



Research article

Bifurcation structure, resonance dynamics, and stabilization in a capacity-limited honeybee–varroa map

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Abstract: We extend a previously studied two-dimensional honeybee–*Varroa destructor* census model by introducing a capacity-limited mite-removal term. For the continuous system, positivity and dissipativity are established, while for the forward Euler map, a conditional invariant simplex and coexistence equilibria are characterized. Boundary analysis distinguishes saddle-node creation of bee-only equilibria from a transcritical mite-invasion threshold. At coexistence, variation of the census interval produces nondegenerate supercritical Neimark–Sacker and period-doubling bifurcations. Two-parameter analysis identifies candidate 1:2, 1:3, and 1:4 strong resonances and their nondegeneracy indicators. An equilibrium-targeting correction provides a local stabilization boundary. Comparisons across census intervals and discretization schemes reveal scheme-dependent global dynamics. The results therefore provide a theoretical bifurcation framework, not a calibrated prediction of colony dynamics.

Keywords: nonlinear systems; bifurcation; strong resonance; sensitivity analysis; simulation

Mathematics Subject Classification: 39A28, 39A30

1. Introduction

Mathematical descriptions of honeybee colonies range from compact population models to age-structured, seasonal, epidemiological, network-based, and individual-based systems. Continuous models have been used to study colony demography [1], parasite invasion [2], virus transmission [3,4], brood-stage structure [5], and seasonal forcing [6]; a honeybee-focused review is available in [7], while a recent systematic survey placed honeybee models within the broader mathematical literature on pollinator dynamics [8]. Network models have also been used to examine testing and hive-removal strategies for the spread of *Varroa destructor* between colonies [9]. Difference-equation models are appropriate when observations or demographic events are organized by census periods. Early examples include the population maps in [10, 11]. More recent computations have revealed periodic windows, multistability, and intricate parameter-plane structures in a closely related bee–mite map [12]. Recent studies of discrete nonlinear maps also emphasized the breadth of dynamics that can arise in low-dimensional and higher-dimensional constructions, including oscillatory memristive hyperchaotic maps and systematic four-dimensional map-generation frameworks [13, 14]. These engineering models are not biological comparators for the present system, but they illustrate the broader current interest in complex dynamics, multistability, and hyperchaotic behavior in discrete nonlinear maps. Consequently, complicated numerical orbits are treated here as background behavior rather than as a novelty claim.

Relative to the bee–mite map of Karmakar and Pal [12], the contribution here is not the mere observation of complicated orbits. The new mathematical content consists of: (i) a capacity-limited mite-removal mechanism and associated positivity/dissipativity results for the continuous system; (ii) a conditional positive invariant simplex for the Euler census map; (iii) center-manifold classifications of the bee-only saddle-node and mite-invasion transcritical thresholds; (iv) explicit codimension-one map normal-form coefficients for Neimark–Sacker and period-doubling bifurcations; (v) a two-parameter calculation of candidate 1:2, 1:3, and 1:4 strong resonances, together with their leading resonant coefficients; and (vi) a local equilibrium-targeting stabilization criterion and a direct comparison of discretization schemes. The biological representation is intentionally low-dimensional: brood stages, age structure, explicit virus transmission, seasonality, environmental stochasticity, observation error, and mechanistic hygienic-behavior pathways are omitted, and the new removal term is phenomenological rather than empirically calibrated. These simplifications are adopted to keep the local bifurcation mechanisms analytically traceable and, therefore, limit the model to qualitative dynamical interpretation.

The present model isolates a particular removal mechanism. Hygienic behavior can reduce mite reproduction [15] or remove damaged brood [16, 17], and empirical studies have documented variation in brood-removal responses [18, 19] and mite-directed behavior [20, 21]. In a previous infestation model that was structured, grooming and hygienic behavior were included [22], whereas, in the current low-dimensional model, a removal function serves as single saturation to allow for bifurcation analysis. The relation between hygienic traits, mite non-reproduction, and infestation of colonies may be tenuous or dependent on specific circumstances [23, 24]. The removal function is thus a capacity constraint and not a behavioral law based on experiments.

The mathematical analysis has four parts. First, the fixed points on the boundary are determined using center-manifold techniques: A recruitment threshold produces two bee-only states via a saddle-

node bifurcation, while an invasion threshold produces a transcritical bifurcation from the top of the mite-free branch to a coexistence branch. Second, the census period is a bifurcation parameter at coexistence, where spectral thresholds and cubic normal-form coefficients are found for Neimark–Sacker and flip bifurcations. Third, the manipulation parameter and the census time step yield a fixed two-parameter unfolding with candidates of 1:2, 1:3, and 1:4 resonant points found numerically. The 1:2 problem is brought down to a cubic Jordan canonical form. Fourth, an equilibrium targeting the correction of all local multipliers gives a stability limit far from the non-hyperbolic equality problem.

The local results are derived according to the map normal-form approach of [25–27]; invariant-manifold definitions follow [28]. Related resonance ratios also arise in modal-coupling problems; for example, auto-parametric longitudinal–transverse resonance in flexible beams has been analyzed through coupled stability and bifurcation calculations [29]. The mechanism there is physically and mathematically different from the strong resonances of the fixed-point multipliers studied here, so the comparison is contextual rather than an identification of the two phenomena. The numerical results are interpreted with caution. A positive finite-time Lyapunov exponent is taken to indicate sensitive dynamics without proving a chaotic invariant set, while high-order resonances of the Euler map are not taken as calibrated colony results.

2. Model and elementary properties

2.1. Continuous model

Let $H(t) \geq 0$ and $M(t) \geq 0$ denote scaled bee and mite abundances. We consider

$$\dot{H} = F(H, M) = \frac{rH^2}{k + H^2} - d_1H - \frac{aHM}{b + cH}, \quad (2.1)$$

$$\dot{M} = G(H, M) = \frac{\sigma aHM}{b + cH} - d_2M - \frac{\gamma HM}{(K_g + H)(1 + \rho M)}. \quad (2.2)$$

The parameters satisfy

$$r, k, d_1, a, b, \sigma, d_2, \gamma, K_g > 0, \quad c, \rho \geq 0.$$

The first term in (2.1) is quadratic at low bee abundance and approaches the recruitment ceiling r . The transfer term is Holling type II. In the final term of (2.2), $H/(K_g + H)$ represents saturating bee participation, while $(1 + \rho M)^{-1}$ makes total removal capacity limited as mite burden increases. Setting $\gamma = 0$ recovers the continuous model underlying the Euler map studied by Karmakar and Pal [12]; setting $\rho = 0$ retains removal without burden-dependent saturation.

Unless stated otherwise, the scaled, dimensionless baseline values are $k = 0.5$, $d_1 = 0.5$, $a = 3.2$, $b = 4.0$, $\sigma = 0.4$, $d_2 = 0.2$, $\gamma = 0.018$, $K_g = 0.35$, and $\rho = 0.45$. The legacy values and the ranges of r and c are retained from Karmakar and Pal [12] to permit benchmark comparison; γ , K_g , and ρ are illustrative values introduced for the capacity-limited removal term. The analysis is exploratory and these parameters are not estimated from colony data.

Theorem 2.1 (Positivity and dissipativity). *For each $(H_0, M_0) \in \mathbb{R}_+^2$, system (2.1)–(2.2) has a unique global non-negative solution. If $W = \sigma H + M$ and $\delta = \min\{d_1, d_2\}$, then*

$$W(t) \leq W(0)e^{-\delta t} + \frac{\sigma r}{\delta}(1 - e^{-\delta t}). \quad (2.3)$$

Proof. The vector field is continuously differentiable in a neighborhood of every point of $\mathbb{R}_+^2$. Since $F(0, M) = 0$ and $G(H, 0) = 0$, uniqueness prevents a non-negative solution from crossing either coordinate axis. Moreover,

$$\begin{aligned}\dot{W} &= \sigma \frac{rH^2}{k+H^2} - \sigma d_1 H - d_2 M - \frac{\gamma HM}{(K_g + H)(1 + \rho M)} \\ &\leq \sigma r - \delta W.\end{aligned}$$

The scalar comparison principle gives (2.3). Boundedness rules out finite-time blow-up and hence extends the local solution to all $t \geq 0$. $\square$

2.2. Census maps

For a census interval $h > 0$, the forward Euler map is

$$H_{n+1} = H_n + hF(H_n, M_n), \quad (2.4)$$

$$M_{n+1} = M_n + hG(H_n, M_n). \quad (2.5)$$

We write $X_{n+1} = \Phi_h(X_n)$, where $X = (H, M)^T$. On bounded time intervals, the Euler method has first-order convergence as $h \rightarrow 0$ [30]. For moderate or large h , however, (2.4)-(2.5) is interpreted as a nonlinear census model rather than as a quantitatively accurate approximation of the flow.

Theorem 2.2 (Conditional invariant simplex). *Let $W_n = \sigma H_n + M_n$, $\bar{W} = \max\{W_0, \sigma r / \delta\}$, and $V = \bar{W}$. If*

$$h\delta \leq 1, \quad h\left(d_1 + \frac{aV}{b}\right) \leq 1, \quad h(d_2 + \gamma) \leq 1,$$

then

$$\Omega_{\bar{W}} = \{(H, M) \in \mathbb{R}_+^2 : \sigma H + M \leq \bar{W}\}$$

is positively invariant under (2.4)-(2.5).

Proof. The first update can be written as

$$H_{n+1} = H_n \left[1 + h \left(\frac{rH_n}{k+H_n^2} - d_1 - \frac{aM_n}{b+cH_n} \right) \right].$$

On $\Omega_{\bar{W}}$, $M_n \leq V$, so the bracket is bounded below by $1 - h(d_1 + aV/b)$. Similarly, the bracket in the mite update is bounded below by $1 - h(d_2 + \gamma)$. Hence, both components remain non-negative. Cancellation of the transfer terms gives

$$W_{n+1} \leq (1 - h\delta)W_n + h\sigma r \leq \bar{W}.$$

The conclusion follows by induction. $\square$

For comparison, we also use the positivity-preserving exponential map

$$H_{n+1} = H_n \exp \left(h \left[\frac{rH_n}{k+H_n^2} - d_1 - \frac{aM_n}{b+cH_n} \right] \right),$$

$$M_{n+1} = M_n \exp \left(h \left[\frac{\sigma a H_n}{b + c H_n} - d_2 - \frac{\gamma H_n}{(K_g + H_n)(1 + \rho M_n)} \right] \right).$$

Besides guaranteeing positivity, this map provides a practical diagnostic benchmark for numerical census dynamics. It prevents sign changes caused solely by a large explicit Euler increment and therefore separates orbit features that persist under a positivity-preserving update from features that are strongly tied to the Euler discretization. We do not claim that the exponential map is a higher-order approximation of the continuous flow; its role is to test the robustness of qualitative classifications when the update rule is changed while the per-capita rates are retained. For strictly positive initial data, both components remain strictly positive at every finite iterate; if either component is initially zero, the multiplicative update leaves that component as zero. The exponential map has the same positive fixed points as the differential system. At each positive equilibrium, the Jacobian of both census maps is $I + hJ_c$, where J_c is the continuous Jacobian. Thus, their local linearizations agree, whereas their global dynamics can differ substantially.

Remark 2.3 (Local bifurcation validity versus global admissibility). *At an interior positive equilibrium, the local bifurcation statements concern the algebraic map in a sufficiently small neighborhood of that equilibrium. Continuity ensures that such a neighborhood can be chosen inside the positive quadrant even when the global sufficient conditions of Theorem 2.2 fail. Hence, a local Neimark–Sacker, period-doubling, or resonance calculation can remain mathematically valid for the discrete map at a large value of h . This does not imply that long forward orbits are biologically admissible: outside the local neighborhood, a large Euler step may generate negative components, numerical blow-up, or other discretization-induced behavior. We therefore distinguish between local map bifurcation validity and global physical admissibility of simulated population trajectories.*

3. Equilibria and linear stability

The extinction equilibrium is $E_0 = (0, 0)$. On the invariant boundary $M = 0$, positive equilibria satisfy

$$d_1 H^2 - rH + d_1 k = 0.$$

Define

$$\Delta_b = r^2 - 4d_1^2 k, \quad H_{\pm} = \frac{r \pm \sqrt{\Delta_b}}{2d_1}, \quad E_{\pm} = (H_{\pm}, 0).$$

They exist as distinct equilibria when $r > 2d_1\sqrt{k}$ and coalesce at equality.

For $H, M > 0$, the first equilibrium equation gives

$$M = B(H) := \frac{b + cH}{a} \left(\frac{rH}{k + H^2} - d_1 \right). \quad (3.1)$$

When $\Delta_b > 0$, $B(H) > 0$ precisely for $H \in I_b = (H_-, H_+)$. Define

$$\begin{aligned} Z(H) &= k + H^2 + \frac{\rho(b + cH)}{a} [rH - d_1(k + H^2)], \\ P(H) &= [\sigma a H - d_2(b + cH)](K_g + H)Z(H) - \gamma H(b + cH)(k + H^2). \end{aligned} \quad (3.2)$$

Theorem 3.1 (Coexistence characterization). Assume $\Delta_b > 0$. A point (H_*, M_*) is a positive equilibrium if and only if

$$H_* \in (H_-, H_+), \quad P(H_*) = 0, \quad M_* = B(H_*) > 0.$$

The polynomial P has degree at most five. Consequently, if P is not the zero polynomial, the system has at most five positive equilibria.

Proof. For $H \in I_b$, the factors $b + cH$, $k + H^2$, $K_g + H$, and $1 + \rho B(H)$ are strictly positive. Substitution of (3.1) into the second per-capita equilibrium equation and multiplication by these positive factors yields (3.2); every algebraic step is reversible on I_b , so no admissible root is introduced or lost. Direct expansion gives $\deg P \leq 5$. A nonzero polynomial of degree at most five has at most five real roots. $\square$

Remark 3.2. The exceptional identity $P \equiv 0$ requires simultaneous cancellation of all six polynomial coefficients and therefore lies in a proper algebraic subset of parameter space. Direct coefficient evaluation excludes it for the baseline parameters and for every parameter path used below.

The continuous Jacobian is

$$J_c(H, M) = \begin{pmatrix} F_H & F_M \\ G_H & G_M \end{pmatrix},$$

where, writing $Q(H) = b + cH$ and

$$q(H, M) = \frac{\sigma a H}{Q(H)} - d_2 - \frac{\gamma H}{(K_g + H)(1 + \rho M)},$$

we have

$$\begin{aligned} F_H &= \frac{2rkH}{(k + H^2)^2} - d_1 - \frac{abM}{Q(H)^2}, \\ F_M &= -\frac{aH}{Q(H)}, \\ G_H &= M \left[\frac{\sigma ab}{Q(H)^2} - \frac{\gamma K_g}{(K_g + H)^2(1 + \rho M)} \right], \\ G_M &= q(H, M) + \frac{\gamma \rho H M}{(K_g + H)(1 + \rho M)^2}. \end{aligned}$$

At a fixed point, $A_h = D\Phi_h = I + hJ_c$. If $T = \text{tr} J_c$ and $D = \det J_c$, then

$$\tau = \text{tr} A_h = 2 + hT, \quad \Delta = \det A_h = 1 + hT + h^2 D.$$

The fixed point is Schur stable if and only if

$$1 - \tau + \Delta > 0, \quad 1 + \tau + \Delta > 0, \quad 1 - \Delta > 0,$$

which is the second-order Jury criterion [31].

4. Boundary bifurcations

4.1. Saddle-node creation of bee-only equilibria

On $M = 0$, the census map is

$$\Psi_h(H; r) = H + h \left(\frac{rH^2}{k + H^2} - d_1 H \right). \quad (4.1)$$

Set

$$r_{\text{SN}} = 2d_1\sqrt{k}, \quad H_{\text{SN}} = \sqrt{k}, \quad \mu = r - r_{\text{SN}}, \quad u = H - H_{\text{SN}}.$$

Theorem 4.1 (Boundary saddle-node). *At $(H, r) = (H_{\text{SN}}, r_{\text{SN}})$, the restriction (4.1) has the local form*

$$u_{n+1} = u_n + a_{\text{SN}}\mu + b_{\text{SN}}u_n^2 + O(|u_n|^3 + |\mu u_n| + \mu^2), \quad (4.2)$$

where

$$a_{\text{SN}} = \frac{h}{2} > 0, \quad b_{\text{SN}} = -\frac{hd_1}{2\sqrt{k}} < 0.$$

Hence, a generic saddle-node bifurcation occurs. If the transverse mite multiplier is not on the unit circle, the same conclusion holds for the full planar map on a one-dimensional center manifold.

Proof. At the critical point, $\Psi_h - H = 0$ and $\partial_H \Psi_h = 1$. Direct differentiation gives

$$\partial_r \Psi_h = \frac{hH^2}{k + H^2} = \frac{h}{2}, \quad \frac{1}{2} \partial_{HH} \Psi_h = -\frac{hd_1}{2\sqrt{k}}.$$

Both coefficients are nonzero, which gives (4.2). For $\mu > 0$, the two branches satisfy

$$u = \pm \sqrt{-\frac{a_{\text{SN}}}{b_{\text{SN}}}\mu} + O(\mu).$$

If the second multiplier is hyperbolic, the center-manifold theorem lifts the scalar saddle-node to the planar map. $\square$

For the baseline parameters, $r_{\text{SN}} = H_{\text{SN}} = 0.70710678$. At $c = 4$ and $h = 1$, the transverse multiplier is 0.92050800, so it is strictly inside the unit circle.

4.2. Transcritical mite invasion

At $E_+ = (H_+, 0)$, the Jacobian is upper triangular. Its multipliers are

$$\lambda_H = 1 + h \frac{d_1(k - H_+^2)}{k + H_+^2}, \quad \lambda_M = 1 + h\iota(H_+, c),$$

where

$$\iota(H, c) = \frac{\sigma a H}{b + cH} - d_2 - \frac{\gamma H}{K_g + H}.$$

The saturation coefficient ρ is absent because the invasion calculation is performed at $M = 0$. The threshold $\lambda_M = 1$ is

$$c_{TC} = \frac{\sigma a}{d_2 + \gamma H_+ / (K_g + H_+)} - \frac{b}{H_+}.$$

Let $\mu = c - c_{TC}$ and translate E_+ to the origin. Denote the first and second state derivatives of the map at the threshold by A and B . Let q and p be right and left eigenvectors for the simple multiplier 1, normalized by $\langle p, q \rangle = 1$.

Theorem 4.2 (Boundary transcritical bifurcation). *Assume $|\lambda_H| \neq 1$ and define*

$$\alpha = \langle p, A_c q \rangle, \quad \beta = \frac{1}{2} \langle p, B(q, q) \rangle.$$

If $\alpha\beta \neq 0$, then a center coordinate v can be chosen so that

$$v_{n+1} = v_n + \alpha\mu v_n + \beta v_n^2 + O(|v_n|^3 + |\mu|v_n^2 + \mu^2|v_n|). \quad (4.3)$$

The mite-free branch $v = 0$ and a nontrivial branch

$$v = -\frac{\alpha}{\beta}\mu + O(\mu^2) \quad (4.4)$$

exchange center stability. If the physical mite component of q is oriented positively, the branch is biologically admissible on the side for which the right-hand side of (4.4) is positive.

Proof. The invariant axis $M = 0$ provides a fixed-point branch for all nearby c , so the reduced fixed-point equation has no parameter-only constant term. The simple unit multiplier and hyperbolicity of the second multiplier yield a one-dimensional center manifold. Projection of the parameter-dependent Taylor expansion onto p gives (4.3). Dividing the nontrivial fixed-point equation by v and applying the implicit-function theorem gives (4.4). The center multiplier on the boundary branch is $1 + \alpha\mu + O(\mu^2)$, whereas on the nontrivial branch, it is $1 - \alpha\mu + O(\mu^2)$; hence, the branches exchange center stability. The sign of the mite component of the oriented eigenvector transfers the sign of v to the physical variable M . $\square$

At $r = 20$ and $h = 1$,

$$H_+ = 39.98749609, \quad c_{TC} = 5.77573795, \quad \lambda_H = 0.50031260.$$

With the center eigenvector oriented toward increasing mite density,

$$\alpha = -0.03707494, \quad \beta = 0.00793285, \quad -\frac{\alpha}{\beta} = 4.67359425.$$

Thus, the admissible coexistence branch enters for $c > c_{TC}$. Figure 1 shows the saddle-node branches, the transcritical intersection, the exchange of critical multipliers, and the local tangent of the physical coexistence branch.

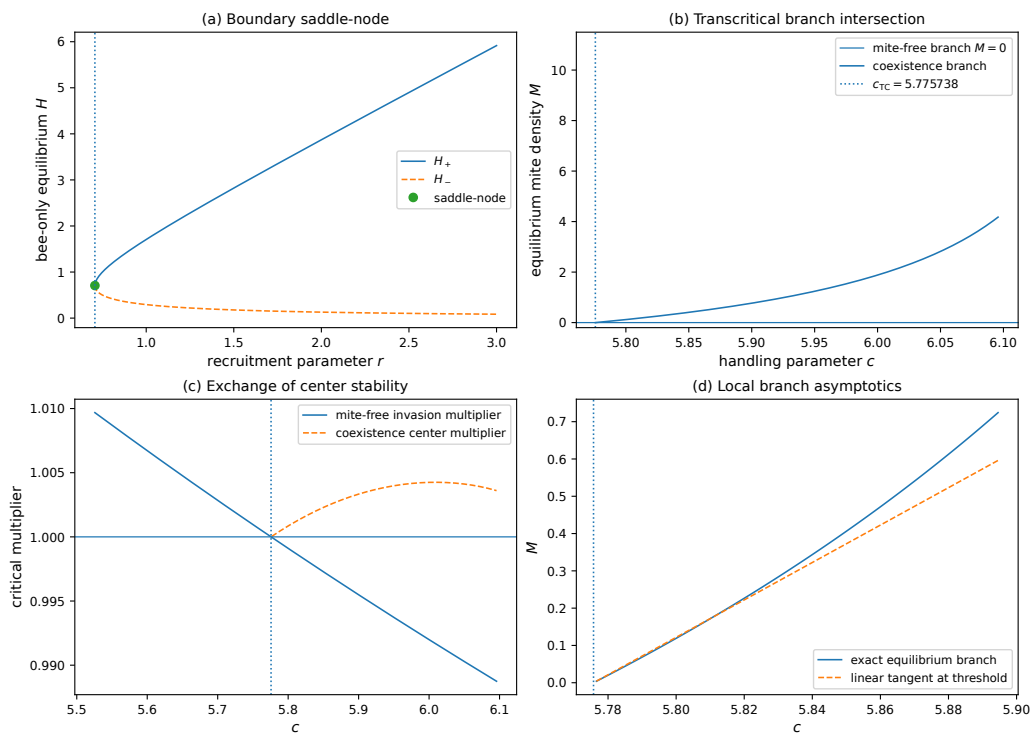


Figure 1. Boundary bifurcations of the census map. (a) Saddle-node creation of the two bee-only equilibria at $r_{\text{SN}} = 2d_1\sqrt{k}$. (b) Intersection of the mite-free and coexistence branches at c_{TC} . (c) Exchange of the critical multiplier across the transcritical threshold. (d) Exact physical branch and its linear tangent at the threshold.

5. Interior codimension-one bifurcations

5.1. Multilinear notation

Let X_* be a coexistence fixed point and write $x = X - X_*$. At a fixed parameter value, expand

$$x' = Ax + \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x\|^4),$$

where $A = D_X\Phi_h(X_*)$ and B, C are symmetric multilinear derivatives. Complex inner products are Hermitian. Nonresonant terms are removed degree by degree through the corresponding homological equations. The eigenvectors are normalized to unit Euclidean norm; for a complex eigenvector q , its phase is fixed by requiring the first component to be real and positive. This convention is essential because individual complex normal-form coefficients are coordinate-dependent [26, 27].

5.2. Neimark–Sacker bifurcation

Suppose $J_c(X_*)$ has a complex conjugate pair and

$$T < 0, \quad D > 0, \quad T^2 < 4D.$$

Since $A_h = I + hJ_c$, the unit-modulus condition $\Delta = 1$ gives

$$h_{\text{NS}} = -\frac{T}{D} > 0.$$

At this value, the multipliers are $\lambda, \bar{\lambda} = e^{\pm i\theta}$ and

$$\left. \frac{d|\lambda(h)|}{dh} \right|_{h=h_{\text{NS}}} = -\frac{T}{2} > 0. \quad (5.1)$$

Let $Aq = \lambda q$, $A^T p = \bar{\lambda} p$, and $\langle p, q \rangle = 1$. Under the non-resonance conditions $\lambda^j \neq 1$ for $j = 1, 2, 3, 4$, define

$$\begin{aligned} h_{20} &= (\lambda^2 I - A)^{-1} B(q, q), \\ h_{11} &= (I - A)^{-1} B(q, \bar{q}), \\ g_{21} &= \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) \rangle, \\ c_1 &= \frac{1}{2} g_{21}, \quad \ell_1 = \text{Re}(\bar{\lambda} c_1). \end{aligned} \quad (5.2)$$

Theorem 5.1 (Neimark–Sacker bifurcation). *Assume a simple unit pair, the transversality condition (5.1), $\lambda^j \neq 1$ for $j = 1, 2, 3, 4$, and $\ell_1 \neq 0$. Then the center-manifold map is locally conjugate to*

$$z' = \lambda(h)z + c_1 z |z|^2 + O(|z|^4 + |h - h_{\text{NS}}| |z|^2).$$

If $d|\lambda|/dh > 0$ and $\ell_1 < 0$, an attracting invariant closed curve is created for $h > h_{\text{NS}}$; if $\ell_1 > 0$, the local invariant curve is repelling and occurs on the opposite side.

Proof. The hypotheses provide a center manifold that is two-dimensional and non-resonant up to degree four. The quadratic and cubic homological equations lead to (5.2), where the radial component of the map has cubic coefficient ℓ_1 . The Neimark–Sacker theorem for maps then provides the existence and stability conclusion; see Chapters 8-9 of [25, 26]. $\square$

For $r = 20$ and $c = 4$, the lower coexistence equilibrium and critical step are

$$X_* = (1.68823672, 32.18691116), \quad h_{\text{NS}} = 1.47694917.$$

The critical multipliers are $0.22176323 \pm 0.97510054i$, so $\theta = 1.34717397$ and no strong resonance of order at most four is present. The crossing derivative is 0.52692184. The normal-form calculation gives

$$c_1 = 0.03173378 - 0.35342252i, \quad \ell_1 = -0.33758511.$$

Thus, this point is a nondegenerate supercritical Neimark–Sacker bifurcation.

5.3. Period-doubling bifurcation

Suppose $J_c(X_*)$ has a negative eigenvalue v_- that reaches the multiplier -1 first. Then,

$$h_{\text{PD}} = -\frac{2}{v_-}, \quad \lambda_1(h_{\text{PD}}) = -1.$$

Let $Aq = -q$, $A^T p = -p$, $\langle p, q \rangle = 1$, and assume that the second multiplier is not ± 1 . Define

$$c_3 = \frac{1}{6} \langle p, C(q, q, q) \rangle + \frac{1}{2} \langle p, B(q, (I - A)^{-1} B(q, q)) \rangle. \quad (5.3)$$

Theorem 5.2 (Period-doubling bifurcation). *Assume a simple multiplier -1 , no other multiplier equal to ± 1 , parameter transversality $\beta = -v_- > 0$, and $c_3 \neq 0$. With $\mu = h - h_{\text{PD}}$, a center coordinate can be chosen so that*

$$x' = -(1 + \beta\mu)x + c_3x^3 + O(|x|^4 + |\mu||x|^2 + \mu^2|x|).$$

If $c_3 > 0$, a period-two orbit is created for $\mu > 0$ and is locally attracting on the center manifold. If, in addition, the noncritical multiplier λ_2 satisfies $|\lambda_2| < 1$ at the bifurcation point, then the two-cycle is locally attracting in the full planar map. If $c_3 < 0$, the flip bifurcation is subcritical.

Proof. The multiplier -1 , which is simple, produces the one-dimensional center manifold. The quadratic part is nonresonant and can be removed with a near-identity change of variables. Projection of the cubic part onto the resulting manifold gives (5.3), whose sign determines the side and center-manifold stability of the bifurcating two-cycle. Full planar attraction additionally requires transverse contraction, $|\lambda_2| < 1$. This is the standard flip normal form for maps [25, 26]. $\square$

For $r = 20$ and $c = 5.1$,

$$X_* = (3.15208682, 34.76221023), \quad h_{\text{PD}} = 2.79053936.$$

The critical multipliers are -1 and 0.14185099 , and

$$\beta = 0.71670732, \quad c_3 = 0.10530525.$$

Since the noncritical multiplier has modulus $0.14185099 < 1$, the flip is nondegenerate and supercritical, and the bifurcating two-cycle is locally attracting in the full planar map. Figure 2 compares the analytical thresholds with orbit and Lyapunov calculations.

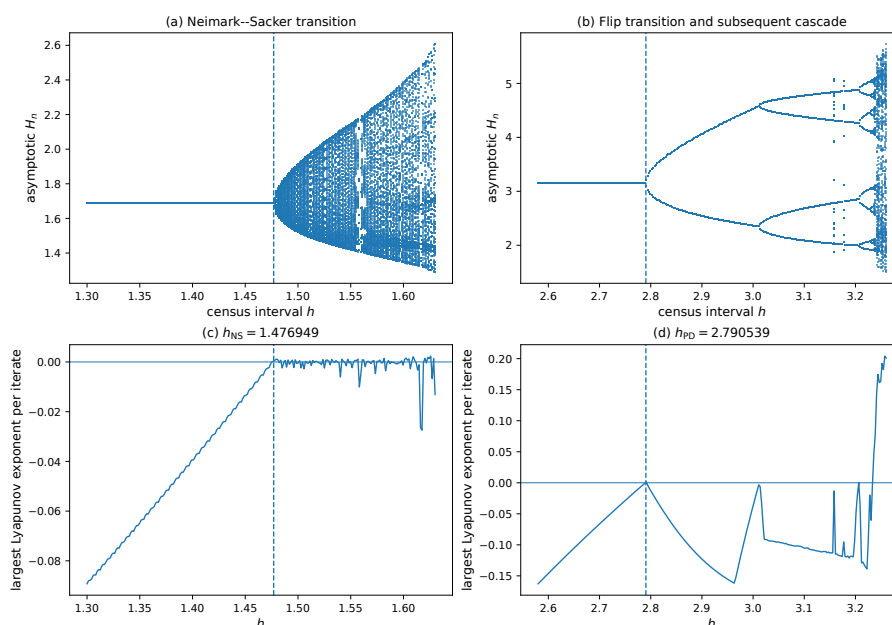


Figure 2. Codimension-one bifurcations with h as the parameter. The left column shows the Neimark–Sacker transition at $(r, c) = (20, 4)$; the right column shows the period-doubling transition and subsequent cascade at $(r, c) = (20, 5.1)$. Dashed lines mark the analytical thresholds. Lyapunov exponents are reported per iterate.

6. Codimension-two resonance calculations

Fix $r = 20$ and use (c, h) as unfolding parameters on the lower coexistence branch. A complex unit pair has $\Delta = 1$ and $\tau = 2 \cos \theta$. The $1:q$ resonances of order $q = 2, 3, 4$ satisfy

resonance	1:2	1:3	1:4
(τ, Δ)	$(-2, 1)$	$(-1, 1)$	$(0, 1)$
T^2/D	4	3	2

The second row follows from $h = -T/D$ and $\tau = 2 - T^2/D$. Define

$$\mathcal{U} = \det \frac{\partial(\tau, \Delta)}{\partial(c, h)}. \quad (6.1)$$

If $\mathcal{U} \neq 0$, trace and determinant are locally independent coordinates. Figure 3 displays the fixed-point classification and the associated paths in the trace–determinant plane.

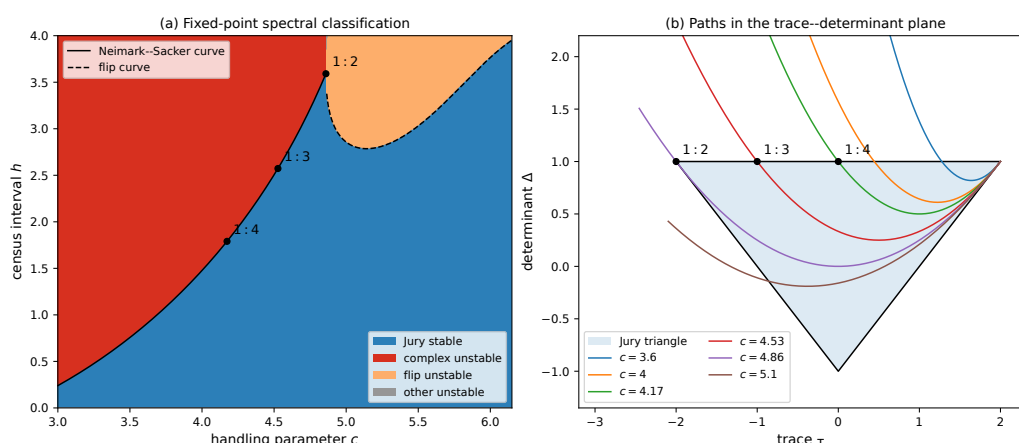


Figure 3. Two-parameter spectral organization for $r = 20$. (a) Fixed-point classification in the (c, h) plane, together with the Neimark–Sacker and flip curves. (b) Corresponding paths in the trace–determinant plane. The marked points satisfy the $1:4$, $1:3$, and $1:2$ resonance conditions.

6.1. $1:4$ resonance

At

$$(c_*, h_*) = (4.17349022, 1.79019937), \quad X_* = (1.82150719, 32.78288089),$$

the multipliers are $\pm i$ and $\mathcal{U} = -2.98288106$. For $\Delta > 0$ near the point, introduce

$$\beta_1 = \sqrt{\Delta} - 1, \quad \beta_2 = \arccos\left(\frac{\tau}{2\sqrt{\Delta}}\right) - \frac{\pi}{2}.$$

Because (6.1) is nonzero, (β_1, β_2) is a regular local reparameterization of $(c - c_*, h - h_*)$. Under the eigenvector convention stated in Section 5, the resonant normal form is

$$z' = (1 + \beta_1)e^{i(\pi/2 + \beta_2)}z + a_{21}z^2\bar{z} + a_{03}\bar{z}^3 + O(|z|^4 + \|\beta\||z|^2),$$

where

$$a_{21} = -0.09293186 - 0.37651585i, \quad a_{03} = -0.08437179 - 0.47101258i.$$

Since $|a_{03}| = 0.47850962 \neq 0$, the lowest-order resonant cubic term does not vanish under the stated normalization. This indicator alone does not constitute a complete global unfolding classification.

6.2. 1:3 resonance

At

$$(c_*, h_*) = (4.52658874, 2.57277943), \quad X_* = (2.17039483, 33.82967512),$$

the multipliers are $e^{\pm 2\pi i/3}$ and $\mathcal{U} = -3.43863389$. Define

$$\beta_1 = \sqrt{\Delta} - 1, \quad \beta_2 = \arccos\left(\frac{\tau}{2\sqrt{\Delta}}\right) - \frac{2\pi}{3}.$$

The local normal form is

$$z' = (1 + \beta_1)e^{i(2\pi/3 + \beta_2)}z + a_{02}\bar{z}^2 + a_{21}z^2\bar{z} + O(|z|^4 + \|\beta\||z|^2),$$

with

$$a_{02} = -0.07134042 - 0.60162430i, \quad a_{21} = 0.08567603 + 0.00033079i.$$

The condition $|a_{02}| = 0.60583930 \neq 0$ excludes vanishing of the lowest-order 1:3 resonant term under the stated normalization. It does not, by itself, classify the global unfolding.

6.3. 1:2 resonance

At

$$(c_*, h_*) = (4.85958450, 3.59135489), \quad X_* = (2.64936698, 34.52477789),$$

the map has a double multiplier -1 with geometric multiplicity one, and $\mathcal{U} = -3.40524986$. Set

$$\beta_1 = -1 - \tau - \Delta, \quad \beta_2 = \tau + 2.$$

These parameters give the characteristic polynomial

$$\lambda^2 - (\beta_2 - 2)\lambda + (1 - \beta_1 - \beta_2) = 0$$

and form a regular local unfolding. Solving the quadratic homological equation removes all quadratic terms. In the real Jordan coordinates determined by the normalized eigenvector and generalized eigenvector, the cubic normal form is

$$\begin{aligned} w'_1 &= -w_1 + w_2, \\ w'_2 &= \beta_1 w_1 + (-1 + \beta_2)w_2 + C_1 w_1^3 + D_1 w_1^2 w_2 + O(\|w\|^4 + \|\beta\|\|w\|^2), \end{aligned}$$

where

$$C_1 = -0.08950908, \quad D_1 = 1.11171615.$$

The standard genericity indicators for this normalization are

$$4C_1 = -0.35803632 \neq 0, \quad -2D_1 - 6C_1 = -1.68637782 \neq 0.$$

Thus, the numerically located 1:2 point satisfies the reported cubic nondegeneracy indicators for the standard map normal form [26, 27]. These local numerical indicators are not an interval-certified existence proof. Table 1 summarizes the numerically located 1:4, 1:3, and 1:2 resonance candidates, along with their parameter values, coexistence equilibria, unfolding determinants, and leading normal-form coefficients.

Table 1. Numerically located candidate strong-resonance points and resonant normal-form coefficients on the lower coexistence branch for $r = 20$. Coefficients use the normalization specified in Section 5.

Resonance	(c, h)	(H_*, M_*)	$\mathcal{U}$	Leading normal-form coefficients
1:4	(4.173490, 1.790199)	(1.821507, 32.782881)	-2.982881	$a_{21} = -0.092932 - 0.376516i$ $a_{03} = -0.084372 - 0.471013i$ $a_{02} = -0.071340 - 0.601624i$
1:3	(4.526589, 2.572779)	(2.170395, 33.829675)	-3.438634	$a_{21} = 0.085676 + 0.000331i$ $C_1 = -0.089509$
1:2	(4.859584, 3.591355)	(2.649367, 34.524778)	-3.405250	$D_1 = 1.111716$

Figure 4 shows the local spectral geometry and the zero sets of the unfolding coordinates. These panels provide numerical evidence of regular local crossings; they are not a continuation of global resonance tongues or secondary periodic-orbit bifurcations.

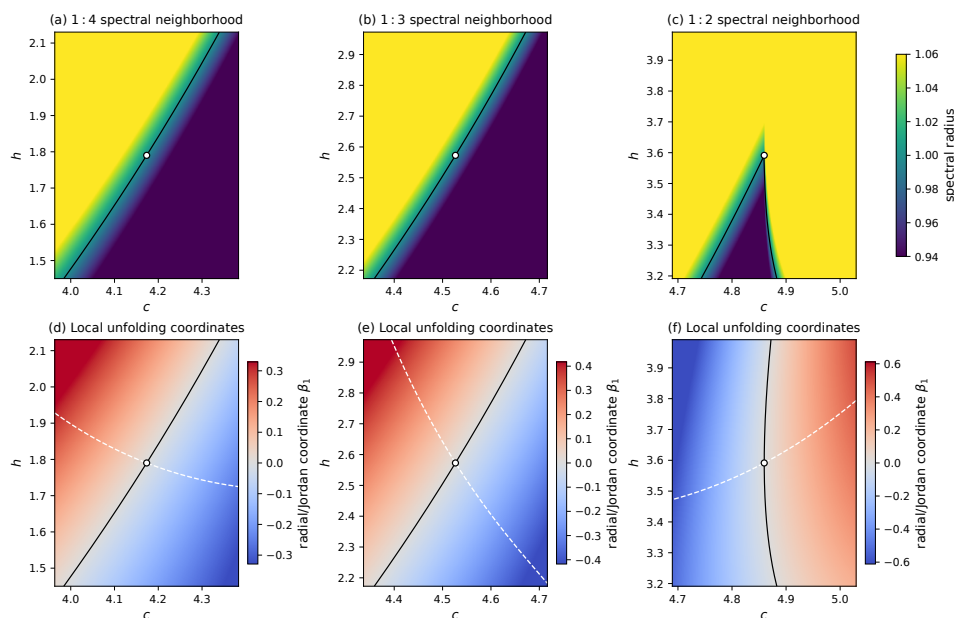


Figure 4. Local neighborhoods of the numerically located 1:4, 1:3, and 1:2 resonance candidates. The upper row shows the spectral radius and its unit contour. The lower row shows the radial or Jordan unfolding coordinate β_1 ; black solid and white dashed contours denote $\beta_1 = 0$ and $\beta_2 = 0$, respectively. Their intersection gives the reported codimension-two approximation.

7. Equilibrium-targeting stabilization

Classical chaos-control procedures stabilize unstable periodic solutions by small state or parameter corrections [32]. Here, the objective is narrower: stabilization of a specified coexistence equilibrium in a numerically sensitive regime. Define

$$\Phi_{h,\kappa}^{\text{ctl}}(X) = (1 - \kappa)\Phi_h(X) + \kappa X_*, \quad 0 \leq \kappa < 1. \quad (7.1)$$

This convex target correction is a mathematical feedback law; it is not identified with a particular apiary intervention.

Theorem 7.1 (Hyperbolic stabilization criterion). *The map (7.1) preserves X_* and satisfies*

$$D\Phi_{h,\kappa}^{\text{ctl}}(X_*) = (1 - \kappa)A_*. \quad (7.2)$$

Consequently, X_* is locally exponentially stable if

$$(1 - \kappa)\rho(A_*) < 1.$$

It is unstable if $(1 - \kappa)\rho(A_*) > 1$; equality is nonhyperbolic, and the linearization is inconclusive. If $\rho(A_*) > 1$, define

$$\kappa_c = 1 - \frac{1}{\rho(A_*)}.$$

Then, $\kappa > \kappa_c$ gives local exponential stability, $\kappa < \kappa_c$ gives instability, and $\kappa = \kappa_c$ is the nonhyperbolic boundary. If Ω is convex, positively invariant under Φ_h , and contains X_* , then it is also invariant under the controlled map.

Proof. Substitution gives $\Phi_{h,\kappa}^{\text{ctl}}(X_*) = X_*$. Differentiation gives (7.2); hence, every multiplier is multiplied by $1 - \kappa$. The strict inequalities follow from the linearization theorem for hyperbolic fixed points, whereas equality leaves a unit-modulus multiplier and requires nonlinear analysis. For $X \in \Omega$, both $\Phi_h(X)$ and X_* belong to Ω . Convexity therefore places their weighted average in Ω . $\square$

At $(r, c, h) = (30, 3.66, 1)$, the target equilibrium is

$$X_* = (1.47106948, 47.11357732),$$

and the uncontrolled multipliers are $0.43541419 \pm 1.02597129i$, so $\rho(A_*) = 1.11454143$ and

$$\kappa_c = 0.10277001.$$

For $\kappa = 0.15$, the finite-time largest Lyapunov exponent per iterate changes from approximately 0.0406 to -0.0532 . Figure 5 compares the uncontrolled and controlled trajectories, while Figure 6 shows the corresponding parameter-plane calculations. These numerical exponents support stabilization of the selected regime but do not constitute a proof of a chaotic invariant set in the uncontrolled map.

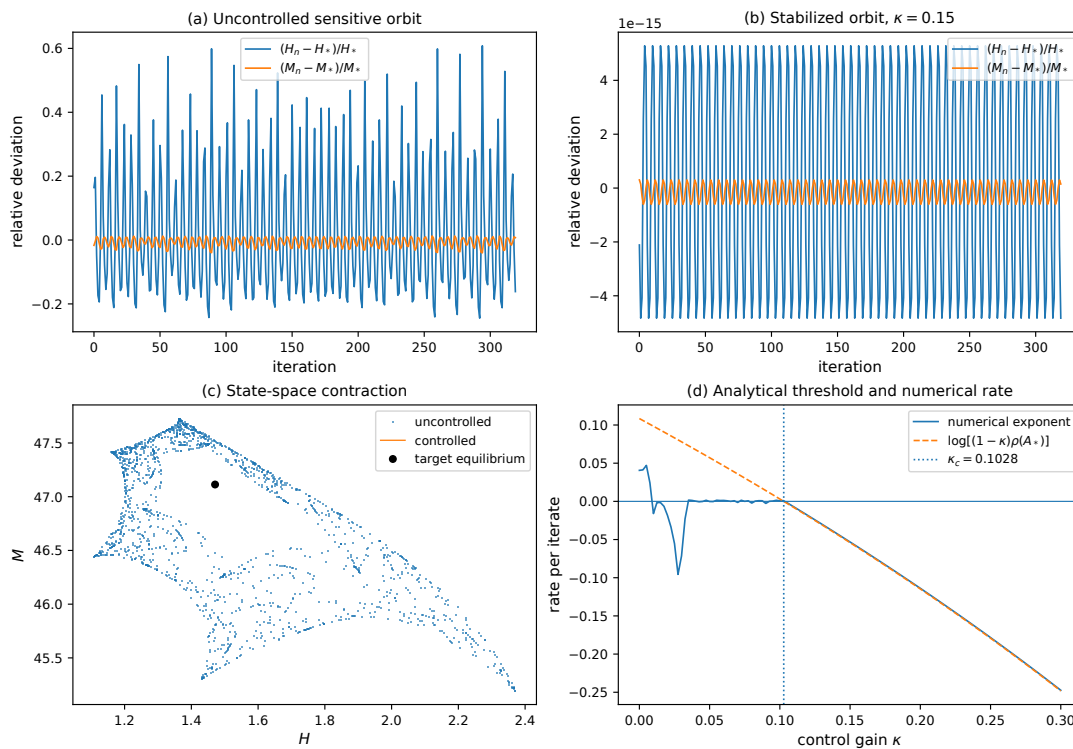


Figure 5. Equilibrium-targeting stabilization at $(r, c, h) = (30, 3.66, 1)$. (a) Relative deviations along the uncontrolled sensitive orbit. (b) Convergence for $\kappa = 0.15$. (c) State-space contraction toward the coexistence target. (d) Numerical Lyapunov exponent and the local rate $\log[(1 - \kappa)\rho(A_*)]$; the vertical line marks κ_c .

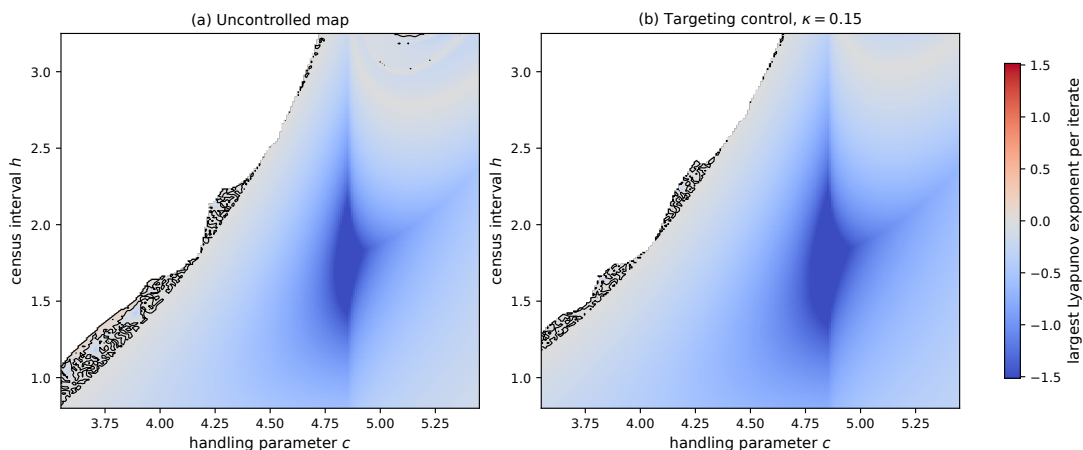


Figure 6. Effect of target correction on the largest Lyapunov exponent. The exponent is shown per iterate in the (c, h) plane for $r = 20$: (a) uncontrolled map and (b) target correction with $\kappa = 0.15$. Each orbit starts near the lower coexistence branch. White regions correspond to inadmissible or failed trajectories, and black contours show the numerically estimated zero-exponent boundary.

8. Numerical parameter-plane results and scheme sensitivity

Coexistence roots were enumerated on a 480×360 grid for $r \in [12, 32]$ and $c \in [3.5, 7]$. At most two admissible roots were found at any sampled parameter pair. The lower- H branch contains the principal Schur-stable region, whereas the higher- H branch is predominantly unstable (typically of saddle type in the sampled region). This finite-grid observation does not exclude additional admissible roots outside the sampled window or between numerical resolution scales; Theorem 3.1 gives the global algebraic upper bound of five positive equilibria when $P \neq 0$.

Figure 7 provides the requested cross-scheme bifurcation and period-classification comparison: it contrasts Euler schemes with $h = 1, 0.5$, and 0.25 with the positive exponential scheme for $h = 1$ over the same displayed parameter rectangle and with the same outcome definitions. In this two-parameter setting, the colored parameter planes serve as bifurcation diagrams because changes of persistent outcome class mark numerically detected transitions in parameter space. “Unresolved” means that the finite numerical classifier found no admissible minimal period $p \leq 120$ within the stated tolerance; it is not, by itself, a classification as quasiperiodic or chaotic. A separate one-dimensional orbit diagram would represent only a single slice of this same comparison and is therefore not used as the primary scheme-sensitivity diagnostic.

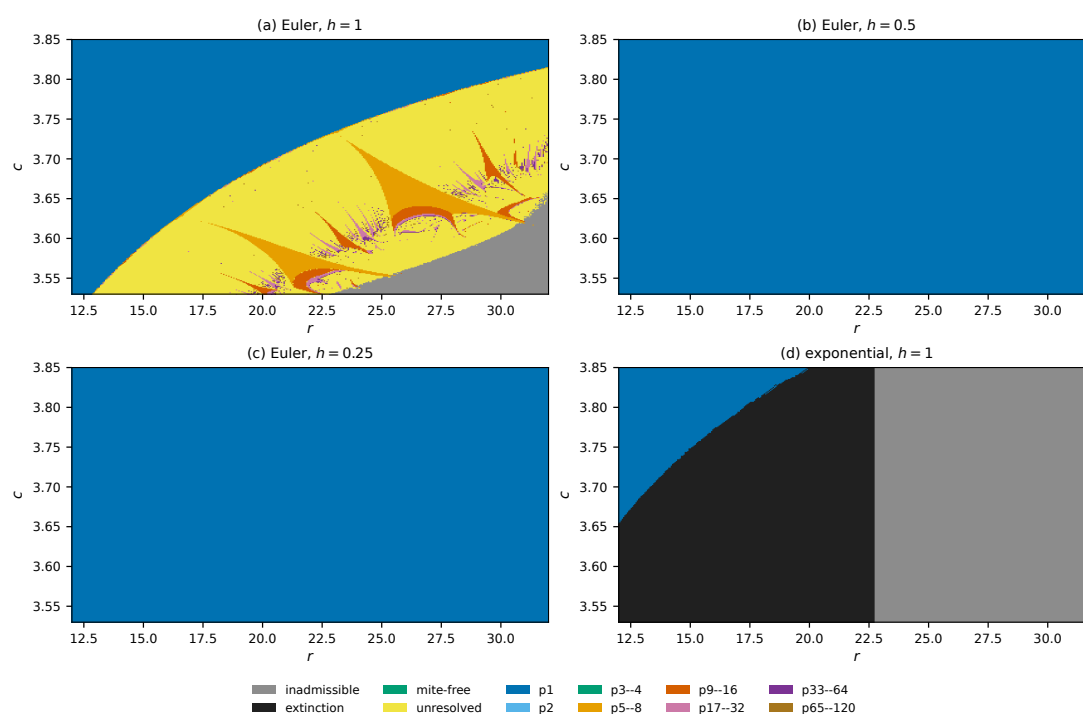


Figure 7. Cross-scheme bifurcation and period-classification comparison over a common parameter plane. The panels compare Euler maps with $h = 1, 0.5$, and 0.25 and the positive exponential map with $h = 1$ using the same outcome definitions. Colors denote persistent outcomes or numerically detected minimal-period classes; “unresolved” means that no admissible period $p \leq 120$ was detected within the stated tolerance, and gray denotes numerical inadmissibility.

Because the Euler and exponential maps have the same Jacobian $I + hJ_c$ at a positive equilibrium, their local fixed-point stability boundary is identical at the linear level for the same equilibrium and parameter values. The scheme sensitivity in Figure 7 therefore concerns the nonlinear post-local and global orbit classifications rather than a discrepancy in the equilibrium Jacobian itself.

To quantify the scheme dependence, we summarize the gridded bifurcation/period classifications displayed in Figure 7. Approximate fractions of the plotted parameter area are reported in Table 2; boundary pixels have a negligible effect on the quoted one-decimal percentages. We also define a pairwise classification-disagreement fraction as the proportion of commonly displayed grid cells assigned to different outcome classes. Relative to Euler $h = 0.5$, the disagreement is approximately 52.5% for Euler $h = 1$, 0.0% for Euler $h = 0.25$, and 89.7% for the exponential map with $h = 1$. Euler $h = 1$ and the exponential map disagree on approximately 82.2% of the displayed cells. Under the stated finite-orbit classifier, the Euler maps with $h = 0.5$ and $h = 0.25$ have identical outcome classifications on this window; this does not imply trajectory-level agreement or numerical convergence. By contrast, the unit-step classification is highly scheme-dependent.

Table 2. Descriptive classification fractions for the parameter plane in Figure 7. Values are percentages of the displayed grid area, rounded to one decimal. The column “period ≥ 5 ” aggregates detected classes $p = 5$ –8, $p = 9$ –16, $p = 17$ –32, $p = 33$ –64, and $p = 65$ –120. The classes $p = 2$, $p = 3$ –4, and mite-free occur with 0.0% on this displayed grid and are therefore omitted as separate columns.

Scheme	$p = 1$	Unresolved	Period $p \geq 5$	Extinction	Inadmissible
Euler, $h = 1$	47.5	39.3	6.1	0.0	7.1
Euler, $h = 0.5$	100.0	0.0	0.0	0.0	0.0
Euler, $h = 0.25$	100.0	0.0	0.0	0.0	0.0
Exponential, $h = 1$	10.3	0.0	0.0	43.2	46.5

The rough high-period structure present at Euler $h = 1$ is therefore largely removed when the Euler step is reduced, and the $h = 1$ exponential update produces a qualitatively different partition of the same parameter plane. This does not undermine the local bifurcation theorems for the specified algebraic map; it limits their interpretation as statements about colony dynamics or about the underlying continuous system. For the baseline $r = 20$ calculation, Theorem 2.2 gives the conservative sufficient bound $h \leq 1/32.5 \approx 0.0308$ when $W_0 \leq \sigma r/\delta$. Therefore, the unit-step calculations and the three resonances in Table 1 are not covered by this sufficient condition for global positivity. As emphasized in Remark 2.3, the local bifurcation calculations near positive equilibria remain statements about the discrete map, whereas unrestricted long orbits at large h cannot be regarded as calibrated population trajectories.

9. Discussion

In boundary analysis, there are two types of transitions that can be confused in one-parameter graphs. At $r = 2d_1\sqrt{k}$, there is a general saddle-node bifurcation, generating bee-only equilibrium points. In contrast to that, the threshold of mite invasion will generate a transcritical transition between the upper mite-free equilibrium and the coexistence point. The effect of removal saturation is not

relevant to coexistence until the mites are positive.

The spectral criteria at coexistence arise from the matrix expression $A_h = I + hJ_c$, where h is the bifurcation parameter. A local Neimark–Sacker or period-doubling bifurcation of the Euler census map can be mathematically valid even when h is too large for Euler trajectories to approximate the continuous flow globally. This distinction is essential: local center-manifold and normal-form results concern the discrete map near its positive equilibrium, while long-orbit physical admissibility requires additional positivity and boundedness control. At $(r, c) = (20, 4)$, the negative first Lyapunov coefficient establishes a supercritical Neimark–Sacker bifurcation of the Euler map.

The points 1:4, 1:3, and 1:2, as specified by the numerical location, define the local spectral geometry of the selected two-parameter census map. The fact that the unfolding determinant is nonzero demonstrates the regularity of the selected parameter coordinates in the vicinity of the given approximations. The resonant coefficients for 1:4 and 1:3, as well as the two cubic terms for 1:2, rule out the corresponding lowest-order conditions of vanishing. They do not provide any information about the global geometry of the resonance tongues, heteroclinic connections, or secondary bifurcations. The targeting correction has a sharp hyperbolic boundary, but its biological interpretation is limited because it presumes knowledge of the equilibrium and acts directly on the state. A field implementation would need bounded, delayed, and measurable interventions such as mite removal or colony support.

The model omits brood-stage structure, explicit virus transmission, seasonality, environmental noise, observation error, and parameter uncertainty, and it is not calibrated to colony data. The capacity-limited removal term is deliberately phenomenological. These limitations preclude quantitative prediction. The results should therefore be interpreted as a bifurcation framework that distinguishes invasion, oscillatory loss of stability, local resonance, scheme-sensitive global dynamics, and equilibrium stabilization within a transparent census model.

10. Methods

Coexistence equilibria are obtained from the polynomial (3.2). A computed root z is treated as real when

$$|\operatorname{Im} z| \leq 2 \times 10^{-7} \max\{1, |\operatorname{Re} z|\}.$$

It is retained only if it lies in I_b with a relative endpoint margin of 10^{-8} , satisfies $B(H) > 10^{-9}$, and has a per-capita residual below 2×10^{-6} . This procedure enumerates floating-point roots; it is not an interval-certified isolation method.

Unless stated otherwise, orbit scans begin at $(H_0, M_0) = (0.5, 0.3)$, discard 5000 iterates, and use the next 2400 iterates. The cross-scheme percentages and disagreement fractions in Table 2 are descriptive summaries of the classifications displayed in Figure 7; they are not convergence-error estimates and are not used to claim fixed-physical-time equivalence across different values of h . The finite-time largest Lyapunov exponent is evaluated from Eq (10.1),

$$\Lambda_N = \frac{1}{N} \sum_{n=0}^{N-1} \log \|D\Phi_h(X_n)v_n\|, \quad (10.1)$$

with the tangent vector normalized after each multiplication. Exponents are calculated per iterate using the conventional renormalization procedure [33]. The last 720 states are checked for periods $p \leq 120$.

A period is recorded only if four successive displacements of length p have a maximum relative residual below 10^{-6} ; candidate periods are tested in increasing order.

The trajectory will be considered numerically inadmissible if any of the two parts is negative, non-finite, or exceeds 10^6 . Inadmissibility is not extinction. Extinction is achieved when both parts stay below 10^{-8} until the last window of observations. In case of a positive exponential map, the breach of the upper limit results in inadmissibility, not extinction.

Derivatives up to cubic order have been determined directly from the map equations and expressed in homogeneous multilinear form. Degree-wise near-identity transformations solve the homological equations without losing the resonant monomials prescribed in Sections 5 and 6. Decimal coefficients and parameter values are only numerical estimates; no verification using interval arithmetic has been done. All images were produced by the authors using the equations, parameter values, and numerical methods outlined in this paper.

11. Conclusions

A local and computational bifurcation analysis has been developed for a capacity-limited honeybee–*Varroa* census map that explicitly extends the model of Karmakar and Pal [12]. Bee-only equilibria arise in a generic saddle-node, whereas mite invasion of the upper bee-only state is governed by a generic transcritical bifurcation. At coexistence, explicit spectral thresholds and cubic normal-form coefficients establish a supercritical Neimark–Sacker bifurcation and a supercritical period-doubling bifurcation.

Three candidate codimension-two strong-resonance points are located numerically in the (c, h) plane. The 1:4 and 1:3 points have nonzero leading complex resonant terms, and the 1:2 point has nonzero cubic Jordan-form indicators. A convex equilibrium-targeting correction supplies a sharp hyperbolic gain boundary and stabilizes the selected sensitive regime numerically. The pronounced step-size and scheme dependence of global orbit classifications limits biological interpretation. The analysis thereby separates proved local statements, numerical evidence, and biological limitations.

Author contributions

Asifa Tassaddiq: Conceptualization, Investigation, Methodology, Writing-original draft. **Muhammad Sajjad Shabbir:** Conceptualization, Investigation, Methodology, Software, Supervision, Writing-original draft. **Rabab Alharbi:** Formal analysis, Software, Validation, Writing-review and editing. **Dalal Khalid Almutairi:** Formal analysis, Software, Visualization, Writing-review and editing. **Rizwan Ahmed:** Methodology, Supervision, Visualization, Writing-review and editing. **JungYoon Kim:** Formal analysis, Software, Writing-review and editing. **Younhyun Jung:** Conceptualization, Investigation, Methodology, Writing- review and editing. **Jawad Khan:** Investigation, Methodology, Software, Writing-review and editing. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest to disclose.

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