



Research article**A self-adaptive inertial approximation method for split equality fixed point problems of k -demimetric mappings in Banach spaces****Chanjuan Pan¹, Yan Zhang¹, Panpan Zhu¹ and Kunyang Wang^{2,*}**¹ School of Computer Science and Technology, Zhejiang University of Water Resources and Electric Power, Hangzhou, 310018, China² Key Laboratory of Rare Earth Optoelectronic Materials and Devices of Zhejiang Province, Institute of Optoelectronic Materials and Devices, China Jiliang University, Hangzhou, 310018, China*** Correspondence:** Email: wangky@cjlu.edu.cn; Tel: +8617855868981.

Abstract: In this paper, we investigate the split equality fixed point problems for k -demimetric mappings in Banach spaces. A self-adaptive inertial approximation method is developed by incorporating an adaptive inertial prediction mechanism, a Landweber-type correction step for the split equality constraint, and a k -demimetric fixed point approximation procedure. Under suitable assumptions, the proposed method generates a sequence that converges strongly to the Bregman projection of the anchor point onto the solution set. Furthermore, the obtained results are applied to split equality variational inclusion problems and split equality equilibrium problems. Numerical experiments are conducted to verify the effectiveness of the proposed method.

Keywords: split equality fixed point problem; self-adaptive inertial algorithm; k -demimetric mappings; strong convergence; Banach space

Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25

1. Introduction

Split-type problems have played an important role in the development of iterative methods for nonlinear operator equations and optimization problems. Among them, the split feasibility problem (SFP), initially introduced by Censor and Elfving [1], has become one of the fundamental models in this area. Let H_1 and H_2 be real Hilbert spaces and $\hat{A} : H_1 \rightarrow H_2$ be a bounded linear operator. The SFP aims to find a point satisfying a prescribed constraint in one space while its image under a given operator satisfies another constraint in a different space. Due to its relevance to inverse problems, including phase retrieval and medical image reconstruction, the SFP has been extensively studied [2,3]. This model has also motivated the formulation of various split-type problems.

A natural extension of the SFP is the split common fixed point problem (SCFPP):

$$\text{find } \hat{x} \in F(U) \text{ such that } \hat{A}\hat{x} \in F(S),$$

where $U : H_1 \rightarrow H_1$ and $S : H_2 \rightarrow H_2$ are nonlinear operators. The problem combines fixed point constraints with operator consistency requirements and has attracted considerable interest in fixed point theory. Many iterative schemes have been developed under different assumptions on the involved spaces and mappings.

Another important extension of split-type problems is the SEP, introduced by Moudafi et al. [4]:

$$\text{find } \hat{x} \in H_1, \hat{y} \in H_2 \text{ such that } A\hat{x} = B\hat{y},$$

where $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ are bounded linear operators. Unlike the classical SFP and SCFPP, the SEP involves two independent variables defined in different spaces and couples them through an equality constraint. Combining the fixed point constraints with the equality relation leads to the split equality fixed point problem (SEFPP):

$$\text{find } \hat{x} \in F(U), \hat{y} \in F(S) \text{ such that } A\hat{x} = B\hat{y}.$$

The SEFPP has attracted increasing attention because of its applications in inverse problems, signal processing, radiation therapy, and other related fields [5, 6].

Various iterative methods have been proposed to approximate solutions of SEFPP [7–9]. Moudafi et al. [4] investigated this problem in Hilbert spaces and obtained a weak convergence result for firmly quasi-nonexpansive mappings. Later, Wang et al. [10] proposed an adapted Mann-type method for SEFPP. Since many Hilbert space techniques rely on inner product structures, extending these methods to Banach spaces has become an important research direction. By using generalized duality mappings and Bregman distances, Izuchukwu et al. [11] investigated SEFPP in Banach spaces and established strong convergence results for Bregman strongly nonexpansive mappings. Although these studies have substantially advanced the theory of SEFPP, most existing results are restricted to nonexpansive-type or Bregman strongly nonexpansive mappings. Moreover, the simultaneous incorporation of more general demimetric mappings, inertial acceleration, and self-adaptive treatment of the split equality constraint in Banach spaces has received comparatively limited attention. This motivates the study of SEFPP involving k -demimetric mappings.

Among various generalized nonlinear mappings, demimetric mappings have attracted considerable attention because they encompass several important classes of fixed point mappings and provide a flexible framework for iterative approximation. Nevertheless, SEFPP involving k -demimetric mappings has been much less investigated in general Banach spaces. Therefore, developing convergent algorithms for this setting constitutes a natural extension of the existing SEFPP theory.

In addition to enlarging the admissible class of mappings, improving the efficiency of iterative algorithms is another important issue. The inertial technique, originating from Polyak's extrapolation idea [12], has been widely incorporated into iterative methods for fixed point problems, variational inequalities, equilibrium problems, and split models [13–17]. For example, Zhou et al. [18] investigated an inertial shrinking projection method for SCFPP in p -uniformly convex and uniformly smooth Banach spaces.

Recently, alternated inertial and adaptive stepsize strategies have received increasing attention. Tan et al. [19] proposed four alternated inertial algorithms with non-monotone adaptive stepsizes for split feasibility problems in real Hilbert spaces. Xu and Li [20] developed a strongly convergent alternated inertial subgradient extragradient method with a non-monotone adaptive stepsize for pseudomonotone equilibrium problems, where the stepsize selection does not require prior knowledge of the Lipschitz constants of the bifunction. Related recent developments include past-iterate extrapolation for projection and contraction methods [21] and alternated inertial subgradient extragradient methods with adaptive halfspace correction [22] for variational inequality problems. These studies demonstrate the potential of inertial extrapolation and adaptive parameter selection to improve computational efficiency. However, they are mainly concerned with split feasibility, equilibrium, or variational inequality problems in Hilbert spaces. Self-adaptive inertial methods for SEFPP involving k -demimetric mappings in general Banach spaces remain comparatively less developed.

Motivated by the above observations, we develop a self-adaptive inertial approximation framework for solving SEFPP involving k -demimetric mappings in p -uniformly convex and uniformly smooth Banach spaces. By reformulating the split equality constraint in the product space $E = E_1 \times E_2$ with the associated operator $G = [A, -B]$, the proposed method combines a dual-space inertial prediction mechanism, a self-adaptive Landweber-type correction, and a k -demimetric fixed point approximation procedure. Unlike the aforementioned alternated inertial and adaptive methods developed mainly for Hilbert space feasibility, equilibrium, and variational inequality problems, our framework addresses SEFPP in a more general Banach space setting and employs an adaptive correction rule that does not require prior knowledge of the operator norm. Under suitable assumptions, strong convergence of the generated sequence to the Bregman projection of the anchor point onto the solution set is established. Furthermore, applications to split equality variational inclusion problems and split equality equilibrium problems are presented, together with numerical experiments illustrating the performance of the proposed method. Thus, the results obtained in this paper extend and complement several existing results (see [8, 11, 18, 23–26]).

2. Preliminaries

Consider a real normed space E paired with its topological dual E^* . For conjugate exponents $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, the generalized dual mapping $J_p^E : E \rightarrow 2^{E^*}$ is defined by

$$J_p^E(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\|_* = \|x\|^{p-1}\}, \forall x \in E.$$

When $p = 2$, this mapping reduces to the classical normalized duality mapping J^E . It is generally known that the dual mapping J_p^E is one-to-one, single-valued, and $J_p^E = (J_q^{E^*})^{-1}$. Assume E is uniformly smooth, then J_p^E is norm-to-norm uniformly continuous on bounded subsets of E ; for more details, see [27].

Definition 2.1. Let C be a nonempty closed convex subset of E , $T : C \rightarrow E$ be a mapping, and I be an identity mapping. The set of fixed points of T is defined by $F(T)$.

- The mapping $I - T$ is called demiclosed at zero if

$$x_n \rightharpoonup x, (I - T)x_n \rightarrow 0, \text{ then } x = Tx.$$

- The mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in E.$$

- The mapping T is called a quasi-nonexpansive mapping if the set $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\|, \forall x \in E, x^* \in F(T).$$

- The mapping T is called a firmly nonexpansive mapping if for any $x, y \in E$,

$$\|Tx - Ty\|^p \leq \langle x - y, J_E^p(Tx - Ty) \rangle.$$

- The mapping T is called firmly quasi-nonexpansive if the set $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^p \leq \langle x - x^*, J_E^p(Tx - x^*) \rangle, \forall x \in E, x^* \in F(T).$$

- The mapping T is called k -demimetric if $k \in (-\infty, 1)$ and the set $F(T) \neq \emptyset$,

$$\frac{1-k}{2} \|x - Tx\|^p \leq \langle x - x^*, J_E^p(x - Tx) \rangle, \forall x \in E, x^* \in F(T).$$

If T is a k -demimetric mapping, then $F(T)$ is closed and convex. We have that a k -strict pseudo-contraction is k -demimetric, a generalized hybrid mapping T with $F(T) \neq \emptyset$ is 0-demimetric, and the resolvents of maximal monotone operators is -1 -demimetric [28]. Thus, the demimetric mapping holds core significance in the field of optimization, as it encompasses numerous common operation types derived from optimization.

Definition 2.2. For the Gâteaux differentiable convex function $g_p(x) = \frac{1}{p}\|x\|^p$, its derivative precisely yields the generalized duality mapping J_p^E , and the Bregman distance of g_p is defined as

$$\begin{aligned} D_p(x, y) &= \frac{1}{q}\|x\|^p - \langle J_p^E x, y \rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_p^E x, x - y \rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_p^E x - J_p^E y, y \rangle. \end{aligned}$$

Definition 2.3. Let C be a nonempty closed convex subset of E . The Bregman projection is defined as:

$$\Pi_C = \operatorname{argmin}_{y \in C} D_p(x, y), x \in E.$$

It can also be characterized by the following variational inequality:

$$\langle z - \Pi_C x, J_p^C x - J_p^C \Pi_C x \rangle \leq 0, \forall z \in C.$$

We now give some relevant lemmas that will be used in the proof of our main theorem.

Lemma 2.4 ([29]). Given that E is p -uniformly convex, for arbitrary elements $x, y \in E$, Bregman distance holds the following relation:

$$\eta\|x - y\|^p \leq D_p(x, y) \leq \langle x - y, J_p^E x - J_p^E y \rangle.$$

Lemma 2.5 ([23]). The function $V_p : E \times E^* \rightarrow \mathbb{R}$ is defined by

$$V_p(x^*, x) = \frac{1}{q} \|x^*\|^q - \langle x^*, x \rangle + \frac{1}{p} \|x\|^p, \forall x \in E, x^* \in E^*.$$

Then V_p is negative and convex with respect to the first variable. The following holds:

(i) $V_p(x^*, x) = D_p(J_q^{E^*} x^*, x)$.

(ii) $V_p(x^*, x) + \langle y^*, J_q^{E^*} x^* - x \rangle \leq V_p(x^* + y^*, x), \forall x \in E, x^*, y^* \in E^*.$

(iii) $D_p(J_q^{E^*} (\sum_{i=1}^N t_i J_p^E(x_i)), z) \leq \sum_{i=1}^N t_i D_p(x_i, z), \forall z \in E$, where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ satisfies $\sum_{i=1}^N t_i = 1$.

Lemma 2.6 ([8]). Let $J_p^E : E \rightarrow 2^{E^*}, 1 < p < \infty$ be the generalized duality mapping. We have

$$\|x + y\|^p \leq \|x\|^p + p \langle y, J_p^E(x + y) \rangle.$$

Lemma 2.7 ([24]). Assume E is a q -uniformly smooth Banach space, then for any $x, y \in E$, one can find $C_q > 0$, and the following norm inequality holds:

$$\|x - y\|^q \leq \|x\|^q - q \langle y, J_q^E x \rangle + C_q \|y\|^q.$$

Lemma 2.8 ([30]). Assume that $\{x_n\}$ and $\{y_n\}$ are bounded sequences within the space E . The condition $\lim_{n \rightarrow \infty} D_p(x_n, y_n) = 0$ is satisfied if and only if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2.9 ([31]). For fixed constants $q \geq 1$ and $r > 0$, it is established that E exhibits uniform convexity precisely when there exists a convex function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, which is continuous and strictly increasing, such that for all pairs x, y within $B_r := \{x \in E, \|x\| \leq r\}$ and $t \in [0, 1]$, the subsequent estimate holds:

$$\|tx + (1 - t)y\|^q \leq t\|x\|^q + (1 - t)\|y\|^q - W_q(t)\varphi(\|x - y\|),$$

where $W_q(t) := t^q(1 - t) + t(1 - t)^q$.

Lemma 2.10 ([9]). Assume that $\{a_n\}$ is a nonempty real sequence, and for all $j \in \mathbb{N}$, there is a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} \leq a_{n_{j+1}}$. Then there exists a nondecreasing sequence $\{m_i\} \subset \mathbb{N}, i \in \mathbb{N}$ such that $m_i \rightarrow \infty$, thus we have

$$a_{m_i} \leq a_{m_{i+1}}, a_i \leq a_{m_{i+1}}.$$

In fact, $m_i = \max\{k \leq i : a_k \leq a_{k+1}\}$.

Lemma 2.11 ([11]). Assume $\{d_n\}$ is a sequence of nonnegative real numbers such that

$$d_{n+1} \leq (1 - \alpha_n)d_n + \alpha_n \sigma_n, n \geq 0,$$

where $\{\alpha_n\} \in (0, 1)$ and $\{\sigma_n\}$ is a real sequence such that

- (1) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\alpha_n \sigma_n| < \infty$.

Then $\lim_{n \rightarrow \infty} d_n = 0$.

3. Main results

In this section, we assume that E_1 , E_2 , and E_3 are three p -uniformly convex and uniformly smooth Banach spaces. Let $A : E_1 \rightarrow E_3$, $B : E_2 \rightarrow E_3$ be two bounded linear operators and A^* , B^* be the adjoint of A and B , respectively. Let $T_1 : E_1 \rightarrow E_1$, $T_2 : E_2 \rightarrow E_2$ be k_1, k_2 -demimetric mappings, respectively. The solution set of the SEFPP is defined by

$$\Gamma = \{(x, y) \in F(T_1) \times F(T_2) \text{ such that } Ax = By\}.$$

Let $E = E_1 \times E_2$, $w = (x, y) \in E$, and define $T : E \rightarrow E$ by $T = [T_1, T_2]$ and $G : E \rightarrow E_3$ by $G = [A, -B]$. Then the SEFPP becomes finding $w = (x, y) \in F(T)$ such that $Gw = 0$. For all $n \geq 0$, $\theta > 0$, θ_n is chosen in such a way that

$$0 \leq \theta_n \leq \overline{\theta}_n = \begin{cases} \min\{\theta, \frac{c_n}{\delta_n}\} & \text{if } \delta_n \neq 0, \\ \theta & \text{otherwise,} \end{cases}$$

where $\delta_n = \|J_p^{E_1} x_n - J_p^{E_1} x_{n-1}\| + \|J_p^{E_2} y_n - J_p^{E_2} y_{n-1}\|$ and $\{c_n\}$ is a sequence of positive numbers. Then we present the following theorem:

Theorem 3.1. Let $v = (v_1, v_2) \in E$, $w_0, w_1 \in E$, and the sequence $\{w_n\}$ be generated by

$$\begin{cases} u_n = J_q^{E^*} [J_p^E w_n + \theta_n (J_p^E w_n - J_p^E w_{n-1})], \\ z_n = J_q^{E^*} [J_p^E u_n - \gamma_n G^* J_p^{E_3} G u_n], \\ w_{n+1} = J_q^{E^*} [\alpha_n J_p^E v + (1 - \alpha_n) [J_p^E z_n - t_n J_p^E (z_n - T z_n)]]. \end{cases} \quad (3.1)$$

Let the sequences $\{\alpha_n\} \subset (0, 1)$, $\xi > 0$ and $\beta = \max\{k_1, k_2\} < 1$, and the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < c_n < \frac{p\eta}{2^{p-1}}$, $c_n = o(\alpha_n)$;
- (iii) $\gamma_n^{q-1} \in (\xi, \frac{q\|Gu_n\|^p}{C_q\|G^* J_p^{E_3} Gu_n\|^q} - \xi)$;
- (iv) $0 < t_n < [\frac{(1-\beta)q}{2C_q}]^{\frac{1}{q-1}}$.

The generated sequence $\{(x_n, y_n)\}$ is strongly convergent to $w^* = \Pi_{\Gamma} v$.

Proof. Given that $w \in \Gamma$, we have $w \in F(T)$ and $Gw = 0$. From Definition 2.2 and Lemma 2.4,

$$\begin{aligned}
 D_p(u_n, w) &= D_p(J_q^{E*} [J_p^E w_n + \theta_n(J_p^E w_n - J_p^E w_{n-1})], w) \\
 &= \frac{1}{p} \|w\|^p - \frac{1}{p} \|u_n\|^p + \langle J_p^E w_n + \theta_n(J_p^E w_n - J_p^E w_{n-1}), u_n - w \rangle \\
 &= \frac{1}{p} \|w\|^p - \frac{1}{p} \|u_n\|^p + \langle J_p^E w_n, u_n - w \rangle + \langle \theta_n(J_p^E w_n - J_p^E w_{n-1}), u_n - w \rangle \\
 &= \frac{1}{p} \|w\|^p - \frac{1}{p} \|w_n\|^p + \langle J_p^E w_n, w_n - w \rangle + \frac{1}{p} \|w_n\|^p - \langle J_p^E w_n, w_n - w \rangle \\
 &\quad - \frac{1}{p} \|u_n\|^p + \langle J_p^E w_n, u_n - w \rangle + \langle \theta_n(J_p^E w_n - J_p^E w_{n-1}), u_n - w \rangle \\
 &= D_p(w_n, w) + \frac{1}{p} \|w_n\|^p - \frac{1}{p} \|u_n\|^p + \langle J_p^E w_n, u_n - w_n \rangle \\
 &\quad + \langle \theta_n(J_p^E w_n - J_p^E w_{n-1}), u_n - w \rangle \\
 &= D_p(w_n, w) - D_p(w_n, u_n) + \langle \theta_n(J_p^E w_n - J_p^E w_{n-1}), u_n - w \rangle \\
 &\leq D_p(w_n, w) - D_p(w_n, u_n) + \theta_n \|J_p^E w_n - J_p^E w_{n-1}\| \|u_n - w\| \\
 &\leq D_p(w_n, w) - D_p(w_n, u_n) + \theta_n \|J_p^E w_n - J_p^E w_{n-1}\| \left[\frac{1}{p} \|u_n - w\| + \frac{1}{q} \right] \\
 &\leq D_p(w_n, w) - D_p(w_n, u_n) + \frac{\theta_n}{q} \|J_p^E w_n - J_p^E w_{n-1}\| \\
 &\quad + \frac{\theta_n}{p} \|J_p^E w_n - J_p^E w_{n-1}\| [2^{p-1} (\|w_n - w\|^p + \|w_n - u_n\|^p)] \\
 &\leq D_p(w_n, w) - D_p(w_n, u_n) + \frac{2^{p-1} c_n}{p\eta} D_p(w_n, w) + \frac{2^{p-1} c_n}{p\eta} D_p(w_n, u_n) + \frac{c_n}{q} \\
 &= (1 + \frac{2^{p-1} c_n}{p\eta}) D_p(w_n, w) - (1 - \frac{2^{p-1} c_n}{p\eta}) D_p(w_n, u_n) + \frac{c_n}{q}.
 \end{aligned} \tag{3.2}$$

If $\|Gu_n\| = 0$, then $G^* J_p^{E_3} Gu_n = 0$. In this case, the stepsize condition in (iii) is not invoked, and the Landweber correction term vanishes. Hence, from the definition of z_n , we have $z_n = u_n$, and therefore

$D_p(z_n, w) = D_p(u_n, w)$. Now suppose that $\|Gu_n\| > 0$. By (3.1) and Lemmas 2.5 and 2.7, we have

$$\begin{aligned}
 D_p(z_n, w) &= D_p(J_q^{E*}[J_p^E u_n - \gamma_n G^* J_p^{E_3} Gu_n], w) \\
 &= V_p(J_p^E u_n - \gamma_n G^* J_p^{E_3} Gu_n, w) \\
 &= \frac{1}{q} \|J_p^E u_n - \gamma_n G^* J_p^{E_3} Gu_n\|^q + \frac{1}{p} \|w\|^p - \langle J_p^E u_n - \gamma_n G^* J_p^{E_3} Gu_n, w \rangle \\
 &\leq \frac{1}{q} \|J_p^E u_n\|^q - \langle \gamma_n G^* J_p^{E_3} Gu_n, u_n \rangle + \frac{C_q}{q} \|\gamma_n G^* J_p^{E_3} Gu_n\|^q \\
 &\quad + \frac{1}{p} \|w\|^p - \langle J_p^E u_n - \gamma_n G^* J_p^{E_3} Gu_n, w \rangle \\
 &= \frac{1}{q} \|u_n\|^p - \langle J_p^E u_n, w \rangle + \frac{1}{p} \|w\|^p - \gamma_n \langle J_p^{E_3} Gu_n, Gu_n - Gw \rangle + \frac{C_q(\gamma_n)^q}{q} \|G^* J_p^{E_3} Gu_n\|^q \\
 &= D_p(u_n, w) - \gamma_n \langle J_p^{E_3} Gu_n, Gu_n - Gw \rangle + \frac{C_q(\gamma_n)^q}{q} \|G^* J_p^{E_3} Gu_n\|^q \\
 &= D_p(u_n, w) - \gamma_n (\|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q} \|G^* J_p^{E_3} Gu_n\|^q).
 \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) with conditions (ii) and (iii), we obtain

$$\begin{aligned}
 D_p(z_n, w) &\leq D_p(u_n, w) \\
 &\leq (1 + \frac{2^{p-1}c_n}{p\eta}) D_p(w_n, w) + \frac{c_n}{q}.
 \end{aligned} \tag{3.4}$$

Put $a_n = J_q^{E*}(J_p^E z_n - t_n J_p^E(z_n - Tz_n))$. Since T_1 and T_2 are k_1 -demimetric and k_2 -demimetric, respectively, the product mapping T is β -demimetric, where $\beta = \max\{k_1, k_2\} < 1$. From Lemma 2.7, we obtain

$$\begin{aligned}
 D_p(a_n, w) &= D_p(J_q^{E*}(J_p^E z_n - t_n J_p^E(z_n - Tz_n)), w) \\
 &= \frac{1}{q} \|J_p^E z_n - t_n J_p^E(z_n - Tz_n)\|^q + \frac{1}{p} \|w\|^p - \langle J_p^E z_n, w \rangle + t_n \langle J_p^E(z_n - Tz_n), w \rangle \\
 &\leq \frac{1}{q} \|J_p^E z_n\|^q - t_n \langle J_p^E(z_n - Tz_n), z_n \rangle + \frac{C_q t_n^q}{q} \|J_p^E(z_n - Tz_n)\|^q \\
 &\quad + \frac{1}{p} \|w\|^p - \langle J_p^E z_n, w \rangle + t_n \langle J_p^E(z_n - Tz_n), w \rangle \\
 &= D_p(z_n, w) - t_n \langle J_p^E(z_n - Tz_n), z_n - w \rangle + \frac{C_q t_n^q}{q} \|J_p^E(z_n - Tz_n)\|^q \\
 &\leq D_p(z_n, w) - t_n \frac{1-\beta}{2} \|z_n - Tz_n\|^p + \frac{C_q t_n^q}{q} \|J_p^E(z_n - Tz_n)\|^q \\
 &= D_p(z_n, w) - t_n (\frac{1-\beta}{2} - \frac{C_q t_n^{q-1}}{q}) \|z_n - Tz_n\|^p.
 \end{aligned}$$

It follows from (3.1) and Lemma 2.5 that

$$\begin{aligned}
 D_p(w_{n+1}, w) &= D_p(J_q^{E^*}(\alpha_n J_p^E v + (1 - \alpha_n) J_p^E a_n), w) \\
 &\leq (1 - \alpha_n) D_p(a_n, w) + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n) D_p(z_n, w) + \alpha_n D_p(v, w) \\
 &\quad - (1 - \alpha_n) t_n \left(\frac{1 - \beta}{2} - \frac{C_q t_n^{q-1}}{q} \right) \|z_n - Tz_n\|^p.
 \end{aligned} \tag{3.5}$$

Since $c_n = o(\alpha_n)$, for $\forall \epsilon \in (0, \frac{p\eta}{2^{p-1}})$, there exist $n_0, \forall n > n_0$ such that $c_n < \epsilon \alpha_n$. Then from (3.4), we have

$$\begin{aligned}
 D_p(w_{n+1}, w) &\leq (1 - \alpha_n) D_p(z_n, w) + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n) \left(1 + \frac{2^{p-1} c_n}{p\eta} \right) D_p(w_n, w) + (1 - \alpha_n) \frac{c_n}{q} + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta} \right]) D_p(w_n, w) + (1 - \alpha_n) \frac{\epsilon \alpha_n}{q} + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta} \right]) D_p(w_n, w) + \alpha_n \left(\frac{\epsilon}{q} + D_p(v, w) \right) \\
 &\leq (1 - \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta} \right]) D_p(w_n, w) + \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta} \right] \frac{M}{1 - \frac{2^{p-1} \epsilon}{p\eta}},
 \end{aligned}$$

where $M = \frac{\epsilon}{q} + D_p(v, w)$. Thus $D_p(w_n, w)$ is bounded, and so are $\{D_p(z_n, w)\}, \{D_p(u_n, w)\}$. From Lemma 2.4, we know that $\eta \|w_n - w\|^p \leq D_p(w_n, w)$, which implies that $\{w_n\}$ is bounded. Therefore, $\{z_n\}$ and $\{u_n\}$ are bounded.

From (3.5) and (3.4), we obtain

$$\begin{aligned}
 D_p(w_{n+1}, w) &\leq (1 - \alpha_n) D_p(z_n, w) + \alpha_n D_p(v, w) - (1 - \alpha_n) t_n \left(\frac{1 - \beta}{2} - \frac{C_q t_n^{q-1}}{q} \right) \|z_n - Tz_n\|^p \\
 &\leq (1 - \alpha_n) \left(1 + \frac{2^{p-1} c_n}{p\eta} \right) D_p(w_n, w) + (1 - \alpha_n) \frac{c_n}{q} + \alpha_n D_p(v, w) \\
 &\quad - (1 - \alpha_n) t_n \left(\frac{1 - \beta}{2} - \frac{C_q t_n^{q-1}}{q} \right) \|z_n - Tz_n\|^p \\
 &\leq (1 - \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta} \right]) D_p(w_n, w) + \alpha_n M - (1 - \alpha_n) t_n \left(\frac{1 - \beta}{2} - \frac{C_q t_n^{q-1}}{q} \right) \|z_n - Tz_n\|^p,
 \end{aligned} \tag{3.6}$$

and using (3.2), we have

$$\begin{aligned}
 D_p(w_{n+1}, w) &\leq (1 - \alpha_n)D_p(z_n, w) + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n)D_p(u_n, w) + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n)\left(1 + \frac{2^{p-1}c_n}{p\eta}\right)D_p(w_n, w) - (1 - \alpha_n)\left(1 - \frac{2^{p-1}c_n}{p\eta}\right)D_p(w_n, u_n) \\
 &\quad + (1 - \alpha_n)\frac{c_n}{q} + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n)\left[1 - \frac{2^{p-1}\epsilon}{p\eta}\right](D_p(w_n, w) + \alpha_n M \\
 &\quad - (1 - \alpha_n)\left(1 - \frac{2^{p-1}c_n}{p\eta}\right)D_p(w_n, u_n)).
 \end{aligned} \tag{3.7}$$

By (3.5) and (3.3), we have

$$\begin{aligned}
 D_p(w_{n+1}, w) &\leq (1 - \alpha_n)D_p(z_n, w) + \alpha_n D_p(v, w) \\
 &\leq (1 - \alpha_n)D_p(u_n, w) + \alpha_n D_p(v, w) - (1 - \alpha_n)\gamma_n(\|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q}\|G^* J_p^{E_3} Gu_n\|^q) \\
 &\leq (1 - \alpha_n)\left(1 + \frac{2^{p-1}c_n}{p\eta}\right)D_p(w_n, w) + (1 - \alpha_n)\frac{c_n}{q} + \alpha_n D_p(v, w) \\
 &\quad - (1 - \alpha_n)\gamma_n(\|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q}\|G^* J_p^{E_3} Gu_n\|^q) \\
 &\leq (1 - \alpha_n)\left[1 - \frac{2^{p-1}\epsilon}{p\eta}\right]D_p(w_n, w) + \alpha_n M \\
 &\quad - (1 - \alpha_n)\gamma_n(\|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q}\|G^* J_p^{E_3} Gu_n\|^q).
 \end{aligned} \tag{3.8}$$

Next, we consider the following two cases:

Case 1. Suppose that $\{D_p(w_n, w)\}$ is decreasing. This implies that $\{D_p(w_n, w)\}$ is convergent. From (3.6)–(3.8), we get

$$\begin{aligned}
 \lim_{n \rightarrow \infty} (1 - \alpha_n)t_n\left(\frac{1 - \beta}{2} - \frac{C_q t_n^{q-1}}{q}\right)\|z_n - Tz_n\|^p &= 0; \\
 \lim_{n \rightarrow \infty} (1 - \alpha_n)\left(1 - \frac{2^{p-1}c_n}{p\eta}\right)D_p(w_n, u_n) &= 0; \\
 \lim_{n \rightarrow \infty} (1 - \alpha_n)\gamma_n(\|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q}\|G^* J_p^{E_3} Gu_n\|^q) &= 0.
 \end{aligned}$$

Then from the conditions, we have that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} D_p(w_n, u_n) &= 0, \\
 \lim_{n \rightarrow \infty} \|z_n - Tz_n\|^p &= 0, \\
 \lim_{n \rightarrow \infty} \|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q}\|G^* J_p^{E_3} Gu_n\|^q &= 0.
 \end{aligned} \tag{3.9}$$

Based on the choice of stepsize, we get

$$\frac{\xi C_q}{q} \|G^* J_p^{E_3} Gu_n\|^q < \|Gu_n\|^p - \frac{C_q(\gamma_n)^{q-1}}{q} \|G^* J_p^{E_3} Gu_n\|^q \rightarrow 0.$$

Thus $\lim_{n \rightarrow \infty} \|G^* J_p^{E_3} Gu_n\| = 0$, and from (3.9), $\lim_{n \rightarrow \infty} \|Gu_n\| = 0$. Further,

$$\begin{aligned} \|J_p^E z_n - J_p^E u_n\| &= \|J_p^E u_n - \gamma_n G^* J_p^{E_3} Gu_n - J_p^E u_n\| \\ &= \gamma_n \|G^* J_p^{E_3} Gu_n\| \\ &= \gamma_n \|G^* J_p^{E_3} Gu_n\| \rightarrow 0. \end{aligned} \quad (3.10)$$

Since J_p^E is norm-to-norm uniformly continuous, from (3.9), (3.10), and Lemma 2.8, we have

$$\|w_n - u_n\| \rightarrow 0, \|z_n - u_n\| \rightarrow 0, \|w_n - z_n\| \rightarrow 0, \|z_n - Tz_n\| \rightarrow 0. \quad (3.11)$$

$\{w_n\}$ is bounded, and there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup w^*$. From (3.11), we have $z_{n_j} \rightharpoonup w^*$ and $\|z_n - Tz_n\| \rightarrow 0$, then $w^* \in F(T)$. Furthermore, $u_{n_j} \rightharpoonup w^*$, and from $\lim_{n \rightarrow \infty} \|Gu_n\| = 0$, we have $\lim_{n \rightarrow \infty} \|Gu_{n_j}\| = 0$. From (3.11), $\|w_n - u_n\| \rightarrow 0$, and by the continuity of G , we get $Gw_{n_j} \rightharpoonup Gw^*$ and $\|Gw_{n_j}\| - \|Gu_{n_j}\| \leq \|G\| \|w_{n_j} - u_{n_j}\| \rightarrow 0$. Hence, $\lim_{n \rightarrow \infty} \|Gw_{n_j}\| = 0$. Furthermore, we get

$$\begin{aligned} \|Gw^*\|^p &= \langle J_p^{E_3} Gw^*, Gw^* \rangle \\ &\leq \lim_{n \rightarrow \infty} \langle J_p^{E_3} Gw^*, Gw_{n_j} \rangle \\ &\leq \lim_{n \rightarrow \infty} \|J_p^{E_3} Gw^*\| \|Gw_{n_j}\| = 0, \end{aligned}$$

thus $Gw^* = 0$.

Besides, using (3.11), we obtain from the definition of a_n and the restricted condition of α_n that

$$\begin{aligned} \|J_p^E a_n - J_p^E z_n\| &= \|J_p^E z_n - t_n J_p^E (z_n - Tz_n) - J_p^E z_n\| \\ &= t_n \|z_n - Tz_n\|^{p-1} \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} \|J_p^E w_{n+1} - J_p^E a_n\| &= \|\alpha_n J_p^E v + (1 - \alpha_n) J_p^E a_n - J_p^E a_n\| \\ &= \alpha_n \|J_p^E v - J_p^E a_n\| \rightarrow 0. \end{aligned}$$

Then from (3.11), we have

$$\|w_{n+1} - w_n\| \leq \|w_{n+1} - a_n\| + \|a_n - z_n\| + \|z_n - u_n\| + \|u_n - w_n\| \rightarrow 0. \quad (3.12)$$

From Lemma 2.5, we have

$$\begin{aligned}
 D_p(w_{n+1}, w) &= D_p(J_q^{E*}(\alpha_n J_p^E v + (1 - \alpha_n) J_p^E a_n), w) \\
 &= V_p(\alpha_n J_p^E v + (1 - \alpha_n) J_p^E a_n, w) \\
 &\leq V_p(\alpha_n J_p^E v + (1 - \alpha_n) J_p^E a_n - \alpha_n [J_p^E v - J_p^E w], w) \\
 &\quad - \langle -\alpha_n [J_p^E v - J_p^E w], J_q^{E*}[\alpha_n J_p^E v + (1 - \alpha_n) J_p^E a_n] - w \rangle \\
 &= V_p(\alpha_n J_p^E w + (1 - \alpha_n) J_p^E a_n, w) + \langle \alpha_n [J_p^E v - J_p^E w], w_{n+1} - w \rangle \\
 &\leq \alpha_n D_p(w, w) + (1 - \alpha_n) D_p(a_n, w) + \langle \alpha_n [J_p^E v - J_p^E w], w_{n+1} - w \rangle \\
 &\leq (1 - \alpha_n) D_p(z_n, w) + \langle \alpha_n [J_p^E v - J_p^E w], w_{n+1} - w \rangle \\
 &\leq (1 - \alpha_n) \left(1 + \frac{2^{p-1} c_n}{p\eta}\right) D_p(w_n, w) + (1 - \alpha_n) \frac{c_n}{q} \\
 &\quad + \langle \alpha_n [J_p^E v - J_p^E w], w_{n+1} - w \rangle \\
 &\leq (1 - \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta}\right]) D_p(w_n, w) + (1 - \alpha_n) \frac{c_n}{q} \\
 &\quad + \langle \alpha_n [J_p^E v - J_p^E w], w_{n+1} - w \rangle \\
 &\leq (1 - \alpha_n \left[1 - \frac{2^{p-1} \epsilon}{p\eta}\right]) D_p(w_n, w) + \alpha_n \left[\frac{c_n}{q\alpha_n} + \langle J_p^E v - J_p^E w, w_{n+1} - w \rangle\right].
 \end{aligned} \tag{3.13}$$

Let $w = \Pi_\Gamma v$. By Definition 2.3, we have

$$\langle l - w, J_p^E v - J_p^E w \rangle \leq 0, \quad \forall l = (l_1, l_2) \in \Gamma.$$

Since $w_{n_j} \rightharpoonup w^*$ and $c_n = o(\alpha_n)$, then from (3.12), we have

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} [\langle J_p^E v - J_p^E w, w_{n+1} - w \rangle + \frac{c_n}{q\alpha_n}] \\
 &\leq \limsup_{j \rightarrow \infty} [\langle J_p^E v - J_p^E w, w_{n_j+1} - w \rangle + \frac{c_{n_j}}{q\alpha_{n_j}}] \\
 &\leq \limsup_{j \rightarrow \infty} \langle J_p^E v - J_p^E w, w^* - w \rangle \\
 &\leq 0.
 \end{aligned} \tag{3.14}$$

Therefore, by (3.13) and Lemma 2.11, we get $\lim_{n \rightarrow \infty} D_p(w_n, w) = 0$. It follows from Lemma 2.8 that $\lim_{n \rightarrow \infty} \|w_n - w\| = 0$. Thus $\{w_n\}$ converges strongly to $w^* = \Pi_\Gamma v$.

Case 2. Assume that $D_p(w_n, w^*)$ is not monotonically decreasing. From Lemma 2.11, there exists a subsequence $\{m_i\}$ as $i \rightarrow \infty$ such that

$$\begin{aligned}
 D_p(w_{m_i}, w^*) &\leq D_p(w_{m_i+1}, w^*); \\
 D_p(w_i, w^*) &\leq D_p(w_{m_i+1}, w^*).
 \end{aligned}$$

Then by the similar proof in Case 1, we have

$$D_p(w_{m_i+1}, w) \leq (1 - \alpha_{m_i} \left[1 - \frac{2^{p-1} \epsilon}{p\eta}\right]) D_p(w_{m_i}, w) + \alpha_{m_i} \left[\frac{c_{m_i}}{q\alpha_{m_i}} + \langle J_p^E v - J_p^E w, w_{m_i+1} - w \rangle\right], \tag{3.15}$$

and

$$\limsup_{i \rightarrow \infty} [\langle J_p^E v - J_p^E w, w_{m_i+1} - w \rangle + \frac{c_{m_i}}{q\alpha_{m_i}}] \leq 0, \quad (3.16)$$

which implies that

$$\alpha_{m_i} \left(1 - \frac{2^{p-1}\epsilon}{p\eta}\right) D_p(w_{m_i}, w^*) \leq D_p(w_{m_i}, w^*) - D_p(w_{m_i+1}, w^*) + \alpha_{m_i} [\langle J_p^E v - J_p^E w, w_{m_i+1} - w \rangle + \frac{c_{m_i}}{q\alpha_{m_i}}].$$

That is,

$$\left(1 - \frac{2^{p-1}\epsilon}{p\eta}\right) D_p(w_{m_i}, w^*) \leq [\langle J_p^E v - J_p^E w, w_{m_i+1} - w \rangle + \frac{c_{m_i}}{q\alpha_{m_i}}].$$

From (3.16), we get $\lim_{i \rightarrow \infty} D_p(w_{m_i}, w^*) = 0$. And by (3.15), we obtain $\lim_{i \rightarrow \infty} D_p(w_{m_i+1}, w^*) = 0$, thus $\lim_{i \rightarrow \infty} D_p(w_i, w^*) = 0$. We conclude that $\{w_n\} \rightarrow w^* \in \Gamma$. This completes the proof. $\square$

Significantly, the algorithmic framework constructed herein maintains its innovative nature within Hilbert space settings. If E_1 , E_2 , and E_3 are assumed to be Hilbert spaces, the Bregman projection Π_Γ reduces to the classical metric projection P_Γ and the function $D_p(x, y) = D_2(x, y) = \frac{1}{2}\|x - y\|^2$. Thus, the subsequent proposition holds:

Theorem 3.2. *Let H_1 , H_2 , and H_3 be three real Hilbert spaces. Suppose that $T_1 : H_1 \rightarrow H_1$ and $T_2 : H_2 \rightarrow H_2$ are k_1, k_2 -demimetric mappings. Let $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ serve as given bounded linear operators. Define $H = H_1 \times H_2$, $w = (x, y) \in H$, and $T : H \rightarrow H$ formulated as $T = [T_1, T_2]$. $G : H \rightarrow H_3$ is defined by $G = [A, -B]$. For $v = (v_1, v_2) \in H$, $w_0, w_1 \in H$, the sequence $\{w_n\}$ is generated by*

$$\begin{cases} u_n = w_n + \theta_n(w_n - w_{n-1}), \\ z_n = u_n - \gamma_n G^* G u_n, \\ w_{n+1} = \alpha_n v + (1 - \alpha_n)[z_n - t_n(z_n - T z_n)], \end{cases}$$

for all $n \geq 0$, $\theta > 0$, and θ_n is chosen in such a way that

$$0 \leq \theta_n \leq \overline{\theta}_n = \begin{cases} \min\{\theta, \frac{c_n}{\delta_n}\} & \text{if } \delta_n \neq 0, \\ \theta & \text{otherwise,} \end{cases}$$

where $\delta_n = \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|$ and $\{c_n\}$ is a sequence of positive numbers. Let the sequences $\{\alpha_n\} \subset (0, 1)$, $\xi > 0$, and $\beta = \max\{k_1, k_2\} < 1$, and the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < c_n < \frac{1}{2}, c_n = o(\alpha_n)$;
- (iii) $\gamma_n \in (\xi, \frac{2}{\|G\|^2} - \xi)$;
- (iv) $0 < t_n < 1 - \beta$.

It follows that $\{(x_n, y_n)\}$ converges strongly to $w^* = P_\Gamma v$.

If we let $E_2 = E_3$ and we take $B = I$, then the SEFPP reduces to the SCFPP in Banach spaces. Consequently, the following theorem can be established:

Theorem 3.3. Assume that E_1 and E_2 denote two p -uniformly convex and uniformly smooth Banach spaces. Let $A : E_1 \rightarrow E_2$ be a bounded linear mapping, whose adjoint operator is denoted A^* . Let $T_1 : E_1 \rightarrow E_1$, $T_2 : E_2 \rightarrow E_2$ be k_1, k_2 -demimetric mappings, respectively. The solution set of the SCFPP is defined by $\Omega = \{x \in F(T_1) \text{ such that } Ax \in F(T_2)\}$. For $u, x_0, x_1 \in E_1$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} e_n = J_q^{E_1}[J_p^{E_1} x_n + \theta_n(J_p^{E_1} x_n - J_p^{E_1} x_{n-1})], \\ h_n = J_q^{E_1}[J_p^{E_1} e_n - \gamma_n A^* J_p^{E_2}(I - T_2)Ae_n], \\ x_{n+1} = J_q^{E_1}[\alpha_n J_p^{E_1} u + (1 - \alpha_n)[J_p^{E_1} h_n - t_n J_p^{E_1}(h_n - T_1 h_n)]. \end{cases}$$

For all $n \geq 0, \theta > 0$, θ_n is chosen in such a way that

$$0 \leq \theta_n \leq \overline{\theta}_n = \begin{cases} \min\{\theta, \frac{c_n}{\|J_p^{E_1} x_n - J_p^{E_1} x_{n-1}\|}\} & \text{if } \delta_n \neq 0, \\ \theta & \text{otherwise,} \end{cases}$$

where $\{c_n\}$ is a sequence of positive numbers. Let the sequences $\{\alpha_n\} \subset (0, 1)$, $\xi > 0$ and $\beta = \max\{k_1, k_2\} < 1$, and the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < c_n < \frac{p\eta}{2^{p-1}}, c_n = o(\alpha_n)$;
- (iii) $\gamma_n^{q-1} \in (\xi, \frac{q}{C_q \|A\|^q} - \xi)$;
- (iv) $0 < t_n < [\frac{(1-\beta)q}{2C_q}]^{\frac{1}{q-1}}$.

It follows that the iterative sequence $\{x_n\}$ converges strongly to $x^* = \Pi_{\Omega} u$.

4. Applications

4.1. Split equality variational inclusion problems

Consider the split equality variational inclusion problem (SEVIP):

$$\text{determine } x \in T_1^{-1}(0), y \in T_2^{-1}(0) \text{ satisfying } A\hat{x} = B\hat{y}.$$

For $r > 0$, the resolvent operator R_r^T corresponding to a maximal monotone mapping T is defined as $R_r^T = (J_p^E + rT)^{-1} J_p^E$. It is well known that R_r^T is single-valued, satisfies $T^{-1}0 = F(R_r^T)$, and is relatively nonexpansive. Therefore, we can apply our results by setting $T_1 = R_r^{T_1}$ and $T_2 = R_r^{T_2}$.

Corollary 4.1. Let E_1, E_2 , and E_3 be Banach spaces that are uniformly smooth and p -uniformly convex. Consider two bounded linear operators $A : E_1 \rightarrow E_3, B : E_2 \rightarrow E_3$. Let $T_1 : E_1 \rightarrow 2^{E_1^*}, T_2 : E_2 \rightarrow 2^{E_2^*}$ be maximal monotone operators such that $\Gamma = \{(x, y) \in T_1^{-1}(0) \times T_2^{-1}(0) \text{ such that } Ax =$

By} is nonempty. Let $E = E_1 \times E_2$, $w = (x, y) \in E$. Define $R_r^T = [R_r^{T_1}, R_r^{T_2}]$ and $G = [A, -B]$. Let $v = (v_1, v_2) \in E$, $w_0, w_1 \in E$, and the sequence $\{w_n\}$ is generated by

$$\begin{cases} u_n = J_q^{E^*} [J_p^E w_n + \theta_n (J_p^E w_n - J_p^E w_{n-1})], \\ z_n = J_q^{E^*} [J_p^E u_n - \gamma_n G^* J_p^{E_3} G u_n], \\ w_{n+1} = J_q^{E^*} [\alpha_n J_p^E v + (1 - \alpha_n) [J_p^E z_n - t_n J_p^E (z_n - R_r^T z_n)]]. \end{cases}$$

For all $n \geq 0$, $\theta > 0$, θ_n is chosen in such a way that

$$0 \leq \theta_n \leq \overline{\theta}_n = \begin{cases} \min\{\theta, \frac{c_n}{\delta_n}\} & \text{if } \delta_n \neq 0, \\ \theta & \text{otherwise,} \end{cases}$$

where $\delta_n = \|J_p^{E_1} x_n - J_p^{E_1} x_{n-1}\| + \|J_p^{E_2} y_n - J_p^{E_2} y_{n-1}\|$ and $\{c_n\}$ is a sequence of positive numbers. Let the sequences $\{\alpha_n\} \subset (0, 1)$, $\xi > 0$, and the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < c_n < \frac{p\eta}{2^{p-1}}, c_n = o(\alpha_n)$;
- (iii) $\gamma_n^{q-1} \in (\xi, \frac{q\|Gu_n\|^p}{C_q\|G^* J_p^{E_3} Gu_n\|^q} - \xi)$;
- (iv) $0 < t_n < [\frac{q}{C_q}]^{\frac{1}{q-1}}$.

The generated sequence $\{(x_n, y_n)\}$ converges strongly to $w^* = \Pi_{\Gamma} v$.

4.2. Split equality equilibrium problems

Let C be a nonempty, closed, and convex subset of a p -uniformly convex and uniformly smooth real Banach space E . Let $f : C \times C \rightarrow \mathbb{R}$ be a bi-function satisfying the following conditions:

- (1) $f(x, x) = 0$ for all $x \in C$;
- (2) $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;
- (3) for each $x, y, z \in C$, $\lim_{t \rightarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (4) for all $x \in C$, $f(x, y)$ is convex and lower semicontinuous.

The associated equilibrium problem (EP) aims to find $x \in C$ such that

$$f(x, y) \geq 0, \forall y \in C,$$

then $EP(f) = F(T_r^f)$, where $EP(f)$ is the solution set of the EP and $T_r^f : E \rightarrow 2^C$ is defined as $T_r^f = \{u \in C, f(u, y) + \frac{1}{r} \langle y - u, J_p^E u - J_p^E x \rangle \geq 0, \forall y \in C\}$. It is well known that the single-valued mapping T_r^f is firmly non-expansive and $EP(f)$ is a closed and convex set. Thus, by setting $T_1 = T_r^f$ and $T_2 = T_r^g$ in Theorem 3.1, we obtain the following result.

Corollary 4.2. Let E_1 , E_2 , and E_3 be p -uniformly convex and uniformly smooth Banach spaces. Suppose $C \subset E_1$ and $Q \subset E_2$ are nonempty, closed, and convex subsets, respectively. Let $A : E_1 \rightarrow E_3$,

$B : E_2 \rightarrow E_3$ be two bounded linear operators. Let $f : C \times C \rightarrow R$ and $g : Q \times Q \rightarrow R$ be bifunctions satisfying conditions (1)–(4) such that $\Omega = \{(x, y) \in EP(f) \times EP(g) \text{ such that } Ax = By\}$ is nonempty. Let $E = E_1 \times E_2$, $w = (x, y) \in E$. Define $T_r = [T_r^f, T_r^g]$ and $G = [A, -B]$. Let $v = (v_1, v_2) \in E$, $w_0, w_1 \in E$, and the sequence $\{w_n\}$ is generated by

$$\begin{cases} u_n = J_q^{E*} [J_p^E w_n + \theta_n (J_p^E w_n - J_p^E w_{n-1})], \\ z_n = J_q^{E*} [J_p^E u_n - \gamma_n G^* J_p^{E_3} G u_n], \\ w_{n+1} = J_q^{E*} [\alpha_n J_p^E v + (1 - \alpha_n) [J_p^E z_n - t_n J_p^E (z_n - T_r z_n)]]. \end{cases}$$

For all $n \geq 0$, $\theta > 0$, θ_n is chosen in such a way that

$$0 \leq \theta_n \leq \overline{\theta}_n = \begin{cases} \min\{\theta, \frac{c_n}{\delta_n}\} & \text{if } \delta_n \neq 0, \\ \theta & \text{otherwise,} \end{cases}$$

where $\delta_n = \|J_p^{E_1} x_n - J_p^{E_1} x_{n-1}\| + \|J_p^{E_2} y_n - J_p^{E_2} y_{n-1}\|$ and $\{c_n\}$ is a sequence of positive numbers. Let the sequences $\{\alpha_n\} \subset (0, 1)$, $\xi > 0$, and the following assumptions hold:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < c_n < \frac{p\eta}{2^{p-1}}$, $c_n = o(\alpha_n)$;
- (iii) $\gamma_n^{q-1} \in (\xi, \frac{q\|Gu_n\|^p}{C_q\|G^* J_p^{E_3} Gu_n\|^q} - \xi)$;
- (iv) $0 < t_n < [\frac{q}{C_q}]^{\frac{1}{q-1}}$.

The generated sequence $\{(x_n, y_n)\}$ is strongly convergent to $w^* = \Pi_{\Omega} v$.

5. Numerical example

In this section, we present several numerical examples to demonstrate the computational performance and applicability of our new algorithm. All experiments are implemented in Python 2020 on a laptop with an Intel(R) Core(TM) i7-4710MQ CPU @ 2.50GHz and RAM 8.00GB.

Example 5.1. Let $E_1 = E_2 = E_3 = l_p$, endowed with the norm $\|x\|_p = (\sum |x_i|^p)^{\frac{1}{p}}$ and let $g_p(x) = \frac{1}{p}\|x\|_p^p$. The corresponding generalized duality mappings are given by $(J_p^E x)_i = |x_i|^{p-2} x_i = \text{sgn}(x_i)|x_i|^{p-1}$ and $(J_q^{E*} u)_i = |u_i|^{q-2} u_i = \text{sgn}(u_i)|u_i|^{q-1}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let $k_1 = \frac{1}{4}$ and $k_2 = \frac{1}{7}$, and choose $x^* = (2, 2, 2, 0, \dots)$, $y^* = (1, 1, 1, 0, \dots)$. Define $T_1(x) = \frac{5}{8}x + \frac{3}{8}x^*$ and $T_2(y) = \frac{4}{7}y + \frac{3}{7}y^*$. More generally, for $k \in (-1, 1)$, consider $T_k(x) = \frac{1+k}{2}x + \frac{1-k}{2}x^*$. Let $d = \frac{1-k}{2}$, then $x - T_k x = d(x - x^*)$. By the homogeneity of the generalized duality mapping, we have

$$J_p^E(x - T_k x) = J_p^E(d(x - x^*)) = d^{p-1} J_p^E(x - x^*).$$

Therefore,

$$\langle x - x^*, J_p^E(x - T_k x) \rangle = d^{p-1} \|x - x^*\|^p,$$

while

$$\frac{1-k}{2}\|x - T_k x\|^p = d^{p+1}\|x - x^*\|^p.$$

Since $0 < d < 1$, it follows that

$$\frac{1-k}{2}\|x - T_k x\|^p \leq \langle x - x^*, J_p^E(x - T_k x) \rangle.$$

Hence T_1 and T_2 are k_1 and k_2 -demimetric mappings, respectively. Moreover, $F(T_1) = \{x^*\}$, $F(T_2) = \{y^*\}$. Define the bounded linear operators $Ax = 3x$, $By = 6y$, then $Ax^* = By^*$, thus $(x^*, y^*) \in \Gamma$. We choose $\alpha_n = \frac{1}{10n+1}$, $c_n = \frac{1}{(n+1)^2}$, and set $\eta = 1$, $\theta = 0.01$, $\beta = \frac{1}{4}$, and $\xi = 0.001$. It is clear that $\alpha_n \rightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Notice that $\frac{c_n}{\alpha_n} = \frac{10n+1}{(n+1)^2} \rightarrow 0$, thus we have $c_n = o(\alpha_n)$. For the self-adaptive equality correction, define $R_n = \frac{q\|Gu_n\|_p^p}{C_q\|G^*J_p^{E_3}Gu_n\|_q^q}$. Whenever $Gu_n \neq 0$, to select a concrete stepsize from the admissible interval in condition (iii), we choose an interior point according to $\gamma_n^{q-1} = \xi + \rho(R_n - 2\xi)$, $\rho \in (0, 1)$. In the numerical experiments, we set $\rho = 0.5$, corresponding to the midpoint of the admissible interval. Hence the parameters satisfy the assumptions of Algorithm (3.1). If $Gu_n = 0$, the Landweber correction vanishes, and we simply set $z_n = u_n$. To examine the robustness of the proposed method with respect to different initial values, we consider four representative configurations involving zero components, relatively large values, substantial differences between the two components, and negative entries. The initial points are chosen as follows:

Case 1: $w_0 = ((3, 2, 0, 1, 0, \dots), (1, -1, 2, 0, \dots))$, $w_1 = ((-3, 1, 2, 0, \dots), (3, 0, 1, 0, \dots))$.

Case 2: $w_0 = ((4, 5, 1, 0, \dots), (1, 2, 1, 0, \dots))$, $w_1 = ((4, 1, 1, 0, \dots), (7, 4, 1, 0, \dots))$.

Case 3: $w_0 = ((1, 8, 1, 0, \dots), (7, 2, 1, 0, \dots))$, $w_1 = ((2, 8, 2, 0, \dots), (7, 4, 1, 0, \dots))$.

Case 4: $w_0 = ((-4, 3, 2, 0, \dots), (-2, 6, 1, 0, \dots))$, $w_1 = ((-1, 5, 3, 0, \dots), (0, 4, 2, 0, \dots))$.

If $p = 2$, then $q = 2$ and $C_q = 1$. For the conditions (iii) and (iv) in Algorithm (3.1), we have $\gamma_n \in (0.001, 0.04344)$, $0 < t_n < \frac{9}{32} \approx 0.281$. To select a concrete stepsize from this admissible interval, we choose its midpoint. Thus, for the present example, $\gamma_n \approx 0.02222$. This choice is obtained directly from the admissible interval in condition (iii) and therefore does not require any prior knowledge of the norm of the operator G . In the numerical implementation, we take $t_n = \frac{t_0}{(n+1)^b}$, where $t_0 = 0.2500$, $b = 0.02$, which meet the specified conditions. The detailed numerical data and convergence behavior for the above four cases are provided in Table 1 and Figure 1, respectively. For the numerical evaluation, we use $Err_n = (\|x - x^*\|_2^2 + \|y - y^*\|_2^2)^{\frac{1}{2}}$ as the approximation error.

Table 1. Numerical errors Err_n for Example 5.1 with $p = 2$.

iter	Case 1	Case 2	Case 3	Case 4
1	3.447899	3.624470	5.693359	4.253577
2	3.117700	3.043521	4.765845	3.632942
10	1.524531	1.116121	1.653411	1.432536
100	0.049404	0.049169	0.049028	0.049214
1000	0.004656	0.004656	0.004656	0.004656
10000	0.000483	0.000483	0.000483	0.000483

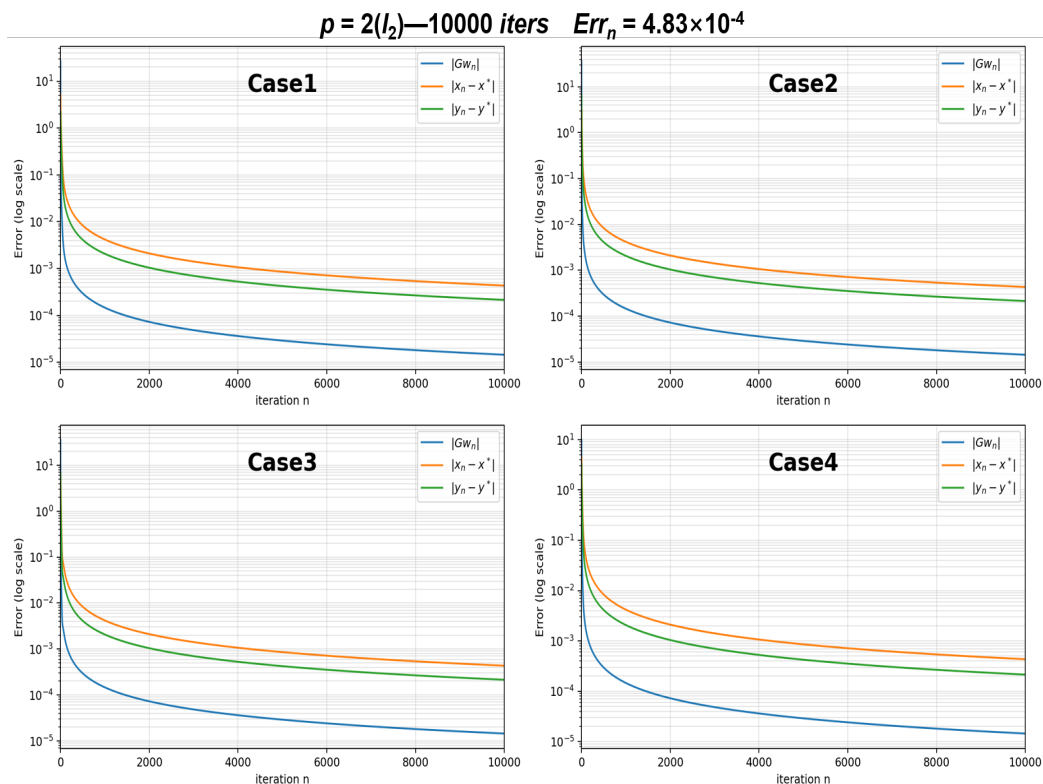


Figure 1. Error plots of each cases with $p = 2$: $\varepsilon = 10^{-4}$.

If $p = 3$, then $q = \frac{3}{2}$ and $C_q = 0.5$. For the conditions (iii) and (iv) in Algorithm (3.1), we have $\gamma_n \in (10^{-6}, 0.022442)$ and $0 < t_n < 0.6495^{0.5} \approx 0.806$. By choosing the midpoint of the admissible interval for $\gamma_n^{\frac{1}{2}}$, we obtain $\gamma_n \approx 0.005686$. Then we take $t_n = \frac{t_0}{(n+1)^b}$, where $t_0 = 0.7000, b = 0.02$, which meet the specified conditions. Table 2 and Figure 2 present the numerical results. For $p = 3$, the approximation error is defined by $Err_n = (\|x - x^*\|_3^3 + \|y - y^*\|_3^3)^{\frac{1}{3}}$.

Table 2. Numerical errors Err_n for Example 5.1 with $p = 3$.

iter	Case 1	Case 2	Case 3	Case 4
1	4.445158	3.920128	5.858748	3.518672
10	3.887642	2.442810	3.749665	2.145545
100	0.384731	0.498607	0.627155	0.477706
500	0.153075	0.132062	0.122565	0.123843
1000	0.109988	0.100867	0.086238	0.097141
10000	0.034170	0.034170	0.034170	0.034170

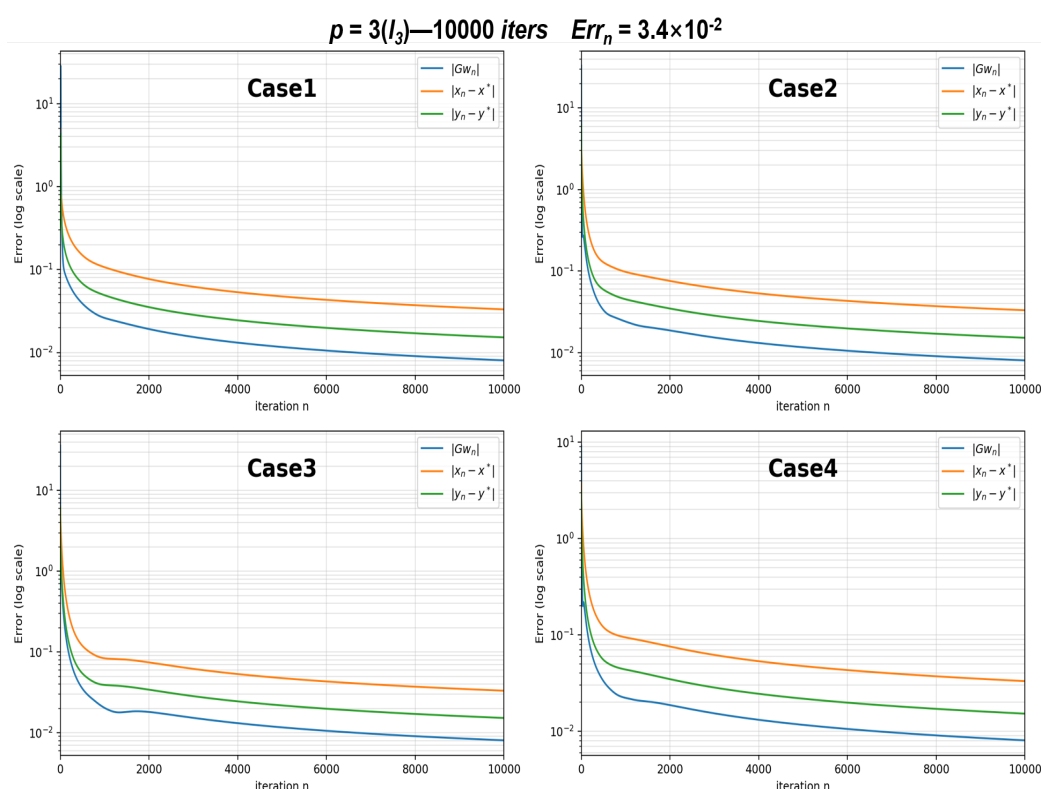


Figure 2. Error plots of each cases with $p = 3$: $\varepsilon = 10^{-2}$.

Comparison with an existing method.

To further evaluate the computational performance of the proposed method, we compare it with the simultaneous iterative algorithm of Wang and Kim [10] for the SEFP of demicontractive mappings. The Wang–Kim method is selected because it addresses the same split equality fixed point structure and establishes strong convergence in real Hilbert spaces. Since the method in [10] is developed in a Hilbert space setting, the comparison is carried out for $p = 2$, which provides a natural common framework for the two methods. We use Case 2 as a representative test problem. The proposed inertial algorithm requires two initial points w_0 and w_1 , whereas the Wang–Kim method requires only one initial point. Therefore, the latter is initialized at w_1 , which is the current iterate used to start the inertial process of the proposed method. The same problem data and the same stopping criterion $Err_n < 10^{-3}$ are adopted for both methods.

For the Wang–Kim method, since $p = 2$, the underlying space reduces to a Hilbert space. Moreover, for $A = 3I$ and $B = 6I$, we have $\lambda_A = 9$, $\lambda_B = 36$. Hence, the admissible stepsize satisfies $0 < \gamma_k < \frac{2}{\lambda_A + \lambda_B} = \frac{2}{45}$. In the numerical implementation, we choose $\gamma_k = 0.02$. In addition, we take $t_k = \frac{1}{k+2}$, $\alpha_k = \beta_k = 0.5$. It is clear that $t_k \rightarrow 0$, $\sum_{k=0}^{\infty} t_k = \infty$, and therefore these parameter choices satisfy the assumptions of the Wang–Kim algorithm. The corresponding comparative numerical results are shown in Table 3. In contrast to the Wang–Kim method, whose admissible stepsize requires spectral information on A^*A and B^*B , the stepsize of the proposed method is determined adaptively without requiring such prior information.

Table 3. Comparison between our proposed method and the Wang–Kim method.

Method	Iterations	Mean CPU (s)	Final error	Final residual
our proposed method	4768	0.470337	0.001	0.000556
Wang–Kim method	16212	0.649618	0.001	0.000621

Sensitivity analysis.

To further investigate the robustness of the proposed algorithm with respect to the choice of parameters, we examine the influence of the inertial upper bound θ and the adaptive stepsize selection parameter ρ . The numerical tests are carried out for Case 2 with $p = 2$, while all the other parameters and the stopping criterion are kept unchanged. In particular, we consider $\theta \in \{0, 0.01, 0.05, 0.10, 0.30\}$ and $\rho \in \{0.25, 0.50, 0.75\}$. The corresponding numerical results are reported in Table 4.

Table 4. Sensitivity of the proposed method with respect to the inertial and adaptive stepsize parameters.

θ	Iterations	Max. θ_n	Final error	Final residual
0.00	4768	0.000000	9.9994×10^{-4}	5.5640×10^{-4}
0.01	4768	0.010000	9.9992×10^{-4}	5.5639×10^{-4}
0.05	4768	0.050000	9.9983×10^{-4}	5.5634×10^{-4}
0.10	4767	0.100000	9.9992×10^{-4}	5.5639×10^{-4}
0.30	4766	0.161785	9.9998×10^{-4}	5.5642×10^{-4}
ρ	Iterations	Final γ_n	Final error	Final residual
0.25	4769	0.011611	9.9995×10^{-4}	5.7905×10^{-4}
0.50	4768	0.022222	9.9992×10^{-4}	5.5639×10^{-4}
0.75	4768	0.032833	9.9983×10^{-4}	5.4744×10^{-4}

Remark 5.2. The parameter choices used in the above numerical experiments are consistent with the assumptions of the proposed convergence theorem, and the numerical results agree well with the theoretical analysis. Tables 1 and 2, together with Figures 1 and 2, show that the sequences generated from the four different initial configurations exhibit stable convergence behavior for both $p = 2$ and $p = 3$. Although the initial errors are substantially different, the four cases display essentially the same asymptotic behavior after a sufficiently large number of iterations, which indicates that the proposed algorithm is robust with respect to the choice of initial points.

For $p = 2$, after 10^4 iterations, the generated solution is close to $w^* = ((2, 2, 2, 0, \dots), (1, 1, 1, 0, \dots))$, and the corresponding approximation error is approximately 4.83×10^{-4} . Moreover, the split equality residual $\|Gw_n\|$ is approximately 1.44×10^{-5} . For $p = 3$, the approximation error at $n = 10^4$ is approximately 3.42×10^{-2} , while the split equality residual is approximately 8.04×10^{-3} . Although the convergence in the Banach space case $p = 3$ is slower than that observed in the Hilbert space case $p = 2$, the monitored quantities $\|Gw_n\|$, $\|x_n - x^*\|$, and $\|y_n - y^*\|$ decrease steadily in both settings. This illustrates the applicability of the proposed method in both Hilbert and Banach spaces.

The comparison results reported in Table 3 further demonstrate the computational effectiveness of the proposed method. Under the same test problem and stopping criterion, the proposed method

reaches the prescribed tolerance after 4768 iterations, whereas the Wang–Kim method requires 16212 iterations. Hence, the number of iterations is reduced by approximately 70.6%. In addition, the final residual obtained by the proposed method is slightly smaller than that of the comparison method. These results indicate that the proposed self-adaptive inertial strategy is competitive for the present test problem.

Finally, the sensitivity results in Table 4 show that moderate variations in the inertial upper bound θ and in the adaptive stepsize selection parameter ρ lead to only minor changes in the iteration numbers and final residuals. This indicates that the proposed algorithm is relatively insensitive to these parameter choices and that the adaptive safeguards provide stable parameter control during the iterative process.

6. Conclusions

This paper develops a self-adaptive inertial approximation framework for solving the SEFPP involving k -demimetric mappings in p -uniformly convex and uniformly smooth real Banach spaces. The proposed framework combines an adaptive inertial prediction mechanism, a Landweber-type correction procedure for the split equality constraint, and a k -demimetric fixed point approximation step. The inertial strategy provides additional information from previous iterations, while the adaptive parameter selection improves the flexibility of the algorithm. Under mild conditions, we establish a strong convergence theorem for the proposed algorithm. Moreover, our approach extends the existing theoretical framework by generalizing the solution space from Hilbert spaces to more general Banach spaces, and by considering a broader class of operators, namely k -demimetric mappings, which unifies many common nonlinear operators such as nonexpansive, quasi-nonexpansive, and strictly pseudocontractive mappings. Importantly, the proposed algorithmic framework is also successfully applied to solve the split equality equilibrium problem and the SEVIP, further demonstrating its wide applicability and theoretical significance. Finally, the numerical experiments have fully verified the stability and convergence of our algorithm.

Author contributions

Chanjuan Pan: Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing-original draft preparation, writing-review and editing, visualization; Yan Zhang: Methodology, formal analysis, writing-review and editing; Panpan Zhu: investigation, resources, supervision, project administration; Kunyang Wang: Software, validation, data curation, visualization, supervision, project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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