



Research article

New oscillation criteria for third-order nonlinear functional differential equations with distributed delay

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Abstract: This paper addresses the oscillatory behavior of solutions to a general class of third-order functional differential equations. The analysis is carried out within a nonstandard framework, where new oscillation criteria are established via multiple analytical techniques, including the Riccati transformation method, the integral method, and comparison principles. Furthermore, refined oscillation conditions are obtained by improving several key inequalities that play a central role in the qualitative analysis of the considered equations. The applicability and sharpness of the proposed results are illustrated through a representative special case, demonstrating enhanced oscillation tests compared to existing criteria. Finally, the obtained results extend and unify several previously reported findings in the literature.

Keywords: oscillation; nonoscillation; delay differential equation; comparison method

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1. Introduction

In this work, we investigate the oscillatory behavior of solutions of third-order differential equation

$$\left(l_2(\tau) \left[\left(l_1(\tau) (v'(\tau))^{g_1} \right)' \right]^{g_2} \right)' + \int_{b_1}^{b_2} q(\tau, s) v^\alpha(a(\tau, s)) ds = 0, \quad \tau \geq \tau_0, \quad (1.1)$$

where $g_i, \alpha \in \mathbb{Q}_{odd}^+$, $i = 1, 2$. Throughout this work, we assume that:

(A1) $l_i, i = 1, 2$, and $q \in C([\tau_0, \infty) \times [b_1, b_2], \mathbb{R}_+)$, with $q(\tau, s) \geq 0$ for all $(\tau, s) \in [\tau_0, \infty) \times [b_1, b_2]$;

(A2) $\int_{\tau_0}^{\infty} l_1^{-\frac{1}{g_1}}(s) ds = \infty$ and $\int_{\tau_0}^{\infty} l_2^{-\frac{1}{g_2}}(s) ds = \infty$;

(A3) $a \in C^1([\tau_0, \infty) \times [b_1, b_2], \mathbb{R})$, with $a(\tau, \varsigma) \leq \tau$ and $a(\tau, \varsigma)$ nondecreasing with respect to ς . Let $a_-(\tau) = a(\tau, b_1)$ and $a_+(\tau) = a(\tau, b_2)$. Then $a_-(\tau) \leq a(\tau, \varsigma) \leq a_+(\tau) \leq \tau$. Assume that $a'_-(\tau) \geq 0$, $a'_+(\tau) \geq 0$ and $\lim_{\tau \rightarrow \infty} a(\tau, \varsigma) = \infty$, for every $\varsigma \in [b_1, b_2]$.

Definition 1.1. A function $v(\tau)$ is said to be a solution of Eq (1.1) if there exists $T_v \geq \tau_0$ such that $v \in ([T_v, \infty))$ and satisfies (1.1) on $[T_v, \infty)$. In what follows, we restrict our attention to only those solutions $v(\tau)$ of (1.1), that is, solutions for which

$$\sup \{|v(\tau)| : \tau \geq T\} > 0 \text{ for all } T > T_v \geq \tau_0.$$

Definition 1.2. A solution $v(\tau)$ is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

Definition 1.3. Equation (1.1) is said to be oscillatory if every nontrivial solution of the equation is oscillatory. Otherwise, the equation is called nonoscillatory.

To facilitate the subsequent analysis, we introduce the auxiliary operators

$$U_1(v(\tau)) = (v'(\tau))^{g_1}$$

and

$$U_2(v(\tau)) = [(l_1(\tau) U_1(v(\tau)))']^{g_2}.$$

Using these operators, Eq (1.1) can be equivalently expressed in the following compact form:

$$(l_2(\tau) U_2(v(\tau)))' + \int_{b_1}^{b_2} q(\tau, \varsigma) v^\alpha(a(\tau, \varsigma)) d\varsigma = 0, \text{ for } \tau \geq \tau_0. \quad (1.2)$$

Differential equations constitute one of the most fundamental mathematical frameworks for modeling dynamic systems and describing the evolution of complex phenomena across science, engineering, and applied technology. Their importance stems from their ability to characterize time-dependent processes governed by interacting mechanisms, thereby providing rigorous tools for both theoretical investigation and practical implementation. In particular, differential equations play an essential role in numerous scientific disciplines, including biomedical sciences, engineering systems, climate modeling, and economic analysis, where they are widely used to describe dynamic behavior, stability, propagation mechanisms, and evolutionary processes [1, 2].

Among the various classes of differential equations, higher-order nonlinear equations have attracted considerable attention due to their broad applicability and rich qualitative structure. In particular, the oscillatory behavior of third-order differential equations has become an active area of research because of its close connection with stability theory, asymptotic analysis, and the qualitative behavior of dynamical systems. [3–5].

Moreover, the rapid advancement of artificial intelligence, computational science, and cyber-physical systems has further increased the importance of differential equation models in modern research. Emerging paradigms, including neural ordinary differential equations, physics-informed machine learning, and data-driven dynamical systems, fundamentally rely on differential equations to describe complex evolving processes [6, 7]. In particular, delay differential equations have attracted considerable attention due to their ability to incorporate memory effects and

delayed responses, where the present state depends not only on current conditions but also on past states. Such delay mechanisms naturally arise in biological regulation, communication systems, control theory, engineering applications, and learning dynamics, often leading to substantially richer and more intricate system behavior [8]. Consequently, the qualitative analysis of delay differential equations, especially the oscillatory behavior of higher-order nonlinear models, remains an important and mathematically challenging topic of contemporary research [9–11].

The oscillatory behavior of classical differential equations can be systematically and effectively investigated by integrating analytical and qualitative techniques. In particular, fixed-point methods and integral inequalities constitute essential tools for addressing nonlinearities and delay effects, especially in neutral and functional differential equations. From a broader perspective, the development of new oscillation criteria often relies on constructing suitable auxiliary functions that capture the intrinsic qualitative dynamics of the underlying system. Iterative schemes further enhance this framework by generating progressively sharper bounds for positive solutions, thereby refining and extending existing oscillation results.

Within this framework, comparison principles and the Riccati transformation constitute fundamental tools in the qualitative analysis of differential equations [12, 13]. Comparison techniques enable the behavior of solutions to be related to that of suitably chosen auxiliary equations, thereby facilitating the derivation of sufficient conditions for oscillation and nonoscillation. On the other hand, the Riccati transformation reformulates the original equation into an associated nonlinear differential inequality, where oscillatory properties can be analyzed more effectively through sign behavior and asymptotic estimates of the transformed variables. The combination of these approaches provides an effective framework for establishing refined oscillation criteria and extending existing results to broader classes of differential equations [14–16].

Later, in a more general framework, the authors in [17] developed new oscillation criteria for the third-order canonical neutral differential equation

$$\left(l_2(\tau) \left[\left(l_1(\tau) ((v(\tau) + p(\tau) v(\tau(\tau)))')^{g_1} \right)' \right]^{g_2} \right)' + q(\tau) v^\alpha(a(\tau)) = 0,$$

ensuring that all its solutions exhibit oscillatory behavior. Their approach is based on a refined analytical strategy that combines the principle of comparison with the theory of first-order differential equations and the Riccati transformation technique, thereby converting the original higher-order neutral equation into an equivalent form that is more amenable to qualitative analysis.

Baculiková and Džurina [10] investigated a class of third-order nonlinear delay differential equations of the form

$$(l_1(\tau) [v''(\tau)]^{g_1})' + q(\tau) v^\alpha(a(\tau)) = 0, \quad (1.3)$$

where $\tau f(\tau) > 0$ for $\tau \neq 0$ and

$$\int_{\tau_0}^{\infty} l_1^{-\frac{1}{g_1}}(s) \, ds = \infty.$$

They applied a comparison theorem involving chosen lower-order equations in order to obtain sufficient conditions for both asymptotic and oscillatory behavior of the solutions. Their results show that Eq (1.3) can be viewed as a special case of the more general framework given in However, the approach also reveals that there is no systematic or unified guideline for selecting the auxiliary function, which plays a crucial role in the establishment of the oscillation criteria for Eq (1.1).

On the other hand, Chatzarakis et al. [18], and Jadlovská and Li [19] investigated the asymptotic and oscillatory behavior of solutions for a class of delay differential equations

$$\left(l_2(\tau) \left[(l_1(\tau) (v'(\tau)))' \right]\right)' + q(\tau) v(a(\tau)) = 0, \quad (1.4)$$

by employing iterative estimates for both increasing and decreasing positive solutions to guarantee the oscillatory nature of the solutions of Eq (1.4). They developed the results mentioned in [10]. Finally, particular attention can be drawn to equation

$$\left(l_2(\tau) \left[(l_1(\tau) (v'(\tau))^{g_1})' \right]^{g_2}\right)' + q(\tau) v^\alpha(a(\tau)) = 0, \quad (1.5)$$

which constitutes a special case of the more general framework developed in this study under specific structural assumptions imposed on the parameters g_1 , g_2 , and α . In this regard, the equation has been extensively studied in the literature under various restrictive settings. For instance, oscillation criteria for the corresponding reduced form were established in [20–22] under the conditions

$$g_1 = g_2 = \alpha = 1.$$

Furthermore, in [23], the oscillatory behavior was investigated under the setting

$$g_1 = 1, g_2 = \alpha,$$

which leads to a quasilinear formulation of the problem. In addition, the authors in [16, 24, 25] examined the same class of equations in the fully linear neutral case characterized by

$$g_1 = g_2 = \alpha = 1,$$

thereby providing further oscillation results under comparable structural assumptions.

The main objective of this paper is to establish new and more general oscillation criteria for a class of nonlinear third-order functional differential equations with deviating arguments. The proposed results extend the existing oscillation theory by deriving sufficient conditions that are less restrictive while remaining widely applicable. In particular, this work builds on [10] by generalizing their framework through a more flexible operator structure and by relaxing key assumptions related to the nonlinear term and the delay structure. It also refines and extends the results in [18], which had already reduced several limitations via comparison principles and integral inequalities. In this study, these restrictions are further weakened by admitting more general nonlinear parameters and introducing broader integral conditions that remain verifiable under milder hypotheses. Consequently, the contribution of this work is a unified oscillation framework that simultaneously extends [10] and [18], covering linear, sublinear, and superlinear cases, and significantly enlarging the class of equations for which oscillation of all solutions can be guaranteed; see, for instance, [21, 26–29]. A notable feature of the proposed approach is the decoupled and adjustable role of the parameters g_1 , g_2 , and α , which enhances the structural flexibility of the model and enables the treatment of a broader range of dynamical behaviors and delay configurations within a single coherent analytical framework.

2. Basic lemmas

To simplify the notation throughout the paper, we introduce the following functions:

$$\Pi(\tau, \tau_0) = \int_{\tau_0}^{\tau} l_2^{-\frac{1}{g_2}}(s) ds, \quad \Pi_1(\tau, \tau_0) = \int_{\tau_0}^{\tau} \left(\frac{\Pi(s, \tau_0)}{l_1(s)} \right)^{1/g_1} ds,$$

and

$$\begin{aligned} \Pi_2(\tau, a_1(\tau)) &= \int_{a_1(\tau)}^{\tau} \Pi_*(\tau, s) ds, \text{ where } \Pi_*(\tau, s) = \left(\frac{\Pi(\tau, s)}{l_1(s)} \right)^{1/g_1}, \\ Q(\tau) &= \int_{b_1}^{b_2} q(\tau, \varsigma) d\varsigma. \end{aligned}$$

To derive the main results, we make use of the following auxiliary lemmas. Hereafter, L_1 and L_2 denote the classes of eventually positive solutions v of Eq (1.1) satisfying $U_1(v(\tau)) \geq 0$ and $U_1(v(\tau)) \leq 0$, respectively, for all sufficiently large τ .

Lemma 2.1. Assume that v is an eventually positive solution of Eq (1.1). Then there exists $\tau_1 \geq \tau_0$ such that, for all $\tau \geq \tau_1$,

$$U_2(v(\tau)) \geq 0,$$

and either

$$\text{Case (1): } v \in L_1, \text{ where } U_1(v(\tau)) \geq 0,$$

or

$$\text{Case (2): } v \in L_2, \text{ where } U_1(v(\tau)) \leq 0.$$

Proof. Let v be an eventually positive solution of Eq (1.1). Since $q(\tau, \varsigma) \geq 0$ and $v(a(\tau, \varsigma)) > 0$, Eq (1.2) implies that

$$(l_2(\tau) U_2(v(\tau)))' \leq 0.$$

Hence, $l_2(\tau) U_2(v(\tau))$ is nonincreasing. Suppose, on the contrary, that $U_2(v(\tau)) < 0$ for all sufficiently large τ . Then, by the monotonicity of $l_2(\tau) U_2(v(\tau))$ and the second condition in (A2), it follows that $l_1(\tau) U_1(v(\tau))$ tends to $-\infty$. Consequently, using the first condition in (A2), $v(\tau)$ must eventually become negative, which contradicts the eventual positivity of v . Therefore, $U_2(v(\tau)) \geq 0$ for all sufficiently large τ .

Furthermore, since $U_2(v(\tau)) \geq 0$, the quantity $l_1(\tau) U_1(v(\tau))$ is nondecreasing. Hence, $U_1(v(\tau))$ is eventually either nonnegative or nonpositive, which yields Cases (1) and (2), respectively. This completes the proof. $\square$

Definition 2.1. An eventually positive solution v of (1.1) is said to have property B_1 if it satisfies Case (1), and property B_2 if it satisfies Case (2).

Lemma 2.2. Assume that $v > 0$ is a nonoscillatory solution of Eq (1.1) such that $v(a(\tau, \varsigma)) > 0$ for all sufficiently large τ , and suppose that it satisfies property B_1 of Lemma 2.1 for all $\tau \geq \tau_1$, where $\tau_1 > \tau_0$. Then

$$\frac{v(\tau)}{\Pi_1(\tau, \tau_1)} \geq (l_2(\tau) U_2(v(\tau)))^{1/g_1 g_2} \text{ for } \tau \geq \tau_1. \quad (2.1)$$

Proof. Since v possesses property B_1 for all $\tau \geq \tau_1$, we have

$$\begin{aligned} U_1(v(\tau)) &\geq \frac{1}{l_1(\tau)} \int_{\tau_1}^{\tau} (l_1(s) U_1(v(s)))' ds \\ &= \frac{1}{l_1(\tau)} \int_{\tau_1}^{\tau} \frac{l_2^{\frac{1}{g_2}}(s) U_2^{\frac{1}{g_2}}(v(s))}{l_2^{\frac{1}{g_2}}(s)} ds, \end{aligned}$$

and so

$$(v'(\tau))^{g_1} \geq \frac{1}{l_1(\tau)} \Pi(\tau, \tau_1) l_2^{\frac{1}{g_2}}(\tau) U_2^{\frac{1}{g_2}}(v(\tau)).$$

That is,

$$v'(\tau) \geq \frac{\Pi^{\frac{1}{g_1}}(\tau, \tau_1)}{l_1^{\frac{1}{g_1}}(\tau)} (l_2(\tau) U_2(v(\tau)))^{\frac{1}{g_1 g_2}}. \quad (2.2)$$

Integrating from τ_1 to τ , we get

$$v(\tau) \geq (l_2(\tau) U_2(v(\tau)))^{\frac{1}{g_1 g_2}} \int_{\tau_1}^{\tau} \frac{\Pi^{\frac{1}{g_1}}(s, \tau_1)}{l_1^{\frac{1}{g_1}}(s)} ds = \Pi_1(\tau, \tau_1) \left((l_2(\tau) U_2(v(\tau)))^{\frac{1}{g_1 g_2}} \right).$$

This completes the proof. $\square$

Lemma 2.3. Assume that $v > 0$ is a nonoscillatory solution of Eq (1.1) such that $v(a(\tau, \varsigma)) > 0$ for all sufficiently large τ , and suppose that it satisfies property B_2 of Lemma 2.1 for all $\tau \geq \tau_1$, where $\tau_1 > \tau_0$. Then

$$\frac{v(a_+(\tau))}{\Pi_2(\tau, a_+(\tau))} \geq (l_2(\tau) U_2(v(\tau)))^{1/g_1 g_2}. \quad (2.3)$$

Proof. Since v possesses property B_2 for all $\tau \geq \tau_1$, we have

$$\begin{aligned} l_1(\tau) U_1(v(\tau)) - l_1(s) U_1(v(s)) &= \int_s^{\tau} (l_1(u) U_1(v(u)))' du \\ &= \int_s^{\tau} \frac{1}{l_2^{\frac{1}{g_2}}(s)} l_2^{\frac{1}{g_2}}(u) U_2^{\frac{1}{g_2}}(v(u)) du. \end{aligned}$$

Thus,

$$-(v'(s))^{g_1} \geq \frac{1}{l_1(s)} \Pi(\tau, s) l_2^{\frac{1}{g_2}}(\tau) U_2^{\frac{1}{g_2}}(v(\tau)),$$

and consequently,

$$-v'(s) \geq \frac{\Pi^{\frac{1}{g_1}}(\tau, s)}{l_1^{\frac{1}{g_1}}(s)} (l_2(\tau) U_2(v(\tau)))^{\frac{1}{g_1 g_2}} \geq \Pi_*(\tau, s) (l_2(\tau) U_2(v(\tau)))^{\frac{1}{g_1 g_2}}. \quad (2.4)$$

Integrating from $a_+(\tau)$ to τ , we get

$$-v(\tau) + v(a_+(\tau)) \geq (l_2(\tau) U_2(v(\tau)))^{\frac{1}{g_1 g_2}} \int_{a_+(\tau)}^{\tau} \Pi_*(\tau, s) ds,$$

that is,

$$\nu(a_+(\tau)) \geq \Pi_2(\tau, a_+(\tau)) (l_2(\tau) U_2(\nu(\tau)))^{\frac{1}{s_1 s_2}}.$$

The proof is complete. $\square$

Remark 2.1. By virtue of Lemma 2.3, together with (1.1) and the fact that ν is decreasing under property B_2 , we have

$$\begin{aligned} -(l_2(\tau) U_2(\nu(\tau)))' &= \int_{b_1}^{b_2} \nu^\alpha(a(\tau, \varsigma)) q(\tau, \varsigma) d\varsigma \\ &\geq \nu^\alpha(a_+(\tau)) \int_{b_1}^{b_2} q(\tau, \varsigma) d\varsigma \\ &\geq Q(\tau) (\Pi_2(\tau, a_+(\tau)))^\alpha (l_2(\tau) U_2(\nu(\tau)))^{\frac{\alpha}{s_1 s_2}}. \end{aligned}$$

Lemma 2.4. Assume that $\nu > 0$ is a nonoscillatory solution of Eq (1.1) such that $\nu(a(\tau, \varsigma)) > 0$, for all sufficiently large τ , suppose that it satisfies property B_2 of Lemma 2.1 for all $\tau \geq \tau_1$, where $\tau_1 > \tau_0$, and assume that (2.3) holds. Further, assume that there exists a constant $\beta \in (1, \infty)$ such that $\frac{\tau}{a_+(\tau)} \geq \beta$ for all sufficiently large τ . Then

$$\frac{\nu(a_+(\tau))}{\Pi_2(\beta a_+(\tau), a_+(\tau))} \geq (l_2(\beta a_+(\tau)) U_2(\nu(\beta a_+(\tau))))^{\frac{1}{s_1 s_2}}. \quad (2.5)$$

Proof. By integrating inequality (2.4) from $a_+(\tau)$ to $\beta a_+(\tau)$, we obtain (2.5). $\square$

3. Oscillation results

Theorem 3.1. Suppose that there exists a constant $\beta \in (1, \infty)$ such that $\tau/a_+(\tau) \geq \beta$ for all sufficiently large τ . If the differential equations

$$Y'(\tau) + (\Pi_1(a_-(\tau), \tau_1))^\alpha Q(\tau) (Y(a_-(\tau)))^{\frac{\alpha}{s_1 s_2}} = 0 \quad (3.1)$$

and

$$Z'(\tau) + (\Pi_2(\beta a_+(\tau), a_+(\tau)))^\alpha Q(\tau) (Z(\beta a_+(\tau)))^{\frac{\alpha}{s_1 s_2}} = 0 \quad (3.2)$$

are oscillatory, then every nontrivial solution of Eq (1.1) is oscillatory.

Proof. Suppose, to the contrary, that Eq (1.1) possesses a nonoscillatory solution ν . Without loss of generality, assume that $\hat{I}^{1/2}$ is eventually positive and that $\nu(a(\tau, \varsigma)) > 0$ for all sufficiently large τ . By Lemma 2.1, ν eventually satisfies either property B_1 or property B_2 . First, suppose that ν satisfies property B_1 . Since ν is increasing under property B_1 and $a_-(\tau) \leq a(\tau, \varsigma)$, it follows that $\nu(a(\tau, \varsigma)) \geq \nu(a_-(\tau))$. Hence, by Eq (1.2), we have

$$\begin{aligned} -(l_2(\tau) U_2(\nu(\tau)))' &= \int_{b_1}^{b_2} q(\tau, \varsigma) \nu^\alpha(a(\tau, \varsigma)) d\varsigma \\ &\geq \nu^\alpha(a_-(\tau)) \int_{b_1}^{b_2} q(\tau, \varsigma) d\varsigma \end{aligned}$$

$$= Q(\tau) v^\alpha(a_-(\tau)).$$

By Lemma 2.2, we obtain

$$v(a_-(\tau)) \geq \Pi_1(a_-(\tau), \tau_1) (l_2(a_-(\tau)) U_2(v(a_-(\tau))))^{\frac{1}{g_1 g_2}}.$$

Consequently,

$$-(l_2(\tau) U_2(v(\tau)))' \geq Q(\tau) \Pi_1(a_-(\tau), \tau_1)^\alpha (l_2(a_-(\tau)) U_2(v(a_-(\tau))))^{\frac{\alpha}{g_1 g_2}}.$$

Setting

$$Y(\tau) = l_2(\tau) U_2(v(\tau)),$$

we obtain

$$Y'(\tau) + Q(\tau) (\Pi_1(a_-(\tau), \tau_1))^\alpha (Y(a_-(\tau)))^{\frac{\alpha}{g_1 g_2}} \leq 0.$$

Hence, by [5, Lemma 2.1], the associated differential equation (3.1) admits a positive solution, contradicting the assumed oscillation of Eq (3.1).

Next, suppose that v satisfies property B_2 . Since v is decreasing under property B_2 and $a(\tau, \varsigma) \leq a_+(\tau)$, it follows that $v(a(\tau, \varsigma)) \geq v(a_+(\tau))$. Hence, by Eq (1.2), we have

$$\begin{aligned} -(l_2(\tau) U_2(v(\tau)))' &= \int_{b_1}^{b_2} q(\tau, \varsigma) v^\alpha(a(\tau, \varsigma)) d\varsigma \\ &\geq v^\alpha(a_+(\tau)) \int_{b_1}^{b_2} q(\tau, \varsigma) d\varsigma \\ &= Q(\tau) v^\alpha(a_+(\tau)). \end{aligned}$$

By Lemma 2.4, we obtain

$$v(a_+(\tau)) \geq \Pi_2(\beta a_+(\tau), a_+(\tau)) (l_2(\beta a_+(\tau)) U_2(v(\beta a_+(\tau))))^{\frac{1}{g_1 g_2}}.$$

Consequently,

$$-(l_2(\tau) U_2(v(\tau)))' \geq Q(\tau) (\Pi_2(\beta a_+(\tau), a_+(\tau)))^\alpha \left((l_2(\beta a_+(\tau)) U_2(v(\beta a_+(\tau))))^{\frac{\alpha}{g_1 g_2}} \right).$$

Setting

$$Z(\tau) = l_2(\tau) U_2(v(\tau)),$$

we obtain

$$Z'(\tau) + (\Pi_2(\beta a_+(\tau), a_+(\tau)))^\alpha Q(\tau) (Z(\beta a_+(\tau)))^{\frac{\alpha}{g_1 g_2}} \leq 0.$$

Therefore, by [5, Lemma 2.1], the associated differential equation (3.2) admits a positive solution, which contradicts the assumed oscillation of Eq (3.2).

Thus, both possible cases lead to contradictions. Hence every nontrivial solution of Eq (1.1) is oscillatory. This completes the proof. $\square$

Theorem 3.2. Suppose that the delay differential equation (3.1) is oscillatory, and that

$$\limsup_{\tau \rightarrow \infty} \int_{a_+(\tau)}^{\tau} Q(s) \Pi_1(a_+(\tau), a_+(s))^\alpha ds > 1, \quad (3.3)$$

where $\alpha = g_1 g_2$. Then every nontrivial solution of Eq (1.1) is oscillatory.

Proof. Suppose, to the contrary, that Eq (1.1) possesses a nonoscillatory solution v . Without loss of generality, assume that v is eventually positive and that $v(a(\tau, \varsigma)) > 0$ for all sufficiently large τ . By Lemma 2.1, v eventually satisfies either property B_1 or property B_2 .

First, suppose that v satisfies property B_1 . Proceeding as in the proof of Theorem 3.1, we obtain a contradiction with the assumed oscillation of Eq (3.1).

Next, suppose that v satisfies property B_2 . For $v \geq u \geq \tau_1$, we have

$$\begin{aligned} l_1(v) U_1(v(v)) - l_1(u) U_1(v(u)) &= \int_u^v (l_1(s) U_1(v(s)))' ds \\ &= \int_u^v \frac{l_2^{\frac{1}{g_2}}(s) U_2^{\frac{1}{g_2}}(v(s))}{l_2^{\frac{1}{g_2}}(s)} ds. \end{aligned}$$

Since $l_2(\tau) U_2(v(\tau))$ is nonincreasing and $l_1(v) U_1(v(v)) < 0$ under property B_2 , it follows that

$$-l_1(u) U_1(v(u)) \geq l_2^{\frac{1}{g_2}}(v) U_2^{\frac{1}{g_2}}(v(v)) \int_u^v \frac{1}{l_2^{\frac{1}{g_2}}(s)} ds.$$

Therefore,

$$-v'(u) \geq (l_2(v) U_2(v(v)))^{\frac{1}{g_1 g_2}} \frac{\left(\int_u^v \frac{1}{l_2^{\frac{1}{g_2}}(s)} ds \right)^{\frac{1}{g_1}}}{l_1^{\frac{1}{g_1}}(u)}.$$

Integrating the preceding inequality from u to v , we obtain

$$v(u) - v(v) \geq (l_2(v) U_2(v(v)))^{\frac{1}{g_1 g_2}} \int_u^v \left(\frac{1}{l_1(y)} \int_y^v \frac{1}{l_2^{\frac{1}{g_2}}(s)} ds \right)^{\frac{1}{g_1}} dy.$$

Hence,

$$v(u) \geq (l_2(v) U_2(v(v)))^{\frac{1}{g_1 g_2}} \Pi_1(v, u).$$

Now, setting $u = a_+(s)$ and $v = a_+(\tau)$ and using the monotonicity of a_+ , we obtain, for $\tau \geq s$ sufficiently large

$$v(a_+(s)) \geq (l_2(a_+(\tau)) U_2(v(a_+(\tau))))^{\frac{1}{g_1 g_2}} \Pi_1(a_+(\tau), a_+(s)). \quad (3.4)$$

On the other hand, integrating Eq (1.1) from $a_+(\tau)$ to τ gives

$$l_2(a_+(\tau)) U_2(v(a_+(\tau))) \geq \int_{a_+(\tau)}^{\tau} \int_{b_1}^{b_2} q(s, \varsigma) v^\alpha(a(s, \varsigma)) d\varsigma ds.$$

Since v is decreasing under property B_2 and $a(s, \varsigma) \leq a_+(s)$, we have $v(a(s, \varsigma)) \geq v(a_+(s))$. Therefore,

$$l_2(a_+(\tau)) U_2(v(a_+(\tau))) \geq \int_{a_+(\tau)}^{\tau} Q(s) v^\alpha(a_+(s)) ds.$$

Using (3.4), we obtain

$$l_2(a_+(\tau)) U_2(v(a_+(\tau))) \geq (l_2(a_+(\tau)) U_2(v(a_+(\tau))))^{\frac{\alpha}{g_1 g_2}} \int_{a_+(\tau)}^{\tau} Q(s) (\Pi_1(a_+(\tau), a_+(s)))^{\alpha} ds.$$

Since $\alpha = g_1 g_2$ and $l_2(a_+(\tau)) U_2(v(a_+(\tau))) > 0$ under property B_2 , dividing by $l_2(a_+(\tau)) U_2(v(a_+(\tau)))$ yields

$$\int_{a_+(\tau)}^{\tau} Q(s) (\Pi_1(a_+(\tau), a_+(s)))^{\alpha} ds \leq 1.$$

This contradicts condition (3.3). Hence, property B_2 is impossible as well.

Thus, both possible cases lead to contradictions. Therefore, every nontrivial solution of Eq (1.1) is oscillatory. This completes the proof. $\square$

Theorem 3.3. Suppose that $\alpha = g_1 g_2 = 1$, and that there exists a nondecreasing function $\phi \in C^1([\tau_0, \infty), (0, \infty))$ such that (3.3) holds and

$$\limsup_{\tau \rightarrow \infty} \int_{\tau_1}^{\tau} \left[Q(s) \phi(s) - \frac{(\phi'(s))^2 l_1^{\frac{1}{g_1}}(s)}{4\alpha a'_-(s) (\phi(s))^{2-\frac{1}{g_1 g_2}} \Pi^{\frac{1}{g_1}}(a_-(s), \tau_1)} \right] ds = \infty. \quad (3.5)$$

Then every nontrivial solution of Eq (1.1) is oscillatory.

Proof. Suppose, to the contrary, that Eq (1.1) possesses a nonoscillatory solution $\hat{y}_{1/2}$. Without loss of generality, assume that $\hat{y}_{1/2}$ is eventually positive and that $v(a(\tau, \varsigma)) > 0$ for all sufficiently large τ . By Lemma 2.1, v eventually satisfies either property B_1 or property B_2 . First, suppose that $\hat{y}_{1/2}$ satisfies property B_1 . Set

$$w(\tau) = \frac{\phi(\tau)}{v^{\alpha}(a_-(\tau))} l_2(\tau) U_2(v(\tau)) > 0.$$

By Lemma 2.2, the decreasing nature of $l_2(\tau) U_2(v(\tau))$, and (2.2), we obtain

$$\begin{aligned} w'(\tau) &= \phi(\tau) \frac{(l_2(\tau) U_2(v(\tau)))'}{v^{\alpha}(a_-(\tau))} + \phi'(\tau) \frac{l_2(\tau) U_2(v(\tau))}{v^{\alpha}(a_-(\tau))} \\ &\quad - \alpha \phi(\tau) \frac{(l_2(\tau) U_2(v(\tau))) v'(a_-(\tau)) a'_-(\tau)}{v^{\alpha+1}(a_-(\tau))} \\ &\leq -\phi(\tau) Q(\tau) + \frac{\phi'(\tau)}{\phi(\tau)} w(\tau) \\ &\quad - \alpha a'_-(\tau) \frac{\Pi^{\frac{1}{g_1}}(a_-(\tau), \tau_1) \phi(\tau) (l_2(\tau) U_2(v(\tau)))^{1+\frac{1}{g_1 g_2}}}{l_1^{\frac{1}{g_1}}(\tau) v^{\alpha+1}(a_-(\tau))} \\ &\leq -\phi(\tau) Q(\tau) + \frac{\phi'(\tau)}{\phi(\tau)} w(\tau) - \frac{\alpha a'_-(\tau)}{\phi^{\frac{1}{g_1 g_2}}(\tau)} \left(\frac{\Pi(a_-(\tau), \tau_1)}{l_1(\tau)} \right)^{\frac{1}{g_1}} w^2(\tau). \end{aligned}$$

Using the inequality $Au - Bu^2 \leq A^2/4B$, we obtain

$$w'(\tau) \leq -\phi(\tau) Q(\tau) + \frac{(\phi'(\tau))^2}{4\alpha a'_-(\tau) (\phi(\tau))^{2-\frac{1}{g_1 g_2}}} \left(\frac{l_1(\tau)}{\Pi(a_-(\tau), \tau_1)} \right)^{\frac{1}{g_1}}.$$

By integrating the above inequality from τ_1 to τ and using condition (3.5), we find that $w(\tau) < 0$ along a sequence tending to infinity, which contradicts $w(\tau) > 0$.

Next, suppose that $\hat{I}_{1/2}$ satisfies property B_2 . Following an argument analogous to that used in the proof of Theorem 3.2, condition (3.3) again leads to a contradiction. The proof is complete. $\square$

4. Examples

Example 4.1. Consider the third-order delay differential equation

$$\left(\tau \left[(\tau^{-2} v'(\tau))'\right]^3\right)' + \int_0^1 b \tau^{-2} v^{\frac{1}{3}} \left(\frac{(2+\zeta)\tau}{6}\right) d\zeta = 0, \quad \tau \geq 3, \quad (4.1)$$

where $b > 0$. In this case,

$$l_1(\tau) = \tau^{-2}, l_2(\tau) = \tau, g_1 = 1, g_2 = 3, \alpha = \frac{1}{3}, q(\tau, \zeta) = b\tau^{-2} > 0, \text{ and } a(\tau, \varsigma) = \frac{(2+\zeta)\tau}{6}.$$

Moreover,

$$a_-(\tau) = a(\tau, 0) = \frac{\tau}{3} \text{ and } a_+(\tau) = a(\tau, 1) = \frac{\tau}{2}.$$

Also,

$$Q(\tau) = \int_0^1 q(\tau, \zeta) d\zeta = b\tau^{-2}.$$

Hence, $a_-(\tau) \leq a(\tau, \zeta) \leq a_+(\tau) \leq \tau$, with $a'_-(\tau) = \frac{1}{3} > 0$, $a'_+(\tau) = \frac{1}{2} > 0$, and $a(\tau, \zeta) \rightarrow \infty$ as $\tau \rightarrow \infty$ for every $\zeta \in [0, 1]$. We verify the hypotheses of Theorem 3.1. Since

$$\Pi(\tau, 1) = \frac{3}{2}\tau^{(2/3)} - 1,$$

we obtain

$$\Pi_1(a_-(\tau), 1) = \frac{\tau^{11/3}}{22(3)^{5/3}} - \frac{\tau^3}{54} + \frac{1}{11}.$$

Hence, the first comparison equation becomes

$$Y'(\tau) + b\tau^{-2} \left(\Pi_1\left(\frac{\tau}{3}, 1\right)\right)^{1/3} Y^{1/9}\left(\frac{\tau}{3}\right) = 0. \quad (4.2)$$

Furthermore, choosing $\beta = \frac{4}{3}$ gives $\beta a_+(\tau) = \frac{2\tau}{3}$. A direct computation yields

$$\Pi_2(\beta a_+(\tau), a_+(\tau)) = C_0 \tau^{11/3},$$

where $C_0 > 0$. Therefore, the second comparison equation takes the form

$$Z'(\tau) + bC_0^{1/3} \tau^{-7/9} Z^{1/9}\left(\frac{2\tau}{3}\right) = 0. \quad (4.3)$$

Define

$$p_1(\tau) = b\tau^{-2} \left(\Pi_1\left(\frac{\tau}{3}, 1\right)\right)^{1/3} \text{ and } p_2(\tau) = bC_0^{1/3} \tau^{-7/9}.$$

Since $\Pi_1\left(\frac{\tau}{3}, 1\right) \geq C\tau^{11/3}$ for sufficiently large τ , we obtain

$$p_1 \geq C_1\tau^{-7/9} \text{ and } p_2 = C_2\tau^{-7/9},$$

where $C_1 > 0, C_2 > 0$.

For both comparison equations (4.2) and (4.3), the corresponding coefficients satisfy

$$\int_{\tau}^{\infty} p_i(s) ds = \infty, \quad i = 1, 2.$$

By the oscillation criterion for first-order nonlinear delay differential equations (see, [31]), both comparison equations (4.2) and (4.3) are oscillatory.

Consequently, all assumptions of Theorem 3.1 are satisfied, and every nontrivial solution of Eq (4.1) is oscillatory.

Example 4.2. Consider the third-order delay differential equation

$$\left(\tau \left[\left(\tau^{-2} v'(\tau) \right)' \right]^3 \right)' + b\tau^{-2} v^{\frac{1}{3}} \left(\frac{1}{3}\tau \right) = 0, \quad \tau \geq 1, \quad (4.4)$$

where

$$l_1(\tau) = \tau^{-2}, l_2(\tau) = \tau, g_1 = 1, g_2 = 3, \alpha = \frac{1}{3}, q(\tau) = b\tau^{-2}, b > 0, \text{ and } a(\tau) = \frac{1}{3}\tau.$$

Clearly, assumptions (A1)–(A3) are satisfied. A straightforward calculation yields

$$\Pi(\tau, 1) = \int_1^{\tau} \frac{1}{s^{\frac{1}{3}}} ds = \frac{3\left(\tau^{\frac{2}{3}} - 1\right)}{2}$$

and

$$\begin{aligned} \Pi_1(a(\tau), 1) &= \int_1^{a(\tau)} \frac{1}{l_1^{\frac{1}{g_1}}(s)} \Pi_{g_1}^{\frac{1}{g_1}}(s, 1) ds = \int_1^{\frac{\tau}{3}} \frac{1}{2} s^2 (3\tau^{\frac{2}{3}} - 3) ds \\ &= \frac{1}{2} \left(\frac{1}{33.3^{\frac{2}{3}}} \tau^{\frac{11}{3}} - \frac{1}{27} \tau^3 + \frac{2}{11} \right). \end{aligned}$$

Hence, Eq (3.1) reduces to

$$\eta'(\tau) + \frac{b}{2} \tau^{-2} \left(\frac{\tau^{\frac{11}{3}}}{33.3^{\frac{2}{3}}} - \frac{\tau^3}{27} + \frac{2}{11} \right)^{\frac{1}{3}} \eta^{\frac{1}{9}} \left(\frac{\tau}{3} \right) = 0. \quad (\text{A})$$

Now, let $\beta = 2$. Then

$$\Pi_*(\tau, s) = \frac{\Pi_{g_1}^{\frac{1}{g_1}}(\tau, s)}{l_1^{\frac{1}{g_1}}(s)} = \tau^2 \int_s^{\tau} u^{-\frac{1}{3}} du = \frac{3}{2} \tau^2 \left(\tau^{\frac{2}{3}} - s^{\frac{2}{3}} \right),$$

and

$$\begin{aligned}\Pi_1(\beta a(\tau), a(\tau)) &= \int_{a(\tau)}^{\beta a(\tau)} \Pi_*(\tau, s) ds = \int_{\frac{\tau}{3}}^{\frac{2\tau}{3}} \frac{3}{2} \tau^2 \left(\tau^{\frac{2}{3}} - s^{\frac{2}{3}} \right) ds \\ &= \frac{1}{2} \tau^{\frac{11}{3}} - \frac{1}{3^{\frac{5}{3}}} \left(2^{\frac{5}{3}} \tau^{\frac{11}{3}} - \tau^{\frac{11}{3}} \right).\end{aligned}$$

This implies that Eq (3.2) becomes

$$P'(\tau) + \frac{b \left(3^{\frac{5}{3}} - \left(2^{\frac{5}{3}} - 1 \right) \right)^{\frac{1}{3}} \tau^{7/9}}{2^{\frac{4}{3}} (3)^{\frac{5}{9}}} P^{\frac{1}{9}} \left(\frac{2}{3} \tau \right) = 0. \quad (\text{B})$$

From [30, Theorem 5], all solutions of (A) and (B) are oscillatory. Consequently, every solution of Eq (4.4) is oscillatory.

Remark 4.1. Figure 1 corresponding to $\theta = g_1 g_2 = 0.5, 1, 2$, and 3 illustrate persistent sign-changing behavior and repeated zero crossings of the solutions of the associated delay equations, confirming their oscillatory character. As (θ) increases, the oscillations become more frequent while remaining bounded, emphasizing the strong influence of the delayed nonlinear terms on the qualitative dynamics of the solutions. Moreover, the consistent oscillatory behavior of both auxiliary equations provides numerical evidence of the established comparison criteria. Consequently, the assumptions of the main theorem are satisfied, ensuring that every solution of (4.4) is oscillatory. These simulations further demonstrate the applicability and effectiveness of the derived conditions for higher-order nonlinear delay differential equations.

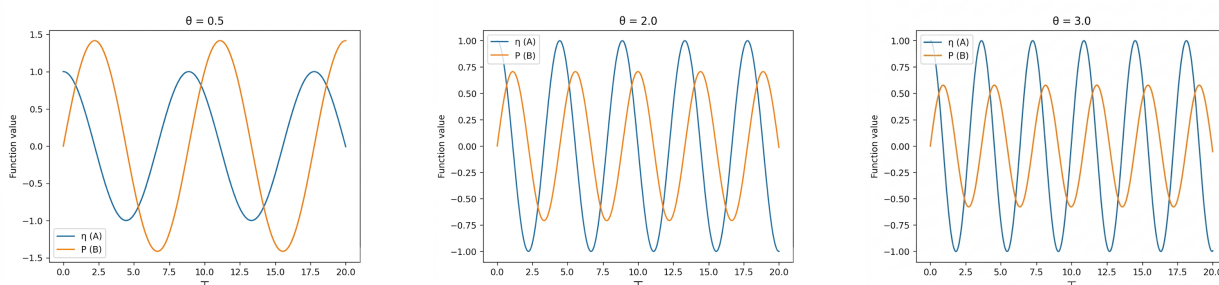


Figure 1. Oscillatory behavior of the associated comparison equations (A) and (B) for varying θ .

Example 4.3. Consider the third-order delay differential equation

$$\left(\tau^{-2} \left[\left(\frac{1}{9} \tau^{-2} (v'(\tau)) \right)^3 \right]' \right) + \delta \tau^{-7} v^3 \left(\frac{\tau}{2} \right) = 0, \quad \tau \geq 1, \quad (4.5)$$

where

$$l_1(\tau) = \frac{1}{9} \tau^{-2}, l_2(\tau) = \tau^{-2}, g_1 = 1, g_2 = 3, \alpha = 3, a(\tau) = 0.5\tau, \text{ and } q(\tau) = \delta \tau^{-7} \text{ for } \delta > 0.$$

It is straightforward to verify that conditions (A1)–(A3) are satisfied. Furthermore, we compute

$$\Pi(\tau, 1) = \int_1^\tau \bar{l}_2^{-\frac{1}{g_2}}(s) ds = \int_1^\tau s^{-2/3} ds = \frac{1}{5} \left(3\tau^{\frac{5}{3}} - 3 \right)$$

and

$$\begin{aligned} \Pi_1(a(\tau), 1) &= \int_1^{a(\tau)} \Pi_{g_1}^{\frac{1}{g_1}}(s, 1) g_1^{\frac{-1}{g_1}}(s) ds = \int_1^{\frac{\tau}{2}} \frac{1}{5} s^2 (3s^{\frac{5}{3}} - 3) ds \\ &= \frac{1}{5} \left(\frac{9\tau^{\frac{14}{3}}}{224(2)^{\frac{2}{3}}} - \frac{\tau^3}{8} - \frac{5}{14} \right). \end{aligned}$$

It is readily observed that (3.1) reduces to

$$\eta'(\tau) + \frac{42}{125} \tau^{-7} \left(\frac{9\tau^{\frac{14}{3}}}{7(2)^{\frac{17}{3}}} - \frac{\tau^3}{8} - \frac{5}{14} \right)^3 \eta(0.5\tau) = 0. \quad (4.6)$$

In view of [12], Eq (4.6) is oscillatory if

$$\lim_{\tau \rightarrow \infty} \int_{\frac{\tau}{2}}^\tau \frac{1}{125} \delta s^{-7} \left(\frac{9s^{\frac{14}{3}}}{7(2)^{\frac{17}{3}}} - \frac{s^3}{8} - \frac{5}{14} \right)^3 ds > \frac{1}{e}. \quad (4.7)$$

Moreover, we find that

$$\Pi(\tau, u) = \int_u^\tau \bar{l}_2^{-\frac{1}{g_2}}(s) ds = \int_u^\tau (s^{-2})^{\frac{-1}{3}} ds = \frac{1}{5} \left(3\tau^{\frac{5}{3}} - 3u^{\frac{5}{3}} \right).$$

That is,

$$\begin{aligned} \Pi_1(a(\tau), a(s)) &= \int_{a(s)}^{a(\tau)} \bar{l}_1^{\frac{-1}{g_1}}(y) \Pi_{g_1}^{\frac{1}{g_1}}(\tau, y) dy = \int_{\frac{s}{2}}^{\frac{\tau}{2}} \frac{3^3}{5} y^2 (\tau^{\frac{5}{3}} - y^{\frac{5}{3}}) dy \\ &= \frac{3^3 \tau^{\frac{5}{3}} (\tau^3 - s^3)}{8(5)^2} - \frac{3^4 (\tau^{\frac{14}{3}} - s^{\frac{14}{3}})}{7.2^{\frac{17}{3}} (5)^2}. \end{aligned}$$

Thus, Eq. (3.3) gives

$$\int_{a(\tau)}^\tau q(s) (\Pi_1(a(\tau), a(s)))^\alpha ds = \int_{\frac{\tau}{2}}^\tau \frac{3^9 \delta}{5^6 s^7} \left(\frac{\tau^{\frac{5}{3}} (\tau^3 - s^3)}{2^3} - \frac{3 (\tau^{\frac{14}{3}} - s^{\frac{14}{3}})}{7.2^{\frac{17}{3}}} \right)^3 ds > 1. \quad (4.8)$$

According to Theorem 3.2, all solutions of (4.5) are oscillatory.

Remark 4.2. As illustrated in Figure 2, the numerical simulations for different values of δ satisfying conditions (4.7) and (4.8) clearly confirm the oscillatory behavior of the associated first-order delay equation through persistent zero crossings and repeated sign changes. As δ increases, the oscillations become more pronounced, with higher frequency and amplified response, highlighting the strengthening effect of the delayed nonlinear term. Since conditions (4.7) and (4.8) ensure oscillation

of Eq (4.5), the observed numerical behavior provides strong computational support for the theoretical result, thereby confirming the oscillatory nature of the solutions to the original equation as well.

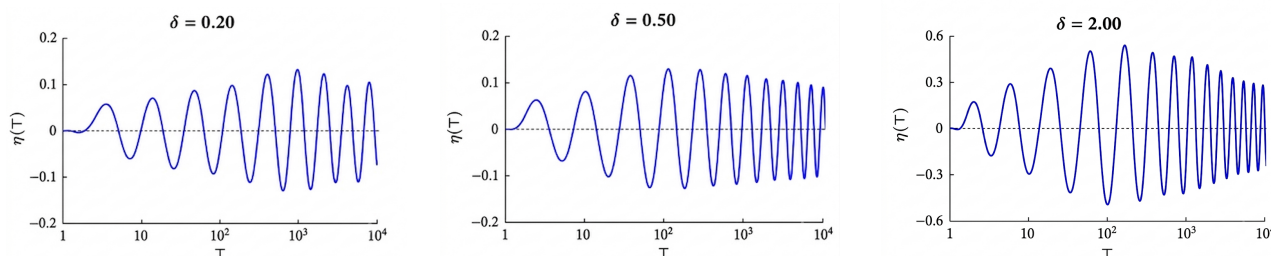


Figure 2. Oscillatory behavior associated with Eq (4.5) for different values of δ .

Example 4.4. Consider the third-order delay differential equation

$$\left(\tau^{-1} \left[\left(\tau^{-1} (\nu'(\tau))^{\frac{1}{3}} \right)' \right]^3 \right)' + \delta \tau^{-3} \nu \left(\frac{1}{3} \tau \right) = 0, \quad \tau \geq 1, \quad (4.9)$$

where

$$l_1(\tau) = \tau^{-1}, l_2(\tau) = \tau^{-1}, g_1 = \frac{1}{3}, g_2 = 3, \alpha = 1, a(\tau) = \frac{\tau}{3}, q(\tau) = \frac{\delta}{\tau^3}, \text{ and } \delta > 0.$$

It is straightforward to verify that assumptions (A1)–(A3) are satisfied. Substituting

$$\Pi(a(s), \tau_1) = \frac{3}{4} \left(3^{-4/3} \tau^{4/3} - 1 \right) \text{ and } \phi(\tau) = \tau^4$$

into (3.5), we obtain

$$\begin{aligned} & \limsup_{\tau \rightarrow \infty} \int_1^\tau \left(q(s) \phi(s) - \frac{(\phi'(s))^2 \Pi^{-1/g_1}(a(s), 1)}{4\alpha a'(s) l_1^{-1/g_1}(s) (\phi(s))^{2-\frac{1}{g_1 g_2}}} \right) ds \\ &= \limsup_{\tau \rightarrow \infty} \int_1^\tau \left(\delta s - \left(\frac{4^3}{3^2} \right) \frac{1}{s} \left(s^{\frac{4}{3}} - 1 \right)^{-3} \right) ds = \infty. \end{aligned}$$

It is readily verified that condition (3.3) is satisfied. Therefore, all the assumptions of Theorem 3.3 are satisfied. Consequently, every solution of Eq (4.9) is oscillatory.

5. Conclusions

This paper develops a unified framework for analyzing the oscillatory behavior of a broad class of nonlinear third-order functional differential equations with deviating arguments of the form (1.1). The main objective is to derive new and more general oscillation criteria that ensure oscillation of all solutions under sufficient conditions that are less restrictive yet widely applicable. By employing comparison principles, integral techniques, and Gronwall-type inequalities, the study establishes explicit and easily verifiable conditions that extend and refine existing results, particularly those

in [10] and [18], through a more flexible operator structure and relaxed assumptions on nonlinear terms and delay forms. The proposed framework accommodates linear, sublinear, superlinear, and quasilinear cases, significantly enlarging the admissible class of equations. A key feature of the approach is the independent and flexible role of the parameters g_1 , g_2 , and α , which enhances the ability to model more complex dynamical structures with various functional forms and delay effects. Overall, the results substantially broaden the scope of oscillation theory and improve existing criteria for higher-order differential systems, as illustrated through representative examples and supported by extensions to broader classes of equations.

Future work may further extend (1.1) to include neutral terms, multiple delays, and advanced arguments while preserving the structural flexibility of the parameters g_1 , g_2 , and α .

Author contributions

Conceptualization: B.Q. and Z.A.; validation: A.A., B.Q. and Z.A.; methodology: A.A., B.Q. and Z.A.; investigation: Z.A., B.Q.; visualization: R.C.S., B.Q. and Z.A.; resources: A.A. and B.Q.; writing—original draft preparation: A.A., Z.A. and B.Q.; data curation: B.Q. and Z.A.; writing—review and editing: A.A., B.Q., R.C.S. and Z.A.; supervision: R.C.S., B.Q. and Z.A.; project administration: B.Q. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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