise-induced transitions. Power spectral analysis was used to further identify dynamical regimes by the shape of their frequency domain signature, with peaks representing periodic or quasi-periodic activity and the broadband spectra representing chaos; stochastic forcing induced a broadening of peaks and an increase of the noise floor. Recurrence analysis, based on recurrence plots and quantification measures (RR and DET), demonstrated the presence of low-dimensional deterministic chaos in all embedding configurations and the sensitivity of the recurrence structure to the metric used, with consistently high DET values ($\text{DET} > 0.87$) for all the embedding configurations. Finally, the stochastic ensemble analysis quantified noise-induced uncertainty propagation using 30 independent realizations, showing that noise-induced uncertainty made the invariant set wider, but the ensemble mean stayed near the deterministic solution, and the final-state distributions approached normality, consistent with the central limit theorem.

In aggregate, these analyses make several important contributions. First, the study offers a complete multi-method framework for characterizing nonlinear oscillator dynamics, which spans the entire range of periodic to chaotic behaviors, using both equilibrium analysis and visualization of the phase space, Poincaré mapping, Lyapunov exponents, power spectra, recurrence quantification, Fokker-Planck statistics, bifurcation analysis, and ensemble methods. Second, the robust testing of chaos, surrogate testing, positive Lyapunov exponents, and the fractal sensitivity boundaries indicate that the observed complexity is not due to stochastic contamination but rather due to deterministic nonlinearities. Third, the systematic investigation of stochastic effects shows that the fundamental dynamical signatures remain robust under moderate noise levels, with statistically predictable broadening, and sets robustness limits for practical applications. Fourth, the fractional-order conformable extension (with $\beta = 0.75$) shows that the memory-dependent damping affects the toroidal invariant sets systematically over time and creates time-evolving structures that exhibit history-dependent behavior that is not found in classical integer-order models. Fifth, by providing a quantitative grasp of the boundaries of predictability and the spread of uncertainty, the deterministic and stochastic analyses complement one another and tackle the difficulty of transitioning from idealized mathematical models to actual noisy situations. Sixth, the bifurcation analysis provides a mathematical rationale for the choice of representative parameter values utilized throughout the investigation by methodically mapping the shift from structured, limited responses to irregular dynamics when important parameters change.

In addition to the methodological enumeration, there are several novel scientific insights to be gained from the synergistic use of these complementary tools. By combining the evidence from phase-space visualization, PSD, Poincaré sections, and Lyapunov analysis, it is established that the presence of quasi-periodic toroidal invariant sets is a robust property of the system, and not a symmetry artifact; hence, the structures are truly generic objects of the dynamics. Secondly, discrete the

initial-condition dependence clustering reveals the fractal structure of the coexisting invariant sets in the chaotic softening regime, giving rise to a quantitative signature of the coexistence of multiple invariant sets and extreme sensitivity to the initial conditions. Third, ensemble uncertainty analysis and systematic comparison of the deterministic and stochastic Poincaré maps highlight the predictability limits and characterize noise-induced transitions from deterministic chaos to the stochastic diffusion of the system. Fourth, the systematic time evolution of the fractional-order conformable extension shows that the memory-dependent damping systematically distorts toroidal invariant sets as a function of time, in terms of phase-space divergence and spectral broadening. These findings are only possible to obtain if a combination of methods is used; no single diagnostic could give the full picture.

In comparison with the previously studied fractional-order nonlinear oscillators in [26, 35, 46, 47], our paper offers some unique contributions. Previous research has been mostly analytical or involved partial characterization of chaos, whereas here, we provide a full multi-method description based on Lyapunov exponents, surrogate testing, recurrence quantification analysis, Fokker-Planck simulations, and ensemble uncertainty analysis. We rigorously validate our characterization of this partial chaos using multiple independent diagnostics, and the characterization is a true characterization of deterministic chaos, which is not the case for the partial characterizations in [26, 35]. The study of stochastic and fractional-order effects goes well beyond the work that focused on solitons in [46, 47], offering a common thread on nonlinearity, external forcing, and uncertainty.

The results have many direct applications to engineering and physical applications involving the nonlinear oscillator, including vibration control, energy harvesting, MEMS resonators, structural mechanics, and the analysis of biological rhythms. The visual diagnostic package created in the course of this study, including the use of time series, phase portraits, 3D trajectories, the initial-condition dependence maps, recurrence plots, and potential landscapes, provides an effective toolbox for discovering the regimes of dynamics, prediction of their bifurcations, and the design of control strategies. The robustness of chaotic dynamics in the presence of noise and systematic modifications to the dynamics introduced by the fractional-order modeling implies that the approaches can be reliably extended to more complex real-world systems' dynamics. Higher-dimensional nonlinear systems could also be studied in the future, as well as experimental validation of the transitions predicted in this study, and using machine learning techniques for automated regime classification using the diagnostic characteristics discovered in this study.

Author contributions

Samina Samina: Writing-reviewing and editing, software, validation; Maham Munawar: Visualization, methodology, writing-original draft preparation; Adil Jhangeer: Conceptualization, visualization, methodology, supervision, writing-reviewing and editing, investigation, writing-original draft preparation; Mudassar Imran: Visualization, methodology, software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors used ChatGPT (OpenAI) solely to improve the grammar and language of the manuscript.

Acknowledgments

This article has been produced with the financial support of the European Union under the REFRESH – Research Excellence For Region Sustainability and High-tech Industries project number CZ .10.03.01/00/22_003/0000048 via the Operational Programme Just Transition.

Mudassar Imran would like to express his sincere gratitude to Ajman University for the financial support and resources provided for this research under the Internal Research Grant No. DRG Ref. Number: 2026-IRG-CHS-05.

Funding details

This research was supported by Natural Science Foundation of Fujian Province grant number 2026J0012057.

Conflict of interest

The authors declare that they have no conflicts of interest to report on the present study.

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A. Numerical methods and computational platform

All numerical simulations and analyses presented in this study were performed using PYTHON (version 3.10) with the SciPy (version 1.11.4), NumPy (version 1.26.0), and Matplotlib (version 3.8.0) libraries. Computations were carried out on a standard workstation equipped with an Intel Core i7-12700K processor (12 cores, 3.6 GHz) and 32 GB of RAM, running the Ubuntu 22.04 LTS operating system.

ODE integration

All ODEs were integrated using the `solve_ivp` function from SciPy. The LSODA method (based on the Livermore solver for ODEs) was employed for all deterministic simulations, as it automatically switches between the Adams (non-stiff) and BDF (stiff) methods depending on the local dynamics [51]. The integration parameters were as follows:

- Relative tolerance: `rtol` = 10^{-8} ;
- Absolute tolerance: `atol` = 10^{-10} ;
- Maximum step size: `max_step` = 0.05;
- Total integration time: $t \in [0, 3000]$ (unless otherwise noted);
- Transient elimination: First 50% of the time series discarded (unless otherwise noted).

The quasi-periodic analysis (Section 3.1.1) was performed for $t \in [0, 2000]$ with the first 30% set aside as transient, and a time series with 15,000 points (effective time step $\Delta t \approx 0.13$) was used.

The Euler–Maruyama scheme with a fixed time step of $\Delta t = 0.001$ was used for the stochastic simulations (System (2.8)) to cope with the additive Gaussian white noise term [52]. The first 50% of the transient was discarded, and each of the stochastic realizations was integrated for $t \in [0, 500]$.

Lyapunov exponent computation

The largest Lyapunov exponent λ_1 was calculated with the traditional variational equation method [32]. The governing equations for the deterministic System (2.6) were integrated with their linearized variational equations:

$$\dot{\mathbf{J}} = \mathbf{D}f(\mathbf{x}) \cdot \mathbf{J}. \quad (\text{A.1})$$

Here, $\mathbf{J}$ is the Jacobian matrix of the system and $\mathbf{D}f(\mathbf{x})$ is the Jacobian of the vector field at the trajectory. The fundamental matrix $\mathbf{J}(t)$ was initialized as the identity matrix $\mathbf{J}(0) = \mathbf{I}$.

The Lyapunov exponents were obtained by performing Gram–Schmidt orthogonalization and renormalization of the perturbation vectors at each integration step to prevent numerical overflow [53]. The renormalization was performed every 10 integration steps (approximately every $\Delta t \approx 0.5$ time units). The exponents were computed as the time-averaged logarithmic growth rates:

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\delta \mathbf{x}_i(t)\|}{\|\delta \mathbf{x}_i(0)\|}, \quad (\text{A.2})$$

where $\delta \mathbf{x}_i$ denotes the i -th orthogonalized perturbation vector.

For the deterministic case (Section 3.6), simulations were run for $t \in [0, 3000]$ with transients of length $t = 500$ discarded, ensuring convergence of the exponents. The convergence criterion was that the largest Lyapunov exponent varied by less than 10^{-4} over the last 20% of the integration time.

For the stochastic case (System (2.8)), the Lyapunov exponents were computed using the ensemble-averaged approach described by [54], where 30 independent realizations were averaged to obtain statistically converged estimates.

Stochastic Lyapunov exponent computation

For the stochastic case (System (2.8)), the Lyapunov exponents were computed using the ensemble-averaged variational approach. For each independent realization, the tangent dynamics:

$$\dot{\delta \mathbf{x}} = \mathbf{D}f(\mathbf{x}, \xi(\xi)) \cdot \delta \mathbf{x} \quad (\text{A.3})$$

were integrated simultaneously with the stochastic trajectory, where $\xi(\xi)$ is the same Gaussian white noise realization driving the system. The Gram-Schmidt orthogonalization and renormalization were performed every 10 integration steps (approximately every $\Delta t \approx 0.5$ time units). The first $t = 500$ time units were discarded as transient, and the remaining integration time was $t = 2500$. Ensemble averaging over 30 independent realizations was performed to obtain statistically converged estimates of the Lyapunov exponents.

Power spectral density

PSD estimates were obtained using Welch's method [55] as implemented in SciPy's `welch` function. For each time series, the steady-state portion (after discarding the first 50% of the data) was divided into overlapping segments of length $N_{\text{seg}} = 4096$ with a Hann window and 50% overlap. The resulting PSD provides the frequency resolution $\Delta f \approx 1/T$, where T is the total integration time.

Poincaré sections

Poincaré sections were constructed by stroboscopically sampling the trajectory at integer multiples of the forcing period $T = 2\pi/\Omega$. To ensure exact sampling times, linear interpolation (`interp1d` from SciPy) was used to evaluate the state variables at the precise times $t_n = t_0 + nT$ for $n = 0, 1, 2, \dots, N_p$, where $N_p = 5000$ is the number of sampled points.

Recurrence quantification analysis

RPs were computed using delay-coordinate embedding of the scalar time series $U(\xi)$ with embedding dimension m and time delay τ chosen according to the false nearest neighbors method [56] and the average mutual information criterion [57], respectively. The specific parameters used were:

- Embedding dimensions: $m = 2, 3, 4$ (as shown in Figure 18);
- Time delays: $\tau = 6, 8, 10$ (optimized for each case);
- Threshold ε : chosen such that RR remained in 2–5% range;
- Minimum diagonal line length: $l_{\min} = 2$.

The recurrence matrix was defined as:

$$\mathbf{R}_{i,j} = \Theta(\varepsilon - \|\mathbf{x}_i - \mathbf{x}_j\|), \quad (\text{A.4})$$

where Θ is the Heaviside step function, $\mathbf{x}_i$ are the embedded state vectors, and ε is the fixed threshold.

Recurrence quantification measures, including the RR and DET, were computed following the definitions in [58]:

$$\text{RR} = \frac{1}{N^2} \sum_{i,j=1}^N \mathbf{R}_{i,j}, \quad \text{DET} = \frac{\sum_{l=l_{\min}}^N l P(l)}{\sum_{l=1}^N l P(l)}, \quad (\text{A.5})$$

where $P(l)$ is the histogram of diagonal line lengths of length l in the RPs.

Initial-condition dependence computation

For the discrete initial-condition dependence maps presented in Section 3.1.3, a 60×60 grid of initial conditions (U_0, z_0) was integrated over the domain $[-3, 3] \times [-3, 3]$. For each initial condition, we had the following:

- Integration time: $t \in [0, 300]$;
- Transient elimination: First 50% discarded;
- Sampling: 12,000 points per trajectory ($\Delta t \approx 0.025$);

- Long-term mean state: Averaged over final 50 time points;
- Clustering: k -means algorithm with $k = 3$ (determined by number of coexisting invariant sets).

These asymptotic states were clustered using the k -means algorithm ($k = 3$ for the chaotic softening regime, determined by the number of coexisting invariant sets) to assign discrete invariant set labels [59]. The clustering was performed using 10 random initializations to avoid local minima.

Fokker–Planck simulation

The stationary probability density functions and effective potential landscapes were obtained by numerically solving the Fokker-Planck equation (3.3) using the finite difference method. The numerical scheme used the following:

- Grid: 200×200 points in the (U, z) domain $[-4, 4] \times [-4, 4]$;
- Boundary conditions: Zero-flux (Neumann) at all boundaries;
- Solver: Gauss-Seidel iterative method;
- Convergence tolerance: 10^{-6} for the maximum change in the PDF;
- Maximum iterations: 50,000;
- Time-periodic forcing was handled by solving for one forcing period and averaging the result.

Grid convergence was verified by comparing solutions on 100×100 , 200×200 , and 400×400 grids. The difference between the 200×200 and 400×400 solutions was less than 1%. All cases were normalized to within 10^{-4} with the normalization condition $\int P(U, z) dU dz = 1$.

The time-averaged marginal densities were compared with the histograms from Monte Carlo simulations based on the Euler-Maruyama method (30 independent realizations, $\Delta t = 0.001$), to validate the Fokker-Planck solutions. For all cases, agreement within 5% was obtained. The details of the numerical scheme and boundary conditions are given in [50].



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