



Research article**On a distance Laplacian analog of Brouwer's conjecture for several classes of graphs****Silin Huang***

College of Information Science and Technology, Jinan University, Guangzhou 510632, China

* **Correspondence:** Email: huangsilin@stu.jnu.edu.cn.

Abstract: Zhou et al. (2025) proposed a distance Laplacian analog of Brouwer's conjecture on partial sums of Laplacian eigenvalues, asserting that for any connected graph G , $\sum_{i=1}^r \partial_i^L(G) \leq W(G) + \binom{r+2}{3}$ holds, where $\partial_i^L(G)$ are the eigenvalues of the distance Laplacian matrix and $W(G)$ is the Wiener index. We prove this inequality for three broad classes of graphs, thereby improving and extending existing results. First, we prove that all connected graphs of diameter at most D satisfy the inequality once the order n satisfies $n \geq \lceil \frac{4}{9}(D+1)^3 \rceil$. Second, we show that the inequality holds for every diameter-2 graph with the only exceptions being $K_{1,3}$ at $r = 2$ and $K_{1,4}$ at $r = 3$. Third, we prove that if the maximum degree is $\Delta(G) = n - k$, then the inequality holds for all $n \geq N(k)$, where $N(2) = 10$ and $N(k) = \lceil 5(k-1)^{3/2} \rceil$ for $k \geq 3$. Our proofs rely on decomposing the distance Laplacian matrix into Laplacian matrices of auxiliary graphs whose edges are vertex pairs at distance at least a prescribed value, together with some classical eigenvalue inequalities.

Keywords: distance Laplacian matrix; Brouwer's conjecture; Wiener index; graph diameter; graph eigenvalue

Mathematics Subject Classification: 05C12, 05C50, 15A42

1. Introduction

Brouwer's conjecture for graph Laplacians asserts that, if $\lambda_1 \geq \cdots \geq \lambda_n = 0$ are the eigenvalues of the Laplacian matrix $L(G)$ of a graph $G(V, E)$ with $|E| = m$ edges, then

$$S_r(G) := \sum_{i=1}^r \lambda_i \leq m + \binom{r+1}{2}. \quad (1.1)$$

This conjecture was first introduced by Brouwer and Haemers in their monograph [1, p. 60], and has generated a large body of work. The cases $r = 2$ and $r = 3$ were proved by Haemers et al. [2] and

Wang et al. [3], respectively. Many further partial results and structural constraints are also known, including work of Du and Zhou [4], Chen [5, 6] and Ganie et al. [7].

Independently, the distance Laplacian matrix was introduced by Aouchiche and Hansen [8] as the Laplacian analog of the distance matrix:

$$D^L(G) = \text{diag}(\text{Tr}_G(v) : v \in V(G)) - D(G). \quad (1.2)$$

Here $\text{Tr}_G(v) = \sum_{u \in V(G)} d_G(u, v)$ is the transmission of the vertex v , and $D(G) = (d_G(u, v))_{u, v \in V(G)}$ is the distance matrix of G , where $d_G(u, v)$ is the distance between vertices u and v in the graph G .

Distance Laplacian spectra have since been studied from several perspectives, including spectral properties, eigenvalue multiplicities, extremal eigenvalues, and cospectrality; see, for example, [9–13] and the survey [14].

In the setting of distance Laplacian matrices, a natural analog of Brouwer's conjecture is to determine an upper bound for

$$U_r(G) := \sum_{i=1}^r \partial_i^L(G), \quad (1.3)$$

where $\partial_1^L(G) \geq \dots \geq \partial_r^L(G)$ are the r largest eigenvalues of $D^L(G)$. Motivated by the identity $\text{tr}(D^L(G)) = 2W(G)$, Zhou et al. [15] proposed the following distance Laplacian analog of Brouwer's conjecture:

$$U_r(G) = \sum_{i=1}^r \partial_i^L(G) \leq W(G) + \binom{r+2}{3}, \quad (1.4)$$

where

$$W(G) = \sum_{\{u, v\} \subseteq V(G)} d_G(u, v) \quad (1.5)$$

is the Wiener index.

Before this conjecture was formulated, Pirzada and Khan [16] had already studied the sum of the largest distance Laplacian eigenvalues and obtained bounds in terms of the order, diameter, and Wiener index. Zhou et al. [15] then proposed this conjecture and proved it for diameter-1 graphs and for some connected diameter-2 graphs with large maximum degree.

Recently, Mushtaq et al. [17] proved it for several diameter-3 and diameter-4 graph families, including graphs in Γ_n (diameter-3 graphs that have a spanning tree of diameter at most 3) and graphs with $\Delta = n - 2$ for $n \geq 95$, sun graphs $S(a, a)$ for $a \geq 12$, and partial sun graphs $S(a, k)$ for $a \geq 450$.

This paper improves and extends some of these results. In the remainder of this paper, we refer to the distance Laplacian analog of Brouwer's conjecture, (1.4), as the target inequality.

The paper is organized as follows.

In Section 2, we collect notation and the eigenvalue inequalities used throughout the paper.

In Section 3, we prove that all connected graphs of order n and diameter at most $D \geq 2$ satisfy the target inequality once

$$n \geq \left\lceil \frac{4}{9}(D+1)^3 \right\rceil. \quad (1.6)$$

In Section 4, we solve the case of diameter-2 graphs completely. We prove that apart from $K_{1,3}$ at $r = 2$ and $K_{1,4}$ at $r = 3$, every connected diameter-2 graph satisfies the target inequality.

In Section 5, we prove that if a graph has maximum degree $\Delta(G) = n - k$, then the target inequality holds once $n \geq N(k)$, where $N(2) = 10$ and $N(k) = \lceil 5(k-1)^{3/2} \rceil$ for $k \geq 3$.

2. Preliminaries

All graphs $G(V, E)$ in this paper are finite, simple, undirected, and connected, unless stated otherwise. For a vertex $u \in V(G)$, let $\deg_G(u)$ denote its degree in G . For vertices $u, v \in V(G)$, let $d_G(u, v)$ denote their distance in G . Then, the distance matrix of G is $D(G) = (d_G(u, v))$, the transmission of v is $\text{Tr}_G(v) = \sum_u d_G(u, v)$, and the distance Laplacian matrix of G is

$$D^L(G) = \text{diag}(\text{Tr}_G(v) : v \in V(G)) - D(G). \quad (2.1)$$

The eigenvalues of $D^L(G)$ are denoted by

$$\partial_1^L(G) \geq \partial_2^L(G) \geq \cdots \geq \partial_n^L(G) = 0, \quad (2.2)$$

where the last equality follows from [9, Theorem 2.2]. The partial sum of these eigenvalues is denoted by

$$U_r(G) = \sum_{i=1}^r \partial_i^L(G), \quad (2.3)$$

while the Wiener index of G is denoted by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v). \quad (2.4)$$

The target inequality, which is an analog of the inequality in Brouwer's conjecture, is

$$U_r(G) \leq W(G) + \binom{r+2}{3} \quad (1 \leq r \leq n). \quad (2.5)$$

We shall use two standard eigenvalue inequalities.

Lemma 2.1 (Ky Fan inequality). *For Hermitian matrices A, B and $1 \leq r \leq n$,*

$$\sum_{i=1}^r \lambda_i(A + B) \leq \sum_{i=1}^r \lambda_i(A) + \sum_{i=1}^r \lambda_i(B). \quad (2.6)$$

Proof. This is a direct corollary of [18, Theorem 1]. □

Lemma 2.2. *Let H be a graph with degree sequence $d_1, \dots, d_n$, $m(H)$ edges, and Laplacian eigenvalues $\lambda_1(H) \geq \cdots \geq \lambda_n(H) = 0$. Then*

$$\sum_{i=1}^r \lambda_i(H) \leq \sum_{v \in V(H)} \min\{d_H(v), r\} \leq m(H) + \frac{nr}{2} \quad (2.7)$$

for $1 \leq r \leq n$.

Proof. The first inequality is the Grone–Merris conjecture [19], which was proved by Bai [20]. The second follows from $\min\{d, r\} \leq (d + r)/2$ and $\sum_v d_H(v) = 2m(H)$. □

Lemma 2.3. Let H be a graph and let $X \subseteq V(H)$ be a vertex cover of H . Then, for $1 \leq r \leq |V(H)|$,

$$\sum_{i=1}^r \lambda_i(L(H)) \leq m(H) + |X|r. \quad (2.8)$$

Proof. By the first inequality in Lemma 2.2,

$$\sum_{i=1}^r \lambda_i(L(H)) \leq \sum_{u \in V(H)} \min\{d_H(u), r\}. \quad (2.9)$$

Since X is a vertex cover, $V(H) \setminus X$ is independent, and hence

$$\sum_{u \notin X} d_H(u) \leq m(H). \quad (2.10)$$

Also $\sum_{u \in X} \min\{d_H(u), r\} \leq |X|r$. The assertion follows. $\square$

For the case $r = n$, the target inequality follows from a simple estimate as soon as the order of the graph is large compared with its diameter.

Lemma 2.4. Let G be a connected graph of order n and diameter at most D . If $n \geq 3D$, then the target inequality holds for $r = n$.

Proof. Since $U_n(G) = \text{tr}(D^L(G)) = 2W(G)$, it is enough to show $W(G) \leq \binom{n+2}{3}$. Indeed,

$$W(G) \leq D \binom{n}{2} \leq \frac{n}{3} \binom{n}{2} = \frac{n^2(n-1)}{6} \leq \binom{n+2}{3}. \quad (2.11)$$

This completes the proof. $\square$

3. Graphs with bounded diameter

In this section we prove the target inequality for all connected graphs whose diameter is bounded by a fixed constant D , provided that the order n is sufficiently large in terms of D . The main idea is to decompose the distance Laplacian matrix into ordinary Laplacian matrices of graphs recording which vertex pairs are at distance at least s , and then reduce the target inequality to a numerical estimate.

For an integer $2 \leq s \leq D$, we define $Y_s = Y_s(G)$ to be a new graph on $V(G)$ in which two vertices u and v are adjacent if and only if $d_G(u, v) \geq s$. Equivalently, the edge set of Y_s consists of all pairs of vertices whose distance in G is at least s . For convenience, we call it the distance-at-least- s graph of G .

Note that the t -th power of a graph G , denoted by G^t , is defined to be a graph with the same vertex set as G in which two distinct vertices are adjacent if and only if their distance in G is at most t . Therefore Y_s is equal to $\overline{G^{s-1}}$, the complement of the $(s-1)$ -th power of G .

We prove that $D(G)$ can be expressed in terms of I (the identity matrix), J (the all-ones matrix) and sums of $A(Y_s)$ (the adjacency matrix of Y_s), while $D^L(G)$ can be expressed in terms of I , J and sums of $L(Y_s)$ (the Laplacian matrix of Y_s).

Lemma 3.1. *If G is connected and $\text{diam}(G) \leq D$, then*

$$D(G) = J - I + \sum_{s=2}^D A(Y_s), \quad D^L(G) = nI - J + \sum_{s=2}^D L(Y_s), \quad (3.1)$$

where $A(H)$ and $L(H)$ denote the adjacency matrix and the Laplacian matrix of a graph H , respectively.

Proof. Let A_i be the i -th distance matrix of G , as in [21, Section 2.3], whose (u, v) -entry is 1 when $d_G(u, v) = i$, and is 0 otherwise. Then

$$J - I = \sum_{i=1}^D A_i, \quad A(Y_s) = \sum_{i=s}^D A_i. \quad (3.2)$$

Therefore

$$J - I + \sum_{s=2}^D A(Y_s) = \sum_{i=1}^D A_i + \sum_{s=2}^D \sum_{i=s}^D A_i = \sum_{i=1}^D iA_i = D(G). \quad (3.3)$$

This is the first identity.

For the second identity, denote by $\mathbf{1}$ the all-ones vector. By definition,

$$D^L(G) = \text{diag} \left(\sum_{u \in V(G)} d_G(u, v) : v \in V(G) \right) - D(G) = \text{diag}(D(G)\mathbf{1}) - D(G). \quad (3.4)$$

The map $f : M \mapsto \text{diag}(M\mathbf{1}) - M$ is linear, so it may be applied to each summand of the first identity. Since $(J - I)\mathbf{1} = (n - 1)\mathbf{1}$, we have

$$f(J - I) = \text{diag}((J - I)\mathbf{1}) - (J - I) = (n - 1)I - (J - I) = nI - J. \quad (3.5)$$

For each s ,

$$f(A(Y_s)) = \text{diag}(A(Y_s)\mathbf{1}) - A(Y_s) = L(Y_s). \quad (3.6)$$

Therefore

$$\begin{array}{rcl} D(G) & = & (J - I) + \sum_{s=2}^D A(Y_s) \\ \downarrow f & & \downarrow f \quad \downarrow f \\ D^L(G) & = & (nI - J) + \sum_{s=2}^D L(Y_s). \end{array} \quad (3.7)$$

This gives the second identity. $\square$

These graphs also give a convenient expression for the Wiener index.

Lemma 3.2. *If G is connected and $\text{diam}(G) \leq D$, then*

$$W(G) = \binom{n}{2} + \sum_{s=2}^D m(Y_s). \quad (3.8)$$

Proof. Each vertex pair $\{u, v\}$ contributes one unit to $W(G)$, and then contributes one additional unit for every $s \geq 2$ not exceeding $d_G(u, v)$. Thus

$$W(G) = \sum_{\{u, v\} \subseteq V(G)} 1 + \sum_{s=2}^D |\{\{u, v\} : d_G(u, v) \geq s\}| = \binom{n}{2} + \sum_{s=2}^D m(Y_s). \quad (3.9)$$

$\square$

The next lemma converts bounds for the Laplacian eigenvalues of the graphs Y_s into the target inequality.

Lemma 3.3. *Let G be connected, have order n , and satisfy $\text{diam}(G) \leq D$. Suppose that for some number B ,*

$$\sum_{s=2}^D \sum_{i=1}^r \lambda_i(L(Y_s)) \leq \sum_{s=2}^D m(Y_s) + Br \quad (3.10)$$

holds for $1 \leq r \leq n-1$. Then the target inequality holds for such r once

$$\binom{n}{2} + \binom{r+2}{3} - (n+B)r \geq 0. \quad (3.11)$$

Proof. By Lemma 3.1, $D^L(G) = nI - J + \sum_{s=2}^D L(Y_s)$. The matrix $nI - J$ has sum of its top r eigenvalues equal to rn for $r \leq n-1$. Lemma 2.1 then gives

$$U_r(G) \leq rn + \sum_{s=2}^D \sum_{i=1}^r \lambda_i(L(Y_s)) \leq \sum_{s=2}^D m(Y_s) + (n+B)r. \quad (3.12)$$

By Lemma 3.2, $W(G) = \binom{n}{2} + \sum_{s=2}^D m(Y_s)$. Therefore

$$\begin{aligned} W(G) - U_r(G) + \binom{r+2}{3} &\geq W(G) - \sum_{s=2}^D m(Y_s) - (n+B)r + \binom{r+2}{3} \\ &= \binom{n}{2} + \binom{r+2}{3} - (n+B)r \geq 0, \end{aligned} \quad (3.13)$$

which is the target inequality. $\square$

By the lemma above, we can derive the main result of this section.

Theorem 3.4. *Let $D \geq 2$. If G is connected, has order n , satisfies $\text{diam}(G) \leq D$, and*

$$n \geq \left\lceil \frac{4}{9}(D+1)^3 \right\rceil, \quad (3.14)$$

then the target inequality holds for every $1 \leq r \leq n$.

Proof. We first treat the range $1 \leq r \leq n-1$. Lemma 2.2 gives

$$\sum_{s=2}^D \sum_{i=1}^r \lambda_i(L(Y_s)) \leq \sum_{s=2}^D m(Y_s) + \frac{(D-1)nr}{2}. \quad (3.15)$$

Write $c = (D+1)/2$. By Lemma 3.3 with $B = (D-1)n/2$, the target inequality follows once

$$\binom{n}{2} + \binom{r+2}{3} - cnr = \frac{n(n-1)}{2} + \frac{r(r+1)(r+2)}{6} - cnr \geq 0 \quad (1 \leq r \leq n-1). \quad (3.16)$$

Let $P_D(n, r)$ denote 6 times the left-hand side of (3.16), that is,

$$P_D(n, r) := 3n(n-1) + r(r+1)(r+2) - 6cnr \quad (3.17)$$

$$= 3n(n-1-2cr) + r(r+1)(r+2) \quad (3.18)$$

$$= 3n^2 - 3n + r^3 + 3r^2 + 2r - 6cnr. \quad (3.19)$$

By (3.14) and $D+1 = 2c$, we obtain

$$n \geq \frac{4}{9}(D+1)^3 = \frac{32}{9}c^3. \quad (3.20)$$

First suppose $r < \sqrt{n}$. Since $D \geq 2$, we have $c \geq 3/2$. Consider the function

$$f(c) := \frac{32}{9}c^3 - \frac{8\sqrt{2}}{3}c^{5/2} - 1, \quad f'(c) = \frac{32}{3}c^2 - \frac{20\sqrt{2}}{3}c^{3/2} \implies f'(c) \geq 0 \text{ when } c \geq \frac{25}{32}. \quad (3.21)$$

Therefore f is increasing for $c \geq 3/2$. Also, $f(3/2) = 11 - 6\sqrt{3} > 0$. So $f(c) > 0$ for $c \geq 3/2$. Consider another function

$$g(x) := x - 2c\sqrt{x}, \quad g'(x) = 1 - \frac{c}{\sqrt{x}} \implies g'(x) \geq 0 \text{ when } x \geq c^2. \quad (3.22)$$

Therefore g is increasing for $x \geq c^2$. Hence

$$n - 1 - 2c\sqrt{n} \geq \frac{32}{9}c^3 - 1 - 2c\sqrt{\frac{32}{9}c^3} = \frac{32}{9}c^3 - \frac{8\sqrt{2}}{3}c^{5/2} - 1 \geq 0, \quad (3.23)$$

where the first inequality follows from the monotonicity of g , since $n \geq 32c^3/9 \geq c^2$, while the last inequality follows from $f(c) > 0$. Therefore

$$P_D(n, r) = 3n(n-1-2cr) + r(r+1)(r+2) \quad (3.24)$$

$$> 3n(n-1-2c\sqrt{n}) + r(r+1)(r+2) \geq 0, \quad (3.25)$$

which proves (3.16) for $r < \sqrt{n}$. Now suppose $r \geq \sqrt{n}$. Since the minimum of the function $x^3 - Bx$ over $x \geq 0$ is $-2B^{3/2}/(3\sqrt{3})$, we have

$$r^3 - 6cnr \geq \frac{-2(6cn)^{3/2}}{3\sqrt{3}} = -4\sqrt{2}c^{3/2}n^{3/2} \quad (3.26)$$

for every $r \geq 0$. Also $n \geq 32c^3/9$ implies

$$n^4 \geq \frac{32c^3n^3}{9} \implies n^2 - \frac{4\sqrt{2}}{3}c^{3/2}n^{3/2} \geq 0 \implies 3n^2 - 4\sqrt{2}c^{3/2}n^{3/2} \geq 0. \quad (3.27)$$

Adding (3.26) and (3.27) gives

$$r^3 + 3n^2 - 6cnr \geq 0 \implies 3n^2 - 6cnr \geq -r^3. \quad (3.28)$$

Thus

$$P_D(n, r) = (3n^2 - 6cnr) + (-3n + r^3 + 3r^2 + 2r) \geq -3n + 3r^2 + 2r \geq 2r \geq 0, \quad (3.29)$$

where the second-to-last inequality follows from $r \geq \sqrt{n} \implies 3r^2 \geq 3n$. This proves (3.16) for $r \geq \sqrt{n}$, and hence the target inequality, for every $1 \leq r \leq n-1$.

Finally, the case $r = n$ follows directly from Lemma 2.4, because

$$D \geq 2 \implies n - 3D \geq \left\lceil \frac{4}{9}(D+1)^3 \right\rceil - 3D \geq \frac{4}{9}(D+1)^3 - 3D \geq \frac{4}{9} \cdot (2+1)^3 - 3 \cdot 2 = 6 > 0, \quad (3.30)$$

where the last inequality follows from the fact that the function $h(D) := \frac{4}{9}(D+1)^3 - 3D$ has positive derivative $h'(D) = \frac{4}{3}(D+1)^2 - 3$ when $D \geq 2$. $\square$

By taking $D = 3, 4, 5$, we have the following corollary.

Corollary 3.5. *The target inequality holds for every connected graph G of order n in each of the following cases:*

- (1) $\text{diam}(G) \leq 3$ and $n \geq 29$;
- (2) $\text{diam}(G) \leq 4$ and $n \geq 56$;
- (3) $\text{diam}(G) \leq 5$ and $n \geq 96$.

Corollary 3.5 considerably strengthens the diameter-3 and diameter-4 results of Mushtaq et al. [17]. Their diameter-3 results apply to the special class Γ_n for $n \geq 95$, while our corollary covers all connected graphs of diameter at most 3 once $n \geq 29$. For diameter 4, they proved the inequality for sun graphs and partial sun graphs under additional parameter restrictions, whereas our corollary applies to every connected graph of diameter at most 4 once $n \geq 56$.

4. Diameter-2 graphs

In this section, we fully determine the diameter-2 case. The key technique is that, for such graphs, the distance Laplacian matrix can be written in terms of the Laplacian matrices of their graph complements. This allows us to reduce most values of r to known cases of the original Brouwer's conjecture and then handle the remaining small cases directly.

We first record the validity of the original Brouwer's conjecture for several values of r , which will be useful later.

Lemma 4.1. *Let H be a graph of order n , size m , and ordinary Laplacian eigenvalues $\lambda_1(H) \geq \dots \geq \lambda_n(H) = 0$. Put $S_r(H) = \sum_{i=1}^r \lambda_i(H)$. Then*

$$S_r(H) \leq m + \binom{r+1}{2} \quad (4.1)$$

holds for $r \in \{2, 3, n-1, n\}$. Moreover, if $n \geq 3$, then this inequality also holds for $r = n-2$; if $n \geq 5$, then this inequality also holds for $r = n-3$.

Proof. The case $r = 2$ was proved by Haemers et al. [2]. The case $r = 3$ was proved by Wang et al. [3]. The cases $r = n-2, n-3$ follow from Chen [6, Corollaries 3.2 and 3.3]. For $r = n-1$, since $\lambda_n(H) = 0$, we have $S_{n-1}(H) = 2m$, and the desired inequality follows from $2m \leq m + \binom{n}{2}$. For $r = n$, we also have $S_n(H) = 2m$, and the desired inequality follows from $2m \leq m + \binom{n+1}{2}$. $\square$

For diameter-2 graphs, the distance Laplacian matrix can be expressed through the Laplacian matrix of their complement graphs.

Lemma 4.2. *Let G be a connected graph of order n and diameter 2, and put $H = \overline{G}$, the complement of G . Then*

$$D(G) = J - I + A(H), \quad D^L(G) = nI - J + L(H). \quad (4.2)$$

Consequently,

$$\partial_i^L(G) = n + \lambda_i(H) \quad (1 \leq i \leq n-1), \quad \partial_n^L(G) = 0. \quad (4.3)$$

Proof. In a graph with diameter 2, nonadjacent vertex pairs are exactly the pairs at distance 2. Hence

$$D(G) = A(G) + 2A(H) = J - I - A(H) + 2A(H) = J - I + A(H). \quad (4.4)$$

Also

$$\text{Tr}_G(v) = \deg_G(v) + 2 \deg_H(v) = n - 1 + \deg_H(v), \quad (4.5)$$

and so

$$D^L(G) = \text{diag}(n - 1 + \deg_H(v)) - (J - I + A(H)) = nI - J + L(H). \quad (4.6)$$

Since J vanishes on $\mathbf{1}^\perp$, the subspace $\mathbf{1}^\perp$ is invariant under $D^L(G)$, and there $D^L(G)$ acts as $nI + L(H)$. On the span of $\mathbf{1}$, it acts as zero. The spectral formula then follows. $\square$

This also gives the relations between $U_r(G)$ and $S_r(H)$.

Lemma 4.3. *Let G be a connected graph of order n and diameter 2, and put $H = \overline{G}$. Then*

$$U_r(G) = rn + S_r(H) \quad (1 \leq r \leq n-1). \quad (4.7)$$

Proof. This follows directly from Lemma 4.2. $\square$

The Wiener index can be expressed through the graph order and the size of the complement graph.

Lemma 4.4. *Let G be a connected graph of order n and diameter 2, and put $H = \overline{G}$ and $m = m(H)$. Then*

$$W(G) = \binom{n}{2} + m. \quad (4.8)$$

Proof. The m edges of H are exactly the nonedges of G , hence exactly the vertex pairs at distance 2 in G . Therefore

$$W(G) = \left(\binom{n}{2} - m \right) + 2m = \binom{n}{2} + m, \quad (4.9)$$

completing the proof. $\square$

We will convert a bound for $S_r(\overline{G})$ into the target inequality by the following lemma.

Lemma 4.5. *Let G be a connected graph of order n and diameter 2, and put $H = \overline{G}$ and $m = m(H)$. Let $1 \leq r \leq n-1$. If*

$$S_r(H) \leq m + T, \quad (4.10)$$

then the target inequality holds for such an r once

$$\binom{n}{2} + \binom{r+2}{3} - rn - T \geq 0. \quad (4.11)$$

Proof. Lemma 4.3 gives $U_r(G) = rn + S_r(H)$ and Lemma 4.4 gives $W(G) = \binom{n}{2} + m$. Hence if (4.11) holds, we have

$$U_r(G) \leq rn + m + T \leq m + \binom{n}{2} + \binom{r+2}{3} = W(G) + \binom{r+2}{3}, \quad (4.12)$$

which is the target inequality. $\square$

The next theorem handles the range $2 \leq r \leq n-1$ once the order is at least 8.

Theorem 4.6. *Let G be a connected diameter-2 graph of order $n \geq 8$. Then the target inequality holds for every $2 \leq r \leq n-1$.*

Proof. Put $H = \overline{G}$ and $m = m(H)$. Lemma 2.2 gives

$$S_r(H) \leq m + \frac{nr}{2}. \quad (4.13)$$

Therefore by Lemma 4.5, it is enough to prove

$$\binom{n}{2} + \binom{r+2}{3} - \frac{3nr}{2} \geq 0. \quad (4.14)$$

Let $F(n, r)$ denote 6 times the left-hand side of this inequality. It then suffices to prove

$$F(n, r) := r^3 + 3r^2 + (2 - 9n)r + 3n^2 - 3n \geq 0. \quad (4.15)$$

For $n \geq 9$, write $x = r + 1$. Then

$$F(n, r) = 3n^2 + 6n + x^3 - (9n + 1)x. \quad (4.16)$$

Since the minimum of $x^3 - Bx$ over $x \geq 0$ is $-2B^{3/2}/(3\sqrt{3})$, it is enough to show

$$H(n) := 3n^2 + 6n - \frac{2(9n + 1)^{3/2}}{3\sqrt{3}} \geq 0. \quad (4.17)$$

In fact, the function H is increasing for $n \geq 9$, because

$$H'(n) = 6n + 6 - 3\sqrt{3}(9n + 1)^{1/2} > 0, \quad (4.18)$$

which follows from

$$(2n + 2)^2 > 3(9n + 1), \quad (4.19)$$

which holds for $n \geq 9$. Also, $H(9) = 297 - 164\sqrt{246}/9 > 0$. Therefore (4.15) is true for all $n \geq 9$. For $n = 8$, direct calculation gives

r	2	3	4	5	6	7
$F(8, r)$	48	12	0	18	72	168

(4.20)

Therefore (4.15) also holds. $\square$

The case $2 \leq r \leq n-1$ with $n \in \{6, 7\}$ is covered by Lemma 4.1.

Theorem 4.7. Let G be a connected diameter-2 graph of order $n \in \{6, 7\}$. Then the target inequality holds for every $2 \leq r \leq n - 1$.

Proof. Put $H = \overline{G}$ and $m = m(H)$. By Lemma 4.1,

$$S_r(H) \leq m + \binom{r+1}{2} \quad (2 \leq r \leq n-1) \quad (4.21)$$

is true for $n \in \{6, 7\}$. Therefore by Lemma 4.5, it remains to check

$$\binom{n}{2} + \binom{r+2}{3} - rn - \binom{r+1}{2} \geq 0. \quad (4.22)$$

In fact, direct calculation gives

$$n = 6 \left| \begin{array}{cccc} r=2 & r=3 & r=4 & r=5 \\ 4 & 1 & 1 & 5 \end{array} \right| \quad n = 7 \left| \begin{array}{ccccc} r=2 & r=3 & r=4 & r=5 & r=6 \\ 8 & 4 & 3 & 6 & 14 \end{array} \right| \quad (4.23)$$

Therefore the target inequality holds. $\square$

We now combine the preceding theorems with an analysis of the small-order graphs to obtain the full diameter-2 graph classification.

Theorem 4.8. Let G be a connected graph of diameter 2. Then the target inequality holds for every valid r , except for $K_{1,3}$ at $r = 2$ and $K_{1,4}$ at $r = 3$ (see Figure 1).

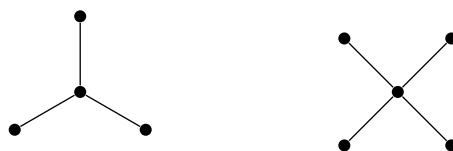


Figure 1. Illustrations of $K_{1,3}$ and $K_{1,4}$, respectively.

Proof. Put $H = \overline{G}$ and $m = m(H)$ and suppose $n \geq 6$.

For $r = 1$, the bound $\lambda_1(H) \leq n$ gives $U_1(G) \leq 2n$. Since a diameter-2 graph is not complete, we have $m \geq 1$. Hence for $n \geq 5$, we have

$$W(G) + \binom{1+2}{3} - U_1(G) \geq \left(\binom{n}{2} + m \right) + 1 - 2n \geq \binom{n}{2} + 2 - 2n = \frac{(n-1)(n-4)}{2} > 0, \quad (4.24)$$

so this proves the $r = 1$ case.

For $2 \leq r \leq n - 1$, the assertion follows from Theorem 4.6 if $n \geq 8$, and from Theorem 4.7 if $n \in \{6, 7\}$.

For $r = n$, the trivial Wiener bound gives

$$W(G) \leq 2 \binom{n}{2} \leq \binom{n+2}{3} = \binom{r+2}{3}. \quad (4.25)$$

Hence $U_n(G) = 2W(G) \leq W(G) + \binom{r+2}{3}$.

It remains only to discuss $n \leq 5$. For $n = 3$, the only diameter-2 graph is $P_3 = K_{1,2}$, whose distance Laplacian spectrum is $\{5, 3, 0\}$. The target inequality follows immediately.

Now let $n = 4$, and again put $H = \overline{G}$ and $m = m(H) \geq 1$. For $r = 1$, the bound $\lambda_1(H) \leq 4$ gives

$$U_1(G) = 4 + \lambda_1(H) \leq 8 \leq \binom{4}{2} + m + 1 = W(G) + 1 = W(G) + \binom{1+2}{3}. \quad (4.26)$$

For $r = 3$, since $\lambda_4(H) = 0$, we have $S_3(H) = \sum_{i=1}^3 \lambda_i(H) = \sum_{i=1}^4 \lambda_i(H) = 2m$. Since G is connected and has order 4, G has at least 3 edges, and so $m \leq \binom{4}{2} - 3 = 3$. Hence

$$U_3(G) = 12 + S_3(H) = 12 + 2m < 16 + m = W(G) + \binom{3+2}{3}. \quad (4.27)$$

The case $r = 4$ follows from $U_4(G) = 2W(G)$ and $W(G) \leq 2\binom{4}{2} < \binom{4+2}{3}$. It remains only to consider $r = 2$. By Lemma 4.5 with $T = 2$, it is enough to prove $S_2(H) \leq m + 2$. If $m \leq 2$, this follows from $S_2(H) \leq 2m$. If $m \geq 3$, then $G = \overline{H}$ has at most 3 edges. Since G is connected and has order 4, it must be a tree; diameter 2 then forces $G = K_{1,3}$, or equivalently $H = K_3 \cup K_1$.

Let $n = 5$. The cases $r = 1, 2, 4$ follow from the preceding arguments: for $r = 1$ use $\lambda_1(H) \leq 5$ and $m \geq 1$, while for $r = 2, 4$ use Lemma 4.1. The case $r = 5$ follows from $U_5(G) = 2W(G)$ and $W(G) \leq 2\binom{5}{2} < \binom{7}{3}$. Thus only $r = 3$ remains. By Lemma 4.5 with $T = 5$, it is enough to prove $S_3(H) \leq m + 5$. If $m \leq 5$, then this follows from $S_3(H) \leq 2m$. If $m \geq 6$, then $G = \overline{H}$ has at most 4 edges. Since G is connected and has order 5, it must be a tree; diameter 2 then forces $G = K_{1,4}$, or equivalently $H = K_4 \cup K_1$.

Thus the only remaining cases are $K_{1,3}$ at $r = 2$ and $K_{1,4}$ at $r = 3$. We now verify directly that these are indeed exceptions. In general, for the star $K_{1,t}$,

$$\text{Spec}(D^L(K_{1,t})) = \{(2t+1)^{(t-1)}, t+1, 0\}, \quad W(K_{1,t}) = t^2. \quad (4.28)$$

The former expression comes from the proof of [15, Theorem 3.6], while the latter expression can be verified by direct calculation. Hence, for $K_{1,3}$,

$$U_2(K_{1,3}) = 7 + 7 = 14 > 13 = 9 + \binom{4}{3} = W(K_{1,3}) + \binom{2+2}{3}. \quad (4.29)$$

For $K_{1,4}$,

$$U_3(K_{1,4}) = 9 + 9 + 9 = 27 > 26 = 16 + \binom{5}{3} = W(K_{1,4}) + \binom{3+2}{3}. \quad (4.30)$$

Therefore these two cases are precisely the exceptions. $\square$

For connected diameter-2 graphs, Zhou et al. [15] proved the target inequality for several classes of such graphs, namely, sufficiently dense graphs, graphs whose complements satisfy the original Brouwer's conjecture, and graphs with maximum degree $n - 1$ or $n - 2$ where $n \geq 21$ or $n \geq 26$, respectively. Here, Theorem 4.8 gives a complete solution for diameter-2 graphs.

5. Graphs with maximum degree $n - k$

In this section we prove the target inequality for graphs whose maximum degree is close to the order. If $\Delta(G) = n - k$, then a vertex of maximum degree leaves only $k - 1$ vertices outside its closed neighborhood. This small vertex subset controls the spectra of the distance-at-least- s graphs for $s \geq 3$.

We first record the precise form of this observation.

Lemma 5.1. *Let $k \geq 2$, and let G be a connected graph of order n with maximum degree $\Delta(G) = n - k$. Let v be a vertex of degree $n - k$ and $X = V(G) \setminus N[v]$. Then $|X| = k - 1$, $\text{diam}(G) \leq k + 1$, and, for every $3 \leq s \leq k + 1$, every edge of Y_s has at least one endpoint in X . Equivalently, X is a vertex cover of Y_s . Consequently, for $1 \leq r \leq n - 1$,*

$$\sum_{i=1}^r \lambda_i(L(Y_s)) \leq m(Y_s) + (k - 1)r \quad (3 \leq s \leq k + 1). \quad (5.1)$$

Proof. Put $C = N[v]$. Then $|X| = k - 1$, and any two vertices of C are within distance at most 2.

Contract C to one vertex c . Since G is connected, the contracted graph is also connected and has $n - |N[v]| + 1 = k$ vertices. Hence the distance between any two of its vertices is at most $k - 1$. For arbitrary $x, y \in V(G)$, let x', y' be their images in the contracted graph, and take a shortest path P from x' to y' in the contracted graph. Now, lift P to G . If P does not contain c , it is already a path in G , so its length in G is the same as that in the contracted graph. If P contains c , the lift replaces c by a path in C joining two appropriate (possibly identical) vertices, using x when $x' = c$ and y when $y' = c$. Since any two vertices of C are at distance at most 2, this increases the path length by at most 2. Thus $d_G(x, y) \leq (k - 1) + 2 = k + 1$. Since x and y were arbitrary, $\text{diam}(G) \leq k + 1$.

Now let $s \geq 3$. Every pair of vertices in C has distance at most 2, so no edge of Y_s can have both endpoints in C . Thus every edge of Y_s is incident with X , and X is a vertex cover of Y_s . The bound then follows from Lemma 2.3. $\square$

We now derive the main result of this section.

Theorem 5.2. *Let $k \geq 2$, and define*

$$N(k) = \begin{cases} 10, & k = 2, \\ \lceil 5(k - 1)^{3/2} \rceil, & k \geq 3. \end{cases} \quad (5.2)$$

If G is a connected graph of order $n \geq N(k)$ and $\Delta(G) = n - k$, then the target inequality holds for every $1 \leq r \leq n$.

Proof. Write $a = (k - 1)^2$. By Lemma 5.1, $\text{diam}(G) \leq k + 1$. Hence, by Lemma 3.1,

$$D^L(G) = nI - J + \sum_{s=2}^{k+1} L(Y_s). \quad (5.3)$$

For $1 \leq r \leq n - 1$, Lemmas 2.2 and 5.1 give

$$\sum_{s=2}^{k+1} \sum_{i=1}^r \lambda_i(L(Y_s)) \leq m(Y_2) + \frac{nr}{2} + \sum_{s=3}^{k+1} (m(Y_s) + (k - 1)r) \quad (5.4)$$

$$= \sum_{s=2}^{k+1} m(Y_s) + \frac{nr}{2} + ar. \quad (5.5)$$

Thus Lemma 3.3, with $B = n/2 + a$, reduces the target inequality to

$$\binom{n}{2} + \binom{r+2}{3} - \frac{3nr}{2} - ar \geq 0 \quad (1 \leq r \leq n-1). \quad (5.6)$$

Let $F_a(n, r)$ denote 6 times the left-hand side of (5.6), that is,

$$F_a(n, r) := 3n^2 - 3n + r^3 + 3r^2 + (2 - 6a)r - 9nr. \quad (5.7)$$

First suppose $k = 2$, $a = 1$ and $n \geq 10$. For fixed r , the quadratic $F_1(n, r)$ reaches its minimum at $n = (3r + 1)/2$. Hence, for $r \geq 6$,

$$F_1(n, r) \geq F_1\left(\frac{3r+1}{2}, r\right) = r^3 - \frac{15}{4}r^2 - \frac{17}{2}r - \frac{3}{4} = r\left(r\left(r - \frac{15}{4}\right) - \frac{17}{2}\right) - \frac{3}{4} > 0, \quad (5.8)$$

where the last expression is increasing and positive for $r \geq 6$. For $1 \leq r \leq 5$, since

$$\frac{\partial F_1(n, r)}{\partial n} = 6n - 3 - 9r \geq 12 > 0, \quad (5.9)$$

the polynomial $F_1(n, r)$ is increasing in n for $n \geq 10$. Direct calculation gives

$$\begin{array}{c|ccccc} r & 1 & 2 & 3 & 4 & 5 \\ \hline F_1(10, r) & 180 & 102 & 42 & 6 & 0 \end{array}, \quad (5.10)$$

so $F_1(10, r) \geq 0$. Thus (5.6) holds for $n \geq 10$ when $k = 2$.

Now suppose $k \geq 3$. Write $b = k - 1$, so $b \geq 2$ and $a = b^2$. Put $x = r + 1$. Since we have the identity

$$r^3 + 3r^2 + 2r = x^3 - x, \quad (5.11)$$

we obtain

$$\begin{aligned} F_a(n, r) &= 3n^2 - 3n + r^3 + 3r^2 + 2r - (9n + 6b^2)r \\ &= 3n^2 + 6n + 6b^2 + x^3 - (9n + 6b^2 + 1)x. \end{aligned} \quad (5.12)$$

For $B > 0$, the minimum of $x^3 - Bx$ over $x \geq 0$ is $-2B^{3/2}/(3\sqrt{3})$. Thus it is enough to show that

$$H(n) := 3n^2 + 6n + 6b^2 - \frac{2(9n + 6b^2 + 1)^{3/2}}{3\sqrt{3}} \geq 0. \quad (5.13)$$

We claim that the function H is increasing for $n \geq 5b^{3/2}$. Indeed,

$$H'(n) = 6n + 6 - 3\sqrt{3}(9n + 6b^2 + 1)^{1/2}, \quad (5.14)$$

and

$$H'(n) > 0 \iff (2n + 2)^2 - 3(9n + 6b^2 + 1) = 4n^2 - 19n - 18b^2 + 1 > 0, \quad (5.15)$$

which is increasing in n for $n \geq 19/8$, and at $n = 5b^{3/2}$ it is

$$100b^3 - 18b^2 - 95b^{3/2} + 1 > (100 - 9 - 34)b^3 + 1 > 0, \quad (5.16)$$

where the second-to-last inequality follows from

$$18b^2 \leq 9b^3, \quad 95b^{3/2} \leq \frac{95}{2^{3/2}}b^3 < 34b^3 \quad (b \geq 2). \quad (5.17)$$

If $k = 3$, $b = 2$, then the assumption gives $n \geq 15$, and

$$H(15) = 789 - \frac{1280\sqrt{30}}{9} > 0. \quad (5.18)$$

Therefore (5.6) holds.

If $k \geq 4$, $b \geq 3$, then evaluate (5.13) at $n = 5b^{3/2}$. Since $45b^{3/2} \leq 26b^2$ if and only if $b \geq (45/26)^2 \approx 2.996$, and $1 \leq b^2$ holds for $b \geq 3$, we have

$$9 \cdot 5b^{3/2} + 6b^2 + 1 \leq (26 + 6 + 1)b^2 = 33b^2 \quad (b \geq 3). \quad (5.19)$$

We then obtain

$$H(5b^{3/2}) \geq 75b^3 + 30b^{3/2} + 6b^2 - \frac{2(33b^2)^{3/2}}{3\sqrt{3}} = (75 - 22\sqrt{11})b^3 + 6b^2 + 30b^{3/2} > 0. \quad (5.20)$$

This proves (5.6) for all $k \geq 2$.

For $r = n$, the assumption $n \geq N(k)$ implies

$$k = 2 \implies n \geq N(2) = 10 \implies n \geq 9 = 3(k+1), \quad (5.21)$$

$$k \geq 3 \implies n \geq \lceil 5(k-1)^{3/2} \rceil \geq 5(k-1)^{3/2} > 3(k+1) \implies n \geq 3(k+1). \quad (5.22)$$

Here, $5(k-1)^{3/2} > 3(k+1)$ for $k \geq 3$ follows by writing $x = k-1 \geq 2$ and deriving $(5x^{3/2})^2 - (3(x+2))^2 = 25x^3 - 9x^2 - 36x - 36 = (x-2)(25x^2 + 41x + 46) + 56 > 0$, where $25x^2 + 41x + 46 > 0$ for all $x \in \mathbb{R}$. Lemma 2.4 therefore applies. $\square$

Theorem 5.2 substantially improves the maximum-degree result of Mushtaq et al. [17]. They proved the target inequality for graphs with $\Delta(G) = n - 2$ under the assumption $n \geq 95$. Theorem 5.2 applies to the same class of graphs already for $n \geq 10$. Moreover, our result is not restricted to the case $\Delta(G) = n - 2$, as it applies to all connected graphs with $\Delta(G) = n - k$ ($k \geq 3$) once $n \geq \lceil 5(k-1)^{3/2} \rceil$.

6. Conclusions

We have proved the distance Laplacian analog of Brouwer's conjecture for three classes of graphs: connected graphs of bounded diameter and sufficiently large order, all diameter-2 graphs except for the two stated star exceptions, and graphs whose maximum degree is $n - k$ while n is large enough in terms of k .

The common tool is the decomposition of the distance Laplacian matrices into Laplacian matrices of distance-at-least- s graphs, which reduces the problem to familiar Laplacian eigenvalue estimates and elementary numerical inequalities.

However, it would be interesting to further sharpen the order thresholds in the bounded-diameter and maximum-degree cases.

Use of Generative-AI tools declaration

The author used GPT-5.5 to improve the grammatical correctness and linguistic coherence of the manuscript. Apart from the aforementioned language improvements, no content in this manuscript was generated by generative AI. All scientific content and results were produced solely by the author, who takes full responsibility for them.

Acknowledgments

The author thanks the reviewers for their helpful comments that improved the quality of this manuscript.

Conflict of interest

The author declares no conflict of interest.

References

1. A. E. Brouwer, W. H. Haemers, *Spectra of graphs*, New York: Springer, 2012. <https://doi.org/10.1007/978-1-4614-1939-6>
2. W. H. Haemers, A. Mohammadian, B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **432** (2010), 2214–2221. <https://doi.org/10.1016/j.laa.2009.03.038>
3. K. Wang, Z. Lin, S. Zhang, C. Ye, A proof of Brouwer's conjecture for $k = 3$, *Linear Algebra Appl.*, **736** (2026), 189–213. <https://doi.org/10.1016/j.laa.2026.01.026>
4. Z. Du, B. Zhou, Upper bounds for the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **436** (2012), 3672–3683. <https://doi.org/10.1016/j.laa.2012.01.007>
5. X. Chen, Improved results on Brouwer's conjecture for sum of the Laplacian eigenvalues of a graph, *Linear Algebra Appl.*, **557** (2018), 327–338. <https://doi.org/10.1016/j.laa.2018.08.003>
6. X. Chen, On Brouwer's conjecture for the sum of k largest Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **578** (2019), 402–410. <https://doi.org/10.1016/j.laa.2019.05.029>
7. H. A. Ganie, S. Pirzada, B. A. Rather, V. Trevisan, Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **588** (2020), 1–18. <https://doi.org/10.1016/j.laa.2019.11.020>
8. M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.*, **439** (2013), 21–33. <https://doi.org/10.1016/j.laa.2013.02.030>
9. M. Aouchiche, P. Hansen, Some properties of the distance Laplacian eigenvalues of a graph, *Czech. Math. J.*, **64** (2014), 751–761. <https://doi.org/10.1007/s10587-014-0129-2>
10. M. Nath, S. Paul, On the distance Laplacian spectra of graphs, *Linear Algebra Appl.*, **460** (2014), 97–110. <https://doi.org/10.1016/j.laa.2014.07.025>
11. H. Lin, B. Wu, Y. Chen, J. Shu, On the distance and distance Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, **492** (2016), 128–135. <https://doi.org/10.1016/j.laa.2015.11.014>

12. R. Fernandes, M. A. A. de Freitas, C. M. da Silva Jr., R. R. Del-Vecchio, Multiplicities of distance Laplacian eigenvalues and forbidden subgraphs, *Linear Algebra Appl.*, **541** (2018), 81–93. <https://doi.org/10.1016/j.laa.2017.11.031>
13. B. Brimkov, K. Duna, L. Hogben, K. Lorenzen, C. Reinhart, S. Y. Song, et al., Graphs that are cospectral for the distance Laplacian, *Electron. J. Linear Algebra*, **36** (2020), 334–351. <https://doi.org/10.13001/ela.2020.4941>
14. B. A. Rather, M. Aouchiche, Distance Laplacian spectra of graphs: A survey, *Discrete Appl. Math.*, **361** (2025), 136–195. <https://doi.org/10.1016/j.dam.2024.10.001>
15. Y. Zhou, L. Wang, Y. Chai, Brouwer type conjecture for the eigenvalues of distance Laplacian matrix of a graph, *Comput. Appl. Math.*, **44** (2025), 138. <https://doi.org/10.1007/s40314-025-03095-0>
16. S. Pirzada, S. Khan, On the sum of distance Laplacian eigenvalues of graphs, *Tamkang J. Math.*, **54** (2023), 83–91. <https://doi.org/10.5556/j.tkjm.54.2023.4120>
17. U. Mushtaq, S. Pirzada, S. Khan, On the sum of the eigenvalues of the distance Laplacian matrix of graphs with diameter three and four, *Rev. Un. Mat. Argentina*, **69** (2026), 193–202. <https://doi.org/10.33044/revuma.5147>
18. K. Fan, On a theorem of Weyl concerning eigenvalues of linear transformations I, *Proc. Natl. Acad. Sci. USA*, **35** (1949), 652–655. <https://doi.org/10.1073/pnas.35.11.652>
19. R. Grone, R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.*, **7** (1994), 221–229. <https://doi.org/10.1137/S0895480191222653>
20. H. Bai, The Grone-Merris conjecture, *Trans. Amer. Math. Soc.*, **363** (2011), 4463–4474. <https://doi.org/10.1090/S0002-9947-2011-05393-6>
21. G. Williams, Smith forms for adjacency matrices of circulant graphs, *Linear Algebra Appl.*, **443** (2014), 21–33. <https://doi.org/10.1016/j.laa.2013.11.006>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)