



Research article**Existence and stability results for a (k, η) -coupled system of nonlinear implicit differential equations with nonlocal integral boundary conditions****Madeaha Alghanmi^{1,*} and Bashir Ahmad²**

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Abstract: This article investigated a new class of coupled systems of nonlinear implicit (k, η) -fractional differential equations complemented with nonlocal integral boundary conditions. Implicit fractional differential systems feature the dependence of the nonlinearities involved in the system on the highest order fractional derivatives and allow encoding instantaneous elastic response and rate-dependent memory response. Examples include nonlinear viscoelastic damping of beams/plates, soft biological tissue models, and fractional resistor–inductor–capacitor (RLC) networks with memristive elements. We first transformed the given coupled system into an equivalent fixed point problem and then established sufficient criteria for existence and uniqueness of its solutions via the fixed point theorems due to Schaefer and Banach. The Hyers-Ulam and generalized Hyers-Ulam stability were also discussed for the proposed system. Examples were given to demonstrate the applicability of the main results.

Keywords: (k, η) -fractional derivative; implicit differential equations; coupled system; existence criteria; stability

Mathematics Subject Classification: 26A33, 34K05

1. Introduction

Differential equations play a crucial role in the study of scientific and engineering phenomena as many chemical and physical processes are characterized by such equations. The significant role of derivative operators in modeling several real-world problems motivated researchers and scientists to devise new forms of such operators to improve the existing modeling techniques. Fractional

derivative operators have emerged as effective and significant modeling tools in contrast to classical derivative operators during the last two decades. The nonlocal nature of fractional order operators takes into account the memory and state effects and makes the resulting mathematical formulation of the phenomena under investigation more realistic than its classical counterpart. Examples include fractional modeling of viscoelasticity in arteries and aneurysms [1], signal processing [2, 3], disease models [4], consensus control [5], polymer rheology [6], adaptive synchronization [7], random networks [8], time-fractional models [9, 10], etc.

The literature on fractional calculus is now much enriched with the development of a variety of fractional order operators such as Riemann–Liouville, Caputo, Erdélyi–Kober, Hadamard, Hilfer, ψ -Riemann–Liouville, ψ -Hilfer; for details, see [11–13]. In 2021, Kucche and Mali proposed a general version of the ψ -Hilfer fractional derivative and called it the (k, ψ) -Hilfer derivative, which specializes to (k, ψ) -Riemann–Liouville and (k, ψ) -Caputo derivatives [14]. The authors in [15] discussed controllability of fractional dynamical systems involving the (k, ψ) -Hilfer fractional derivative. A (k, ψ) -Hilfer fractional Cauchy problem of Sobolev type was studied in [16]. In [17], the authors investigated the existence of solutions for a coupled system of (k, ψ) -Hilfer fractional differential equations with nonlocal integral multi-point coupled boundary conditions given by

$$\begin{cases} {}^{k,H}D^{\varrho_1, \tau_1; \psi} u(t) = \chi(t, u(t), v(t)), & t \in (a, b], \\ {}^{k,H}D^{\varrho_2, \tau_2; \psi} v(t) = \omega(t, u(t), v(t)), & t \in (a, b], \\ u(a) = 0, & \int_a^b \psi'(s) u(s) ds = \sum_{j=1}^m \kappa_j v(\sigma_j), \\ v(a) = 0, & \int_a^b \psi'(s) v(s) ds = \sum_{i=1}^n \lambda_i u(\zeta_i), \end{cases}$$

where ${}^{k,H}D^{\varrho_1, \tau_1; \psi}$, ${}^{k,H}D^{\varrho_2, \tau_2; \psi}$ denote the (k, ψ) -Hilfer fractional derivative operators of orders $\varrho_1, \varrho_2 \in (1, 2)$ and parameters $\tau_1, \tau_2 \in [0, 1]$, respectively, $\chi, \omega : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $\kappa_j, \lambda_i \in \mathbb{R}$, $a < \sigma_j, \zeta_i < b$, $j = 1, 2, \dots, m$, $i = 1, 2, \dots, n$, and ψ is an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. For a detailed description of boundary value problems for coupled systems of fractional differential equations involving the (k, ψ) -Hilfer fractional derivative and other such operators, we refer the reader to the book [18].

Implicit differential equations (IDEs) are the ones which cannot be solved for the leading order derivative in it. One can find some interesting recent results on implicit fractional differential equations in [19–22].

The research on coupled systems of fractional IDEs complemented with different kinds of initial and boundary conditions (BCs) received considerable attention as such equations model the interaction of more than one species. In [23], the authors studied a coupled system of fractional IDEs equipped with periodic and anti-periodic BCs. A ψ -fractional coupled system of IDEs supplemented with nonlocal BCs was investigated in [24]. The Banach contraction mapping principle and Krasnoselskii's fixed point theorem were applied to explore the existence and uniqueness criteria for solutions of a coupled system of ϱ -fractional IDEs with anti-periodic BCs in [25]. Although the coupled systems of IDEs are appearing frequently in applied science, there is not enough work on such systems especially involving (k, η) -fractional order derivative operators.

The notion, presently known as Ulam–Hyers stability, dates back to a problem posed by Ulam [26] and later developed by Hyers [27]. Rassias [28] applied it to a broader family of functional equations and Jung [29] discussed the Hyers–Ulam–Rassias stability for a wider class of functional equations. For application of this concept of stability to fractional boundary value problems, for instance, see [30–32].

In this paper, inspired by the aforementioned studies on nonlinear boundary value problems for (k, ψ) -Hilfer fractional differential systems, we introduce a new class of coupled systems involving nonlinear implicit (k, η) -fractional differential equations and nonlocal integral boundary conditions given by

$$\begin{cases} {}^k D_{0+}^{\varrho_1, \eta} x(t) = \mathcal{J}(t, z(t), {}^k D_{0+}^{\varrho_1, \eta} x(t)), & t \in [0, T], \\ {}^k D_{0+}^{\varrho_2, \eta} z(t) = \mathcal{K}(t, x(t), {}^k D_{0+}^{\varrho_2, \eta} z(t)), & t \in [0, T], \\ (\eta_0(t))^{1-\frac{\varrho_1}{k}} x(t)|_{t=0} + (\eta_0(t))^{1-\frac{\varrho_1}{k}} x(t)|_{t=T} = \sigma_1 {}^k I_{0+}^{\varsigma_1, \eta} z(\mu_1), \\ (\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)|_{t=0} + (\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)|_{t=T} = \sigma_2 {}^k I_{0+}^{\varsigma_2, \eta} x(\mu_2), \end{cases} \quad (1.1)$$

where ${}^k D_{0+}^{\varrho_1, \eta}$ and ${}^k D_{0+}^{\varrho_2, \eta}$ denote the Riemann–Liouville (k, η) - fractional derivatives of orders $\varrho_i, k > 0$, such that $\frac{\varrho_i}{k} \in (0, 1)$, $(i = 1, 2)$, $\sigma_i \in \mathbb{R}$, $(i = 1, 2)$, ${}^k I_{0+}^{\varsigma_i, \eta}$ denote (k, η) - fractional integral of order $\varsigma_i > 0$, $i = 1, 2$, $\eta_0(t) = \eta(t) - \eta(0)$, and $\mathcal{J}, \mathcal{K} \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$.

The main objective of this work is to establish the solvability and stability criteria for the problem (1.1). The fixed point theorems due to Schaefer and Banach will be utilized to obtain the existence and uniqueness results for the given problem. The stability for the system (1.1) in terms of Ulam–Hyers and generalized Ulam–Hyers will also be discussed. Here, we emphasize that the nonlocal integral boundary conditions give the system more realistic description to model many processes in scientific and engineering applications, where their measurements depend on the interior points of the domain. As a special case, when $k = 1$ and $\sigma_1 = \sigma_2 = 0$, system (1.1) reduced to the one studied in [25], and the theoretical analysis presented in this paper can be applied to it.

In the rest of the article, we set the material as follows. The preliminary concepts and an auxiliary lemma for the linear variant of problem (1.1) are presented in Section 2. The existence and uniqueness results for problem (1.1) are proved in Section 3, while the Ulam–Hyers and generalized Ulam–Hyers stability for (1.1) are analyzed in Section 4. Finally, illustrative examples are presented in Section 5 to demonstrate the applicability of the obtained results.

2. Preliminaries

Let us start this section by recalling some definitions and lemmas related to this work. For a continuous function $\eta \in C^1((0, T])$, which is increasing on $(0, T]$ and $\eta'(t) \neq 0$ for all $t \in (0, T)$, we have the following definitions and properties from [14] with $\eta_v(t) = \eta(t) - \eta(v)$.

Definition 2.1. ([14]) For $\varrho, k > 0$, the left-sided (k, η) -Riemann Liouville fractional integral for a continuous function g is given by

$$({}^k I_{0+}^{\varrho, \eta} g)(t) = \frac{1}{k \Gamma_k(\varrho)} \int_0^t g(v) \eta'(v) (\eta_v(t))^{\frac{\varrho}{k}-1} dv,$$

where k -gamma function $\Gamma_k(\cdot)$ is defined by $\Gamma_k(\varrho) = \int_0^\infty x^{\varrho-1} e^{-\frac{x^k}{k}} dx$ and it satisfies the properties

- (1) $\Gamma_k(w+k) = w\Gamma_k(w)$;
 (2) $\Gamma_k(k) = 1$;
 (3) $\Gamma_k(w) = k^{\frac{w}{k}-1}\Gamma(\frac{w}{k})$;
 (4) $B_k(w, \zeta) = \frac{\Gamma_k(w)\Gamma_k(\zeta)}{\Gamma_k(w+\zeta)}$, with k -beta function $B_k(.,.)$ given by $B_k(w, \zeta) = \frac{1}{k} \int_0^1 r^{\frac{w}{k}-1} (1-r)^{\frac{\zeta}{k}-1} dr$.

Lemma 2.1. ([14]) Let $\varrho, \varsigma, k > 0$. Then,

$${}^k I_{0+}^{\varrho, \eta} {}^k I_{0+}^{\varsigma, \eta} = {}^k I_{0+}^{\varrho+\varsigma, \eta}.$$

Definition 2.2. ([14]) For $\varrho, k \in (0, \infty)$, and $n \in \mathbb{N}$ such that $n-1 < \frac{\varrho}{k} < n$, the left-sided (k, η) -Riemann fractional derivative for a function $g \in C^n([0, b], \mathbb{R})$ is given by

$$({}^k D_{0+}^{\varrho, \eta} g)(t) = \frac{1}{\Gamma_k(nk - \varrho)} \left(\frac{k}{\eta'(t)} \frac{d}{dt} \right)^n \int_0^t g(v) \eta'(v) (\eta_v(t))^{\frac{nk-\varrho}{k}-1} dv. \quad (2.1)$$

Remark 2.1. For dynamical systems, the fractional derivative operators distinguish themselves from the classical ones by taking into consideration of the past history of behavior of these systems. On the other hand, the fractional derivative with respect to another monotonic function η allows us to describe memory/kernel behavior into a model that matches a natural internal clock of the system rather than physical time. For instance, the function η can reflect a nonlinear time scale such as accumulated stress, biological age, and log-time for scale-invariant processes by connecting the fractional-order memory kernel to that of internal clock instead of uniform physical time. This situation appears in anomalous diffusion models, epidemic models with age-structured memory, viscoelasticity with variable time scales, etc. The extra factor k in the above definitions acts as a scaling factor for relaxing memory effects.

Lemma 2.2. ([14]) Let $\varrho, k > 0$ and $\varsigma \in \mathbb{R}$ such that $\frac{\varsigma}{k} > -1$. Then,

$${}^k I_{a+}^{\varrho, \eta} (\eta_a(t))^{\frac{\varsigma}{k}} = \frac{\Gamma_k(\varsigma + k)}{\Gamma_k(\varsigma + k + \varrho)} (\eta_a(t))^{\frac{\varsigma+\varrho}{k}}, \quad (2.2)$$

and

$${}^k D_{a+}^{\varrho, \eta} (\eta_a(t))^{\frac{\varsigma}{k}} = \frac{\Gamma_k(\varsigma + k)}{\Gamma_k(\varsigma + k - \varrho)} (\eta_a(t))^{\frac{\varsigma-\varrho}{k}}. \quad (2.3)$$

Remark 2.2. For the reader's convenience, we outline the derivation of (2.2) as follows. Using the substitution $u = \frac{\eta(v) - \eta(a)}{\eta(t) - \eta(a)}$ and the definition of k -beta function (property (4)), we obtain

$$\begin{aligned} {}^k I_{a+}^{\varrho, \eta} (\eta_a(t))^{\frac{\varsigma}{k}} &= \frac{1}{k\Gamma_k(\varrho)} \int_0^t \eta'(v) (\eta(t) - \eta(v))^{\frac{\varrho}{k}-1} (\eta(v) - \eta(a))^{\frac{\varsigma}{k}} dv \\ &= \frac{1}{k\Gamma_k(\varrho)} \int_0^t \eta'(v) (\eta(t) - \eta(a) - (\eta(v) - \eta(a)))^{\frac{\varrho}{k}-1} (\eta(v) - \eta(a))^{\frac{\varsigma}{k}} dv \\ &= (\eta(t) - \eta(a))^{\frac{\varrho}{k}-1+\frac{\varsigma}{k}+1} \frac{1}{k\Gamma_k(\varrho)} \int_0^1 (1-u)^{\frac{\varrho}{k}-1} u^{\frac{\varsigma}{k}} du \\ &= (\eta(t) - \eta(a))^{\frac{\varrho+\varsigma}{k}} \frac{1}{\Gamma_k(\varrho)} B_k(\varrho, \varsigma + k) \\ &= \frac{\Gamma_k(\varsigma + k)}{\Gamma_k(\varsigma + k + \varrho)} (\eta(t) - \eta(a))^{\frac{\varrho+\varsigma}{k}}. \end{aligned}$$

Lemma 2.3. ([14]) Let $\varrho > 0, k > \delta > 0, n - 1 < \frac{\varrho}{k} < n, f \in C_{\frac{\delta}{k}, \eta}^{\delta}[a, d]$, and ${}^k I_{a+}^{nk-\varrho, \eta} f \in C_{\frac{\delta}{k}, \eta}^n[a, d]$. Then,

$${}^k I_{a+}^{\varrho, \eta} {}^k D_{a+}^{\varrho, \eta} f(x) = f(x) - \sum_{j=1}^n \frac{(\eta_a(x))^{\frac{\varrho}{k}-j}}{\Gamma_k(\varrho - jk + k)} \left[\left(\frac{k}{\eta'(x)} \frac{d}{dx} \right)^{n-j} {}^k I_{a+}^{nk-\varrho, \eta} f(x) \right]_{x=a}, \quad (2.4)$$

where

$$C_{\frac{\delta}{k}, \eta}^{\delta}[a, d] = \left\{ f : [a, d] \rightarrow \mathbb{R} : (\eta_a(\cdot))^{\frac{\delta}{k}} f(\cdot) \in C([a, d], \mathbb{R}) \right\},$$

$$C_{\frac{\delta}{k}, \eta}^n[a, d] = \left\{ f : [a, d] \rightarrow \mathbb{R} : \left(\frac{k}{\eta'(t)} \frac{d}{dt} \right)^{n-1} f \in C([a, d], \mathbb{R}), \left(\frac{k}{\eta'(t)} \frac{d}{dt} \right)^n f \in C_{\frac{\delta}{k}, \eta}^{\delta}([a, d]) \right\}.$$

The next lemma, which plays a key role to transform the nonlinear problem (1.1) into a system of integral equations, deals with the linear variant of (1.1).

Lemma 2.4. Let $\psi_1, \psi_2 \in C([0, T])$, $x, {}^k I_{0+}^{k-\varrho_1, \eta} x \in C_{\frac{\varrho_1}{k}, \eta}^{\varrho_1}[0, T]$, $z, {}^k I_{0+}^{k-\varrho_2, \eta} z \in C_{\frac{\varrho_2}{k}, \eta}^{\varrho_2}[0, T]$. Then, for $\Lambda \neq 0$, a solution of the linear problem

$$\begin{cases} {}^k D_{0+}^{\varrho_1, \eta} x(t) = \psi_1(t), & t \in [0, T], \\ {}^k D_{0+}^{\varrho_2, \eta} z(t) = \psi_2(t), & t \in [0, T], \\ (\eta_0(t))^{1-\frac{\varrho_1}{k}} x(t)|_{t=0} + (\eta_0(t))^{1-\frac{\varrho_1}{k}} x(t)|_{t=T} = \sigma_1 {}^k I_{0+}^{S_1, \eta} z(\mu_1), \\ (\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)|_{t=0} + (\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)|_{t=T} = \sigma_2 {}^k I_{0+}^{S_2, \eta} x(\mu_2), \end{cases} \quad (2.5)$$

is given by

$$\begin{aligned} x(t) = & \frac{1}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v) (\eta_v(t))^{\frac{\varrho_1}{k}-1} \psi_1(v) dv \\ & + \frac{(\eta_0(t))^{\frac{\varrho_1}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_1}{k}-1} \psi_1(v) dv \right. \\ & + \frac{2\sigma_1}{k\Gamma_k(\varrho_2 + S_1)} \int_0^{\mu_1} \eta'(v) (\eta_v(\mu_1))^{\frac{\varrho_2+S_1}{k}-1} \psi_2(v) dv \\ & - \frac{\sigma_1 \Delta_1 (\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_2}{k}-1} \psi_2(v) dv \\ & \left. + \frac{\sigma_1 \sigma_2 \Delta_1}{k\Gamma_k(\varrho_1 + S_2)} \int_0^{\mu_2} \eta'(v) (\eta_v(\mu_2))^{\frac{\varrho_1+S_2}{k}-1} \psi_1(v) dv \right\}, \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} z(t) = & \frac{1}{k\Gamma_k(\varrho_2)} \int_0^t \eta'(v) (\eta_v(t))^{\frac{\varrho_2}{k}-1} \psi_2(v) dv \\ & + \frac{(\eta_0(t))^{\frac{\varrho_2}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_2}{k}-1} \psi_2(v) dv \right. \\ & + \frac{2\sigma_2}{k\Gamma_k(\varrho_1 + S_2)} \int_0^{\mu_2} \eta'(v) (\eta_v(\mu_2))^{\frac{\varrho_1+S_2}{k}-1} \psi_1(v) dv \\ & - \frac{\sigma_2 \Delta_2 (\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_1}{k}-1} \psi_1(v) dv \\ & \left. + \frac{\sigma_1 \sigma_2 \Delta_2}{k\Gamma_k(\varrho_2 + S_1)} \int_0^{\mu_1} \eta'(v) (\eta_v(\mu_1))^{\frac{\varrho_2+S_1}{k}-1} \psi_2(v) dv \right\}, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned}\Lambda &= 4 - \sigma_1 \sigma_2 \Delta_1 \Delta_2, \\ \Delta_1 &= \frac{\Gamma_k(\varrho_2)}{\Gamma_k(\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}, \\ \Delta_2 &= \frac{\Gamma_k(\varrho_1)}{\Gamma_k(\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{\varrho_1 + \varsigma_2}{k} - 1}.\end{aligned}\quad (2.8)$$

Proof. Let us operate ${}^k I_{0+}^{\varrho_1, \eta}$ and ${}^k I_{0+}^{\varrho_2, \eta}$ on the first and second equations in (2.5), respectively, and use Lemma 2.3 to obtain

$$x(t) = \frac{1}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} \psi_1(v) dv + c_1 (\eta_0(t))^{\frac{\varrho_1}{k}-1}, \quad (2.9)$$

$$z(t) = \frac{1}{k\Gamma_k(\varrho_2)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_2}{k}-1} \psi_2(v) dv + c_2 (\eta_0(t))^{\frac{\varrho_2}{k}-1}, \quad (2.10)$$

where $c_1, c_2 \in \mathbb{R}$ are unknown constants. Using (2.9) and (2.10) in the boundary conditions of (2.5), and replacing $\frac{\varsigma}{k}$ in (2.2) with $\frac{\varsigma}{k} - 1$, we obtain

$$c_1 + (\eta_0(T))^{1-\frac{\varrho_1}{k}} {}^k I_{0+}^{\varrho_1, \eta} \psi_1(T) + c_1 = \sigma_1 {}^k I_{0+}^{\varrho_2 + \varsigma_1, \eta} \psi_2(\mu_1) + \frac{\sigma_1 \Gamma_k(\varrho_2)(\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}}{\Gamma_k(\varrho_2 + \varsigma_1)} c_2, \quad (2.11)$$

$$c_2 + (\eta_0(T))^{1-\frac{\varrho_2}{k}} {}^k I_{0+}^{\varrho_2, \eta} \psi_2(T) + c_2 = \sigma_2 {}^k I_{0+}^{\varrho_1 + \varsigma_2, \eta} \psi_1(\mu_2) + \frac{\sigma_2 \Gamma_k(\varrho_1)(\eta_0(\mu_2))^{\frac{\varrho_1 + \varsigma_2}{k} - 1}}{\Gamma_k(\varrho_1 + \varsigma_2)} c_1. \quad (2.12)$$

Rewriting (2.11) and (2.12), we have

$$2c_1 - \frac{\sigma_1 \Gamma_k(\varrho_2)(\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}}{\Gamma_k(\varrho_2 + \varsigma_1)} c_2 = -(\eta_0(T))^{1-\frac{\varrho_1}{k}} {}^k I_{0+}^{\varrho_1, \eta} \psi_1(T) + \sigma_1 {}^k I_{0+}^{\varrho_2 + \varsigma_1, \eta} \psi_2(\mu_1), \quad (2.13)$$

$$2c_2 - \frac{\sigma_2 \Gamma_k(\varrho_1)(\eta_0(\mu_2))^{\frac{\varrho_1 + \varsigma_2}{k} - 1}}{\Gamma_k(\varrho_1 + \varsigma_2)} c_1 = -(\eta_0(T))^{1-\frac{\varrho_2}{k}} {}^k I_{0+}^{\varrho_2, \eta} \psi_2(T) + \sigma_2 {}^k I_{0+}^{\varrho_1 + \varsigma_2, \eta} \psi_1(\mu_2). \quad (2.14)$$

Multiplying (2.13) by 2 and (2.14) by $\frac{\sigma_1 \Gamma_k(\varrho_2)(\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}}{\Gamma_k(\varrho_2 + \varsigma_1)}$ and adding the resulting equations yields

$$\begin{aligned}& c_1 \left[4 - \sigma_1 \sigma_2 \left(\frac{\Gamma_k(\varrho_1)(\eta_0(\mu_2))^{\frac{\varrho_1 + \varsigma_2}{k} - 1}}{\Gamma_k(\varrho_1 + \varsigma_2)} \right) \left(\frac{\Gamma_k(\varrho_2)(\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}}{\Gamma_k(\varrho_2 + \varsigma_1)} \right) \right] \\&= -2(\eta_0(T))^{1-\frac{\varrho_1}{k}} {}^k I_{0+}^{\varrho_1, \eta} \psi_1(T) + 2\sigma_1 {}^k I_{0+}^{\varrho_2 + \varsigma_1, \eta} \psi_2(\mu_1) \\&\quad - \frac{\sigma_1 \Gamma_k(\varrho_2)(\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}}{\Gamma_k(\varrho_2 + \varsigma_1)} (\eta_0(T))^{1-\frac{\varrho_2}{k}} {}^k I_{0+}^{\varrho_2, \eta} \psi_2(T) \\&\quad + \frac{\sigma_1 \Gamma_k(\varrho_2)(\eta_0(\mu_1))^{\frac{\varrho_2 + \varsigma_1}{k} - 1}}{\Gamma_k(\varrho_2 + \varsigma_1)} \sigma_2 {}^k I_{0+}^{\varrho_1 + \varsigma_2, \eta} \psi_1(\mu_2).\end{aligned}$$

Using the notation (2.8) in the left-hand side of the above equation, we find that

$$\begin{aligned} c_1 = & \frac{1}{\Lambda} \left\{ -2(\eta_0(T))^{1-\frac{\varrho_1}{k}} I_{0+}^{\varrho_1, \eta} \psi_1(T) + 2\sigma_1^k I_{0+}^{\varrho_2+\varsigma_1, \eta} \psi_2(\mu_1) \right. \\ & - \sigma_1 \frac{\Gamma_k(\varrho_2)}{\Gamma_k(\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} (\eta_0(T))^{1-\frac{\varrho_2}{k}} I_{0+}^{\varrho_2, \eta} \psi_2(T) \\ & \left. + \sigma_1 \sigma_2 \frac{\Gamma_k(\varrho_2)}{\Gamma_k(\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} I_{0+}^{\varrho_1+\varsigma_2, \eta} \psi_1(\mu_2) \right\}, \end{aligned}$$

In a similar manner, eliminating c_1 from (2.13) and (2.14) and using the notation (2.8), the value of c_2 is found to be

$$\begin{aligned} c_2 = & \frac{1}{\Lambda} \left\{ -2(\eta_0(T))^{1-\frac{\varrho_2}{k}} I_{0+}^{\varrho_2, \eta} \psi_1(T) + 2\sigma_2^k I_{0+}^{\varrho_1+\varsigma_2, \eta} \psi_1(\mu_2) \right. \\ & - \sigma_2 \frac{\Gamma_k(\varrho_1)}{\Gamma_k(\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} (\eta_0(T))^{1-\frac{\varrho_1}{k}} I_{0+}^{\varrho_1, \eta} \psi_1(T) \\ & \left. + \sigma_1 \sigma_2 \frac{\Gamma_k(\varrho_1)}{\Gamma_k(\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} I_{0+}^{\varrho_2+\varsigma_1, \eta} \psi_2(\mu_1) \right\}. \end{aligned}$$

Inserting the values of c_1 and c_2 in (2.9) and (2.10), respectively, leads to the solution (2.6) and (2.7). The converse of the lemma can be obtained by directly substituting (2.6) and (2.7) in (2.5). $\square$

The above lemma plays a pivotal role in analyzing the existence, uniqueness, and Ulam-Hyers stability of solutions to the nonlinear implicit system (1.1) by converting it into an equivalent fixed point problem as described in the next section.

3. Existence results

Let the weighted space $C_{1-\frac{\gamma}{k}, \eta}([0, T], \mathbb{R})$ of continuous functions with $0 < \gamma < 1$ be defined by

$$C_{1-\frac{\gamma}{k}, \eta}([0, T], \mathbb{R}) = \{x : [0, T] \rightarrow \mathbb{R} | (\eta_0(t))^{1-\frac{\gamma}{k}} x \in C([0, T], \mathbb{R})\}.$$

Equipping this space with the norm $\|x\|_\gamma = \sup_{t \in [0, T]} |(\eta_0(t))^{1-\frac{\gamma}{k}} x(t)|$ makes it a Banach space as argued in [24]. Consequently, $\Upsilon = C_{1-\frac{\varrho_1}{k}, \eta}([0, T], \mathbb{R}) \times C_{1-\frac{\varrho_1}{k}, \eta}([0, T], \mathbb{R})$ is a Banach space endowed with the norm $\|(x, z)\|_\Upsilon = \|x\|_{\varrho_1} + \|z\|_{\varrho_2}$, for $(x, z) \in \Upsilon$.

By Lemma 2.4, we convert the nonlinear problem (1.1) into an equivalent fixed-point problem $(x, z) = \mathcal{E}(x, z)$, where the operator $\mathcal{E} : \Upsilon \rightarrow \Upsilon$ is defined by

$$\mathcal{E}(x, z)(t) := (\mathcal{E}_1(x, z)(t), \mathcal{E}_2(x, z)(t)), \quad (3.1)$$

with

$$\mathcal{E}_1(x, z)(t) = \frac{1}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v) (\eta_v(t))^{\frac{\varrho_1}{k}-1} \beta(v) dv$$

$$\begin{aligned}
& + \frac{(\eta_0(t))^{\frac{\varrho_1}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^\tau \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} \beta(v) dv \right. \\
& + \frac{2\sigma_1}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} \chi(v) dv \\
& - \frac{\sigma_1\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} \chi(v) dv \\
& \left. + \frac{\sigma_1\sigma_2\Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} \beta(v) dv \right\}, \tag{3.2}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_2(x, z)(t) &= \frac{1}{k\Gamma_k(\varrho_2)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_2}{k}-1} \chi(v) dv \\
& + \frac{(\eta_0(t))^{\frac{\varrho_2}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} \chi(v) dv \right. \\
& + \frac{2\sigma_2}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} \beta(v) dv \\
& - \frac{\sigma_2\Delta_2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} \beta(v) dv \\
& \left. + \frac{\sigma_1\sigma_2\Delta_2}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} \chi(v) dv \right\}, \tag{3.3}
\end{aligned}$$

where $\beta(t) = \mathcal{J}(t, z(t), \beta(t))$ and $\chi(t) = \mathcal{K}(t, x(t), \chi(t))$ are implicit functions. Obviously, the existence of a fixed point of the operator $\mathcal{E} : \Upsilon \rightarrow \Upsilon$ corresponds to that of the solution of problem (1.1).

Next, we introduce the hypotheses needed to establish the main results in this article.

(d_1) Assume $p_j, q_j \in C((0, T], \mathbb{R}^+)$, $j = 1, 2, 3$, such that

$$|\mathcal{J}(t, x, z)| \leq p_1(t) + p_2(t)|x| + p_3(t)|z|,$$

$$|\mathcal{K}(t, x, z)| \leq q_1(t) + q_2(t)|x| + q_3(t)|z|,$$

for all $t \in (0, T]$ and $x, z \in \mathbb{R}$, with

$$\begin{aligned}
\bar{p}_1 &= \sup_{t \in (0, T]} \left\{ p_1(t)(\eta_0(t))^{1-\frac{\varrho_1}{k}} \right\} < \infty, \quad \widehat{p}_2 = \sup_{t \in (0, T]} \left\{ p_2(t)(\eta_0(t))^{\frac{\varrho_2-\varrho_1}{k}} \right\} < \infty, \\
p_3^* &= \sup_{t \in (0, T]} p_3(t) < 1, \\
\bar{q}_1 &= \sup_{t \in (0, T]} \left\{ q_1(t)(\eta_0(t))^{1-\frac{\varrho_2}{k}} \right\} < \infty, \quad \widehat{q}_2 = \sup_{t \in (0, T]} \left\{ q_2(t)(\eta_0(t))^{\frac{\varrho_1-\varrho_2}{k}} \right\} < \infty, \\
q_3^* &= \sup_{t \in (0, T]} q_3(t) < 1; \tag{3.4}
\end{aligned}$$

(d_2) Assume that $C_j, L_j \in C((0, T], \mathbb{R}^+)$, $j = 1, 2$, such that

$$|\mathcal{J}(t, x_1, z_1) - \mathcal{J}(t, x_2, z_2)| \leq C_1(t)|x_1 - x_2| + C_2(t)|z_1 - z_2|,$$

$$|\mathcal{K}(t, x_1, z_1) - \mathcal{K}(t, x_2, z_2)| \leq L_1(t)|x_1 - x_2| + L_2(t)|z_1 - z_2|,$$

for all $t \in (0, T]$ and $x_j, z_j \in \mathbb{R}$, $j = 1, 2$, with

$$\begin{aligned} C_1^* &= \sup_{t \in (0, T]} \left\{ C_1(t)(\eta_0(t))^{\frac{\varrho_2 - \varrho_1}{k}} \right\} < \infty, \quad C_2^* = \sup_{t \in (0, T]} C_2(t) < 1, \\ L_1^* &= \sup_{t \in (0, T]} \left\{ L_1(t)(\eta_0(t))^{\frac{\varrho_1 - \varrho_2}{k}} \right\} < \infty, \quad L_2^* = \sup_{t \in (0, T]} L_2(t) < 1. \end{aligned} \quad (3.5)$$

Now we present the values of the integrals appearing in the forthcoming analysis in a simplified form (see Step 1 in the proof of Theorem 3.1). We mention the integrals involved in Φ_1 , while the values of Φ_2 , Φ_3 , and Φ_4 follow in a similar manner.

$$\begin{aligned} \Phi_1 &= \frac{(\eta_0(t))^{1 - \frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k} - 1} (\eta_0(v))^{\frac{\varrho_1}{k} - 1} dv \\ &+ \frac{1}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1 - \frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k} - 1} (\eta_0(v))^{\frac{\varrho_1}{k} - 1} dv \right. \\ &\quad \left. + \frac{|\sigma_1 \sigma_2| \Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1 + \varsigma_2}{k} - 1} (\eta_0(v))^{\frac{\varrho_1}{k} - 1} dv \right\} \\ &= \Gamma_k(\varrho_1) \left[\frac{1}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} \left(1 + \frac{2}{|\Lambda|} \right) + \frac{|\sigma_1 \sigma_2| \Delta_1}{|\Lambda| \Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1 + \varsigma_2}{k} - 1} \right], \\ \Phi_2 &= \frac{|\sigma_1| \Gamma_k(\varrho_2)}{|\Lambda|} \left[\frac{2}{\Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2 + \varsigma_1}{k} - 1} + \frac{\Delta_1}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \right], \\ \Phi_3 &= \Gamma_k(\varrho_2) \left[\frac{1}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \left(1 + \frac{2}{|\Lambda|} \right) + \frac{|\sigma_1 \sigma_2| \Delta_2}{|\Lambda| \Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2 + \varsigma_1}{k} - 1} \right], \\ \Phi_4 &= \frac{|\sigma_2| \Gamma_k(\varrho_1)}{|\Lambda|} \left[\frac{2}{\Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1 + \varsigma_2}{k} - 1} + \frac{\Delta_2}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} \right]. \end{aligned} \quad (3.6)$$

The existence criteria for solutions of problem (1.1) will be established with the aid of the following fixed-point result.

Lemma 3.1. (Schaefer [33]) *Let $\mathcal{G} : \mathcal{H} \rightarrow \mathcal{H}$ be a compact and continuous mapping on a Banach space $\mathcal{H}$ and let the set of the solutions of $x = \lambda \mathcal{G}x$ be bounded when $0 < \lambda < 1$. Then, $\mathcal{G}$ possesses a fixed point $\bar{x} \in \mathcal{H}$.*

Theorem 3.1. *Assume $\mathcal{J}, \mathcal{K} : ([0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R})$ are continuous functions and the hypothesis (d_1) is satisfied. Then, there exists at least one solution on $[0, T]$ to problem (1.1), if*

$$\frac{\hat{q}_2}{1 - q_3^*} [\Phi_2 + \Phi_3] < 1 \quad \text{and} \quad \frac{\hat{p}_2}{1 - p_3^*} [\Phi_1 + \Phi_4] < 1, \quad (3.7)$$

where Φ_i ($i = 1, 2, 3, 4$) are defined in (3.6).

Proof. First, we prove that $\mathcal{E} : \Upsilon \rightarrow \Upsilon$ introduced by (3.1) is continuous and completely continuous. Continuity of the operator $\mathcal{E}$ is an immediate consequence of that of $\mathcal{J}$ and $\mathcal{K}$. Next, we consider a bounded set $\Omega_\alpha = \{(x, z) \in \Upsilon : \|(x, z)\|_\Upsilon \leq \alpha\} \subset \Upsilon$. It follows by condition (d_1) that

$$|(\eta_0(t))^{1 - \frac{\varrho_1}{k}} \beta(t)| = |(\eta_0(t))^{1 - \frac{\varrho_1}{k}} \mathcal{J}(t, z(t), \beta(t))|$$

$$\leq (\eta_0(t))^{1-\frac{\varrho_1}{k}} p_1(t) + p_2(t)(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\eta_0(t))^{\frac{\varrho_2}{k}-1} |(\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)| \\ + p_3(t) |(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)|,$$

which can be expressed as

$$|(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t) - p_3(t) |(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)| \leq (\eta_0(t))^{1-\frac{\varrho_1}{k}} p_1(t) + p_2(t)(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\eta_0(t))^{\frac{\varrho_2}{k}-1} |(\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)|.$$

Rewriting the above inequality as

$$|(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)| (1 - p_3^*) \leq |(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)| (1 - p_3(t)) \\ \leq (\eta_0(t))^{1-\frac{\varrho_1}{k}} p_1(t) + p_2(t)(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\eta_0(t))^{\frac{\varrho_2}{k}-1} |(\eta_0(t))^{1-\frac{\varrho_2}{k}} z(t)|,$$

we get

$$|(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)| \leq \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*}, \quad (3.8)$$

where $\widehat{p}_2$, p_3^* , and $\bar{p}_1$ are defined in (3.4). Similarly, we find that

$$|(\eta_0(t))^{1-\frac{\varrho_2}{k}} \chi(t)| = |(\eta_0(t))^{1-\frac{\varrho_2}{k}} \mathcal{K}(t, \chi(t), x(t))| \\ \leq (\eta_0(t))^{1-\frac{\varrho_2}{k}} q_1(t) + q_2(t)(\eta_0(t))^{1-\frac{\varrho_2}{k}} (\eta_0(t))^{\frac{\varrho_1}{k}-1} |(\eta_0(t))^{1-\frac{\varrho_1}{k}} x(t)| \\ + q_3(t) |(\eta_0(t))^{1-\frac{\varrho_2}{k}} \chi(t)|,$$

which yields

$$|(\eta_0(t))^{1-\frac{\varrho_2}{k}} \chi(t)| \leq \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*}, \quad (3.9)$$

where $\widehat{q}_2$, q_3^* , and $\bar{q}_1$ are defined in (3.4).

Now, we accomplish the applicability of Lemma 3.1 in three steps.

Step 1. $\mathcal{E}$ is uniformly bounded in Ω_α . For any $(x, z) \in \Omega_\alpha$, by condition (3.8), we have

$$|(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\mathcal{E}_1(x, z)(t))| \\ \leq \left| \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} \beta(v) dv \right. \\ \left. + \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\eta_0(t))^{\frac{\varrho_1}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} \beta(v) dv \right. \right. \\ \left. + \frac{2\sigma_1}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} \chi(v) dv - \frac{\sigma_1 \Delta_1 (\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} \chi(v) dv \right. \\ \left. \left. + \frac{\sigma_1 \sigma_2 \Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} \beta(v) dv \right\} \right| \\ \leq \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_1}{k}} \beta(v)| dv$$

$$\begin{aligned}
& + \frac{1}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1}(\eta_0(v))^{\frac{\varrho_1}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_1}{k}} \beta(v)| dv \right. \\
& + \frac{2|\sigma_1|}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1}(\eta_0(v))^{\frac{\varrho_2}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_2}{k}} \chi(v)| dv \\
& + \frac{|\sigma_1|\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1}(\eta_0(v))^{\frac{\varrho_2}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_2}{k}} \chi(v)| dv \\
& \left. + \frac{|\sigma_1\sigma_2|\Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1}(\eta_0(v))^{\frac{\varrho_1}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_1}{k}} \beta(v)| dv \right\} \\
& \leq \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1}(\eta_0(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} dv \\
& + \frac{1}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1}(\eta_0(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} dv \right. \\
& + \frac{2|\sigma_1|}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1}(\eta_0(v))^{\frac{\varrho_2}{k}-1} \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*} dv \\
& + \frac{|\sigma_1|\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1}(\eta_0(v))^{\frac{\varrho_2}{k}-1} \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*} dv \\
& \left. + \frac{|\sigma_1\sigma_2|\Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1}(\eta_0(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} dv \right\} \\
& \leq \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} \left\{ \frac{\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} + \frac{1}{|\Lambda|} \left[\frac{2\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} + \frac{|\sigma_1\sigma_2|\Delta_1\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1+\varsigma_2}{k}-1} \right] \right\} \\
& + \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*} \left\{ \frac{1}{|\Lambda|} \left[\frac{2|\sigma_1|\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2+\varsigma_1}{k}-1} + \frac{|\sigma_1|\Delta_1\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \right] \right\},
\end{aligned}$$

which implies that $\|\mathcal{E}_1(x, z)\|_{\varrho_1} \leq \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} \Phi_1 + \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*} \Phi_2$.

Similarly, we can find, with the aid of (3.9), that

$$\begin{aligned}
\|\mathcal{E}_2(x, z)\|_{\varrho_2} & \leq \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*} \left\{ \frac{\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} + \frac{1}{|\Lambda|} \left[\frac{2\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \right. \right. \\
& \left. \left. + \frac{|\sigma_1\sigma_2|\Delta_2\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2+\varsigma_1}{k}-1} \right] \right\} \\
& + \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} \left\{ \frac{1}{|\Lambda|} \left[\frac{2|\sigma_2|\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1+\varsigma_2}{k}-1} + \frac{|\sigma_2|\Delta_2\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} \right] \right\} \\
& = \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1 - q_3^*} \Phi_3 + \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1 - p_3^*} \Phi_4.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|\mathcal{E}(x, z)\|_{\Upsilon} & = \|\mathcal{E}_1(x, z)\|_{\varrho_1} + \|\mathcal{E}_2(x, z)\|_{\varrho_2} \\
& \leq \max \left\{ \frac{\bar{p}_2}{1 - p_3^*} (\Phi_1 + \Phi_4), \frac{\widehat{q}_2}{1 - q_3^*} (\Phi_2 + \Phi_3) \right\} (\|x\|_{\varrho_1} + \|z\|_{\varrho_2})
\end{aligned}$$

$$+ \frac{\bar{p}_1}{1-p_3^*} \{\Phi_1 + \Phi_4\} + \frac{\bar{q}_1}{1-q_3^*} \{\Phi_2 + \Phi_3\} = M,$$

which implies that $\mathcal{E}$ is uniformly bounded

Step 2. $\mathcal{E}$ is equicontinuous on Ω_α . Letting $\zeta_1, \zeta_2 \in (0, T]$ with $\zeta_1 < \zeta_2$, we obtain

$$\begin{aligned} & |\mathcal{E}_1(x, z)(\zeta_2) - \mathcal{E}_1(x, z)(\zeta_1)| \\ & \leq \frac{1}{k\Gamma_k(\varrho_1)} \int_0^{\zeta_1} \eta'(v) |(\eta_v(\zeta_2))^{\frac{\varrho_1}{k}-1} - (\eta_v(\zeta_1))^{\frac{\varrho_1}{k}-1}| (\eta_v(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1-p_3^*} dv \\ & \quad + \frac{1}{k\Gamma_k(\varrho_1)} \int_{\zeta_1}^{\zeta_2} \eta'(v) (\eta_v(\zeta_2))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1-p_3^*} dv \\ & \quad + \frac{|(\eta_0(\zeta_2))^{\frac{\varrho_1}{k}-1} - (\eta_0(\zeta_1))^{\frac{\varrho_1}{k}-1}|}{|\Lambda|} \times \\ & \quad \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1-p_3^*} dv \right. \\ & \quad + \frac{2|\sigma_1|}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v) (\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} (\eta_0(v))^{\frac{\varrho_2}{k}-1} \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1-q_3^*} dv \\ & \quad + \frac{|\sigma_1|\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_2}{k}-1} (\eta_0(v))^{\frac{\varrho_2}{k}-1} \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1-q_3^*} dv \\ & \quad \left. + \frac{|\sigma_1\sigma_2|\Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v) (\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1-p_3^*} dv \right\} \\ & \leq \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1-p_3^*} \left[\frac{1}{k\Gamma_k(\varrho_1)} \int_0^{\zeta_1} \eta'(v) |(\eta_v(\zeta_2))^{\frac{\varrho_1}{k}-1} - (\eta_v(\zeta_1))^{\frac{\varrho_1}{k}-1}| (\eta_0(v))^{\frac{\varrho_1}{k}-1} dv \right. \\ & \quad + \frac{1}{k\Gamma_k(\varrho_1)} \int_{\zeta_1}^{\zeta_2} \eta'(v) (\eta_v(\zeta_2))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} dv \Big] \\ & \quad + \frac{|(\eta_0(\zeta_2))^{\frac{\varrho_1}{k}-1} - (\eta_0(\zeta_1))^{\frac{\varrho_1}{k}-1}|}{|\Lambda|} \left\{ \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\varrho_2}}{1-p_3^*} \left[\frac{2\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} \right. \right. \\ & \quad + \frac{|\sigma_1\sigma_2|\Delta_1\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1+\varsigma_2}{k}-1} \Big] \\ & \quad \left. + \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\varrho_1}}{1-q_3^*} \left[\frac{2|\sigma_1|\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2+\varsigma_1}{k}-1} + \frac{|\sigma_1|\Delta_1\Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \right] \right\}, \end{aligned}$$

where $\|x\|_{\varrho_1}$ and $\|z\|_{\varrho_2}$ are bounded by α on Ω_α , $|(\eta_v(\zeta_2))^{\frac{\varrho_1}{k}-1} - (\eta_v(\zeta_1))^{\frac{\varrho_1}{k}-1}| \rightarrow 0$ and $|(\eta_0(\zeta_2))^{\frac{\varrho_1}{k}-1} - (\eta_0(\zeta_1))^{\frac{\varrho_1}{k}-1}| \rightarrow 0$ as $\zeta_2 - \zeta_1 \rightarrow 0$, because η is a continuous function for $0 < v < \zeta_1 < \zeta_2$.

Thus, the above inequality does not depend on $(x, z) \in \Omega_\alpha$ and tends to zero as $\zeta_2 - \zeta_1 \rightarrow 0$. In a similar manner, it can be shown that

$$|\mathcal{E}_2(x, z)(\zeta_2) - \mathcal{E}_2(x, z)(\zeta_1)| \rightarrow 0$$

as $\zeta_2 - \zeta_1 \rightarrow 0$ independently of $(x, z) \in \Omega_\alpha$. Thus, $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$ is equicontinuous on Ω_α , and hence $\mathcal{E}$ is relatively compact on Ω_α . Thus, by an application of the Arzelá-Ascoli theorem, the operator $\mathcal{E} : \Upsilon \rightarrow \Upsilon$ is completely continuous.

Step 3. Introduce a set Θ as

$$\Theta = \{(x, z) \in \Upsilon | (x, z) = \lambda \mathcal{E}(x, z), \lambda \in (0, 1)\},$$

and show that it is bounded. For $(x, z) \in \Theta$, we have $(x, z) = \lambda \mathcal{E}(x, z)$, that is,

$$x(t) = \lambda \mathcal{E}_1(x(t), z(t)), \quad z(t) = \lambda \mathcal{E}_2(x(t), z(t)), \quad \forall t \in [0, T].$$

As in step 1, using notation in (3.6), we have

$$|(\eta_0(t))^{1-\frac{q_1}{k}} x(t)| = \lambda |(\eta_0(t))^{1-\frac{q_1}{k}} \mathcal{E}_1(x, z)(t)| \leq \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\mathcal{Q}_2}}{1 - p_3^*} \Phi_1 + \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\mathcal{Q}_1}}{1 - q_3^*} \Phi_2,$$

$$|(\eta_0(t))^{1-\frac{q_2}{k}} z(t)| = \lambda |(\eta_0(t))^{1-\frac{q_2}{k}} \mathcal{E}_2(x, z)(t)| \leq \frac{\bar{q}_1 + \widehat{q}_2 \|x\|_{\mathcal{Q}_1}}{1 - q_3^*} \Phi_3 + \frac{\bar{p}_1 + \widehat{p}_2 \|z\|_{\mathcal{Q}_2}}{1 - p_3^*} \Phi_4.$$

Consequently, we get

$$\begin{aligned} \|x\|_{\mathcal{Q}_1} + \|z\|_{\mathcal{Q}_2} &\leq \frac{\widehat{q}_2 \|x\|_{\mathcal{Q}_1}}{1 - q_3^*} [\Phi_2 + \Phi_3] + \frac{\widehat{p}_2 \|z\|_{\mathcal{Q}_2}}{1 - p_3^*} [\Phi_1 + \Phi_4] \\ &\quad + \frac{\bar{q}_1}{1 - q_3^*} [\Phi_2 + \Phi_3] + \frac{\bar{p}_1}{1 - p_3^*} [\Phi_1 + \Phi_4], \end{aligned}$$

which, by the condition (3.7), implies that

$$\|(x, z)\|_{\Upsilon} \leq \frac{\frac{\bar{q}_1}{1 - q_3^*} [\Phi_2 + \Phi_3] + \frac{\bar{p}_1}{1 - p_3^*} [\Phi_1 + \Phi_4]}{W},$$

where

$$W = \min \left\{ 1 - \left(\frac{\widehat{q}_2}{1 - q_3^*} [\Phi_2 + \Phi_3] \right), 1 - \left(\frac{\widehat{p}_2}{1 - p_3^*} [\Phi_1 + \Phi_4] \right) \right\},$$

and it is easy to see that $W > 0$ under the condition (3.7). Since $(x, z) \in \Theta$ is arbitrary, therefore, the set Θ is bounded. In view of Steps 1–3, the conclusion of Lemma 3.1 applies and, hence, there exists at least one fixed point to the operator $\mathcal{E}$ on $[0, T]$, which is indeed a solution to problem (1.1). $\square$

The next result, based on the Banach fixed point theorem [33], provides the uniqueness criterion for solutions to problem (1.1).

Theorem 3.2. Let $\mathcal{J}, \mathcal{K} : ([0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous and satisfy the hypothesis (d_2) . Then, problem (1.1) has a unique solution on $[0, T]$, if

$$\max \left\{ \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \right\} < 1, \quad (3.10)$$

where $C_1^*, C_2^*, L_1^*, L_2^*$, are defined in (3.5) while Φ_i ($i = 1, 2, 3, 4$) are given in (3.6).

Proof. Let $\mathcal{J}_0(t) = \mathcal{J}(t, 0, 0)$ and $\mathcal{K}_0(t) = \mathcal{K}(t, 0, 0)$, then by choosing

$$\alpha > \frac{\omega_1(\Phi_1 + \Phi_4) + \omega_2(\Phi_2 + \Phi_3)}{1 - \left(\max \left\{ \frac{C_1^*}{1-C_2^*}(\Phi_1 + \Phi_4), \frac{L_1^*}{1-L_2^*}(\Phi_2 + \Phi_3) \right\} \right)},$$

with

$$\omega_1 = \frac{\sup_{t \in (0, T]} |(\eta_0(T))^{1-\frac{\varrho_1}{k}} \mathcal{J}_0(t)|}{1 - C_2^*} < \infty, \quad \omega_2 = \frac{\sup_{t \in (0, T]} |(\eta_0(T))^{1-\frac{\varrho_2}{k}} \mathcal{K}_0(t)|}{1 - L_2^*} < \infty,$$

we define a set $\Omega_\alpha = \{(x, z) \in \Upsilon : \|(x, z)\| \leq \alpha\}$, and prove that $\mathcal{E}\Omega_\alpha \subset \Omega_\alpha$, where $\mathcal{E} : \Upsilon \rightarrow \Upsilon$ is introduced in (3.1). In view of (d_2) , we have

$$\begin{aligned} |(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)| &\leq (\eta_0(t))^{1-\frac{\varrho_1}{k}} [|\mathcal{J}(t, z(t), \beta(t)) - \mathcal{J}_0(t)| + |\mathcal{J}_0(t)|] \\ &\leq (\eta_0(t))^{1-\frac{\varrho_1}{k}} [C_1(t)|z(t)| + C_2(t)|\beta(t)|] + |(\eta_0(t))^{1-\frac{\varrho_1}{k}} \mathcal{J}_0(t)|, \end{aligned}$$

which can be expressed as

$$|(\eta_0(t))^{1-\frac{\varrho_1}{k}} \beta(t)| \leq \frac{C_1^*}{1 - C_2^*} \|z\|_{\mathcal{Q}_2} + \omega_1.$$

Likewise, it can be found that

$$|(\eta_0(t))^{1-\frac{\varrho_2}{k}} \chi(t)| \leq \frac{L_1^*}{1 - L_2^*} \|x\|_{\mathcal{Q}_1} + \omega_2.$$

Therefore, for $(x, z) \in \Omega_\alpha$, we obtain

$$\begin{aligned} &|(\eta_0(t))^{1-\frac{\varrho_1}{k}} \mathcal{E}_1(x, y)(t)| \\ &= \left| \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} \beta(v) dv \right. \\ &+ \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\eta_0(t))^{\frac{\varrho_1}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} \beta(v) dv \right. \\ &+ \frac{2\sigma_1}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} \chi(v) dv - \frac{\sigma_1 \Delta_1 (\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} \chi(v) dv \\ &\left. + \frac{\sigma_1 \sigma_2 \Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} \beta(v) dv \right\} \Big| \\ &\leq \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_1}{k}} \beta(v)| dv \\ &+ \frac{1}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_1}{k}} \beta(v)| dv \right. \\ &\left. + \frac{2|\sigma_1|}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} (\eta_0(v))^{\frac{\varrho_2}{k}-1} |(\eta_0(v))^{1-\frac{\varrho_2}{k}} \chi(v)| dv \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{|\sigma_1| \Delta_1 (\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k \Gamma_k(\varrho_2)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_2}{k}-1} (\eta_0(v))^{\frac{\varrho_2}{k}-1} \left| (\eta_0(v))^{1-\frac{\varrho_2}{k}} \chi(v) \right| dv \\
& + \frac{|\sigma_1 \sigma_2| \Delta_1}{k \Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v) (\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \left| (\eta_0(v))^{1-\frac{\varrho_1}{k}} \beta(v) \right| dv \Big\} \\
\leq & \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k \Gamma_k(\varrho_1)} \int_0^t \eta'(v) (\eta_v(t))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) dv \\
& + \frac{1}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k \Gamma_k(\varrho_1)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_1}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) dv \right. \\
& + \frac{2|\sigma_1|}{k \Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v) (\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} (\eta_0(v))^{\frac{\varrho_2}{k}-1} \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) dv \\
& + \frac{|\sigma_1| \Delta_1 (\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k \Gamma_k(\varrho_2)} \int_0^T \eta'(v) (\eta_v(T))^{\frac{\varrho_2}{k}-1} (\eta_0(v))^{\frac{\varrho_2}{k}-1} \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) dv \\
& + \frac{|\sigma_1 \sigma_2| \Delta_1}{k \Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v) (\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} (\eta_0(v))^{\frac{\varrho_1}{k}-1} \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) dv \Big\} \\
\leq & \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) \left\{ \frac{\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} + \frac{1}{|\Lambda|} \left[\frac{2\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} \right. \right. \\
& + \left. \left. \frac{|\sigma_1 \sigma_2| \Delta_1 \Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1+\varsigma_2}{k}-1} \right] \right\} \\
& + \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) \left\{ \frac{1}{|\Lambda|} \left[\frac{2|\sigma_1| \Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2+\varsigma_1}{k}-1} + \frac{|\sigma_1| \Delta_1 \Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \right] \right\} \\
& + \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) \Phi_1 + \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) \Phi_2,
\end{aligned}$$

which, on taking the norm for $t \in [0, T]$, yields

$$\|\mathcal{E}_1(x, z)\|_{\varrho_1} \leq \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) \Phi_1 + \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) \Phi_2. \quad (3.11)$$

In a similar manner, one can find that

$$\|\mathcal{E}_2(x, z)\|_{\varrho_2} \leq \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) \Phi_3 + \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) \Phi_4.$$

In consequences, for any $(x, z) \in \Omega_\alpha$, we have

$$\begin{aligned}
\|\mathcal{E}(x, z)\|_{\Upsilon} &= \|\mathcal{E}_1(x, z)\|_{\varrho_1} + \|\mathcal{E}_2(x, z)\|_{\varrho_2} \\
&\leq \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) \Phi_1 + \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) \Phi_2 \\
&\quad + \left(\frac{L_1^*}{1-L_2^*} \|x\|_{\varrho_1} + \omega_2 \right) \Phi_3 + \left(\frac{C_1^*}{1-C_2^*} \|z\|_{\varrho_2} + \omega_1 \right) \Phi_4 \\
&\leq \max \left\{ \frac{C_1^*}{1-C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1-L_2^*} (\Phi_2 + \Phi_3) \right\} (\|x\|_{\varrho_1} + \|z\|_{\varrho_2})
\end{aligned}$$

$$+\omega_1(\Phi_1 + \Phi_4) + \omega_2(\Phi_2 + \Phi_3) < \alpha,$$

which shows that $\mathcal{E}$ maps Ω_α into itself.

Now, we prove the contraction property of $\mathcal{E}$. Letting $(x_1, z_1), (x_2, z_2) \in \Omega_\alpha$, we obtain, by using (d_2) , that

$$\begin{aligned} |(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\beta_1(t) - \beta_2(t))| &= |(\eta_0(t))^{1-\frac{\varrho_1}{k}}[\mathcal{J}(t, z_1(t), \beta_1(t)) - \mathcal{J}(t, z_2(t), \beta_2(t))]| \\ &\leq C_1(t)(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\eta_0(t))^{\frac{\varrho_2-1}{k}}|(\eta_0(t))^{1-\frac{\varrho_2}{k}}(z_1(t) - z_2(t))| \\ &\quad + C_2(t)|(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\beta_1(t) - \beta_2(t))| \\ &= C_1(t)(\eta_0(t))^{\frac{\varrho_2-\varrho_1}{k}}|(\eta_0(t))^{1-\frac{\varrho_2}{k}}(z_1(t) - z_2(t))| \\ &\quad + C_2(t)|(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\beta_1(t) - \beta_2(t))|, \end{aligned}$$

which implies that

$$|(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\beta_1(t) - \beta_2(t))| \leq \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\mathcal{Q}_2}. \quad (3.12)$$

Analogously, we have

$$|(\eta_0(t))^{1-\frac{\varrho_2}{k}}(\chi_1(t) - \chi_2(t))| \leq \frac{L_1^*}{1 - L_2^*} \|x_1 - x_2\|_{\mathcal{Q}_1}, \quad (3.13)$$

where C_1^* , C_2^* , L_1^* , and L_2^* are defined in (3.5), $\beta_1(t) = \mathcal{J}(t, z_1, \beta_1(t))$, $\beta_2(t) = \mathcal{J}(t, z_2, \beta_2(t))$, $\chi_1(t) = \mathcal{K}(t, x_1, \chi_1(t))$, and $\chi_2(t) = \mathcal{K}(t, x_2, \chi_2(t))$. In view of (3.12) and (3.13), we obtain

$$\begin{aligned} &|(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\mathcal{E}_1(x_1, z_1)(t) - \mathcal{E}_1(x_2, z_2)(t))| \\ &= \left| \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\mathcal{Q}_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1}(\beta_1(v) - \beta_2(v))dv \right. \\ &\quad + \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}(\eta_0(t))^{\frac{\varrho_1}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\mathcal{Q}_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1}(\beta_1(v) - \beta_2(v))dv \right. \\ &\quad + \frac{2\sigma_1}{k\Gamma_k(\mathcal{Q}_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1}(\chi_1(v) - \chi_2(v))dv \\ &\quad - \frac{\sigma_1\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\mathcal{Q}_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1}(\chi_1(v) - \chi_2(v))dv \\ &\quad \left. \left. + \frac{\sigma_1\sigma_2\Delta_1}{k\Gamma_k(\mathcal{Q}_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1}(\beta_1(v) - \beta_2(v))dv \right\} \right| \\ &\leq \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\mathcal{Q}_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\mathcal{Q}_2} dv \\ &\quad + \frac{1}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\mathcal{Q}_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\mathcal{Q}_2} dv \right. \\ &\quad + \frac{2|\sigma_1|}{k\Gamma_k(\mathcal{Q}_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} \frac{L_1^*}{1 - L_2^*} \|x_1 - x_2\|_{\mathcal{Q}_1} dv \\ &\quad \left. + \frac{|\sigma_1|\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\mathcal{Q}_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} \frac{L_1^*}{1 - L_2^*} \|x_1 - x_2\|_{\mathcal{Q}_1} dv \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{|\sigma_1 \sigma_2| \Delta_1}{k \Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v) (\eta_v(\mu_2))^{\frac{\varrho_1 + \varsigma_2}{k} - 1} \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\varrho_2} dv \Big\} \\
& \leq \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\varrho_2} \left\{ \frac{\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} + \frac{1}{|\Lambda|} \left[\frac{2\Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1)} (\eta_0(T))^{\frac{\varrho_1}{k}} \right. \right. \\
& \quad \left. \left. + \frac{|\sigma_1 \sigma_2| \Delta_1 \Gamma_k(\varrho_1)}{\Gamma_k(2\varrho_1 + \varsigma_2)} (\eta_0(\mu_2))^{\frac{2\varrho_1 + \varsigma_2}{k} - 1} \right] \right\} \\
& \quad + \frac{L_1^*}{1 - L_2^*} \|x_1 - x_2\|_{\varrho_1} \left\{ \frac{1}{|\Lambda|} \left[\frac{2|\sigma_1| \Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2 + \varsigma_1)} (\eta_0(\mu_1))^{\frac{2\varrho_2 + \varsigma_1}{k} - 1} + \frac{|\sigma_1| \Delta_1 \Gamma_k(\varrho_2)}{\Gamma_k(2\varrho_2)} (\eta_0(T))^{\frac{\varrho_2}{k}} \right] \right\} \\
& = \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\varrho_2} \Phi_1 + \frac{L_1^*}{1 - L_2^*} \|x_1 - x_2\|_{\varrho_1} \Phi_2.
\end{aligned}$$

Likewise, we have

$$|\mathcal{E}_2(x_1, y_1)(t) - \mathcal{E}_2(x_2, y_2)(t)| \leq \frac{L_1^*}{1 - L_2^*} \|x_1 - x_2\|_{\varrho_1} \Phi_3 + \frac{C_1^*}{1 - C_2^*} \|z_1 - z_2\|_{\varrho_2} \Phi_4.$$

Hence, it follows that

$$\begin{aligned}
& \|\mathcal{E}(x_1, z_1) - \mathcal{E}(x_2, z_2)\|_{\Upsilon} \\
& = \|\mathcal{E}_1(x_1, z_1) - \mathcal{E}_1(x_2, z_2)\|_{\varrho_1} + \|\mathcal{E}_2(x_1, z_1) - \mathcal{E}_2(x_2, z_2)\|_{\varrho_2} \\
& \leq \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4) \|x_1 - x_2\|_{\varrho_2} + \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \|z_1 - z_2\|_{\varrho_1} \\
& \leq \max \left\{ \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \right\} [\|x_1 - x_2\|_{\varrho_1} + \|z_1 - z_2\|_{\varrho_2}] \\
& = \max \left\{ \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \right\} \|(x_1 - x_2, z_1 - z_2)\|_{\Upsilon},
\end{aligned}$$

which, by (3.10), shows that $\mathcal{E}$ is a contraction mapping. Thus, by an application of the Banach contraction mapping principle, there exists a unique fixed point to the operator $\mathcal{E}$, which corresponds to a unique solution to (1.1) on $[0, T]$. $\square$

4. Ulam-Hyers Stability

Let us first recall the core distinction between Hyers Ulam stability and other stability concepts for nonlinear fractional differential equations. A question relating to the closeness of an approximate solution associated with a system of nonlinear equations perturbed up to a small error with an exact solution of the system is termed as Ulam-type family (Ulam-Hyers, Hyers-Ulam-Rassias, generalized Ulam-Hyers). On the other hand, the question of obtaining an approximate solution for a system of equations by perturbing its initial condition or the trajectory itself close to the exact solution of the system gives rise to the concept of classical Lyapunov family (Lyapunov stability, asymptotic stability, Mittag-Leffler stability).

Here, we develop the Ulam-Hyers stability criteria for solutions to problem (1.1). For a pair $(x^*, z^*) \in \Upsilon$, let us consider the following system of inequalities with the same boundary conditions as

considered in the original system (1.1)

$$\left\{ \begin{array}{l} |{}^k D_{0+}^{\varrho_1, \eta} x^*(t) - \mathcal{J}(t, z^*(t), D_{0+}^{\varrho_1, \eta} x^*(t))| \leq \epsilon_1, \quad t \in [0, T], \\ |{}^k D_{0+}^{\varrho_2, \eta} z^*(t) - \mathcal{K}(t, x^*(t), D_{0+}^{\varrho_2, \eta} z^*(t))| \leq \epsilon_2, \quad t \in [0, T], \\ [\eta_0(t)]^{1-\frac{\varrho_1}{k}} x^*(t)|_{t=0} + [\eta_0(t)]^{1-\frac{\varrho_1}{k}} x^*(t)|_{t=T} = \sigma_1 {}^k I_{0+}^{S_1, \eta} z^*(\mu_1), \\ [\eta_0(t)]^{1-\frac{\varrho_2}{k}} z^*(t)|_{t=0} + [\eta_0(t)]^{1-\frac{\varrho_2}{k}} z^*(t)|_{t=T} = \sigma_2 {}^k I_{0+}^{S_2, \eta} x^*(\mu_2). \end{array} \right. \quad (4.1)$$

Definition 4.1. The implicit coupled system (1.1) is Ulam-Hyers stable if there exists a constant $\xi = (\xi_1, \xi_2)$, with $\xi_i > 0, i = 1, 2$, such that for each pair $\epsilon = (\epsilon_1, \epsilon_2)$, with $\epsilon_i > 0, i = 1, 2$, and a solution $(x^*, z^*) \in \Upsilon$ satisfying (4.1), there exists a unique solution $(x, z) \in \Upsilon$ of (1.1) such that

$$\|(x, z) - (x^*, z^*)\|_{\Upsilon} \leq \xi \epsilon^T,$$

where $\xi \epsilon^T = \xi_1 \epsilon_1 + \xi_2 \epsilon_2 > 0$.

Definition 4.2. The implicit coupled system (1.1) is generalized Ulam-Hyers stable if there exists a continuous function φ , with $\varphi : R^+ \times R^+ \rightarrow R^+, i = 1, 2$, such that for each pair $\epsilon = (\epsilon_1, \epsilon_2)$, with $\epsilon_i > 0, i = 1, 2$, and a solution $(x^*, z^*) \in \Upsilon$ satisfying (4.1), there exists a unique solution $(x, z) \in \Upsilon$ of (1.1) such that

$$\|(x, z) - (x^*, z^*)\|_{\Upsilon} \leq \varphi(\epsilon_1, \epsilon_2).$$

One can notice that the error of approximate and exact solutions of the given system of nonlinear fractional differential equations relates to a general function of the perturbation parameters ϵ_1 and ϵ_2 in case of generalized Ulam-Hyers stability, while it depends on the perturbation parameters ϵ_1 and ϵ_2 linearly in case of Ulam-Hyers stability.

Remark 4.1. For a solution (x^*, z^*) of (4.1), there exist two functions $\lambda_1(t), \lambda_2(t)$ with $|\lambda_1(t)| \leq \epsilon_1, |\lambda_2(t)| \leq \epsilon_2$, for all $t \in [0, T]$, satisfying

$$\left\{ \begin{array}{l} {}^k D_{0+}^{\varrho_1, \eta} x^*(t) - \mathcal{J}(t, z^*(t), {}^k D_{0+}^{\varrho_1, \eta} x^*(t)) = \lambda_1(t), \quad t \in [0, T], \\ {}^k D_{0+}^{\varrho_2, \eta} z^*(t) - \mathcal{K}(t, x^*(t), {}^k D_{0+}^{\varrho_2, \eta} z^*(t)) = \lambda_2(t), \quad t \in [0, T], \\ [\eta_0(t)]^{1-\frac{\varrho_1}{k}} x^*(t)|_{t=0} + [\eta_0(t)]^{1-\frac{\varrho_1}{k}} x^*(t)|_{t=\tau} = \sigma_1 {}^k I_{0+}^{S_1, \eta} z^*(\mu_1), \\ [\eta_0(t)]^{1-\frac{\varrho_2}{k}} z^*(t)|_{t=0} + [\eta_0(t)]^{1-\frac{\varrho_2}{k}} z^*(t)|_{t=\tau} = \sigma_2 {}^k I_{0+}^{S_2, \eta} x^*(\mu_2). \end{array} \right. \quad (4.2)$$

Theorem 4.1. Suppose that $\mathcal{J}, \mathcal{K} : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions satisfying condition (d_2) and (3.10) holds. Then, the solution to problem (1.1) is Ulam-Hyers stable and generalized Ulam-Hyers stable on $[0, T]$.

Proof. Let (x^*, z^*) be a solution to problem (4.2). Then, according to Lemma 2.4, we have

$$x^*(t) = \frac{1}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} [\beta^*(v) + \lambda_1(v)] dv$$

$$\begin{aligned}
& + \frac{(\eta_0(t))^{\frac{\varrho_1}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} [\beta^*(v) + \lambda_1(v)] dv \right. \\
& + \frac{2\sigma_1}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} [\chi^*(v) + \lambda_2(v)] dv \\
& - \frac{\sigma_1\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} [\chi^*(v) + \lambda_2(v)] dv \\
& \left. + \frac{\sigma_1\sigma_2\Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} [\beta^*(v) + \lambda_1(v)] dv \right\}, \tag{4.3}
\end{aligned}$$

and

$$\begin{aligned}
z^*(t) &= \frac{1}{k\Gamma_k(\varrho_2)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_2}{k}-1} [\chi^*(v) + \lambda_2(v)] dv \\
& + \frac{(\eta_0(t))^{\frac{\varrho_2}{k}-1}}{\Lambda} \left\{ \frac{-2(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} [\chi^*(v) + \lambda_2(v)] dv \right. \\
& + \frac{2\sigma_2}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} [\beta^*(v) + \lambda_1(v)] dv \\
& - \frac{\sigma_2\Delta_2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(r)(\eta_v(T))^{\frac{\varrho_1}{k}-1} [\beta^*(v) + \lambda_1(v)] dv \\
& \left. + \frac{\sigma_1\sigma_2\Delta_2}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} [\chi^*(v) + \lambda_2(v)] dv \right\}, \tag{4.4}
\end{aligned}$$

where $\beta^*(t) = \mathcal{J}(t, z^*(t), \beta^*(t))$, $\chi^*(t) = \mathcal{K}(t, x^*(t), \chi^*(t))$.

Consequently, for $\mathcal{E} : \Upsilon \rightarrow \Upsilon$ which is defined by (3.1), we find that

$$\begin{aligned}
& |(\eta_0(t))^{1-\frac{\varrho_1}{k}} (x^* - \mathcal{E}_1(x^*, z^*)(t))| \\
& \leq \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^t \eta'(v)(\eta_v(t))^{\frac{\varrho_1}{k}-1} |\lambda_1(v)| dv \\
& + \frac{(\eta_0(t))^{1-\frac{\varrho_1}{k}} (\eta_0(t))^{\frac{\varrho_1}{k}-1}}{|\Lambda|} \left\{ \frac{2(\eta_0(T))^{1-\frac{\varrho_1}{k}}}{k\Gamma_k(\varrho_1)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_1}{k}-1} |\lambda_1(v)| dv \right. \\
& + \frac{2|\sigma_1|}{k\Gamma_k(\varrho_2 + \varsigma_1)} \int_0^{\mu_1} \eta'(v)(\eta_v(\mu_1))^{\frac{\varrho_2+\varsigma_1}{k}-1} |\lambda_2(v)| dv \\
& + \frac{|\sigma_1|\Delta_1(\eta_0(T))^{1-\frac{\varrho_2}{k}}}{k\Gamma_k(\varrho_2)} \int_0^T \eta'(v)(\eta_v(T))^{\frac{\varrho_2}{k}-1} |\lambda_2(v)| dv \\
& \left. + \frac{|\sigma_1\sigma_2|\Delta_1}{k\Gamma_k(\varrho_1 + \varsigma_2)} \int_0^{\mu_2} \eta'(v)(\eta_v(\mu_2))^{\frac{\varrho_1+\varsigma_2}{k}-1} |\lambda_1(v)| dv \right\} \\
& \leq \epsilon_1\Phi_1 + \epsilon_2\Phi_2,
\end{aligned}$$

which, on taking the norm for $t \in [0, T]$, yields

$$\|x^* - \mathcal{E}_1(x^*, z^*)\|_{\varrho_1} \leq \epsilon_1\Phi_1 + \epsilon_2\Phi_2.$$

Similarly, one can find that

$$\|z^* - \mathcal{E}_2(x^*, z^*)\|_{\varrho_2} \leq \epsilon_1\Phi_4 + \epsilon_2\Phi_3.$$

Now, as the hypotheses of Theorem 3.2 are satisfied, problem (1.1) has a unique solution, say (x, z) . Then, we want to prove that

$$\|(x^*, z^*) - (x, z)\|_{\Upsilon} \leq \xi \epsilon^T.$$

So, employing the argument used in the proof of Theorem 3.2, we get

$$\|\mathcal{E}_1(x^*, z^*) - \mathcal{E}_1(x, z)\|_{\mathcal{Q}_1} \leq \frac{L_1^*}{1 - L_2^*} \|x^* - x\|_{\mathcal{Q}_1} \Phi_2 + \frac{C_1^*}{1 - C_2^*} \|z^* - z\|_{\mathcal{Q}_2} \Phi_1,$$

$$\|\mathcal{E}_2(x^*, z^*) - \mathcal{E}_2(x, z)\|_{\mathcal{Q}_2} \leq \frac{L_1^*}{1 - L_2^*} \|x^* - x\|_{\mathcal{Q}_2} \Phi_3 + \frac{C_1^*}{1 - C_2^*} \|z^* - z\|_{\mathcal{Q}_2} \Phi_4.$$

Therefore, we have

$$\begin{aligned} \|x^* - x\|_{\mathcal{Q}_1} &= \|x^* - \mathcal{E}_1(x^*, z^*) + \mathcal{E}_1(x^*, z^*) - \mathcal{E}_1(x, z)\|_{\mathcal{Q}_1} \\ &\leq \|x^* - \mathcal{E}_1(x^*, z^*)\|_{\mathcal{Q}_1} + \|\mathcal{E}_1(x^*, z^*) - \mathcal{E}_1(x, z)\|_{\mathcal{Q}_1} \\ &\leq \epsilon_1 \Phi_1 + \epsilon_2 \Phi_2 + \frac{L_1^*}{1 - L_2^*} \|x^* - x\|_{\mathcal{Q}_1} \Phi_2 + \frac{C_1^*}{1 - C_2^*} \|z^* - z\|_{\mathcal{Q}_2} \Phi_1, \end{aligned}$$

and

$$\begin{aligned} \|z^* - z\|_{\mathcal{Q}_2} &= \|z^* - \mathcal{E}_2(x^*, z^*) + \mathcal{E}_2(x^*, z^*) - \mathcal{E}_2(x, z)\|_{\mathcal{Q}_2} \\ &\leq \|z^* - \mathcal{E}_2(x^*, z^*)\|_{\mathcal{Q}_2} + \|\mathcal{E}_2(x^*, z^*) - \mathcal{E}_2(x, z)\|_{\mathcal{Q}_2} \\ &\leq \epsilon_1 \Phi_4 + \epsilon_2 \Phi_3 + \frac{L_1^*}{1 - L_2^*} \|x^* - x\|_{\mathcal{Q}_2} \Phi_3 + \frac{C_1^*}{1 - C_2^*} \|z^* - z\|_{\mathcal{Q}_2} \Phi_4. \end{aligned}$$

Adding the last two inequalities, we obtain

$$\begin{aligned} &\|(x^*, z^*) - (x, z)\|_{\Upsilon} \\ &= \|(x^* - x, z^* - z)\|_{\Upsilon} = \|x^* - x\|_{\mathcal{Q}_1} + \|z^* - z\|_{\mathcal{Q}_2} \\ &\leq \epsilon_1 \Phi_1 + \epsilon_2 \Phi_2 + \frac{L_1^*}{1 - L_2^*} \|x^* - x\|_{\mathcal{Q}_1} \Phi_2 + \frac{C_1^*}{1 - C_2^*} \|z^* - z\|_{\mathcal{Q}_2} \Phi_1 \\ &\quad + \epsilon_1 \Phi_4 + \epsilon_2 \Phi_3 + \frac{L_1^*}{1 - L_2^*} \|x^* - x\|_{\mathcal{Q}_2} \Phi_3 + \frac{C_1^*}{1 - C_2^*} \|z^* - z\|_{\mathcal{Q}_2} \Phi_4 \\ &\leq \epsilon_1 (\Phi_1 + \Phi_4) + \epsilon_2 (\Phi_2 + \Phi_3) \\ &\quad + \max \left\{ \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \right\} \|(x^*, z^*) - (x, z)\|_{\Upsilon}. \end{aligned}$$

Defining

$$\Omega = 1 - \max \left\{ \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \right\},$$

which is a positive constant in view of (3.10), the above inequality takes the form

$$\|(x^*, z^*) - (x, z)\|_{\Upsilon} \leq \epsilon_1 \frac{(\Phi_1 + \Phi_4)}{\Omega} + \epsilon_2 \frac{(\Phi_2 + \Phi_3)}{\Omega} = \xi \epsilon^T,$$

where

$$\xi = \left(\frac{(\Phi_1 + \Phi_4)}{\Omega}, \frac{(\Phi_2 + \Phi_3)}{\Omega} \right).$$

So, in view of Definition 4.1, we deduce that system (1.1) is Ulam-Hyers stable on $[0, T]$. Furthermore, for $\varphi(\epsilon_1, \epsilon_2) = \frac{(\Phi_1 + \Phi_4)}{\Omega} \epsilon_1 + \frac{(\Phi_2 + \Phi_3)}{\Omega} \epsilon_2$, the system (1.1) satisfies Definition 4.2 and, hence, it is generalized Ulam-Hyers stable on $[0, T]$.

Remark 4.2. *Ulam-Hyers type stability definitions state if we have an approximate solution of the system, then its exact solution will be very close to its approximate solution. In our work we have derived Ulam-Hyers and generalized Ulam-Hyers criteria for stability of solutions, whereas one can easily investigate another type of stability, which is so-called Ulam-Hyers-Rassias, by replacing ϵ_1 and ϵ_2 in (4.1) with $\varphi(t)\epsilon_1$ and $\varphi(t)\epsilon_2$, respectively, to estimate that $\|(x, z) - (x^*, z^*)\|_T \leq \xi(\varphi(t)\epsilon_1, \varphi(t)\epsilon_2)^T$, for some $\varphi \in C([0, T], \mathbb{R})$.*

5. Examples

This section is devoted to illustration of the results accomplished in Sections 3 and 4.

Example 5.1. Consider the following problem

$$\begin{cases} {}^{3/2}D_{0+}^{1/4, \eta} x(t) = \mathcal{J}(t, y(t), {}^{3/2}D_{0+}^{1/4, \eta} x(t)), & t \in [0, 2], \\ {}^{3/2}D_{0+}^{2/5, \eta} z(t) = \mathcal{K}(t, x(t), {}^{3/2}D_{0+}^{2/5, \eta} z(t)), & t \in [0, 2], \\ (\eta_0(t))^{5/6} x(t)|_{t=0} + (\eta_0(t))^{5/6} x(t)|_{t=2} = \frac{3}{5} {}^{3/2}I_{0+}^{5/7, \eta} z(1), \\ (\eta_0(t))^{11/15} z(t)|_{t=0} + (\eta_0(t))^{11/15} z(t)|_{t=2} = 2 {}^{3/2}I_{0+}^{4/3, \eta} x(\frac{4}{3}), \end{cases} \quad (5.1)$$

where $k = 3/2$, $\varrho_1 = 1/4$, $\varrho_2 = 2/5$, and so $\frac{\varrho_1}{k} = \frac{1}{6} \in (0, 1]$ and $\frac{\varrho_2}{k} = \frac{4}{15} \in (0, 1]$, $\sigma_1 = 3/5$, $\sigma_2 = 2$, $\varsigma_1 = 5/7$, $\varsigma_2 = 4/7$, $\mu_1 = 1$, $\mu_2 = 4/3$, $T = 2$, $\eta(t) = t^3 + 2t$, and $\mathcal{J}(t, x, z)$ and $\mathcal{K}(t, x, z)$ will be introduced later. Obviously $\eta(t)$ is increasing on $(0, 2]$ and $\eta'(t) = 3t^2 + 2$ is a continuous on $(0, 2)$ with $\eta'(t) \neq 0$. According to the given data, we find that $\Phi_1 = 5.568105$, $\Phi_2 = 2.309732$, $\Phi_3 = 6.565949$, and $\Phi_4 = 5.9205$, where Φ_i ($i = 1, 2, 3, 4$) are given in (3.6).

(i) For demonstrating application of Theorem 3.1, let us take

$$\mathcal{J}(t, x, z) = \frac{1}{2\sqrt{900+t}} (\sin x + (\cos t)z + e^t) \text{ and } \mathcal{K}(t, x, z) = \frac{1}{(8+t)^2} ((t^3 + 2t)x + \frac{z}{3} + \cos t). \quad (5.2)$$

It easy to verify that $\mathcal{J}$ and $\mathcal{K}$ satisfy condition (d_1) with

$$\begin{aligned} p_1(t) &= \frac{e^t}{2\sqrt{900+t}}, \quad p_2(t) = \frac{1}{2\sqrt{900+t}}, \quad p_3(t) = \frac{|\cos t|}{2\sqrt{900+t}}, \\ q_1(t) &= \frac{|\cos t|}{(8+t)^2}, \quad q_2(t) = \frac{(t^3 + 2t)}{(8+t)^2}, \quad q_3(t) = \frac{1}{3(8+t)^2}. \end{aligned}$$

Clearly $\bar{p}_1 = 0.975604$, $\widehat{p}_2 = 0.021344$, $p_3^* = 0.016667$, $\bar{q}_1 = 0.0155955$, $\bar{q}_2 = 0.093597$, $\bar{q}_3 = 0.005208$. Furthermore,

$\frac{\hat{q}_2}{1-q_3^*}[\Phi_2 + \Phi_3] \approx 0.835089 < 1$, and $\frac{\hat{p}_2}{1-p_3^*}[\Phi_1 + \Phi_4] \approx 0.249374 < 1$. Thus, all the assumptions of Theorem 3.1 hold true, and hence problem (5.1) with $\mathcal{J}(t, x, y)$ and $\mathcal{K}(t, x, y)$ given in (5.2) has at least one solution on $[0, 2]$.

(ii) For illustrating Theorem 3.2, let us choose

$$\mathcal{J}(t, x, z) = \frac{e^{-t}}{60} \left(\frac{|x|}{1+|x|} + \tan^{-1} z + 1 \right) \text{ and } \mathcal{K}(t, x, z) = \frac{1}{(t^2 + 6)^2} \left((t^3 + 2t) \sin x + \frac{|z|}{2 + |z|} + t \right), \quad (5.3)$$

and note that $\mathcal{J}$ and $\mathcal{K}$ satisfy the condition (d_2) with

$$C_1(t) = C_2 = \frac{e^{-t}}{60}, \quad L_1(t) = \frac{(t^3 + 2t)}{(t^2 + 6)^2}, \quad L_2(t) = \frac{1}{2(t^2 + 6)^2},$$

and $C_1^* = 0.0128452$, $C_2^* = 0.016667$, $L_1^* = 0.0935973$, $L_2^* = 0.0138889$. Also,

$$\max \left\{ \frac{C_1^*}{1 - C_2^*} (\Phi_1 + \Phi_4), \frac{L_1^*}{1 - L_2^*} (\Phi_2 + \Phi_3) \right\} \approx 0.8424399 < 1.$$

Hence, all the conditions of Theorem 3.2 are satisfied, and, consequently, problem (5.1) with $\mathcal{J}(t, x, y)$ and $\mathcal{K}(t, x, y)$ given in (5.3) has a unique solution on $[0, 2]$.

(iii) For illustration of Theorem 4.1, we consider $\mathcal{J}$ and $\mathcal{K}$ given in (5.3). Then, the conditions (d_2) and (3.10) are satisfied as shown in (ii). Consequently, Definition 4.1 holds true with $\xi = \left(\frac{(\Phi_1 + \Phi_4)}{\Omega}, \frac{(\Phi_2 + \Phi_3)}{\Omega} \right) \approx (72.916, 56.333)$, where $\Omega = 0.15756009$, which implies that problem (5.1) is Ulam-Hyers stable on $(0, 2]$. Moreover, problem (5.1) is generalized Ulam-Hyers stable on $[0, 2]$, with $\varphi(\epsilon_1, \epsilon_2) = (73\epsilon_1, 57\epsilon_2)$.

6. Conclusions

This paper presents a novel contribution arising from the study of an implicit coupled system of (k, η) -fractional differential equations complemented with nonlocal integral BCs. The existence and uniqueness criteria for solutions to the given problem are obtained by utilizing Schaefer's fixed point theorem and Banach contraction mapping principle. The Ulam-Hyers stability analysis is also carried out for the problem at hand. Examples illustrating the applicability of main results are also constructed. It is imperative to mention that our results will produce several new results as special cases by fixing the values of k and η . For instance, our results will correspond to the η -Riemann–Liouville IDEs when $k = 1$, and k -fractional IDEs when $\eta(t) = t$, which are indeed new in the given configuration. Thus, the work accomplished in this paper is a useful contribution to the existing literature on the topic.

Author contributions

Each of the authors, M. A and B. A, contributed equally to each part of this work. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interests in this paper.

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