



*Research article***Numerical solutions and convergence analysis of (k, ρ, ϕ) -Caputo fractional differential equations with applied science applications****Weerawat Sudsutad¹, Jutarat Kongson² and Chatthai Thaiprayoon^{2,*}**¹ Department of Statistics, Faculty of Science, Ramkhamhaeng University, Bangkok 10240, Thailand² Research Group of Theoretical and Computation in Applied Science, Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, Thailand*** Correspondence:** Email: chatthai@go.buu.ac.th.

Abstract: In this study, we addressed the initial value problem in the context of the (k, ρ, ϕ) -Caputo fractional operators. The existence and uniqueness results were established via the fixed-point theory of Banach's and Krasnoselskii's types. A comprehensive qualitative analysis was conducted, including the investigation of Ulam-Hyers-Mittag-Leffler (UHML) stability. A modified numerical scheme, based on a two-step Lagrange interpolation polynomial, was developed for the (k, ρ, ϕ) -Caputo fractional derivative operator. A convergence analysis of the piecewise Lagrange interpolation method is presented, establishing the theoretical order of convergence. The applicability and effectiveness of the proposed method were demonstrated through selected real-world problems. Furthermore, numerical results were compared with those obtained from various fractional derivative operators and with observed data. The accuracy and performance of the analytic and proposed methods were quantitatively assessed applying L^2 -norm error, mean square error (MSE), and root mean square error (RMSE).

Keywords: (k, ρ, ϕ) -Caputo fractional derivative; fractional differential equations; two-step Lagrange interpolation polynomial

Mathematics Subject Classification: 26A33, 34A08, 44A35, 65L05

1. Introduction

Fractional calculus has evolved into a significant area of mathematical research in which differentiation and integration are extended beyond the classical integer-order setting to arbitrary orders. The beginnings of this field can be traced back to a question posed by Leibniz in the seventeenth century, and subsequent developments by Riemann, Liouville, and Caputo established the mathematical foundations of the subject. These contributions led to the formulation of fundamental

operators such as the Riemann-Liouville (RL) and Caputo derivatives, which remain central to theoretical investigations and practical applications [1–3]. Among these, the Caputo derivative [4] is particularly advantageous in modeling applications, as it accommodates standard initial conditions in a natural way. In addition to these classical definitions, a wide variety of fractional derivative operators have been developed to better capture complex dynamical behaviors. These include the Hadamard and Erdélyi-Kober derivatives [5, 6], as well as more recent generalizations such as the Katugampola [7], Hilfer [8], conformable [9], and so on. Each of these operators possesses distinct characteristics and is suited to modeling different types of memory and nonlocal effects. A further level of generalization has been achieved through the introduction of fractional operators defined with respect to an auxiliary function ϕ , which enables additional adaptability in modeling nonlocal effects. This approach has led to the development of ϕ -based operators, including the ϕ -RL [10], ϕ -Hilfer derivative [11], and (k, ϕ) -Hilfer derivative [12]. Furthermore, proportional fractional derivatives have been proposed to incorporate proportional dynamics into fractional models [13]. By combining these concepts, more comprehensive frameworks have emerged, among which the (k, ϕ) -type fractional operators provide a unified structure that includes several classical and modern operators as particular cases.

In recent decades, fractional differential equations (FDEs) have gained increasing attention due to their ability to describe systems whose behavior depends not only on present conditions but also on their past evolution. This inherent memory effect, arising from the nonlocal nature of fractional operators, makes such equations especially suitable for modeling complex phenomena encountered in engineering, physics, biology, and other applied sciences. Consequently, substantial efforts have been devoted to establishing theoretical results, including existence, uniqueness, and stability of solutions, as well as to designing numerical methods capable of accurately approximating these models. Building on these developments, studies have highlighted significant progress in the development, analysis, and numerical approximation of generalized fractional operators. In particular, several researchers have focused on the formulation and generalization of fractional operators. For instance, Sudsutad et al. [14] proposed a novel class of generalized (k, ϕ) -Hilfer proportional fractional operators ($((k, \rho, \phi)$ -Hilfer) that unify the (k, ϕ) -RL and (k, ϕ) -Caputo operators. The corresponding ϕ -Laplace transform and its properties were developed and utilized to the solution of Cauchy problems. Subsequently, Sudsutad et al. [15] investigated a Cauchy-type initial value problem for (k, ϕ) -Caputo proportional fractional derivative operator ($((k, \rho, \phi)$ -Caputo-FDO) and proposed a decomposition-based numerical method that was validated through applications, including blood alcohol level models. In a similar direction, Mallah et al. [16] investigated ϕ -Hilfer generalized proportional fractional derivative operator ($((\rho, \phi)$ -Hilfer-FDO) and analyzed their fundamental properties. Earlier contributions have also entailed generalized fractional integrals and their analytical bounds, such as [17], where the researchers established Ostrowski and Hadamard inequalities for generalized k -RL-fractional integrals. Moreover, the researchers in [18] further examined the Langevin-type differential problems and extended analytical frameworks for fractional operators. In parallel, considerable attention has been considered to the development of efficient numerical schemes for fractional models. Ramalakshmi and Sundaravadivoo [19] studied optimal control problems with the ϕ -Hilfer-FDO and proposed a two-step Lagrange interpolation-based numerical scheme validated through numerical experiments. Moreover, Shapour and Katani [20] conducted a convergence analysis of a piecewise Lagrange interpolation method applied to a fractal fractional model of tumor immune interactions and established its order of convergence. For additional applications of the FDOs, we refer to the following models and references:

tumor-immune [21], bi-model COVID-19 [22], HIV [23], COVID-19 [24], reaction-diffusion [25, 26], and the references cited therein.

Fractional order operators provide an additional degree of freedom, enabling systems to more accurately capture the memory and hereditary effects inherent in many physical and real-world processes. In particular, fractional derivatives have demonstrated superior accuracy and efficiency in modeling complex phenomena, especially in systems where early states significantly power present-time dynamics. Within this framework, the (k, ρ, ϕ) -Caputo-FDO offers enhanced modeling flexibility by combining a proportional structure with a general function ϕ , enabling a more precise representation of nonlocal and memory-dependent behaviors. Moreover, the (k, ρ, ϕ) -Caputo-FDO serves as a unifying framework, as it borders famous fractional operators as special cases under suitable selectiveness of the parameters k and the function ϕ . Several classical differential equation models considered in this study have been extensively investigated in the literature. These models are therefore not introduced as new physical models in this work. Instead, they are employed as representative benchmark and application-oriented problems for examining the proposed numerical framework. Motivated by the developments in [14, 19, 20], particularly the advancement of generalized fractional operators and their successful applications to real-world models, we aim to further explore the qualitative results and numerical methods aspects of the (k, ρ, ϕ) -Caputo-FDO, together with rigorous convergence analysis, which remain relatively limited. In particular, the combined effects of parameters k, ρ , and the transformation function ϕ on the numerical approximation and convergence behavior require further investigation. This motivates our study. The major contributions of this work can be concluded as follows:

- (i) We consider a Cauchy problem involving the (k, ρ, ϕ) -Caputo-FDO (3.1).
- (ii) We prove the existence and uniqueness results for the given initial value problem by applying the fixed-point techniques of Banach's and Krasnoselskii's types and establish UHML stability results.
- (iii) We derive a numerical scheme based on a two-step Lagrange interpolation for a class of FDEs involving the (k, ρ, ϕ) -Caputo-FDO.
- (iv) We provide a convergence and error analysis of the proposed numerical method under the stated assumptions.
- (v) We investigate the effects of the fractional and operator parameters on the numerical solution and validate the proposed method using known analytical solutions as reference solutions.
- (vi) We demonstrate the applicability of the proposed approach through real-world models and validate the results via comparisons with observational data using appropriate error measures.

In this work, we develop and rigorously analyze a numerical framework for a generalized (k, ρ, ϕ) -Caputo-FDO; the existing classical models and known exact solutions are used as benchmark/application settings to establish consistency, accuracy, and convergence. The framework of this work is outlined as follows: In Section 2, we introduce the key definitions and main properties of the (k, ρ, ϕ) -fractional integral and derivative operators ((k, ρ, ϕ) -FIO/ (k, ρ, ϕ) -FDO), which provide the foundation for the posterior developments. In Section 3, we investigate the main qualitative results, including the existence and uniqueness results obtained through the Banach's and Krasnoselskii's fixed-point theorems, together with the analysis of UHML stability. In Section 4, we are concerned with the construction of a numerical scheme for (k, ρ, ϕ) -Caputo-FDEs using a two-step Lagrange interpolation polynomial, along with a detailed study of its convergence and error analysis. In

Section 5, we present applications to real-world problems, including the generalized Newton's law of cooling, world population growth, and blood alcohol level models. Comparisons between numerical solutions generated by fractional derivative operators and observational data are conducted for each case study, with the accuracy and performance of the proposed method assessed via L^2 -norm error, MSE, and RMSE. Finally, In Section 6, we summarize the main findings and conclusions of this study.

2. Preliminaries

Suppose that a strictly increasing continuous function $\phi : [a, b] \rightarrow \mathbb{R}$, which corresponds to $\phi'(t) \neq 0$, $\mathbb{X} = C([a, b], \mathbb{R})$ is a Banach space of all continuous function $x : [a, b] \rightarrow \mathbb{R}$ supplemented with the norm $\|x\|_{\mathbb{X}} = \sup_{t \in [a, b]} \{x(t)\}$. Next, we will define the weighted spaces that will be applied in this work. Suppose that $\phi \in C^n([a, b])$ so that $\phi'(t) > 0$ on $[a, b]$. Then,

$$\begin{aligned}\mathcal{AC}_{\phi}^n([a, b]) &:= \left\{ y : [a, b] \rightarrow \mathbb{R} \text{ and } y^{[n-1]} = \left(\frac{1}{\phi'(t)} \cdot \frac{d}{dt} \right)^{n-1} y \right\}, \\ C_{\alpha, \phi}([a, b]) &:= \left\{ y : (a, b] \rightarrow \mathbb{R} \text{ so that } (\phi(t) - \phi(a))^{\alpha} y(t) \in C([a, b]) \right\}, \\ &\quad \text{where } C_{0, \phi}([a, b]) = C_{\phi}([a, b]), \\ C_{\alpha, \phi}^n([a, b]) &:= \left\{ y : (a, b] \rightarrow \mathbb{R} \text{ so that } y^{[n-1]} \in C([a, b]) \text{ and } y^{[n]} \in C_{\alpha, \phi}([a, b]) \right\}, \\ &\quad \text{where } C_{0, \phi}^n([a, b]) = C_{\phi}^n([a, b]).\end{aligned}$$

Next, some definitions and properties of the (k, ρ, ϕ) -FIO/ (k, ρ, ϕ) -FDO are defined in this subsection, which are applied in this work. For more details, we provide some materials to the readers; see [14, 27].

Definition 2.1 ([14]). Suppose that $\alpha > 0$, $k > 0$, $\rho \in (0, 1]$, $k \geq \alpha$, and $f \in L^1([a, b], \mathbb{R})$. Then, the (k, ρ, ϕ) -RL-FIO of a function y of order α is provided by

$${}_a \mathcal{I}_{k, \phi}^{\alpha, \rho} y(t) = \frac{1}{\rho^{\frac{\alpha}{k}} k \Gamma_k(\alpha)} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \phi'(s) y(s) ds, \quad (2.1)$$

where $\Gamma_k(u) = \int_0^{\infty} s^{u-1} e^{-\frac{s}{k}} ds$, $u > 0$, and

$$\Gamma_k(u+k) = u \Gamma_k(u), \quad \Gamma_k(k) = 1, \quad \Gamma_k(u) = k^{\frac{u}{k}-1} \Gamma\left(\frac{u}{k}\right), \quad \Gamma(u) = \lim_{k \rightarrow 1} \Gamma_k(u). \quad (2.2)$$

Definition 2.2 ([14]). Suppose that $\alpha > 0$, $k > 0$, $\rho \in (0, 1]$, $y \in C^n([a, b], \mathbb{R})$, $\phi(t) \in C^n([a, b], \mathbb{R})$, with $\phi'(t) \neq 0$, $n = 1, 2, \dots$, so that $n = \lfloor \alpha/k \rfloor + 1$ with $k \geq \alpha$. Then, the (k, ρ, ϕ) -Caputo-FDO of y of order α is provided by

$${}_a^C \mathfrak{D}_{k, \phi}^{\alpha, \rho} y(t) = \frac{1}{\rho^{\frac{nk-\alpha}{k}} k \Gamma_k(nk - \alpha)} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{nk-\alpha}{k}-1} \phi'(s) (\mathfrak{D}_{k, \phi}^{n, \rho} y(s)) ds,$$

where $\mathfrak{D}_{k, \phi}^{n, \rho} = \underbrace{\mathfrak{D}_{k, \phi}^{\rho} \mathfrak{D}_{k, \phi}^{\rho} \cdots \mathfrak{D}_{k, \phi}^{\rho}}_{n \text{ times}}$ and $\mathfrak{D}_{k, \phi}^{1, \rho} y(t) = \mathfrak{D}_{k, \phi}^{\rho} y(t) = (1 - \rho)y(t) + k\rho \frac{y'(t)}{\phi'(t)}$.

The formulation considered in this work is restricted to the temporal variable t . In particular, the transformation involving $\phi(t)$ is introduced with respect to the temporal coordinate and is used to construct the corresponding time-fractional operator. The formulation does not claim a direct extension to spatial fractional operators.

Next, we provide the ϕ -Laplace transform and its properties.

Definition 2.3 ([27]). Suppose that $y : [a, \infty) \rightarrow \mathbb{R}$ and $\phi \in C([a, \infty), \mathbb{R})$ with $\phi'(t) > 0$. Then, the ϕ -Laplace transform of y is provided by

$$\mathcal{L}_\phi\{y(t)\} = \int_a^\infty e^{-s(\phi(s)-\phi(a))} y(t) \phi'(t) dt, \quad \forall s \leq t, \quad (2.3)$$

under $\int_a^\infty e^{-s(\phi(s)-\phi(a))} y(t) \phi'(t) dt$ is valid.

Definition 2.4 ([27]). If there exist numbers $p \geq 0$, $M \geq 0$, and $b \geq 0$ so that $|x(t)| \leq M \exp(p\phi(t))$ for all $t \geq b$, then a function $x : [0, \infty) \rightarrow \mathbb{R}$ is called the ϕ -exponential-order.

Definition 2.5. Suppose that two functions f and g are piece-wise continuous on $[a, b]$ and of ϕ -exponential-order. Then, the ϕ -convolution of f and g is provided by

$$(f *_\phi g)(t) = \int_a^t f(s) g(\phi^{-1}(\phi(t) + \phi(a) - \phi(s))) \phi'(s) ds, \text{ where } (f *_\phi g)(t) = (g *_\phi f)(t).$$

Some fundamental properties of the ϕ -Laplace transform are defined in Lemma 2.1.

Lemma 2.1 ([27]). Assume that $\alpha > 0$, $\beta > 0$, and $|\lambda/s^\alpha| < 1$. Hence,

- (i) $\mathcal{L}_\phi\{1\} = s^{-1}$, $s > 0$.
- (ii) $\mathcal{L}_\phi\{(\phi(t) - \phi(a))^\beta\} = s^{-(\beta+1)} \Gamma(\beta + 1)$, $s > 0$.
- (iii) $\mathcal{L}_\phi\{e^{\lambda(\phi(t)-\phi(a))}\} = (s - \lambda)^{-1}$, $s > \lambda$.
- (iv) $\mathcal{L}_\phi\{e^{\lambda(\phi(t)-\phi(a))} f(t)\} = \mathcal{L}_\phi\{f(t)\}(s - \lambda)$.
- (v) $\mathcal{L}_\phi\{(\phi(t) - \phi(a))^{\beta-1} \mathbb{E}_{\alpha,\beta}(\lambda(\phi(t) - \phi(a))^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}$ where $\mathbb{E}_{\alpha,\beta}(z) = \sum_{i=0}^\infty \frac{z^i}{\Gamma(i\alpha + \beta)}$.

Theorem 2.1 ([27]). Assume that $\alpha > 0$, $k > 0$, $\alpha/k \in (n-1, n)$, $k \geq \alpha$, $\rho \in (0, 1]$, $x \in \mathcal{AC}_{\alpha,\phi}^n([a, b], \mathbb{R})$, $\phi \in C^n([a, b], \mathbb{R})$ such that $\phi'(t) > 0$, and $\mathfrak{D}_{k,\phi}^{i,\rho} x$ is of ϕ -exponential-order, $i = 0, 1, \dots, n-1$, $n \in \mathbb{N}$. Then,

$$\mathcal{L}_\phi\left\{{}_a^C \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t)\right\} = (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}} \left[\mathcal{L}_\phi\{x(t)\} - \sum_{i=0}^{n-1} \frac{1}{(k\rho)^i} \left(s - \frac{\rho-1}{k\rho}\right)^{-i-1} (\mathfrak{D}_{k,\phi}^{i,\rho} x(a))\right].$$

Theorem 2.2. (Generalized Gronwall inequality [15]). Assume that $\alpha > 0$, $k > 0$, $k \geq \alpha$, $\rho \in (0, 1]$, and $\phi \in C^1([a, b], \mathbb{R})$, $\phi'(t) \neq 0$ for $t \in [a, b]$. Let the following assumptions hold:

- (G₁) Two non-negative functions $x(t)$ and $y(t)$ are integrable on $[a, b]$.
- (G₂) A function $z(t)$ is a non-negative, non-decreasing and continuous defined on $[a, b]$ such that $z(t) \leq z^*$, where $z^* \in \mathbb{R}^+$.

If

$$x(t) \leq y(t) + \frac{\Gamma_k(\alpha)}{k} z(t)_a \mathcal{I}_{k,\phi}^{\alpha,\rho} f(x), \quad (2.4)$$

then,

$$x(t) \leq y(t) + \int_a^t \sum_{n=1}^{\infty} \frac{[\Gamma_k(\alpha)z(t)]^n}{\rho^{\frac{n\alpha}{k}} k^{n+1} \Gamma_k(n\alpha)} e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{n\alpha}{k}-1} y(s) \phi'(s) ds. \quad (2.5)$$

Corollary 2.1. (Generalized Gronwall inequality [15]). Assume that all condition in Theorem 2.2 hold and a function $y(t)$ is a non-decreasing on $[a, b]$. Then, for any $\alpha, \beta, k > 0$,

$$x(t) \leq y(t) \mathbb{E}_{k,\alpha,k} \left(\frac{\Gamma_k(\alpha)}{\rho^{\frac{\alpha}{k}} k} z(t) (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \right), \quad \mathbb{E}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(n\alpha + \beta)}. \quad (2.6)$$

3. Qualitative properties

In this section, we analyze the existence, uniqueness, and UHML stability results of the initial value problem in the context of the (k, ρ, ϕ) -Caputo-FDO.

Now, we investigate an essential property of the initial value problem (3.1) under the (k, ρ, ϕ) -Caputo-FDO, which is given in Lemma 3.1.

Lemma 3.1. Suppose that $\alpha > 0, k > 0, \rho \in (0, 1], \alpha/k \in (0, 1), k \geq \alpha, \lambda < 0$, and $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$. Then, the proposed problem

$$\begin{cases} {}^C \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t) = \lambda(k\rho)^{\frac{\alpha}{k}} x(t) + f(t, x(t)), & t \in [a, b], \\ x(a) = x_a, & x_a \in \mathbb{R}, \end{cases} \quad (3.1)$$

has a unique solution

$$\begin{aligned} x(t) = & x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t) - \phi(0))^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} \\ & \times (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, x(s)) ds. \end{aligned} \quad (3.2)$$

Proof. Assume that x is a solution of the proposed nonlinear problem (3.1). Taking Theorem 2.1 with $n = 1, x(a) = x_a$, for $t \in [a, b]$, we achieve the following results:

$$(k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} \mathcal{L}_{\phi} \{x(t)\} - (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1} x_a = \lambda(k\rho)^{\frac{\alpha}{k}} \mathcal{L}_{\phi} \{x(t)\} + \mathcal{L}_{\phi} \{f(t, x(t))\},$$

or

$$(k\rho)^{\frac{\alpha}{k}} \left[\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - \lambda \right] \mathcal{L}_{\phi} \{x(t)\} = (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1} x_a + \mathcal{L}_{\phi} \{f(t, x(t))\}.$$

Then,

$$x(t) = x_a \mathcal{L}_{\phi}^{-1} \left\{ \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - \lambda} \right\} + (k\rho)^{-\frac{\alpha}{k}} \mathcal{L}_{\phi}^{-1} \left\{ \frac{1}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - \lambda} \mathcal{L}_{\phi} \{f(t, x(t))\} \right\}. \quad (3.3)$$

Applying Lemma 2.1 implies that

$$\mathcal{L}_\phi \left\{ e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} \left(\lambda (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) \right\} = \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - \lambda}, \quad (3.4)$$

$$\mathcal{L}_\phi \left\{ e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} (\phi(t) - \phi(0))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}} \left(\lambda (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) \right\} = \frac{1}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - \lambda}. \quad (3.5)$$

Taking the relations (3.4) and (3.5) into (3.3), it follows that

$$\begin{aligned} x(t) = & x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} (\lambda (\phi(t) - \phi(0))^{\frac{\alpha}{k}}) \\ & + (k\rho)^{-\frac{\alpha}{k}} \mathcal{L}_\phi^{-1} \left\{ \mathcal{L}_\phi \left\{ e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} (\phi(t) - \phi(0))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}} (\lambda (\phi(t) - \phi(0))^{\frac{\alpha}{k}}) \right\} \mathcal{L}_\phi \{f(t, x(t))\} \right\}. \end{aligned}$$

Using Definition 2.5 in the above relation, we achieve relation (3.2). $\square$

From Lemma 3.1, we provide an operator $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}$ by

$$\begin{aligned} (\mathcal{Q}x)(t) = & x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} (\lambda (\phi(t) - \phi(0))^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} \\ & \times (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}} (\lambda (\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, x(s)) ds. \end{aligned} \quad (3.6)$$

3.1. Existence and uniqueness properties

Now, we prove the uniqueness result for the proposed problem (3.1), by applying the Banach contraction mapping principle [28].

Theorem 3.1. Assume that $\alpha > 0$, $\rho \in (0, 1]$, $k > 0$, $\alpha/k \in (0, 1)$, $k \geq \alpha$, and $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$. Suppose that

(H₁) There exists a real constant $L > 0$, so that

$$|f(t, x) - f(t, y)| \leq L|x - y|, \quad t \in [a, b], \quad x, y \in \mathbb{R}.$$

Hence, proposed problem (3.1) has a unique solution on $[a, b]$, defined that

$$L(\phi(b) - \phi(a))^{\frac{\alpha}{k}} < \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k). \quad (3.7)$$

Proof. Let $B_r = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} \leq r\}$ be a closed and bounded ball with

$$r \geq \left(x_a + \frac{\mathcal{M}(\phi(b) - \phi(a))^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} \right) \left(1 - \frac{L(\phi(b) - \phi(a))^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} \right)^{-1} \text{ where } \mathcal{M} = \sup_{t \in [a, b]} \{|f(t, 0)|\} < +\infty.$$

Since $\Gamma(1) = 1$ and $\mathbb{E}_{x,y}(z) \leq 1/\Gamma(y)$ with (H₁), yields that

$$\begin{aligned} & |(\mathcal{Q}x)(t)| \\ = & \sup_{t \in [a, b]} \left\{ x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} (\lambda (\phi(t) - \phi(0))^{\frac{\alpha}{k}}) \right. \end{aligned}$$

$$\begin{aligned}
& + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, x(s)) ds \Big\} \\
& \leq x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \left| \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda(\phi(t) - \phi(0))^{\frac{\alpha}{k}}) \right| + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \\
& \quad \times \left| \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \right| \phi'(s) \left[|f(s, x(s)) - f(s, 0)| + |f(s, 0)| \right] ds \\
& \leq x_a e^{\frac{\rho-1}{k\rho}(\phi(b)-\phi(0))} + \frac{Lr + \mathcal{M}}{(k\rho)^{\frac{\alpha}{k}} \Gamma(\frac{\alpha}{k})} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \phi'(s) ds \\
& \leq x_a + \frac{Lr + \mathcal{M}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} (\phi(b) - \phi(a))^{\frac{\alpha}{k}} \leq r.
\end{aligned}$$

We can conclude that $QB_r \subseteq B_r$. Next, let $x, y \in C([a, b], \mathbb{R})$. Hence, for any $t \in [a, b]$, we have

$$\begin{aligned}
|(Qx)(t) - (Qy)(t)| & \leq \sup_{t \in [a, b]} \left\{ (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \right. \\
& \quad \times \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \phi'(s) [f(s, x(s)) - f(s, y(s))] ds \Big\} \\
& \leq (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \\
& \quad \times \left| \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \right| \phi'(s) L |x(s) - y(s)| ds \\
& \leq \frac{L \|x - y\|}{(k\rho)^{\frac{\alpha}{k}} \Gamma(\frac{\alpha}{k})} \int_a^t (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \phi'(s) ds \\
& = \frac{(\phi(b) - \phi(a))^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} L \|x - y\|.
\end{aligned}$$

This implies $\|Qx - Qy\|$. It is easy to see that Eq (3.7) holds. Then, we obtain Q , which has a contraction's property, which implies that Q has a unique fixed-point (by [28]). Therefore, proposed problem (3.1) has a unique solution on a closed interval $[a, b]$. $\square$

Now, we utilize the Krasnoselskii's fixed-point theorem to demonstrate the existence result for the proposed problem (3.1).

Lemma 3.2. ([29]) Assume that B is a bounded, convex, closed, and non-empty subset of the Banach space $\mathbb{X}$. Assume that Q_1 and Q_2 are two operators, such that (C_1) $Q_1x + Q_2y \in B$ whenever $x, y \in B$; (C_2) Q_1 is compact and continuous; (C_3) Q_2 is a contraction mapping. Hence, there is $z \in B$, such that $z = Q_1(z) + Q_2(z)$.

Theorem 3.2. Suppose that $\alpha > 0$, $k > 0$, $\rho \in (0, 1]$, $\alpha/k \in (0, 1)$, $k \geq \alpha$, $\lambda < 0$, and $v \in C([a, b], \mathbb{R})$, satisfying (H_2) . In addition, we provide that:

(H_2) For any $t \in [a, b]$ and $x \in \mathbb{R}$, there is a function $v \in C([a, b], \mathbb{R}^+)$ so that

$$|f(t, x(t))| \leq v(t).$$

Hence, proposed problem (3.1) has at least one solution on $[a, b]$, defined that $L(\phi(b) - \phi(a))^{\frac{\alpha}{k}} < \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)$.

Proof. Suppose that $\|v\| = \sup_{t \in [a,b]} \{ |v(t)| \}$ and the closed ball $B_{r_2} = \{x \in \mathbb{R} : \|x\|_{\mathbb{X}} \leq r_2\}$ via $r_2 \geq \|v\|(\phi(b) - \phi(a))^{\frac{\alpha}{k}} / \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k) + x_a$. Let us define $Q = Q_1 + Q_2$ on B_{r_2} , where Q_1 and Q_2 are given by the following:

$$(Q_1 x)(t) = x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t) - \phi(0))^{\frac{\alpha}{k}}), \quad t \in [a, b], \quad (3.8)$$

$$(Q_2 x)(t) = (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, x(s)) ds, \quad t \in [a, b]. \quad (3.9)$$

Next, we will demonstrate that Q_1 and Q_2 satisfy all assumptions of Lemma 3.2. The procedure process is organized into three parts.

Step I. We demonstrate that $Q_1 x + Q_2 y \in B_{r_2}$.

For every $x, y \in B_{r_2}$, we can compute

$$\begin{aligned} |(Q_1 x)(t) + (Q_2 y)(t)| &\leq (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \left| \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \right| \\ &\quad \times \phi'(s) |f(s, x(s))| ds + x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \left| \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t) - \phi(0))^{\frac{\alpha}{k}}) \right| \\ &\leq \frac{\|v\|}{(k\rho)^{\frac{\alpha}{k}} \Gamma_k(\frac{\alpha}{k})} \int_a^t (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \phi'(s) ds + x_a \\ &= \frac{\|v\|(\phi(b) - \phi(a))^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)} + x_a. \end{aligned}$$

Then, $\|Q_1 x + Q_2 y\| \leq r_2$, which yields $Q_1 x + Q_2 y \in B_{r_2}$.

Step II. We demonstrate that Q_1 is compact and continuous.

From the continuous property of f , we have that Q_1 is consequently continuous. For any $x \in B_{r_2}$, we get

$$|(Q_1 x)(t)| \leq x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \left| \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t) - \phi(0))^{\frac{\alpha}{k}}) \right| \leq x_a.$$

Consequently, $\|Q_1 x\| \leq x_a$, this yields that Q_1 is uniformly bounded on B_{r_2} . Assume that $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. For every $x \in B_{r_2}$, implies that

$$|(Q_1 x)(t_2) - (Q_1 x)(t_1)| \leq x_a \left| e^{\frac{\rho-1}{k\rho}(\phi(t_2)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t_2) - \phi(0))^{\frac{\alpha}{k}}) - e^{\frac{\rho-1}{k\rho}(\phi(t_1)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t_1) - \phi(0))^{\frac{\alpha}{k}}) \right|,$$

which yields that $|(Q_1 x)(t_2) - (Q_1 x)(t_1)| \rightarrow 0$ when $t_1 \rightarrow t_2$, independently of $x \in B_{r_2}$. Hence, Q_1 is equicontinuous. Thus, by the Arzelà-Ascoli theorem, Q_1 is compact on B_{r_2} .

Step III. We demonstrate that Q_2 is a contraction mapping.

Applying (3.7) and (H_1) , for each $t \in [a, b]$, and $x, y \in B_{r_2}$, yields that

$$\begin{aligned} &|(Q_2 x)(t) - (Q_2 y)(t)| \\ &\leq \frac{1}{(k\rho)^{\frac{\alpha}{k}}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \left| \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t) - \phi(s))^{\frac{\alpha}{k}}) \phi'(s) \right| \times |f(s, x(s)) - f(s, y(s))| ds \\ &\leq \frac{L\|x - y\|}{(k\rho)^{\frac{\alpha}{k}} \Gamma_k(\frac{\alpha}{k})} \int_a^t (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \phi'(s) ds = \frac{L\|x - y\|(\phi(b) - \phi(a))^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)}. \end{aligned}$$

Then, $\|Q_2 x - Q_2 y\| \leq L\|x - y\|(\phi(b) - \phi(a))^{\frac{\alpha}{k}} / \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)$ yields that Q_2 is a contraction.

Therefore, all assumptions in Lemma 3.2 are true, which implies that proposed problem (3.1) has at least one solution on $[a, b]$. $\square$

3.2. UHML stability results

Now, we analyze all kinds of the UHML stable of the proposed problem (3.1). Next, we define some sufficient materials that are used in this work. Let $\epsilon > 0$, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g(0) = 0$, and we define the inequalities as follows:

$$\left| {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t) - \lambda(k\rho)^{\frac{\alpha}{k}} z(t) - h(t) \right| \leq \epsilon, \quad (3.10)$$

$$\left| {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t) - \lambda(k\rho)^{\frac{\alpha}{k}} z(t) - h(t) \right| \leq g(t), \quad (3.11)$$

$$\left| {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t) - \lambda(k\rho)^{\frac{\alpha}{k}} z(t) - h(t) \right| \leq \epsilon \chi(t), \quad (3.12)$$

$$\left| {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t) - \lambda(k\rho)^{\frac{\alpha}{k}} z(t) - h(t) \right| \leq \chi(t). \quad (3.13)$$

Definition 3.1. The proposed problem (3.1) is called to be UHML stable if there is a positive number $\mathfrak{B}_g > 0$ so that for any $\epsilon > 0$ and $z \in C([a, b], \mathbb{R}^+)$ of inequality (3.10), there exists a function $x \in C([a, b], \mathbb{R}^+)$ of the proposed problem (3.1) under

$$|x(t) - z(t)| \leq \mathfrak{B}_g \mathbb{E}_{k,\alpha,k}(\kappa_g (\phi(t) - \phi(a))^{\frac{\alpha}{k}}), \quad \kappa_g \geq 0, \quad t \in [a, b]. \quad (3.14)$$

Definition 3.2. The proposed problem (3.1) is called to be generalized UHML stable if there is a function $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $g(0) = 0$, so that for any $\epsilon > 0$ and $z \in C([a, b], \mathbb{R}^+)$ of inequality (3.11), there exists a function $x \in C([a, b], \mathbb{R}^+)$ of the proposed problem (3.1) under

$$|x(t) - z(t)| \leq g(\epsilon) \mathbb{E}_{k,\alpha,k}(\kappa_g (\phi(t) - \phi(a))^{\frac{\alpha}{k}}), \quad \kappa_g \geq 0, \quad t \in [a, b]. \quad (3.15)$$

Definition 3.3. The proposed problem (3.1) is called to be Ulam-Hyers-Rassias-Mittag-Leffler (UHRML) stable with respect to another function $\chi(t)$ if there is a positive number $\mathfrak{B}_{g_\chi} > 0$ so that for any $\epsilon > 0$ and $z \in C([a, b], \mathbb{R}^+)$ of inequality (3.12), there exists a function $x \in C([a, b], \mathbb{R}^+)$ of the proposed problem (3.1) under

$$|x(t) - z(t)| \leq \mathfrak{B}_{g_\chi} \epsilon \chi(t) \mathbb{E}_{k,\alpha,k}(\kappa_{g_\chi} (\phi(t) - \phi(a))^{\frac{\alpha}{k}}), \quad \kappa_{g_\chi} \geq 0, \quad t \in [a, b]. \quad (3.16)$$

Definition 3.4. The proposed problem (3.1) is called to be generalized UHRML stable with respect to another function $\chi(t)$ so that for any $\epsilon > 0$ and $z \in C([a, b], \mathbb{R}^+)$ of inequality (3.13), there exists a function $x \in C([a, b], \mathbb{R}^+)$ of the proposed problem (3.1) under

$$|x(t) - z(t)| \leq \mathfrak{B}_{g_\chi} \chi(t) \mathbb{E}_{k,\alpha,k}(\kappa_{g_\chi} (\phi(t) - \phi(a))^{\frac{\alpha}{k}}), \quad \kappa_{g_\chi} \geq 0, \quad t \in [a, b]. \quad (3.17)$$

Let the following conditions

- (C₁) $|u(t)| \leq \epsilon$ for each $u \in C([a, b], \mathbb{R})$, and $t \in [a, b]$.
- (C₂) ${}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} z(t) = \lambda(k\rho)^{\frac{\alpha}{k}} z(t) + h(t) + u(t)$ for each $t \in [a, b]$.
- (C₃) $|v(t)| \leq \epsilon g(t)$ for each $v \in C([a, b], \mathbb{R})$, and $t \in [a, b]$.
- (C₄) ${}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} z(t) = \lambda(k\rho)^{\frac{\alpha}{k}} z(t) + h(t) + v(t)$ for each $t \in [a, b]$.

Theorem 3.3. Let $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$, so (H_1) and $(\phi(b) - \phi(s))^{\frac{\alpha}{k}} < \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)$ hold. Hence, the proposed problem (3.1) is UHML stable and consequently generalized UHML stable on $[a, b]$.

Proof. Suppose that $\epsilon > 0$ and $z \in C([a, b], \mathbb{R})$ is a solution of (3.10). Applying (C_2) , we have the following problem

$$\begin{cases} {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} z(t) = \lambda(k\rho)^{\frac{\alpha}{k}} z(t) + f(t, z(t)) + u(t), & t \in [a, b], \\ z(a) = z_a, & z_a \in \mathbb{R}. \end{cases} \quad (3.18)$$

From (3.2) in Lemma 3.1, problem (3.18) has the solution as follows:

$$\begin{aligned} z(t) &= z_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}) \\ &\quad + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, z(s)) ds \\ &\quad + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \phi'(s) u(s) ds. \end{aligned} \quad (3.19)$$

Let $x \in C([a, b], \mathbb{R})$ be the solution of (3.1). Then,

$$\begin{aligned} &|x(t) - z(t)| \\ &\leq |x_a - z_a| e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \left| \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}) \right| + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \\ &\quad \times \left| \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \right| \phi'(s) |f(s, x(s)) - f(s, z(s))| ds \\ &\quad + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \left| \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \right| \phi'(s) |u(s)| ds. \end{aligned} \quad (3.20)$$

Applying (H_1) and (C_1) , the above inequality (3.20) can be recomputed as

$$\begin{aligned} |x(t) - z(t)| &\leq \frac{L}{(k\rho)^{\frac{\alpha}{k}} \Gamma_k(\frac{\alpha}{k})} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \phi'(s) |x(s) - z(s)| ds \\ &\quad + \frac{\epsilon}{(k\rho)^{\frac{\alpha}{k}} \Gamma_k(\frac{\alpha}{k})} \int_a^t (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \phi'(s) ds \\ &= \frac{(\phi(b)-\phi(a))^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} \epsilon + \frac{\Gamma_k(\alpha)}{k} \left(\frac{kL}{\Gamma_k(\alpha)} \right) {}_a I_{k,\phi}^{\alpha,\rho} |x(t) - z(t)|. \end{aligned} \quad (3.21)$$

Applying Theorem 2.2 with Lemma 2.1, the result (3.21) can be obtained as

$$|x(t) - z(t)| \leq \mathfrak{B}_g \epsilon \mathbb{E}_{k,\alpha,k} \left(\kappa_g (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \right),$$

where $\mathfrak{B}_g = (\phi(b) - \phi(a))^{\frac{\alpha}{k}} / \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k)$ and $\kappa_g = L \rho^{-\frac{\alpha}{k}}$. Then, problem (3.1) is UHML stable. Furthermore, by taking $g(\epsilon) = \mathfrak{B}_g \epsilon$ such that $g(0) = 0$, problem (3.1) is generalized UHML stable on $[a, b]$ under

$$|x(t) - z(t)| \leq g(\epsilon) \mathbb{E}_{k,\alpha,k} \left(\kappa_g (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \right).$$

□

Before proving Theorem 3.4, we give the following necessary condition as below:

(C₅) Assume that a non-decreasing function $\chi \in C([a, b], \mathbb{R})$. There exists a constant $\omega > 0$, so that for $t \in [a, b]$,

$$\int_a^t e^{\frac{\rho-1}{k\rho}(\phi(s)-\phi(a))}(\phi(s)-\phi(a))^{\frac{\alpha}{k}-1}\phi'(s)\chi(s)ds \leq \omega\chi(t). \quad (3.22)$$

Theorem 3.4. Let $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$, (H_1) and $(\phi(b) - \phi(s))^{\frac{\alpha}{k}} < \rho^{\frac{\alpha}{k}}\Gamma_k(\alpha + k)$ hold. Hence, problem (3.1) is UHRML stable and consequently generalized UHRML stable on $[a, b]$.

Proof. Assume that $\epsilon > 0$ and $z \in C([a, b], \mathbb{R})$ is a solution of inequality (3.10). Using (C₄), we obtain the following problem

$$\begin{cases} {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} z(t) = \lambda(k\rho)^{\frac{\alpha}{k}} z(t) + f(t, z(t)) + v(t), & t \in [a, b], \\ z(a) = z_a, & z_a \in \mathbb{R}. \end{cases} \quad (3.23)$$

Applying (3.2) in Lemma 3.1, problem (3.23) has the solution as follows:

$$\begin{aligned} z(t) &= z_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}) \\ &\quad + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))}(\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, z(s)) ds \\ &\quad + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))}(\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \phi'(s) v(s) ds. \end{aligned} \quad (3.24)$$

Assume that $x \in C([a, b], \mathbb{R})$ is the solution of the proposed problem (3.1). Thus,

$$\begin{aligned} &|x(t) - z(t)| \\ &\leq |x_a - z_a| e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \left| \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}) \right| + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} \\ &\quad \times (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \left| \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \right| \phi'(s) |f(s, x(s)) - f(s, z(s))| ds \\ &\quad + (k\rho)^{-\frac{\alpha}{k}} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))}(\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \left| \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \right| \phi'(s) |v(s)| ds. \end{aligned} \quad (3.25)$$

Applying (H_1) and (C₃), inequality (3.25) can be obtained as

$$\begin{aligned} |x(t) - z(t)| &\leq \frac{L}{(k\rho)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k})} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))}(\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \phi'(s) |x(s) - z(s)| ds \\ &\quad + \frac{\epsilon}{(k\rho)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k})} \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))}(\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \phi'(s) \chi(s) ds \\ &= \frac{\epsilon\omega\chi(t)}{(k\rho)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k})} + \frac{\Gamma_k(\alpha)}{k} \left(\frac{kL}{\Gamma_k(\alpha)} \right) {}^a\mathcal{I}_{k,\phi}^{\alpha,\rho} |x(t) - z(t)|. \end{aligned} \quad (3.26)$$

Applying Theorem 2.2 with Lemma 2.1, the result (3.26) can be achieved as

$$|x(t) - z(t)| \leq \mathfrak{B}_{g_\chi} \epsilon \chi(t) \mathbb{E}_{k,\alpha,k} \left(\kappa_{g_\chi} (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \right),$$

where $\mathfrak{B}_{g_\chi} = \omega/(k\rho)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k})$ and $\kappa_{g_\chi} = L\rho^{-\frac{\alpha}{k}}$. Then, the proposed problem (3.1) is UHRML stable. Furthermore, by taking $\epsilon = 1$, the proposed problem (3.1) is generalized UHRML stable on $[a, b]$ under the following assumption:

$$|x(t) - z(t)| \leq \mathfrak{B}_{g_\chi} \chi(t) \mathbb{E}_{k,\alpha,k} \left(\kappa_g (\phi(t) - \phi(s))^{\frac{\alpha}{k}-1} \right).$$

□

In this study, the qualitative analysis is restricted to the fractional-order regime $\alpha \in (0, 1]$. This restriction is adopted consistently in the theoretical analysis and numerical scheme developed in this work. According to the definition $n = \lfloor \alpha \rfloor + 1$, the order of the underlying proportional differential operator depends on α and k . In particular, for $k = 1$, the condition $\alpha \in (0, 1]$ yields $n = 1$, and hence $\mathfrak{D}_{1,\phi}^{n,\rho} = \mathfrak{D}_{1,\phi}^\rho$. Moreover, if $\alpha > 1$, the value of n may increase. For example, when $k = 1$ and $\alpha \in (1, 2]$, $n = 2$, and therefore $\mathfrak{D}_{1,\phi}^{n,\rho} = \mathfrak{D}_{1,\phi}^\rho \mathfrak{D}_{1,\phi}^\rho$. Thus, the higher-order fractional case involves a different underlying differential operator and may require additional initial conditions and a separate qualitative and convergence analysis. Therefore, the results established in Section 3 are claimed only for $\alpha \in (0, 1]$, and no direct extension of these conclusions to $\alpha > 1$ is made in this manuscript. The investigation of higher-order fractional regimes is left as a potential direction for future research.

Additionally, FDEs with orders greater than one are also mathematically and practically meaningful. In general, such models can be used to describe systems whose dynamics involve higher-order temporal effects together with memory and hereditary properties. In the (k, ρ, ϕ) -Caputo-FDO, however, the case $\alpha > 1$ may introduce a higher-order proportional differential operator through the integer n . Consequently, its modeling interpretation and numerical treatment should be developed consistently with the corresponding higher-order dynamics and initial conditions. Such an extension is beyond our objectives of this work and is therefore reserved for future investigation.

4. Numerical scheme for the (k, ρ, ϕ) -Caputo-FDE

In this section, we develop a numerical scheme for solving the considered (k, ρ, ϕ) -Caputo-FDE. The proposed formulation is constructed by discretizing the integral representation of the fractional operator and approximating the unknown solution over a uniform temporal mesh. To obtain the numerical approximation, a two-step Lagrange interpolation polynomial is employed to approximate the solution within each time interval. The resulting discrete formulation provides a computationally tractable approximation of the nonlocal fractional operator. Subsequently, the convergence properties of the proposed numerical scheme are established by deriving appropriate error estimates under the stated assumptions.

4.1. A two-step Lagrange interpolation polynomial

In this subsection, we propose a modified numerical technique to solve the (k, ρ, ϕ) -Caputo-FDE, as presented in (4.1). Consider the following nonlinear (k, ρ, ϕ) -Caputo-FDE:

$$\begin{cases} {}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho} x(t) = \lambda(k\rho)^{\frac{\alpha}{k}} x(t) + f(t, x(t)), & t \in [a, T], \quad \alpha/k \in (0, 1], \\ x(t_a) = x_a, \quad x_a \in \mathbb{R}^+, \quad T \in \mathbb{R}^+, \end{cases} \quad (4.1)$$

where ${}^C_a \mathfrak{D}_{k,\phi}^{\alpha,\rho}$ denotes the (k, ρ, ϕ) -Caputo-FDO, which is defined in Definition 2.2. Applying Theorem 2.1 into the problem (4.1), the solution of the problem (4.1) is defined as

$$\begin{aligned} x(t) = & x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(t_a))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(t_a))^{\frac{\alpha}{k}}) \\ & + (k\rho)^{-\frac{\alpha}{k}} \int_{t_a}^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \phi'(s) f(s, x(s)) ds. \end{aligned}$$

Changing variable $\tau = \phi(s)$ and $ds = (1/\phi'(s))d\tau$, we obtain

$$\begin{aligned}
x(t) &= x_a e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(t_a))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(t_a))^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_{\phi(t_a)}^{\phi(t)} e^{\frac{\rho-1}{k\rho}(\phi(t)-\tau)} (\phi(t)-\tau)^{\frac{\alpha}{k}-1} \\
&\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\tau)^{\frac{\alpha}{k}}) f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) d\tau.
\end{aligned} \tag{4.2}$$

Taking $\phi(t) = \phi_{n+1}$ for $n = 0, 1, 2, \dots$ where $\phi_0 = \phi(t_a)$, $\phi_{n+1} = \phi_n + h$ with step size $h = (\phi_{n+1} - \phi_0)/N$, the Eq (4.2) can be reform as follows:

$$\begin{aligned}
x(\phi^{-1}(\phi_{n+1})) &= x_a e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_{\phi_0}^{\phi_{n+1}} e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} \\
&\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}}) f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
x(\phi^{-1}(\phi_{n+1})) &= x_a e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \int_{\phi_i}^{\phi_{i+1}} e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} \\
&\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}}) f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) d\tau.
\end{aligned} \tag{4.3}$$

Approximating the function $e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} f(\phi^{-1}(\tau), x(\phi^{-1}(\tau)))$ by a two-step Lagrange interpolation polynomial, we have

$$e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) = \frac{F(t_i, x_i)(\tau - \phi_{i-1}) - F(t_{i-1}, x_{i-1})(\tau - \phi_i)}{\phi_i - \phi_{i-1}}, \tag{4.4}$$

where $F(t_i, x_i) = e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_i)} f(\phi^{-1}(\phi_i), x(\phi^{-1}(\phi_i)))$. Substituting (4.4) into (4.3), implies

$$\begin{aligned}
x(\phi^{-1}(\phi_{n+1})) &= x_a e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) \\
&\quad + (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_i, x_i)}{\phi_i - \phi_{i-1}} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} (\tau - \phi_{i-1}) \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}}) d\tau \\
&\quad - (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_{i-1}, x_{i-1})}{\phi_i - \phi_{i-1}} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} (\tau - \phi_i) \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}}) d\tau \\
&= x_a e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) \\
&\quad + (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_i, x_i)}{h} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} (\tau - \phi_{i-1}) \sum_{j=0}^{\infty} \frac{(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}})^j}{\Gamma(j\frac{\alpha}{k} + \frac{\alpha}{k})} d\tau \\
&\quad - (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_{i-1}, x_{i-1})}{h} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} (\tau - \phi_i) \sum_{j=0}^{\infty} \frac{(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}})^j}{\Gamma(j\frac{\alpha}{k} + \frac{\alpha}{k})} d\tau \\
&= x_a e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) \\
&\quad + (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_i, x_i)}{h} \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j\frac{\alpha}{k} + \frac{\alpha}{k})} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{(j+1)\frac{\alpha}{k}-1} (\tau - \phi_{i-1}) d\tau \\
&\quad - (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_{i-1}, x_{i-1})}{h} \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j\frac{\alpha}{k} + \frac{\alpha}{k})} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{(j+1)\frac{\alpha}{k}-1} (\tau - \phi_i) d\tau.
\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}
 & \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1} - \tau)^{(j+1)\frac{\alpha}{k}-1} (\tau - \phi_{i-1}) d\tau \\
 = & \frac{(\phi_i - \phi_{i-1})(\phi_{n+1} - \phi_i)^{(j+1)\frac{\alpha}{k}} - (\phi_{i+1} - \phi_{i-1})(\phi_{n+1} - \phi_{i+1})^{(j+1)\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} + \frac{(\phi_{n+1} - \phi_i)^{(j+1)\frac{\alpha}{k}+1} - (\phi_{n+1} - \phi_{i+1})^{(j+1)\frac{\alpha}{k}+1}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} \\
 = & \frac{h((n+1-i)h)^{(j+1)\frac{\alpha}{k}} - 2h((n-i)h)^{(j+1)\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} + \frac{((n+1-i)h)^{(j+1)\frac{\alpha}{k}+1} - ((n-i)h)^{(j+1)\frac{\alpha}{k}+1}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} \\
 = & h((n+1-i)h)^{\frac{\alpha}{k}} \left(\frac{((n+1-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} \right) + ((n+1-i)h)^{\frac{\alpha}{k}+1} \left(\frac{((n+1-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} \right) \\
 & - 2h((n-i)h)^{\frac{\alpha}{k}} \left(\frac{((n-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} \right) - ((n-i)h)^{\frac{\alpha}{k}+1} \left(\frac{((n-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} \right), \tag{4.6}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1} - \tau)^{(j+1)\frac{\alpha}{k}-1} (\tau - \phi_i) d\tau \\
 = & \frac{(\phi_{n+1} - \phi_i)^{(j+1)\frac{\alpha}{k}+1} - (\phi_{n+1} - \phi_{i+1})^{(j+1)\frac{\alpha}{k}+1}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} - \frac{(\phi_{i+1} - \phi_i)(\phi_{n+1} - \phi_{i+1})^{(j+1)\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} \\
 = & \frac{((n+1-i)h)^{(j+1)\frac{\alpha}{k}+1} - ((n-i)h)^{(j+1)\frac{\alpha}{k}+1}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} - \frac{h((n-i)h)^{(j+1)\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} \\
 = & ((n+1-i)h)^{\frac{\alpha}{k}+1} \left(\frac{((n+1-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} \right) - ((n-i)h)^{\frac{\alpha}{k}+1} \left(\frac{((n-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k} \left((j+1)\frac{\alpha}{k} + 1 \right)} \right) \\
 & - h((n-i)h)^{\frac{\alpha}{k}} \left(\frac{((n-i)h)^{j\frac{\alpha}{k}}}{(j+1)\frac{\alpha}{k}} \right). \tag{4.7}
 \end{aligned}$$

Inserting the relations (4.6) and (4.7) into (4.5) and using $\mathbb{E}_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+\beta)}$ implies that

$$\begin{aligned}
 & x(\phi^{-1}(\phi_{n+1})) \\
 = & x_a e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) \\
 & + (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_i, x_i)}{h} \left[h((n+1-i)h)^{\frac{\alpha}{k}} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}+1}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right. \\
 & + ((n+1-i)h)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) - 2h((n-i)h)^{\frac{\alpha}{k}} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \\
 & \left. - ((n-i)h)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right] \\
 & - (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \frac{F(t_{i-1}, x_{i-1})}{h} \left[((n+1-i)h)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right.
 \end{aligned}$$

$$\begin{aligned}
& -((n-i)h)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) - h((n-i)h)^{\frac{\alpha}{k}} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \Big] \\
= & x_a e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) \\
& + \left(\frac{h}{k\rho}\right)^{\frac{\alpha}{k}} \sum_{i=0}^n \left[F(t_i, x_i) \left((n+1-i)^{\frac{\alpha}{k}} \left[\mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right. \right. \right. \\
& + (n+1-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \Big] \\
& - (n-i)^{\frac{\alpha}{k}} \left[2 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + (n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right] \Big) \\
& - F(t_{i-1}, x_{i-1}) \left((n+1-i)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right. \\
& \left. \left. - (n-i)^{\frac{\alpha}{k}} \left[(n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right] \right) \right].
\end{aligned}$$

Then,

$$x_{n+1} = x_a e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) + \left(\frac{h}{k\rho}\right)^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} f(t_i, x_i) - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n+2-i)h} f(t_{i-1}, x_{i-1}) \right], \quad (4.8)$$

where $t_i = \phi^{-1}(\phi_i)$, for $i = 0, 1, \dots, n$, and

$$\begin{aligned}
\Omega(n, i) = & (n+1-i)^{\frac{\alpha}{k}} \left[\mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) + (n+1-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right] \\
& - (n-i)^{\frac{\alpha}{k}} \left[2 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + (n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right], \quad (4.9)
\end{aligned}$$

$$\begin{aligned}
\Upsilon(n, i) = & (n+1-i)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \\
& - (n-i)^{\frac{\alpha}{k}} \left[(n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right]. \quad (4.10)
\end{aligned}$$

Throughout the derivation, ϕ is considered a transformation of the temporal variable. Consequently, all numerical approximations developed below concern time-fractional differential equations.

The two-step Lagrange interpolation introduced above provides the approximation required to discretize the (k, ρ, ϕ) -Caputo-FD $\mathbb{O}$. Based on this approximation, the resulting numerical scheme is constructed in terms of the discrete solution values at the prescribed time levels. The accuracy of the proposed scheme is subsequently examined through a rigorous convergence analysis. In particular, we establish an error estimate between the exact solution and the numerical approximation and determine the convergence behavior with respect to the temporal discretization step.

4.2. Convergence analysis

This part proves the error analysis for the numerical scheme of the (k, ϕ) -Caputo-FDE. Now, we define an essential lemma that will be applied in our theoretical analysis.

Lemma 4.1 (Discrete Gronwall's Inequality [30]). *Let $\{r_j\}_{j=1}^{\infty}$ be a non-negative sequence, and let ϵ_n be a sequence such that $\epsilon_0 \leq \rho_0$, and $\epsilon_n \leq \rho_0 + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} r_j \epsilon_j$, $n \geq 1$, with $\rho_0 \leq 0$ and $q_j \leq 0$. Then*

$$\epsilon_n \leq \left(\rho_0 + \sum_{j=0}^{n-1} q_j \right) \exp \left(\sum_{j=0}^{n-1} r_j \right), \quad n \geq 1.$$

Theorem 4.1. Suppose that $x(t)$ is an exact solution of (4.2) and x_{b+1} is the numerical solution, which is given by (4.8) in the point t_{b+1} . Suppose that $f(t, x(t))$ satisfies the Lipschitz condition. Then, the proposed numerical scheme is convergent and for sufficiently smooth f , its order of convergence is at least $2 + \alpha/k$.

Proof. Using the Lipschitz condition of $f(t, x(t))$, that is $|f(t, x_2(t)) - f(t, x_1(t))| \leq L_f |x_2(t) - x_1(t)|$ with $L_f \geq 0$, for all $x_1, x_2 \in \mathbb{R}$, we obtain that

$$\begin{aligned} |\mathbf{e}_{n+1}| &= |x(t_{n+1}) - x_{n+1}| \\ &\leq \frac{1}{(k\rho)^{\frac{\alpha}{k}}} \sum_{i=0}^n \left| \int_{\phi_i}^{\phi_{i+1}} e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} (\phi_{n+1} - \tau)^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda(\phi_{n+1} - \tau)^{\frac{\alpha}{k}} \right) f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) d\tau \right. \\ &\quad \left. - h^{\frac{\alpha}{k}} \Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} f(t_i, x_i) + h^{\frac{\alpha}{k}} \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} f(t_{i-1}, x_{i-1}) \right| \\ &\leq \frac{1}{(k\rho)^{\frac{\alpha}{k}}} \sum_{i=0}^n \left| \int_{\phi_i}^{\phi_{i+1}} e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} (\phi_{n+1} - \tau)^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda(\phi_{n+1} - \tau)^{\frac{\alpha}{k}} \right) f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) d\tau \right. \\ &\quad \left. + h^{\frac{\alpha}{k}} \Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} L_f |x(t_i) - x_i| - h^{\frac{\alpha}{k}} \Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} f(t_i, x(t_i)) \right. \\ &\quad \left. + h^{\frac{\alpha}{k}} \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} L_f |x(t_{i-1}) - x_{i-1}| + h^{\frac{\alpha}{k}} \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} f(t_{i-1}, x(t_{i-1})) \right|. \end{aligned}$$

Since

$$\begin{aligned} &-h^{\frac{\alpha}{k}} \Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} f(t_i, x(t_i)) + h^{\frac{\alpha}{k}} \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} f(t_{i-1}, x(t_{i-1})) \\ &= \frac{1}{h} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1} - \tau)^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda(\phi_{n+1} - \tau)^{\frac{\alpha}{k}} \right) \left[e^{\frac{\rho-1}{k\rho}(n-i+2)h} f(t_{i-1}, x(t_{i-1})) (\tau - \phi_i) \right. \\ &\quad \left. - e^{\frac{\rho-1}{k\rho}(n-i+1)h} f(t_i, x(t_i)) (\tau - \phi_{i-1}) \right] d\tau. \end{aligned}$$

Setting $\mathbf{e}_i = x(t_i) - x_i$ and $\mathbf{e}_{i-1} = x(t_{i-1}) - x_{i-1}$, implies that

$$\begin{aligned} |\mathbf{e}_{n+1}| &\leq \frac{1}{(k\rho)^{\frac{\alpha}{k}}} \sum_{i=0}^n \left| \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1} - \tau)^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda(\phi_{n+1} - \tau)^{\frac{\alpha}{k}} \right) \left[e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) \right. \right. \\ &\quad \left. \left. + \frac{e^{\frac{\rho-1}{k\rho}(n-i+2)h} f(t_{i-1}, x(t_{i-1})) (\tau - \phi_i) - e^{\frac{\rho-1}{k\rho}(n-i+1)h} f(t_i, x(t_i)) (\tau - \phi_{i-1})}{h} \right] d\tau \right. \\ &\quad \left. + h^{\frac{\alpha}{k}} \Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} L_f |\mathbf{e}_i| + h^{\frac{\alpha}{k}} \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} L_f |\mathbf{e}_{i-1}| \right| \\ &\leq \frac{\|I\|}{(k\rho)^{\frac{\alpha}{k}}} \sum_{i=0}^n \left| \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1} - \tau)^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda(\phi_{n+1} - \tau)^{\frac{\alpha}{k}} \right) d\tau \right. \\ &\quad \left. + h^{\frac{\alpha}{k}} \Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} L_f |\mathbf{e}_i| + h^{\frac{\alpha}{k}} \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} L_f |\mathbf{e}_{i-1}| \right|, \end{aligned}$$

where

$$\|I\| = \left\| \frac{e^{\frac{\rho-1}{k\rho}(n-i+2)h} f(t_{i-1}, x(t_{i-1})) (\tau - \phi_i) - e^{\frac{\rho-1}{k\rho}(n-i+1)h} f(t_i, x(t_i)) (\tau - \phi_{i-1})}{h} + e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} f(\phi^{-1}(\tau), x(\phi^{-1}(\tau))) \right\|.$$

Since,

$$\begin{aligned} & \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1} - \tau)^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda (\phi_{n+1} - \tau)^{\frac{\alpha}{k}} \right) d\tau \\ &= [(n+1-i)h]^{\frac{\alpha}{k}} E_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(\lambda [(n+1-i)h]^{\frac{\alpha}{k}} \right) - [(n-i)h]^{\frac{\alpha}{k}} E_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(\lambda [(n-i)h]^{\frac{\alpha}{k}} \right), \end{aligned}$$

we have

$$\begin{aligned} |\mathbf{e}_{n+1}| &\leq \left(\frac{h}{k\rho} \right)^{\frac{\alpha}{k}} \|I\| \sum_{i=0}^n \left[(n+1-i)^{\frac{\alpha}{k}} E_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(\lambda [(n+1-i)h]^{\frac{\alpha}{k}} \right) - (n-i)^{\frac{\alpha}{k}} E_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(\lambda [(n-i)h]^{\frac{\alpha}{k}} \right) \right] \\ &\quad + \left(\frac{h}{k\rho} \right)^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n-i+1)h} L_f |\mathbf{e}_i| + \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i+2)h} L_f |\mathbf{e}_{i-1}| \right]. \end{aligned} \quad (4.11)$$

The inequality (4.11) can be rewritten as follows:

$$\mathbf{e}_{n+1} \leq B_1 \left(\frac{h}{k\rho} \right)^{\frac{\alpha}{k}} \|I\| + \sum_{i=0}^n B_2(h, i) |\mathbf{e}_i|, \quad (4.12)$$

where B_1 and B_2 are constants, that are

$$\begin{aligned} B_1 &= \sum_{i=0}^n \left[(n+1-i)^{\frac{\alpha}{k}} E_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(\lambda [(n+1-i)h]^{\frac{\alpha}{k}} \right) - (n-i)^{\frac{\alpha}{k}} E_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(\lambda [(n-i)h]^{\frac{\alpha}{k}} \right) \right], \\ B_2(h, i) &= L_f \left(\frac{h}{k\rho} \right)^{\frac{\alpha}{k}} \begin{cases} [\Omega(n, i) + \Upsilon(n, i+1)] e^{\frac{\rho-1}{k\rho}(n-i+1)h}, & i = 0, 1, \dots, n-1, \\ \Omega(n, n) e^{\frac{\rho-1}{k\rho}h}, & i = n. \end{cases} \end{aligned}$$

Applying Lemma 4.1 into inequality (4.12) yields that

$$\mathbf{e}_{n+1} \leq C_1 \left(\frac{h}{k\rho} \right)^{\frac{\alpha}{k}} \|I\| \exp \left(\sum_{i=0}^n C_2(h, i) \right).$$

Therefore, taking $h \rightarrow 0$, we have $\mathbf{e}_{n+1} \rightarrow 0$. Since $\|I\|$ is of order h^2 , then the convergence order is $O(h^{2+\frac{\alpha}{k}})$. $\square$

Based on the discrete formulation derived in the preceding subsection, the numerical solution can be computed recursively at each temporal node. The complete computational procedure for the proposed (k, ρ, ϕ) -Caputo-FDE is summarized in Algorithm 1. The algorithm provides a step-by-step implementation of the numerical scheme, including the construction of the temporal mesh, evaluation of the fractional kernel, calculation of the history-dependent contribution, and computation of the numerical solution at each time level.

The proposed direct implementation involves a history-dependent summation at each time level. Since the number of previously computed terms increases with the time level, the total number of history evaluations is $\sum_{n=0}^{N-1} (n+1) = N(N+1)/2$. Therefore, the computational complexity of the direct implementation is $O(N^2)$, assuming that the evaluation of the Mittag-Leffler functions and the coefficient terms requires $O(1)$ operations.

Algorithm 1 is used to implement all numerical experiments reported in the following section. The computational time is measured under the same hardware and software environment for all fractional definitions. The same temporal grids, initial conditions, model parameters, and stopping criteria are employed to ensure a fair comparison of the computational efficiency.

Require: $a, T, N, \alpha, k, \rho, h, \phi(t), \lambda, x_a, f(t, x)$.

Ensure: Numerical solution $\{x_n\}_{n=0}^N$.

```

1: Set  $h = (\phi(T) - \phi(t_0))/N$ , for  $n = 0, 1, \dots, N$ .
2: Set the initial values  $t_0 = a$  and  $x_0 = x_a$ .
3: Compute  $\phi_i = \phi(t_i)$  and temporal mesh  $t_{i+1} = \phi^{-1}(\phi_i + h)$ , for  $i = 0, 1, \dots, N$ .
4: for  $n = 0, 1, \dots, N - 1$  do
5:   Compute the initial contribution according to Eq (4.8).
6:   for  $i = 0, 1, \dots, n$  do
7:      $t_{i+1} = \phi^{-1}(\phi(t(i)) + h)$ 
8:     Evaluate  $\Omega(n, i)$  using Eq (4.9).
9:     Evaluate  $\Upsilon(n, i)$  using Eq (4.10).
10:    Evaluate the nonlinear function  $f_i = f(t_i, x_i)$ .
11:    Compute the history contribution.
12:  end for
13:  Compute  $x_{n+1}$  from Eq (4.8).
14: end for
15: return  $\{x_n\}_{n=0}^N$ .
```

Algorithm 1. Computational procedure for the proposed numerical scheme.

5. Numerical examples

In this section, we offer the numerical results and graphical simulations of the proposed technique, which are compared to approaches such as the ϕ -Laplace transform [27] and use real-world data for each genuine situation. Our objective is to present the practicality of the proposed method and validate the numerical scheme by solving and displaying results for three real-world problems, such as Newton's law of cooling [31], population growth [32], and blood alcohol levels [33]. At the end of this subsection, we present an example of nonlinear problems to indicate the efficiency of the proposed technique in this work. In addition, the error measurement instruments commonly used to evaluate the performance of the proposed numerical scheme match experimental data, including the L^2 -norm error, MSE, and RMSE. They are defined by

$$\|x_i - \tilde{x}_i\|_2 = \sqrt{\sum_{i=1}^n (x_i - \tilde{x}_i)^2}, \quad \text{MSE} = \frac{1}{n} \sum_{i=1}^n (x_i - \tilde{x}_i)^2, \quad \text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \tilde{x}_i)^2}, \quad (5.1)$$

where x_i and $\tilde{x}_i$ denote the experiment data and approximated solution, respectively, and n denotes the number of points. All numerical experiments are implemented in MATLAB (R2018b) software. The computations are performed on a computer equipped with Intel(R) Core(TM) i5-10300H CPU@2.50GHz, and RAM 16.0 GB. All competing fractional formulations are implemented under the same computational environment and numerical settings.

5.1. Generalization of Newton's law of the cooling model

According to Newton's rule of cooling, the rate at which a body's temperature reduces over time is proportional to the temperature difference between the body and its environment. This rule is based on several assumptions, the most important being the stability of heat capacity and small temperature differences. A temperature gradient within the cooling body is discovered to impede a satisfactory fit of experimental data to the rule in this basic version. This can be presented in the form of an initial value problem as follows:

$$\begin{cases} \frac{dT(t)}{dt} = \lambda(T(t) - T_e), & \lambda < 0, \\ T(0) = T_0, \end{cases} \quad (5.2)$$

where $T(t)$ is the body's temperature at time t , T_e is the temperature of surrounding environment, T_0 is the initial temperature, and λ is the cooling coefficient (or convective), which is defined by $\lambda = hA/mC$, the units of s^{-1} , the heat transfer coefficient is denoted by h , the heat transfer surface area is denoted by A , the body's mass is denoted by m , and the thermal heat capacity is denoted by C . Applying the classical Laplace transform into problem (5.2), we achieve the solution of problem (5.2) as

$$T(t) = T_e + (T_0 - T_e)e^{\lambda t}. \quad (5.3)$$

Notice that Newton's law of cooling has been used in a variety of applications, including materials science, high-temperature superconductivity, and atmospheric physics [31]. Next, we find the solution of problem (5.4) by using the ϕ -Laplace transform technique and the proposed numerical scheme (two-step Lagrange interpolation polynomial).

The consistency of the proposed (k, ρ, ϕ) -Caputo-FD $\odot$ with the classical first-order derivative can be demonstrated through an appropriate special case. First, the generalized k -Gamma function used in this work is defined by $\Gamma_k(u) = \int_0^\infty s^{u-1} e^{-\frac{s^k}{k}} ds$. By using the transformation $x = s^k/k$ and $ds = s^{1-k} dx$, it follows that $\Gamma_k(u) = k^{\frac{u}{k}-1} \Gamma\left(\frac{u}{k}\right)$. Hence, $\Gamma_1(u) = \Gamma(u)$. Consider the proposed (k, ρ, ϕ) -Caputo-FD $\odot$ and impose the special choices $k = 1$, $\rho = 1$, and $\phi(t) = t$. Since $n = \lfloor \alpha/k \rfloor + 1$, for $\alpha \in (0, 1]$ and $k = 1$, we have $n = 1$. Furthermore, the first-order proportional operator becomes

$$\mathfrak{D}_{1,t}^{1,1} y(t) = (1-1)y(t) + (1)(1) \frac{y'(t)}{1} = y'(t).$$

Consequently, the proposed fractional operator reduces to

$${}_a^C \mathfrak{D}_{1,t}^{\alpha,1} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} y'(s) ds,$$

which is the standard Caputo fractional derivative of order $\alpha \in (0, 1]$. Moreover, under the classical-order limit $\alpha \rightarrow 1^-$, and assuming sufficient regularity of y , the above operator satisfies

$$\lim_{\alpha \rightarrow 1^-} {}_a^C \mathfrak{D}_{1,t}^{\alpha,1} y(t) = y'(t), \quad \text{implies that} \quad \lim_{\alpha \rightarrow 1^-} {}_a^C \mathfrak{D}_{k,t}^{\alpha,1} T(t) = \frac{dT(t)}{dt}.$$

This result demonstrates that the proposed fractional formulation is consistent with the corresponding classical first-order model under the appropriate special-case and limiting conditions. Importantly, the physical state variable $T(t)$ is unchanged throughout this generalization; $T(t)$ continues to denote the

temperature of the body, while only the mathematical operator governing its temporal evolution is generalized.

To indicate the effectiveness of the proposed numerical scheme, we consider the generalized Newton's law of the cooling model under the (k, ρ, ϕ) -Caputo-FDO:

$$\begin{cases} {}^C_0\mathfrak{D}_{k,\phi}^{\alpha,\rho}T(t) = \lambda(k\rho)^{\frac{\alpha}{k}}(T(t) - T_e), & \alpha \in (0, 1], \\ T(0) = T_0, & \lambda < 0, \quad \rho \in (0, 1], \quad k > 0. \end{cases} \quad (5.4)$$

To obtain the exact solution of problem (5.4), we use the ϕ -Laplace transform in Definition 2.3 and Lemma 2.1. Taking the ϕ -Laplace transform on the problem (5.4) and applying Theorem 2.1, we achieve the following results:

$$\begin{aligned} (k\rho)^{\frac{\alpha}{k}}\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}}\mathcal{L}_\phi\{T(t)\} - (k\rho)^{\frac{\alpha}{k}}\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}-1}T(0) &= \lambda(k\rho)^{\frac{\alpha}{k}}\mathcal{L}_\phi\{T(t) - T_e\}, \\ \left[\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}} - \lambda\right]\mathcal{L}_\phi\{T(t)\} &= \left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}-1}T_0 - \frac{\lambda T_e}{s}. \end{aligned}$$

Then,

$$\mathcal{L}_\phi\{T(t)\} = \frac{\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}} - \lambda}T_0 - \frac{1}{s\left[\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}} - \lambda\right]}\lambda T_e. \quad (5.5)$$

Applying Lemma 2.1, it follows that

$$\mathcal{L}_\phi\left\{e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}\mathbb{E}_{\frac{\alpha}{k},1}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)\right\} = \frac{\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}} - \lambda}, \quad (5.6)$$

$$\mathcal{L}_\phi\left\{e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}(\phi(t)-\phi(0))^{\frac{\alpha}{k}-1}\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)\right\} = \frac{1}{\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}} - \lambda}. \quad (5.7)$$

Applying the relations (5.6) and (5.7), Eq (5.5) can be computed as

$$\begin{aligned} \mathcal{L}_\phi\{T(t)\} &= \mathcal{L}_\phi\left\{e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}\mathbb{E}_{\frac{\alpha}{k},1}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)\right\}T_0 \\ &\quad - \lambda T_e \mathcal{L}_\phi\{1\} \mathcal{L}_\phi\left\{e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}(\phi(t)-\phi(0))^{\frac{\alpha}{k}-1}\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)\right\}. \end{aligned}$$

Using the ϕ -inverse Laplace transform, which implies that

$$\begin{aligned} T(t) &= e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}\mathbb{E}_{\frac{\alpha}{k},1}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)T_0 \\ &\quad - \lambda T_e \mathcal{L}_\phi^{-1}\left\{\mathcal{L}_\phi\{1\} \mathcal{L}_\phi\left\{e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}(\phi(t)-\phi(0))^{\frac{\alpha}{k}-1}\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)\right\}\right\}. \end{aligned}$$

Hence, the analytic solution of problem (5.4) is given by

$$T(t) = e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))}\mathbb{E}_{\frac{\alpha}{k},1}\left(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}\right)T_0$$

$$-\lambda T_e \int_a^t e^{\frac{\rho-1}{k\rho}(\phi(s)-\phi(a))} (\phi(s)-\phi(a))^{\frac{\alpha}{k}-1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left(\lambda (\phi(s)-\phi(a))^{\frac{\alpha}{k}} \right) \phi'(s) ds. \quad (5.8)$$

Next, the numerical solution of problem (5.4) is provided by applying formula (4.8) with $f(t, T(t)) = \lambda(k\rho)^{\frac{\alpha}{k}}(T(t) - T_e)$ as follows:

$$\begin{aligned} T_{n+1} = & T_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) + \lambda h^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} (T_i - T_e) \right. \\ & \left. - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n+2-i)h} (T_{i-1} - T_e) \right], \end{aligned} \quad (5.9)$$

where $\Omega(n, i)$ and $\Upsilon(n, i)$ are given by

$$\begin{aligned} \Omega(n, i) = & (n+1-i)^{\frac{\alpha}{k}} \left[\mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) + (n+1-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right] \\ & - (n-i)^{\frac{\alpha}{k}} \left[2 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + (n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right], \end{aligned} \quad (5.10)$$

$$\begin{aligned} \Upsilon(n, i) = & (n+1-i)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \\ & - (n-i)^{\frac{\alpha}{k}} \left[(n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \right]. \end{aligned} \quad (5.11)$$

Applying formula (5.8), the numerical results of the observed data for 40 mL water and oil [31] and the analytic solutions for various fractional derivative operators (Classical: $\alpha = k = \rho = 1$ and $\phi(t) = t$; Caputo: $\alpha = 0.9994$, $k = \rho = 1$, and $\phi(t) = t$; k -Caputo: $\alpha = 0.9994$, $k = 1.002$, $\rho = 1$, and $\phi(t) = t$; ρ -Caputo: $\alpha = 0.9994$, $k = 1$, $\rho = 0.9998$, and $\phi(t) = t$, and (k, ρ) -Caputo: $\alpha = 0.9994$, $k = 1.002$, $\rho = 0.9998$ and $\phi(t) = 0.9998t$) are shown in Tables 1 and 2, respectively.

From Figures 1 and 2, it can be observed that the exact solution yields the minimum value of all errors for the observed cooling data 40 mL water and oil, respectively. Then, we can conclude that the proposed model (5.4) performs better than other fractional derivative operators.

Table 1. Comparison of the observed cooling data for 40 mL water with the exact solution (5.8).

Time (min)	Observed	Classical	Caputo	k -Caputo	ρ -Caputo	(k, ρ) -Caputo
0	92.7508	92.7508	92.7508	92.7508	92.7508	92.7508
5	71.5597	80.5139	80.5215	80.5501	80.4411	80.4721
10	59.6864	69.8967	69.9173	69.9971	69.7776	69.8615
15	52.7337	60.6848	60.7166	60.8453	60.5348	60.6687
20	47.8517	52.6921	52.7325	52.9035	52.5223	52.6989
25	44.0043	45.7574	45.8037	46.0092	45.5756	45.7870
30	40.9361	39.7406	39.7903	40.0229	39.5528	39.7911
35	38.1243	34.5202	34.5712	34.8239	34.3307	34.5889
40	36.8658	29.9908	30.0414	30.3080	29.8028	30.0746
45	35.6100	26.0609	26.1096	26.3851	25.8766	26.1567
$\ x - \tilde{x}\ _2$	-	20.6645	20.6528	20.5501	20.6587	20.5401
MSE	-	42.7020	42.6539	42.2306	42.6782	42.1896
RMSE	-	6.5347	6.5310	6.4985	6.5329	6.4954
time(s)	-	0.000384	24.134284	24.878243	26.891761	23.184952

Table 2. Comparison of the observed cooling data for oil with the exact solution (5.8).

Time (min)	Observed	Classical	Caputo	k -Caputo	ρ -Caputo	(k, ρ) -Caputo
0	105.2900	105.2900	105.2900	105.2900	105.2900	105.2900
5	82.4516	90.7349	90.7443	90.7782	90.6536	90.6904
10	68.5161	78.1976	78.2229	78.3172	78.0667	78.1658
15	59.2258	67.3984	67.4377	67.5888	67.2358	67.3929
20	51.8710	58.0963	58.1465	58.3460	57.9146	58.1206
25	46.4516	50.0838	50.1417	50.3800	49.8920	50.1370
30	41.0323	43.1821	43.2448	43.5128	42.9866	43.2612
35	38.3226	37.2372	37.3022	37.5917	37.0427	37.3383
40	35.2280	32.1165	32.1817	32.4854	31.9261	32.2355
45	33.6774	27.7057	27.7694	28.0814	27.5217	27.8387
50	32.1290	23.9063	23.9673	24.2824	23.7301	24.0496
55	30.9677	20.6337	20.6910	21.0051	20.4661	20.7839
$\ x - \tilde{x}\ _2$	-	22.5084	22.4893	22.4810	22.3227	20.5401
MSE	-	42.2192	42.1475	42.1163	42.1738	41.5254
RMSE	-	6.4976	6.4921	6.4897	6.4941	6.4440
time(s)	-	0.000391	25.354053	26.792274	27.214784	23.618804

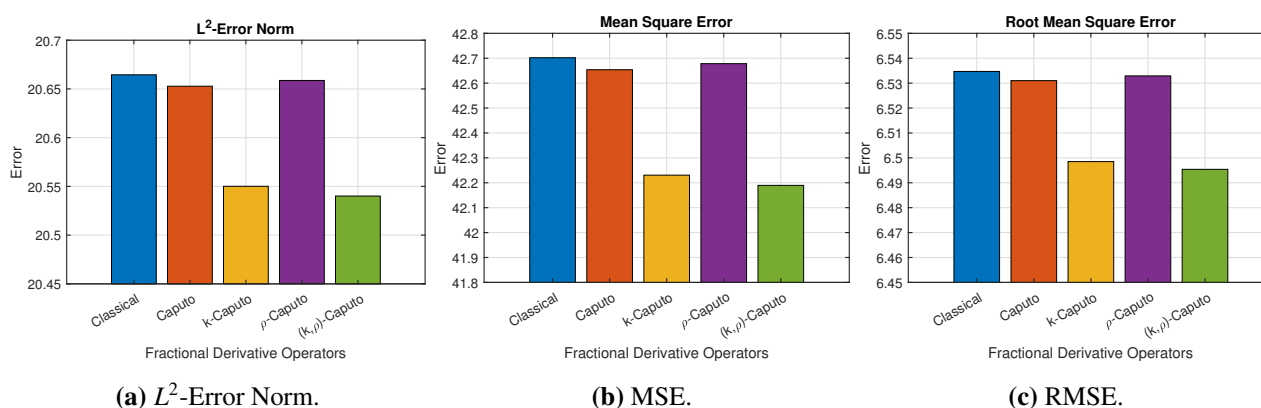
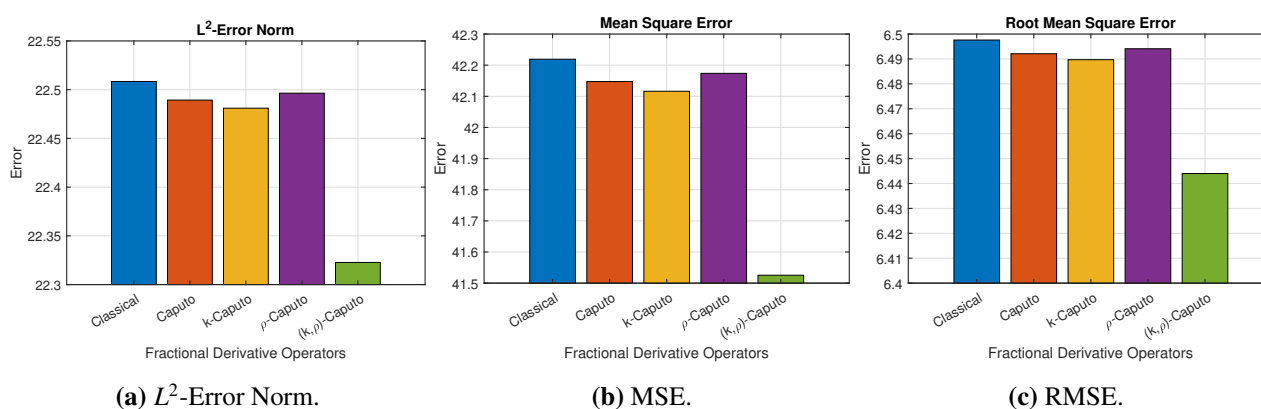
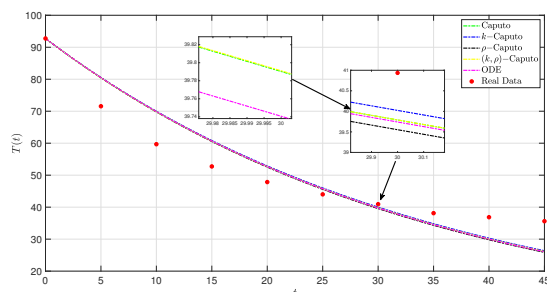
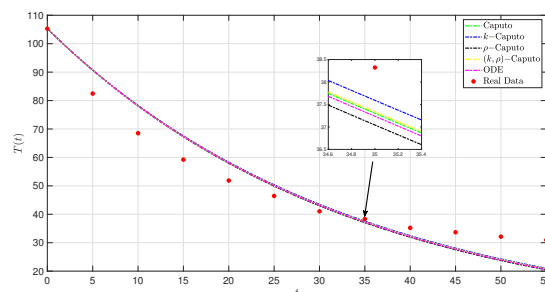
**Figure 1.** The L^2 -error norm, MSE, and RMSE values for the observed cooling data for 40 mL water under various fractional derivative operators.**Figure 2.** The L^2 -error norm, MSE, and RMSE values for the observed cooling data for oil under various fractional derivative operators.

Figure 3 displays the exact solution curves with the observed cooling data 40 mL of water (Figure 3a) and oil (Figure 3b), respectively, and the obtained results are compared with the numerical results under various fractional derivative operators.



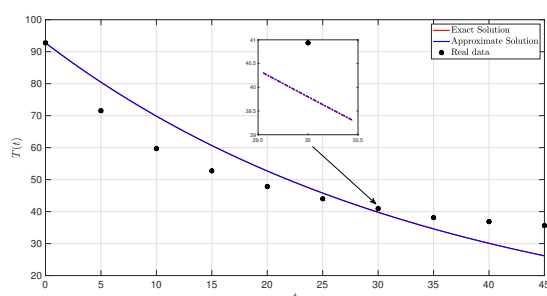
(a) 40 mL water.



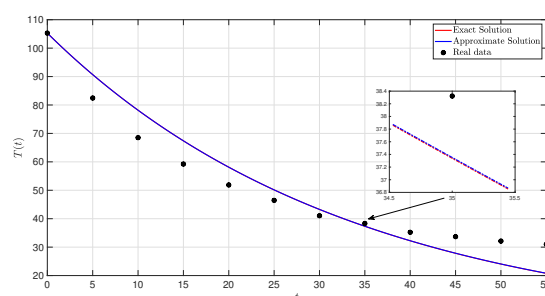
(b) Oil.

Figure 3. Comparison of the analytic solution curves with observed cooling data for 40 mL water and oil under various fractional derivative operators.

The numerical solution can be computed using Eqs (5.9)–(5.11), with its order of convergence at least $2 + \alpha/k = 2 + 0.9994/1.002 \approx 2.9974$. In addition, Figure 4 shows a comparison between the exact solution achieved from the ϕ -Laplace transform and the approximated solution obtained from the proposed method for the proposed model (5.4) against the observed cooling data for 40 mL water and oil, respectively. Error values between the exact and approximated solutions for 40 mL water (Figure 5a) and oil (Figure 5b), respectively, as shown in Figure 5.

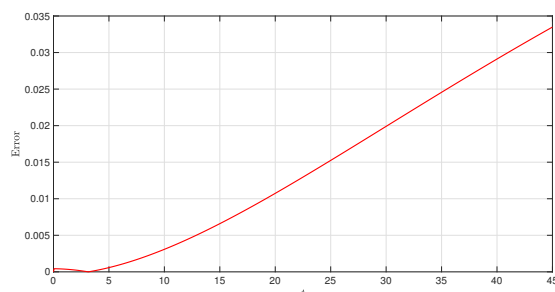


(a) 40 mL water.

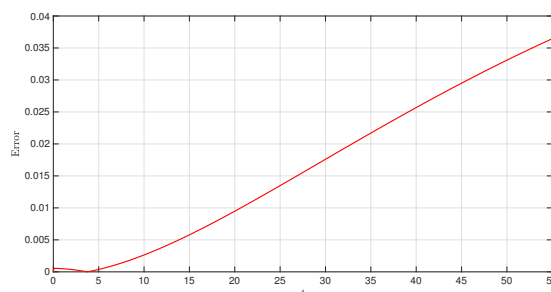


(b) Oil.

Figure 4. Comparison of the exact and approximated solutions with the observed cooling data for 40 mL water and oil, respectively.



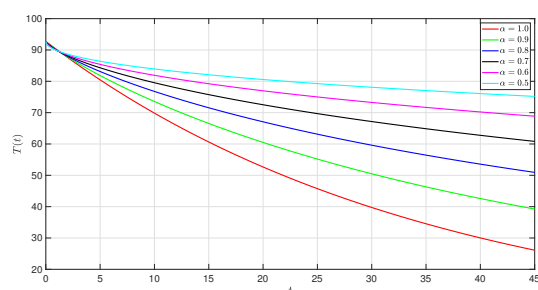
(a) 40 mL water.



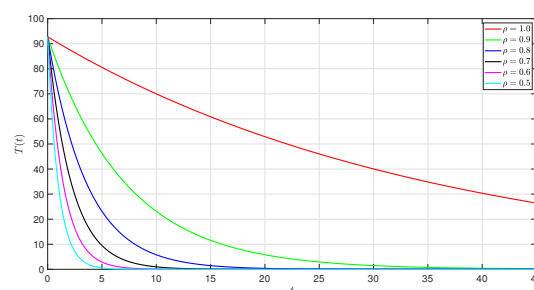
(b) Oil.

Figure 5. Error between the exact and approximated solutions for 40 mL water and oil, respectively.

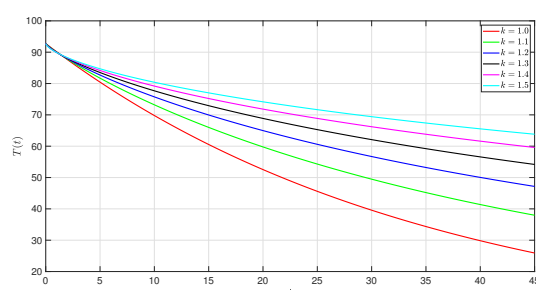
Figures 6 and 7 display the impact of α , ρ , and k in the approximate solutions for 40 mL water and oil, respectively; they show that when α increases from 0.5 to 1.0, then the approximate solutions decrease (Figures 6a and 7a). ρ increases from 0.5 to 1.0, and the approximate solution increases, as shown in Figures 6b and 7b, respectively. In addition, k increase from 1.0 to 1.5, and the approximate solution increases, as shown in Figures 6c and 7c.



(a) $\alpha \in [0.5, 1.0]$.

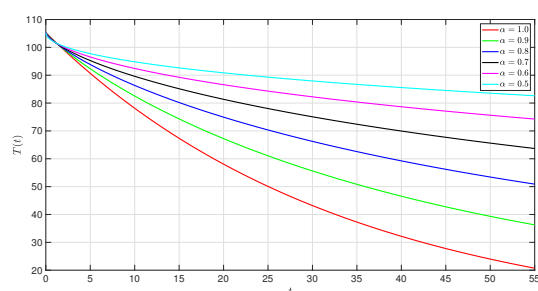


(b) $\rho \in [0.5, 1.0]$.

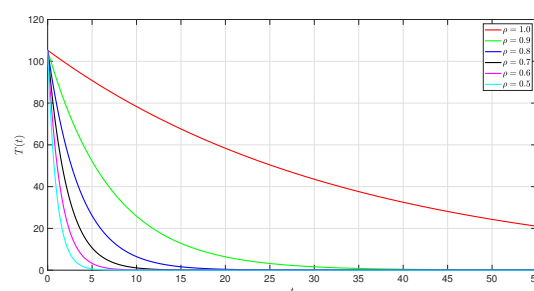


(c) $k \in [1.0, 1.5]$.

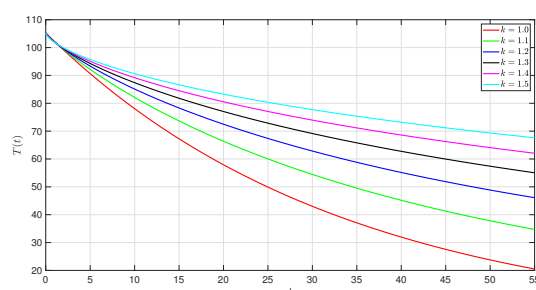
Figure 6. Impact of fractional-order α , proportional ρ , and k for 40 mL water.



(a) $\alpha \in [0.5, 1.0]$.



(b) $\rho \in [0.5, 1.0]$.



(c) $k \in [1.0, 1.5]$.

Figure 7. Impact of fractional-order α , proportional ρ , and k for oil.

5.2. Generalized world population growth model

For over half a century, many researchers have attempted to explain the growth of the world's population [32]. The Malthusian rule of population growth is the most basic model applied to forecast populations under ideal circumstances. To present population growth with a differential equation, we must first add variables and important terms. The population model in (5.12) is derived from a balance between population gains and losses. Assume that $P(t)$ is the population's size at time t , b , d denote the birth and death rates, respectively, r is the production rate, and b and d are arbitrary constants. During a sufficiently small time interval dt , the expected population gain due to births is $bP(t)dt$, whereas the corresponding population loss due to deaths is $dP(t)dt$. Consequently, the net population variation is

$$dP(t) = bP(t)dt - dP(t)dt,$$

and hence, the net growth rate may then be calculated using

$$\frac{dP}{dt} = (b - d)P(t) = rP(t), \quad r = b - d. \quad (5.12)$$

Applying the classical Laplace transform into problem (5.12), the solution of problem (5.12) is given by

$$P(t) = P_0 e^{Pt}, \quad t \geq 0, \quad (5.13)$$

where P_0 denotes the initial population at $t = 0$. It is well known that solution (5.13) is called the exponential growth model. Next, we establish the solution of the problem (5.14) using various techniques, including the ϕ -Laplace transform technique and the proposed numerical scheme (two-step Lagrange interpolation polynomial).

To incorporate nonlocal temporal effects into the population dynamics, we generalize the local temporal evolution operator in (5.12) while retaining $P(t)$ as the population size. Accordingly, the fractional population model is formulated as

$${}_0^C \mathfrak{D}_{k,\phi}^{\alpha,\rho} P(t) = rP(t), \quad 0 < \alpha \leq 1.$$

Here, the fractional operator represents a generalized temporal evolution mechanism that accounts for the dependence of the population dynamics on its preceding states. Importantly, the physical interpretation of $P(t)$ is unchanged: It remains the population size at time t . The fractionalization is applied to the evolution law rather than to the population variable. The consistency of this formulation with the classical population model can be verified through the classical limit of the proposed operator. Under $k = 1$, $\rho = 1$, $\phi(t) = t$, the first-order proportional operator satisfies

$$\mathfrak{D}_{1,t}^{1,1} P(t) = \frac{dP(t)}{dt}.$$

Moreover, since $\Gamma_k(u) = k^{\frac{u}{k}-1} \Gamma\left(\frac{u}{k}\right)$, we have $\Gamma_1(u) = \Gamma(u)$. Hence, for $k = 1$, $\rho = 1$, and $\phi(t) = t$, the proposed fractional operator reduces to the standard Caputo derivative. In the limiting case $\alpha \rightarrow 1^{-1}$,

$${}_a^C \mathfrak{D}_{1,t}^{\alpha,1} P(t) \longrightarrow \frac{dP(t)}{dt}.$$

Then, the classical population growth equation is recovered, demonstrating the consistency of the fractional formulation with the original population balance law. For the fractional formulation, r is interpreted as the effective net growth parameter associated with the generalized temporal operator. Thus, its role is to characterize the net population growth tendency, while its dimensional interpretation is understood consistently with the adopted fractional-order formulation.

To establish the effectiveness of the proposed numerical scheme to the (k, ρ, ϕ) -Caputo-FDO, we consider the generalized population growth model under the (k, ρ, ϕ) -Caputo-FDO:

$$\begin{cases} {}^C_0 \mathfrak{D}_{k,\phi}^{\alpha,\rho} P(t) = (k\rho)^{\frac{\alpha}{k}} r P(t), & \alpha \in (0, 1], \\ P(0) = P_0, & \rho \in (0, 1], \quad k > 0. \end{cases} \quad (5.14)$$

Applying the ϕ -Laplace transform in Definition 2.3, Lemma 2.1, and Theorem 2.1 into the proposed problem (5.14), we obtain the following results:

$$\begin{aligned} (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} \mathcal{L}_\phi \{P(t)\} - (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1} P(0) &= (k\rho)^{\frac{\alpha}{k}} r \mathcal{L}_\phi \{P(t)\} \\ \left[\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - r \right] \mathcal{L}_\phi \{P(t)\} &= \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1} P_0. \end{aligned}$$

Then,

$$\mathcal{L}_\phi \{P(t)\} = \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} - r} P_0. \quad (5.15)$$

Substituting $\lambda = r$ into (5.6) with helping the ϕ -inverse Laplace transform, the relation (5.15) can be rewritten as

$$P(t) = e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} \left(r(\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) P_0. \quad (5.16)$$

The numerical solution of the problem (5.14) is provided by applying the formula (4.8) with $f(t, P(t)) = (k\rho)^{\frac{\alpha}{k}} r P(t)$ as follows:

$$P_{n+1} = P_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k},1} \left(r((n+1)h)^{\frac{\alpha}{k}} \right) + rh^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} P_i - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n+2-i)h} P_{i-1} \right], \quad (5.17)$$

where $\Omega(n, i)$ and $\Upsilon(n, i)$ are defined by

$$\begin{aligned} \Omega(n, i) &= (n+1-i)^{\frac{\alpha}{k}} \left[\mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(r((n+1-i)h)^{\frac{\alpha}{k}} \right) + (n+1-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2} \left(r((n+1-i)h)^{\frac{\alpha}{k}} \right) \right] \\ &\quad - (n-i)^{\frac{\alpha}{k}} \left[2 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(r((n-i)h)^{\frac{\alpha}{k}} \right) + (n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2} \left(r((n-i)h)^{\frac{\alpha}{k}} \right) \right], \end{aligned} \quad (5.18)$$

$$\begin{aligned} \Upsilon(n, i) &= (n+1-i)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2} \left(r((n+1-i)h)^{\frac{\alpha}{k}} \right) \\ &\quad - (n-i)^{\frac{\alpha}{k}} \left[(n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2} \left(r((n-i)h)^{\frac{\alpha}{k}} \right) + \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(r((n-i)h)^{\frac{\alpha}{k}} \right) \right]. \end{aligned} \quad (5.19)$$

Applying formula (5.16), the numerical results of the world's population [32] and the exact solutions for various types of fractional derivative operators (Classical: $\alpha = k = \rho = 1$ and $\phi(t) = t$; Caputo: $\alpha = 0.999999$, $k = \rho = 1$, and $\phi(t) = t$; k -Caputo: $\alpha = 0.999999$, $k = 1.000001$, $\rho = 1$, and $\phi(t) = t$;

ρ -Caputo: $\alpha = 0.999999$, $k = 1$, $\rho = 0.99998$, and $\phi(t) = t$, and (k, ρ) -Caputo: $\alpha = 0.999999$, $k = 1.000001$, $\rho = 0.99998$ and $\phi(t) = t^{0.9997}$ are shown in Table 3.

Table 3. Comparison of the world's population data with the exact solution $\alpha = 0.999999$, $\rho = 0.99998$, $k = 1.000001$, and $\phi(t) = t^{0.9997}$.

Time (year)	Observed ($\times 10^9$)	Classical	Caputo	k -Caputo	ρ -Caputo	(k, ρ) -Caputo
0	2.493	2.4930	2.4930	2.4930	2.4930	2.4930
5	2.740	2.7182	2.7182	2.7182	2.7180	2.7179
10	3.015	2.9638	2.9638	2.9638	2.9633	2.9629
15	3.335	3.2316	3.2316	3.2316	3.2307	3.2299
20	3.695	3.5236	3.5236	3.5235	3.5222	3.5210
25	4.071	3.8420	3.8419	3.8418	3.8400	3.8383
30	4.448	4.1891	4.1890	4.1889	4.1866	4.1842
35	4.869	4.5676	4.5675	4.5673	4.5644	4.5612
40	5.328	4.9803	4.9802	4.9799	4.9763	4.9722
45	5.759	5.4303	5.4301	5.4298	5.4253	5.4202
50	6.172	5.9209	5.9207	5.9203	5.9149	5.9085
55	6.587	6.4559	6.4557	6.4551	6.4487	6.4408
60	7.022	7.0391	7.0389	7.0383	7.0306	7.0211
65	7.470	7.6751	7.6749	7.6741	7.6651	7.6536
70	7.887	8.3686	8.3683	8.3674	8.3568	8.3430
$\ x - \tilde{x}\ _2$	-	0.914851	0.914925	0.914818	0.915147	0.914788
MSE	-	0.055797	0.055806	0.055793	0.055833	0.055789
RMSE	-	0.236214	0.236233	0.236205	0.236290	0.236197
time(s)	-	0.000476	0.013811	0.007216	0.005672	0.009542

From Figure 8, the exact solution yields the minimum value of all errors for the world's population data. Then, model (5.14) performs better than other fractional derivative operators.

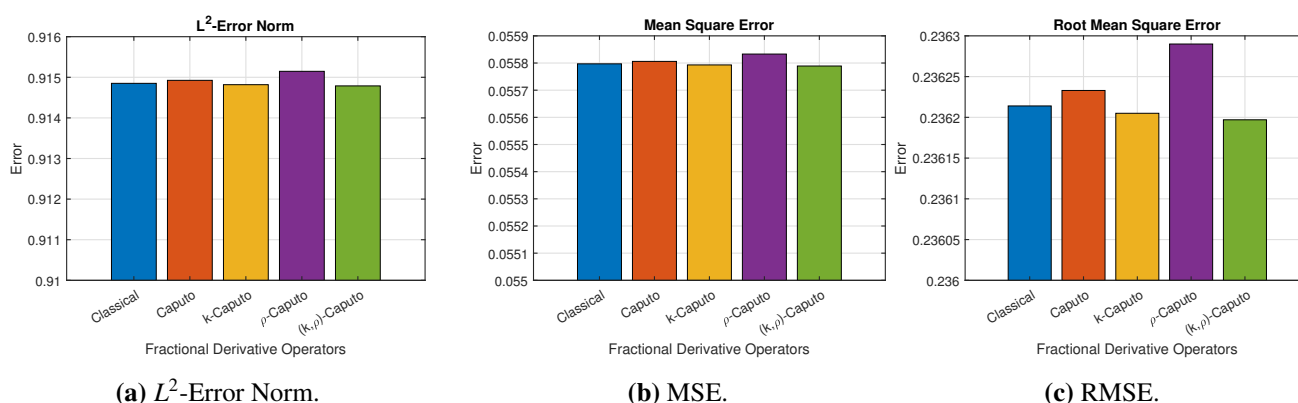


Figure 8. The L^2 -error norm, MSE, and RMSE values for the world's population data under various fractional derivative operators.

Figure 9 displays the analytical solution curves with world population data, which compared with the numerical results under various fractional derivative operators.

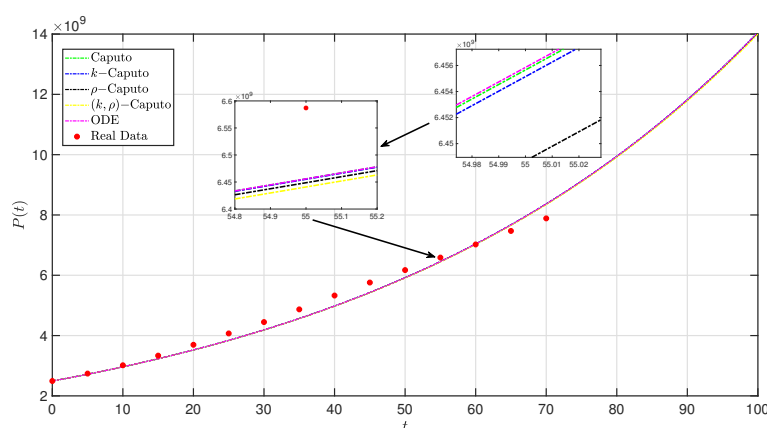


Figure 9. Comparison of the exact solution curves with the world's population data under various fractional derivative operators.

The numerical solution can be computed using Eqs (5.17)–(5.19), with its order of convergence at least $2 + \alpha/k = 2 + 0.999999/1.000001 \approx 2.999998$. In addition, Figure 10 displays a comparison between the exact solution obtained from the ϕ -Laplace transform and the approximated solution obtained from the proposed method for the proposed model (5.14) against the world's population data. Error values between the exact and approximated solutions are shown in Figure 11.

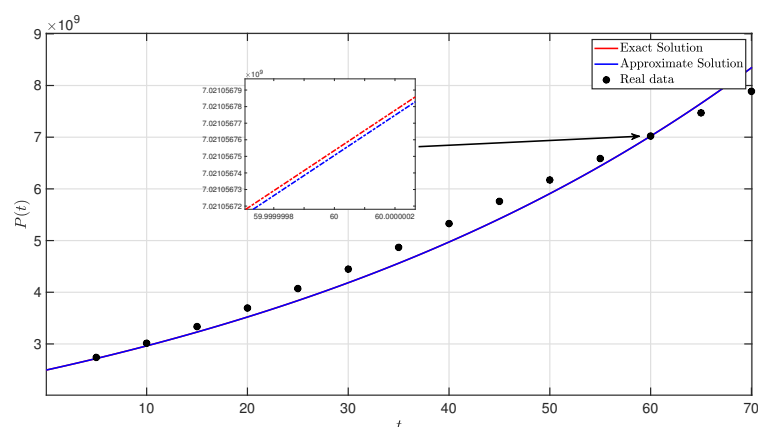


Figure 10. Comparison of the exact and approximated solutions with the world population data.

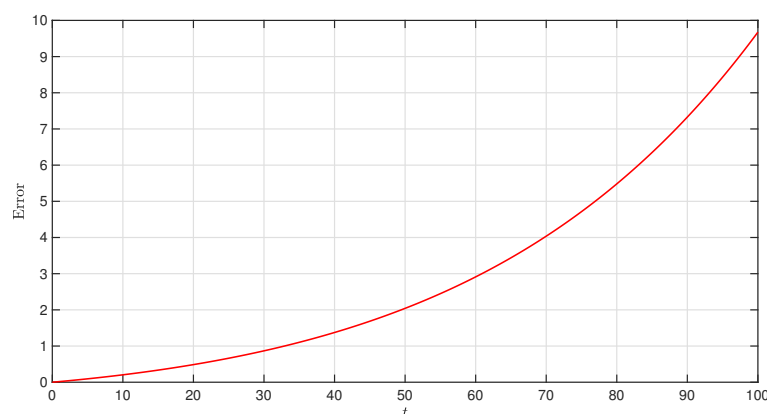
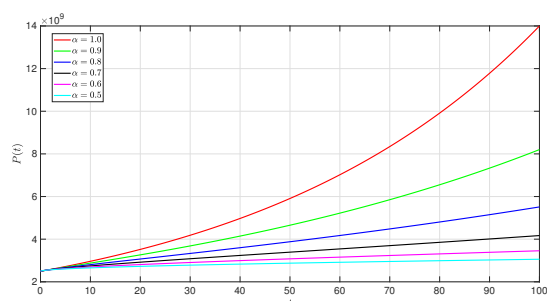
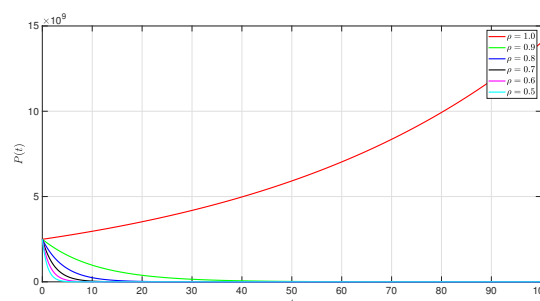


Figure 11. Error between the exact and approximated solutions for the world population.

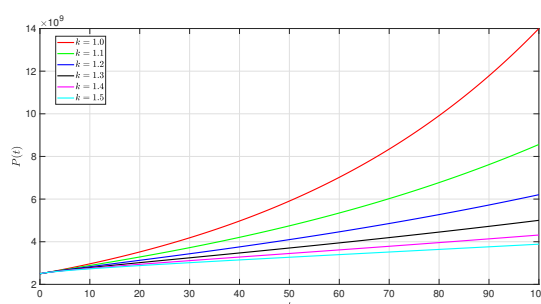
Figure 12 presents the impact of α , ρ , and k in the approximate solutions for the world population model; it is observed that when k increases from 0.5 to 1.0, then the approximate solutions decrease (Figure 12c). Moreover, α and ρ increase from 0.5 to 1.0, and the approximate solution increases, as shown in Figures 12a and 12b, respectively.



(a) $\alpha \in [0.5, 1.0]$.



(b) $\rho \in [0.5, 1.0]$.



(c) $k \in [1.0, 1.5]$.

Figure 12. Impact of fractional-order α , proportional ρ , and k for the approximate solutions of the world population model.

5.3. Generalized blood alcohol level model

The coupled system (5.20) represents a coupled two-compartment kinetic system. Let $A(t)$ and $B(t)$ denote the amounts or concentrations of a substance in compartments A and B , respectively. The first compartment undergoes a first-order depletion process with rate constant K_1 , whereas material transferred from compartment A enters compartment B . The material in compartment B is subsequently removed through a first-order process with rate constant K_2 . Accordingly, the balance law for compartment A is

$$\frac{dA}{dt} = -K_1 A(t),$$

because the only process affecting A is its first-order depletion. For compartment B , the rate of change is determined by two competing processes. The first is the input from compartment A , that is $K_1 A(t)$, whereas the second is the removal from compartment B , that is $K_2 B(t)$. Therefore,

$$\frac{dB}{dt} = K_1 A(t) - K_2 B(t).$$

Together with $A(0) = A_0$ and $B(0) = B_0$, this yields the coupled system (5.20). Investigating impacts is most important, as they contribute to advantage and disadvantage significance in the body's health, and

it can be better understood how they impact the body's health and make reported decisions on alcohol consumption. In 2011, Ludwin [33] studied the model of processing of ethanol in the human body in the context of first-order differential equations:

$$\begin{cases} \frac{dA}{dt} = -K_1 A(t), & \frac{dB}{dt} = K_1 A(t) - K_2 B(t), \\ A(0) = A_0, & B(0) = B_0, \end{cases} \quad (5.20)$$

where A and B denote the concentration of alcohol in the stomach and blood at any time t (mg/l), respectively. K_1 and K_2 present nonzero numbers (min^{-1}). A_0 and B_0 denote the initial concentration of alcohol in the stomach and blood (mg/l), respectively. Applying the classical Laplace transform into problem (5.20), the solution of problem (5.20) is given by

$$\begin{cases} A(t) = A_0 e^{-K_1 t}, \\ B(t) = A_0 \left(\frac{K_1}{K_2 - K_1} \right) (e^{-K_1 t} - e^{-K_2 t}). \end{cases} \quad (5.21)$$

Next, we investigate the solution of problem (5.22) by using various techniques, including the ϕ -Laplace transform technique and the proposed numerical scheme (two-step Lagrange interpolation polynomial).

To introduce memory effects into the two-compartment dynamics, we generalize the temporal evolution operator while preserving the underlying transfer structure between the two compartments. The resulting fractional model under the (k, ρ, ϕ) -Caputo-FDO is formulated as:

$$\begin{cases} {}^C_0 \mathfrak{D}_{k, \phi}^{\alpha, \rho} A(t) = -(k\rho)^{\frac{\alpha}{k}} K_1 A(t), & \alpha \in (0, 1], \\ {}^C_{0, k} \mathfrak{D}^{\alpha, \rho; \phi} B(t) = (k\rho)^{\frac{\alpha}{k}} (K_1 A(t) - K_2 B(t)), & \alpha \in (0, 1], \\ A(0) = A_0, & B(0) = B_0, \quad \rho \in (0, 1], \quad k > 0. \end{cases} \quad (5.22)$$

The fractionalization does not alter the physical roles of the state variables or the transfer pathways. Specifically, $A(t)$ continues to represent the amount or concentration in compartment A , and $B(t)$ continues to represent the corresponding quantity in compartment B . The term $K_1 A(t)$ retains its interpretation as the transfer/depletion flux from compartment A , while $K_2 B(t)$ represents the removal flux from compartment B .

The modification introduced by the fractional operator concerns the temporal response of each compartment. In contrast to the classical model, in which the instantaneous rate of change depends only on the current state, the fractional formulation introduces a history-dependent temporal response through the integral kernel of the (k, ρ, ϕ) -Caputo proportional operator. Thus, the model preserves the original physical pathways while enabling the current dynamics to depend on the preceding evolution of the system.

To achieve the exact solution of the problem (5.22), we use the ϕ -Laplace transform in Definition 2.3, Lemma 2.1, and Theorem 2.1, then

$$(k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho - 1}{k\rho} \right)^{\frac{\alpha}{k}} \mathcal{L}_{\phi} \{A(t)\} - (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho - 1}{k\rho} \right)^{\frac{\alpha}{k} - 1} A(0) = -(k\rho)^{\frac{\alpha}{k}} K_1 \mathcal{L}_{\phi} \{A(t)\}$$

$$\left[\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_1 \right] \mathcal{L}_\phi \{A(t)\} = \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1} A_0.$$

Then,

$$\mathcal{L}_\phi \{A(t)\} = \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_1} A_0. \quad (5.23)$$

Substituting $\lambda = -K_1$ into (5.6) with applying the ϕ -inverse Laplace transform, relation (5.23) can be rewritten as

$$A(t) = e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} \left(-K_1 (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) A_0. \quad (5.24)$$

Inserting (5.24) into the second equation of the proposed problem (5.22), we obtain that

$${}_0^C \mathfrak{D}_{k,\phi}^{\alpha,\rho} B(t) = (k\rho)^{\frac{\alpha}{k}} \left(K_1 e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} \left(-K_1 (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) A_0 - K_2 B(t) \right). \quad (5.25)$$

Taking the ϕ -Laplace transform into (5.25) with Theorem 2.1, which implies that

$$\begin{aligned} & (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} \mathcal{L}_\phi \{B(t)\} - (k\rho)^{\frac{\alpha}{k}} \left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1} B(0) \\ &= (k\rho)^{\frac{\alpha}{k}} K_1 A_0 \mathcal{L}_\phi \left\{ e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(a))} \mathbb{E}_{\frac{\alpha}{k},1} \left(-K_1 (\phi(t) - \phi(a))^{\frac{\alpha}{k}} \right) \right\} - (k\rho)^{\frac{\alpha}{k}} K_2 \mathcal{L}_\phi \{B(t)\}. \end{aligned}$$

Then,

$$\begin{aligned} & \mathcal{L}_\phi \{B(t)\} \\ &= \frac{K_1 A_0}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_2} \mathcal{L}_\phi \left\{ e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(a))} \mathbb{E}_{\frac{\alpha}{k},1} \left(-K_1 (\phi(t) - \phi(a))^{\frac{\alpha}{k}} \right) \right\} + \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_2} B_0 \\ &= \frac{K_1 A_0}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_2} \cdot \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_1} + \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_2} B_0. \end{aligned}$$

Applying the partial fraction technique, the above equation can be reformulated as

$$\mathcal{L}_\phi \{B(t)\} = \frac{K_1 A_0}{K_1 - K_2} \left(-K_2 \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_2} + K_1 \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_1} \right) + \frac{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}-1}}{\left(s - \frac{\rho-1}{k\rho} \right)^{\frac{\alpha}{k}} + K_2} B_0.$$

Taking the ϕ -inverse Laplace transform into the above result yields that

$$\begin{aligned} B(t) &= B_0 e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1} \left(-K_2 (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) + \frac{K_1 A_0}{K_1 - K_2} e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \\ &\quad \times \left[K_1 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(-K_1 (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) - K_2 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1} \left(-K_2 (\phi(t) - \phi(0))^{\frac{\alpha}{k}} \right) \right]. \end{aligned} \quad (5.26)$$

To achieve the numerical solution of the generalized of BAL model (5.22), we use the first equation of the generalized of BAL model (5.22) under the initial condition as follows:

$$\begin{cases} {}_0^C \mathfrak{D}_{k,\phi}^{\alpha,\rho} A(t) = \lambda (k\rho)^{\frac{\alpha}{k}} A(t) + f_1(s, A(s), B(s)), & \alpha \in (0, 1], \\ \rho \in (0, 1], & k > 0. \end{cases} \quad (5.27)$$

Applying Theorem 2.1 into the problem (5.27), we have

$$\begin{aligned} A(t) &= A_0 e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_0^t e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1} \\ &\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\phi(s))^{\frac{\alpha}{k}}) \phi'(s) f_1(s, A(s), B(s)) ds. \end{aligned} \quad (5.28)$$

Changing variable $\tau = \phi(s)$ and $ds = (1/\phi'(s))d\tau$, we get

$$\begin{aligned} A(t) &= A_0 e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(0))} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi(t)-\phi(0))^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_{\phi(0)}^{\phi(t)} e^{\frac{\rho-1}{k\rho}(\phi(t)-\tau)} (\phi(t)-\tau)^{\frac{\alpha}{k}-1} \\ &\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi(t)-\tau)^{\frac{\alpha}{k}}) f_1(\phi^{-1}(\tau), A(\phi^{-1}(\tau), B(\phi^{-1}(\tau))) d\tau. \end{aligned} \quad (5.29)$$

Taking $\phi(t) = \phi_{n+1}$ for $n = 0, 1, 2, \dots$ where $\phi_0 = \phi(0)$, $\phi_{n+1} = \phi_n + h$ with step size $h = (\phi_{n+1} - \phi_0)/N$, the formula (5.29) can be reformulated as follows:

$$\begin{aligned} A(\phi^{-1}(\phi_{n+1})) &= A_0 e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \int_{\phi_0}^{\phi_{n+1}} e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} \\ &\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}}) f_1(\phi^{-1}(\tau), A(\phi^{-1}(\tau), B(\phi^{-1}(\tau))) d\tau. \end{aligned}$$

Then,

$$\begin{aligned} A(\phi^{-1}(\phi_{n+1})) &= A_0 e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) + (k\rho)^{-\frac{\alpha}{k}} \sum_{i=0}^n \int_{\phi_i}^{\phi_{i+1}} e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} (\phi_{n+1}-\tau)^{\frac{\alpha}{k}-1} \\ &\quad \times \mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}(\lambda(\phi_{n+1}-\tau)^{\frac{\alpha}{k}}) f_1(\phi^{-1}(\tau), A(\phi^{-1}(\tau), B(\phi^{-1}(\tau))) d\tau. \end{aligned} \quad (5.30)$$

The function $e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} f_1(\phi^{-1}(\tau), A(\phi^{-1}(\tau), B(\phi^{-1}(\tau)))$ is approximated by applying a two-step Lagrange interpolation polynomial, and it follows that

$$e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\tau)} f_1(\phi^{-1}(\tau), A(\phi^{-1}(\tau), B(\phi^{-1}(\tau))) = \frac{F(t_i, A_i, B_i)(\tau - \phi(t_{i-1})) - F(t_{i-1}, A_{i-1}, B_{i-1})(\tau - \phi(t_i))}{\phi(t_i) - \phi(t_{i-1})}, \quad (5.31)$$

where $F(t_i, A_i, B_i) = e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi(t_i))} f_1(t_i, A_i, B_i)$. Inserting (5.31) into (5.30), we have

$$\begin{aligned} A(\phi^{-1}(\phi_{n+1})) &= A_0 e^{\frac{\rho-1}{k\rho}(\phi_{n+1}-\phi_0)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda(\phi_{n+1}-\phi_0)^{\frac{\alpha}{k}}) \\ &\quad + \sum_{i=0}^n \frac{F(t_i, A_i, B_i)}{(k\rho)^{\frac{\alpha}{k}} h} \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j\frac{\alpha}{k} + \frac{\alpha}{k})} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{(j+1)\frac{\alpha}{k}-1} (\tau - \phi(t_{i-1})) d\tau \\ &\quad - \sum_{i=0}^n \frac{F(t_{i-1}, A_{i-1}, B_{i-1})}{(k\rho)^{\frac{\alpha}{k}} h} \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j\frac{\alpha}{k} + \frac{\alpha}{k})} \int_{\phi_i}^{\phi_{i+1}} (\phi_{n+1}-\tau)^{(j+1)\frac{\alpha}{k}-1} (\tau - \phi_i) d\tau. \end{aligned} \quad (5.32)$$

Applying (4.6) and (4.7), Eq (5.32) can be rewritten as

$$\begin{aligned} A(\phi^{-1}(\phi_{n+1})) &= A_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k},1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) \\ &\quad + \left(\frac{h}{k\rho}\right)^{\frac{\alpha}{k}} \sum_{i=0}^n \left[F(t_i, A_{i-1}, B_i) \left((n+1-i) \left[\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}+1}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
& + (n+1-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \Big] \\
& - (n-i)^{\frac{\alpha}{k}} \Big[2 \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + (n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \Big] \\
& - F(t_{i-1}, A_{i-1}, B_{i-1}) \Big((n+1-i)^{\frac{\alpha}{k}+1} \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n+1-i)h)^{\frac{\alpha}{k}}) \\
& - (n-i)^{\frac{\alpha}{k}} \Big[(n-i) \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+2}(\lambda((n-i)h)^{\frac{\alpha}{k}}) + \mathbb{E}_{\frac{\alpha}{k}, \frac{\alpha}{k}+1}(\lambda((n-i)h)^{\frac{\alpha}{k}}) \Big] \Big).
\end{aligned}$$

Then,

$$\begin{aligned}
A_{n+1} = & A_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) + \left(\frac{h}{k\rho}\right)^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} f_1(t_i, A_i, B_i) \right. \\
& \left. - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n+2-i)h} f_1(t_{i-1}, A_{i-1}, B_{i-1}) \right].
\end{aligned} \quad (5.33)$$

where $t_i = \phi^{-1}(\phi_i)$ for $i = 0, 1, \dots, n$. In the same way, we can achieve the second numerical solution of the generalized of BAL model (5.22) as follows:

$$\begin{aligned}
B_{n+1} = & B_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) + \left(\frac{h}{k\rho}\right)^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} f_2(t_i, A_i, B_i) \right. \\
& \left. - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n+2-i)h} f_2(t_{i-1}, A_{i-1}, B_{i-1}) \right],
\end{aligned} \quad (5.34)$$

Substituting $f_1(t, A(t), B(t)) = -(k\rho)^{\frac{\alpha}{k}} K_1 A(t)$ and $f_2(t, A(t), B(t)) = (k\rho)^{\frac{\alpha}{k}} (K_1 A(t) - K_2 B(t))$ into (5.33) and (5.34), we obtain the numerical solutions of the generalized of BAL model (5.22)

$$A_{n+1} = A_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) - K_1 h^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} A_i - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i)h} A_{i-1} \right], \quad (5.35)$$

$$\begin{aligned}
B_{n+1} = & B_0 e^{\frac{\rho-1}{k\rho}((n+1)h)} \mathbb{E}_{\frac{\alpha}{k}, 1}(\lambda((n+1)h)^{\frac{\alpha}{k}}) + h^{\frac{\alpha}{k}} \sum_{i=0}^n \left[\Omega(n, i) e^{\frac{\rho-1}{k\rho}(n+1-i)h} (K_1 A_i - K_2 B_i) \right. \\
& \left. - \Upsilon(n, i) e^{\frac{\rho-1}{k\rho}(n-i)h} (K_1 A_{i-1} - K_2 B_{i-1}) \right],
\end{aligned} \quad (5.36)$$

where $\Omega(n, i)$ and $\Upsilon(n, i)$ are given by (4.9) and (4.10), respectively. Applying formulas (5.24) and (5.26), the numerical results of the observed BAL data [33] and the exact solution for various fractional derivative operators (Classical: $\alpha = k = \rho = 1$ and $\phi(t) = t$; Caputo: $\alpha = 0.9999982$, $k = \rho = 1$, and $\phi(t) = t$; k -Caputo: $\alpha = 0.9999982$, $k = 0.9999985$, $\rho = 1$, and $\phi(t) = t$; ρ -Caputo: $\alpha = 0.9999982$, $k = 1$, $\rho = 0.9999994$, and $\phi(t) = t$, and (k, ρ) -Caputo: $\alpha = 0.9999982$, $k = 0.9999985$, $\rho = 0.9999994$, and $\phi(t) = 1.002t$) are shown in Table 4.

Table 4. Comparison of the observed BAL data and model predictions using different fractional derivatives with $\alpha = 0.9999982$, $\rho = 0.9999994$, $k = 0.9999985$, and $\phi(t) = 1.002t$.

Time (min)	BAL (mg/L)	Classical	Caputo	k -Caputo	ρ -Caputo	(k, ρ) -Caputo
0	0	0.0000	0.0000	0.0000	0.0000	0.0000
10	150	147.5379	147.5376	147.5378	147.5367	147.6646
20	200	172.9499	172.9497	172.9499	172.9476	172.9455
30	160	161.3813	161.3814	161.3813	161.3785	161.2672
45	130	130.0021	130.0026	130.0022	129.9991	129.8089
80	70	71.0018	71.0026	71.0019	70.9992	70.7981
90	60	59.4910	59.4918	59.4911	59.4886	59.2987
110	40	41.7418	41.7426	41.7420	41.7399	41.5768
170	20	14.4100	14.4106	14.4101	14.4091	14.3220
$\ x - \tilde{x}\ _2$	-	27.8428	27.8430	27.8428	27.8450	27.8370
MSE	-	86.1358	86.1369	86.1359	86.1495	86.0997
RMSE	-	9.2809	9.2810	9.2809	9.2817	9.2790
time(s)	-	0.014567	0.023912	0.013491	0.014607	0.030102

From Figure 13a, it can be observed that the exact solution yields the minimum value of all errors for the observed BAL data. Then, we can conclude that the proposed model (5.22) performs better than other fractional derivative operators.

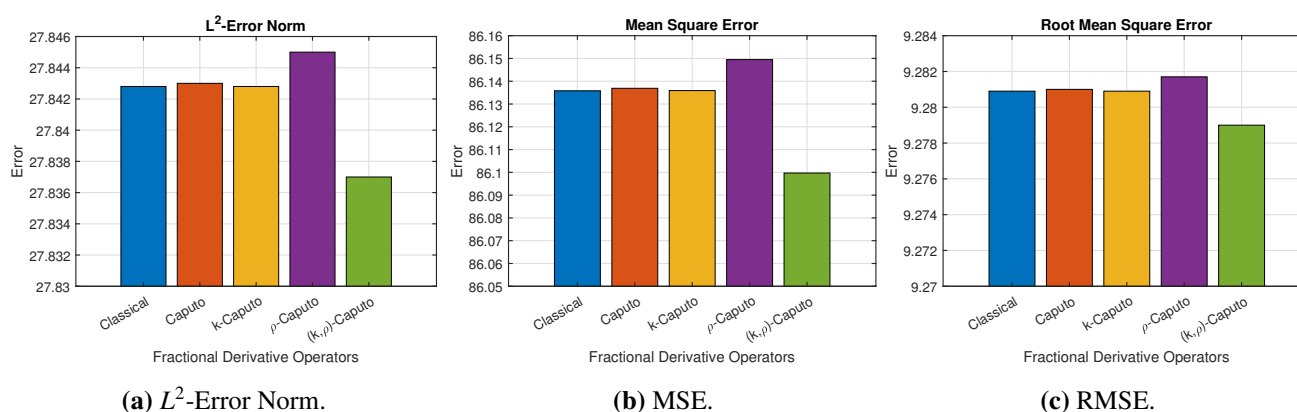


Figure 13. The L^2 -error norm, MSE, and RMSE values for the observed BAL data under various fractional derivative operators.

Figure 14 displays the exact solution curves with the observed BAL data, and these are compared with the numerical results under various fractional derivative operators.

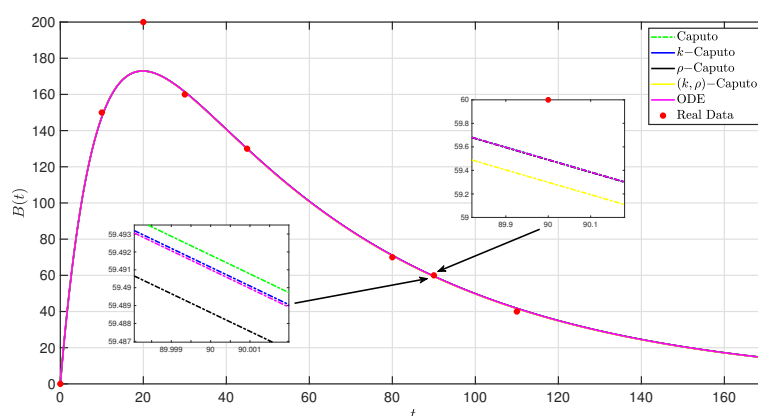


Figure 14. Comparison of the exact solution curves with the observed BAL data under various fractional derivative operators.

The numerical solution can be computed using Eqs (5.35) and (5.36), with its order of convergence at least $2 + \alpha/k = 2 + 0.9999982/0.9999985 \approx 2.9999997$. In addition, Figure 15 displays a comparison between the exact solution obtained from the ϕ -Laplace transform and the approximated solution obtained from the proposed method for proposed model (5.22) against the observed BAL data. Error values between the exact and approximated solutions are shown in Figure 16.

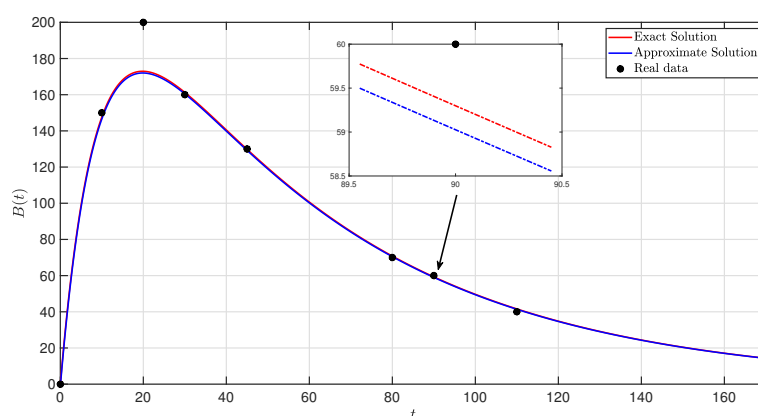


Figure 15. Comparison of the exact and approximated solutions with the observed BAL data.

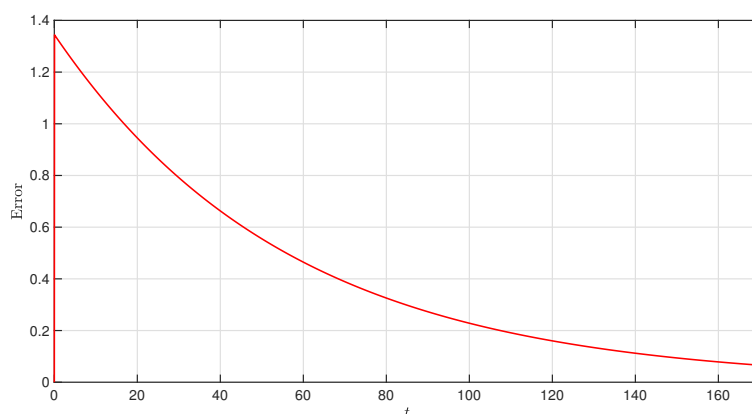
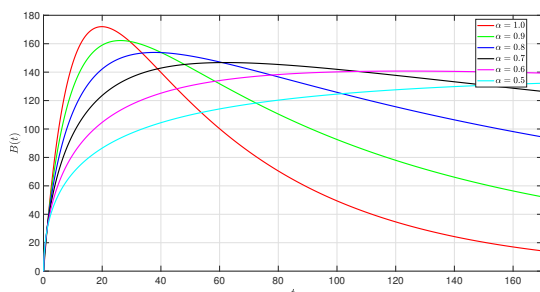
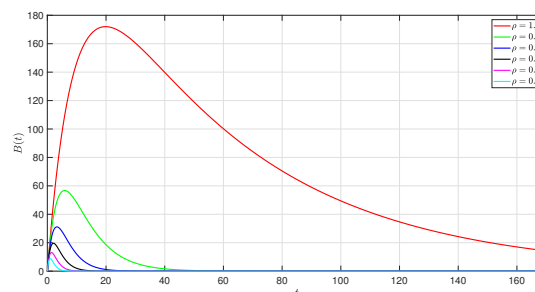


Figure 16. Error between the exact and approximated solutions for the observed BAL data.

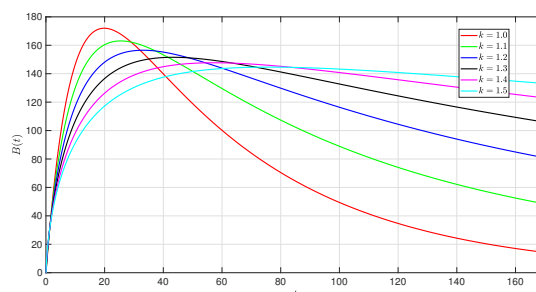
Figure 17 presents the impact of α , ρ , and k in the approximate solutions for the BAL model; it is observed that when α and ρ increase from 0.5 to 1.0, then the approximate solutions increase (Figures 17a and 17b). Moreover, k increases from 1.0 to 1.5, and the approximate solution decreases, as shown in Figure 17c.



(a) $\alpha \in [0.5, 1.0]$.



(b) $\rho \in [0.5, 1.0]$.



(c) $k \in [1.0, 1.5]$.

Figure 17. Impact of fractional-order α , proportional ρ , and k for the approximate solutions of the BAL model.

It should be emphasized that the fractionalization adopted in Subsection 5.1–5.3 does not redefine the physical state variables or the underlying interaction mechanisms. In each application, the variables $T(t)$, $P(t)$, $A(t)$, and $B(t)$ retain their original physical meanings as temperature, population size, and compartmental quantities, respectively. Likewise, the parameters describing the corresponding physical mechanisms retain their original roles. The proposed (k, ρ, ϕ) -Caputo-FD $\odot$ is introduced to generalize the temporal evolution law and to incorporate nonlocal memory effects. The consistency with the corresponding classical models follows from the special case $k = 1$, $\rho = 1$, and $\phi(t) = t$, followed by the classical-order limit $\alpha \rightarrow 1^-$, under which the fractional operator reduces to the ordinary first-order temporal derivative. Therefore, the proposed models should be interpreted as fractional generalizations of the corresponding classical physical models rather than as changes in the definitions of their physical state variables.

In addition, Table 5 reports the computational time required for the numerical evaluation of the four scientific models, namely the Newton-Oil, Newton-Water, Population, and BAL models, under the (k, ρ, ϕ) -Caputo-FD $\odot$. The results are presented for different numbers of discretization points, N , together with the corresponding step size h . As expected, increasing N results in a smaller step size and consequently a finer numerical discretization. Furthermore, the computational time increases for all four models. For example, for the Newton-Oil model, the computational time increases from 19.10

seconds at $N = 200$ to 110.44 seconds at $N = 1000$. A similar trend is observed for the Newton-Water model, for which the computational time increases from 19.35 to 110.63 seconds over the same range of N . These results indicate the expected trade-off between numerical resolution and computational cost. Among the considered models, the Population model requires the largest computational time, increasing from approximately 31.53 seconds at $N = 200$ to 193.07 seconds at $N = 1000$. This higher computational cost is associated with the numerical evaluation of the corresponding fractional formulation and its solution structure. The BAL model also exhibits an increasing computational cost as the discretization becomes finer, although its computational time remains lower than that of the Population model for the considered values of N . The Newton-Oil and Newton-Water models exhibit very similar computational times for all discretization levels. This behavior is consistent with the fact that both models are governed by closely related Newton-type fractional formulations and have comparable numerical structures. Overall, the results demonstrate that the proposed numerical framework remains computationally feasible for the considered scientific models, while the increase in computational cost with N should be taken into account when higher numerical resolution is required.

Table 5. Computational time (in seconds) comparison for the different models considered in Tables 2–4 under the (k, ρ, ϕ) -Caputo-FDO.

N	Newton-Oil (5.4)		Newton-Water (5.4)		Population (5.14)		BAL (5.22)	
	h	time(s)	h	time(s)	h	time(s)	h	time(s)
200	0.2750	19.1007	0.2250	19.3463	0.4993	31.5281	0.8517	22.6461
400	0.1375	39.0802	0.1125	39.1166	0.2497	65.5403	0.4259	51.1026
600	0.0917	61.0783	0.0750	61.0916	0.1664	103.978	0.2839	87.1789
800	0.0687	84.69113	0.0562	85.36611	0.1248	146.5026	0.2129	131.1566
1000	0.0550	110.4413	0.0450	110.6277	0.0999	193.0715	0.1703	183.7249

From the perspective of scientific applications, the computational results further demonstrate that the proposed fractional framework can be applied to models originating from different scientific contexts, including heat transfer, population dynamics, and biochemical processes. Although this study is theoretical and does not involve direct experimental measurements, the considered models retain the physical meaning of their corresponding state variables and governing mechanisms. The numerical simulations therefore serve as theoretical demonstrations of the applicability of the proposed fractional framework to established scientific models. In particular, the computational results confirm that the analytical and numerical procedures can be implemented consistently across models with different physical interpretations.

To further clarify the physical and engineering relevance of the proposed fractional framework, Table 6 summarizes the physical meaning of the scientific models considered in this study and their correspondence with the associated classical models. In each case, the state variable retains its standard physical interpretation, while the classical first-order temporal derivative is generalized by the (k, φ) -Hilfer proportional fractional operator. This formulation provides a mathematical mechanism for incorporating nonlocal and memory-dependent effects into the temporal evolution of the corresponding physical systems.

As shown in Table 6, the proposed fractional formulations preserve the physical interpretation of the corresponding state variables while extending their temporal dynamics through a nonlocal fractional

operator. For the Newton cooling model, the formulation describes the relaxation of temperature toward the ambient temperature, whereas the RC and RL models represent generalized charge and current responses, respectively. Similarly, the population and BAL models describe the evolution of population and alcohol concentration under memory-dependent dynamics. Therefore, the proposed framework provides a unified mathematical formulation for modeling physical, engineering, and scientific processes in which the current state may depend not only on the present conditions but also on the evolution of the system over time.

Table 6. Physical/Engineering Interpretation.

Model	State Variable	Classical Interpretation	Fractional Interpretation	Application Area
Generalized kinetic/ population	$N(t)$	Population or concentration evolution	Memory-dependent population dynamics with external input	Population dynamics /kinetics
Newton's law of cooling	$T(t)$	Temperature relaxation toward ambient temperature	Generalized thermal relaxation with memory effects	Heat transfer/thermal processes
RC circuit	$Q(t)$	Electric charge during charging/discharging	Memory-dependent charge dynamics	Electrical engineering
RL circuit	$i(t)$	Transient current response	Generalized current relaxation with memory effects	Electrical engineering
BAL model	$B(t)$	Alcohol concentration and elimination over time	Nonlocal concentration dynamics incorporating memory effects	Biochemical /physiological modeling

5.4. A nonlinear example

Here, we apply the proposed numerical scheme to nonlinear problem (5.37). To further demonstrate the applicability of the proposed fractional numerical method to nonlinear dynamical systems, we consider the following nonlinear problem under the (k, ρ, ϕ) -Caputo-FDO:

$$\begin{cases} {}^C_0\mathfrak{D}_{k,\phi}^{\alpha,\rho}x(t) = -(k\rho)^{\frac{\alpha}{k}}0.0524x(t) + \frac{5e^{-t}\sin(\pi x(t))}{t+20} - \frac{|x(t)|}{|x(t)|+28}, & t \in (0, 8], \\ x(0) = 10. \end{cases} \quad (5.37)$$

Here, $x(t)$ is interpreted as a normalized state variable describing the deviation or magnitude of a measurable quantity in a dissipation nonlinear system. The model is phenomenological and is intended to represent a class of nonlinear systems involving relaxation, time-dependent external forcing, and bounded nonlinear saturation. Thus, the proposed problem (5.37) is not intended to represent a

unique physical system, but rather a mathematically structured model that captures several mechanisms commonly encountered in nonlinear dynamical systems.

The right-hand side of the proposed problem (5.37) consists of three distinct contributions. The first term,

$$-(k\rho)^{\frac{\alpha}{k}} 0.0524x(t),$$

represents a linear relaxation or dissipation mechanism. Its negative sign indicates that this mechanism tends to reduce the magnitude of the state variable and drive the system toward its reference state. The relaxation coefficient 0.0524 controls the strength of this linear relaxation, while the factor $(k\rho)^{\alpha/k}$ arises from the scaling associated with the proposed fractional operator. The second term

$$\frac{5e^{-t} \sin(\pi x(t))}{t + 20},$$

represents a bounded nonlinear external excitation whose intensity decreases with time. Factor e^{-t} introduces temporal attenuation, while the denominator $t + 20$ further reduces the forcing magnitude as time increases. The factor $\sin(\pi x(t))$ introduces nonlinear dependence on the system state and enables the forcing to change sign according to the state of the system. Such a structure provides a simple representation of a nonlinear, time-dependent excitation acting on a dissipation system. The third term

$$-\frac{|x(t)|}{|x(t)| + 28},$$

represents a bounded nonlinear loss or saturation effect. In particular,

$$0 \leq -\frac{|x(t)|}{|x(t)| + 28} < 1,$$

for finite $x(t)$. Hence, this nonlinear contribution remains bounded even when the magnitude of the state variable becomes large. This type of nonlinear saturation is useful for representing processes in which the effective loss does not increase indefinitely with the state magnitude. Finally, the initial condition $x(0) = 10$ specifies the initial state of the normalized state variable. It is selected to provide a non-equilibrium initial state and thereby enable the numerical method to capture the subsequent transient dynamics of the nonlinear system. Moreover, proposed problem (5.37) is considered a phenomenological nonlinear benchmark model rather than as a governing equation for a specific physical system.

In the fractional formulation, the state variable $x(t)$ retains the same physical interpretation as in the corresponding classical dynamical system. The role of the (k, ρ, ϕ) -Caputo-FD $\odot$ is to generalize the temporal evolution of the state variable by incorporating a nonlocal dependence on its previous states. Therefore, the fractional formulation does not change the individual mechanisms represented by the three terms on the right-hand side; rather, it modifies the temporal response of the system to these mechanisms.

The connection with the corresponding classical nonlinear dynamical system can be established by considering $k = 1$, $\rho = 1$, and $\phi(t) = t$. Under these conditions, the proposed operator reduces to the standard Caputo derivative. Furthermore, as $\alpha \rightarrow 1^-$, the fractional derivative converges to the ordinary first-order derivative, and the proposed problem (5.37) reduces to

$$\begin{cases} \frac{dx}{dt} = -0.0524x(t) + \frac{5e^{-t} \sin(\pi x(t))}{t + 20} - \frac{|x(t)|}{|x(t)| + 28}, & t \in (0, 8], \\ x(0) = 10. \end{cases} \quad (5.38)$$

Thus, the fractional model can be regarded as a nonlocal temporal generalization of the corresponding classical nonlinear dynamical system.

Form the considered problem (5.37), we set $\alpha = 0.96$, $\rho = 0.98$, $k = 1.02$, and $\phi = 0.45t$, with its order of convergence at least $2 + \alpha/k = 2 + 0.96/1.02 \approx 2.94$. The following nonlinear function is given by

$$f(t, x) = \frac{5e^{-t} \sin(\pi x(t))}{t + 20} - \frac{|x(t)|}{|x(t)| + 28}.$$

For every $x_i \in \mathbb{R}$, $i = 1, 2$, $t \in [0, 8]$, one has $|f(t, x_1) - f(t, x_2)| \leq (2/7)|x_1 - x_2|$. It is easy to see that (H_1) is fulfilled under $L = 2/7$, yielding that

$$L(\phi(b) - \phi(a))^{\frac{\alpha}{k}} = 0.953917 < 0.976172 = \rho^{\frac{\alpha}{k}} \Gamma_k(\alpha + k).$$

Thus, all the conditions of Theorem 3.1 are fulfilled. Moreover, problem (5.37) has a unique solution on $[0, 8]$. Additionally, we have $\mathfrak{B}_g = 3.431252$ and $\kappa_g = 0.291199$. Then, by Theorem 3.3, problem (5.37) is UHML stable on $[0, 8]$. Additionally, if we take $\chi(t) = e^{\frac{-0.02}{0.9996}(\phi(t) - \phi(0))}(\phi(t) - \phi(0))^{\frac{\pi}{1.02} - 1}$ in (3.22), this implies that

$$\int_0^t e^{\frac{-0.02}{0.9996}(\phi(s) - \phi(0))}(\phi(s) - \phi(0))^{\frac{0.96}{1.02} - 1} \phi'(s) \chi(s) ds \leq k \mathbb{B}_k(0.96, \pi)(\phi(8) - \phi(0))^{\frac{0.96}{1.02}} \chi(t). \quad (5.39)$$

This yields that $\omega = k \mathbb{B}_k(0.96, \pi)(\phi(8) - \phi(0))^{\frac{0.96}{1.02}} = 1.211997$, where $\mathbb{B}_k(0.96, \pi) = \frac{\Gamma_{1.02}(0.96) \Gamma_{1.02}(\pi)}{\Gamma_{1.02}(\pi + 0.96)}$. Hence, by Theorem 3.4, problem (5.37) is UHRML stable on $[0, 8]$. To understand the impact of each parameter, we display the graphical simulation of the solution for the proposed coupled system (5.37), as illustrated in Figures 18–20. Figure 18 presents the impact of α in the approximate solutions for the BAL model; it observed that when α increase from 0.90 to 1.00, then the approximate solutions increase. Moreover, ρ increases from 0.60 to 1.00, and the approximate solution decreases, as shown in Figure 19. Additionally, if k increases from 1.00 to 1.40, the approximate solution decreases, as shown in Figure 20.

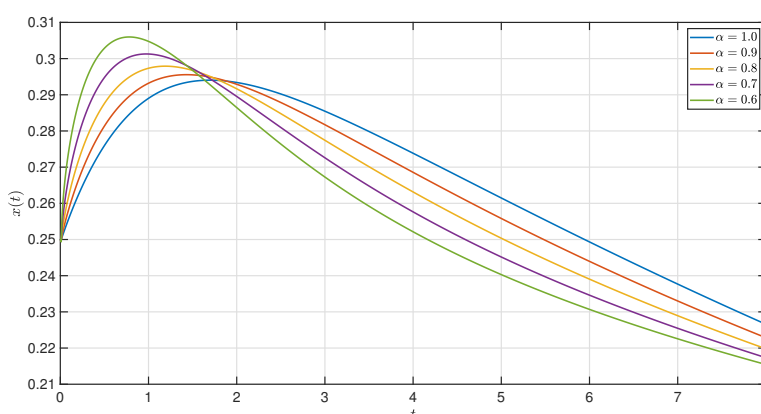


Figure 18. The approximated solution of problem (5.37) under various of $\alpha = \{0.90, 0.93, 0.96, 0.99, 1.00\}$.

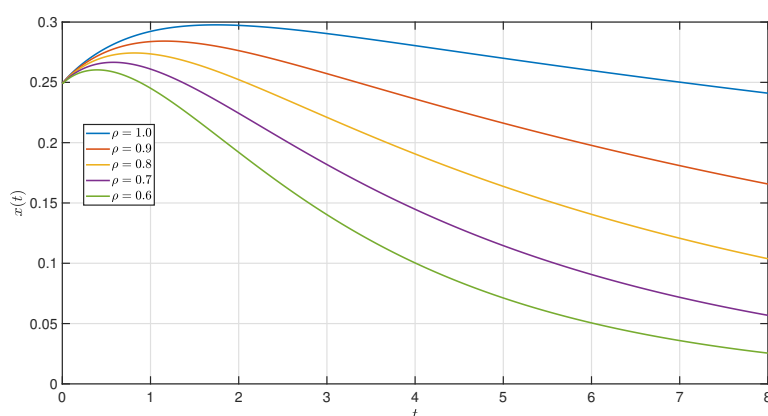


Figure 19. The approximated solution of problem (5.37) under various of $\rho = \{0.60, 0.70, 0.80, 0.90, 1.00\}$.

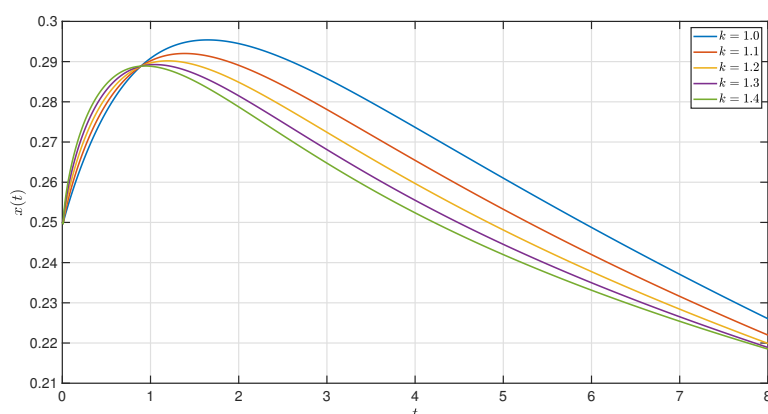


Figure 20. The approximated solution of problem (5.37) under various of $k = \{1.00, 1.10, 1.20, 1.30, 1.40\}$.

6. Conclusions

In this work, we demonstrate that the study of fractional dynamical systems can be approached from a unified analytical, computational, and data-oriented perspective. Rather than treating generalized fractional operators solely as mathematical extensions, we show how the (k, ρ, ϕ) -Caputo-FDO framework can be incorporated into a Cauchy-type initial value problem that includes theoretical analysis, numerical approximation, and validation against observed data. The analytical results guarantee that the considered initial value problem admits a mathematically reliable solution and that its qualitative behavior remains stable under perturbations. These properties provide the foundation upon which the numerical methodology is built. The proposed two-step piecewise Lagrange interpolation scheme is shown to be convergent and capable of efficiently approximating solutions associated with the generalized fractional operator. Moreover, the obtained existence and uniqueness results guarantee that the considered initial value problems are mathematically well posed, while the UHML stability analysis guarantees that approximate solutions remain close to the corresponding exact

solution, thereby supporting the reliability of the analytical and numerical findings. Our numerical experiments demonstrate the effectiveness of the proposed method in approximating solutions of real-world problems. As the results show, a nonlinear fractional initial value problem is considered to present the applicability of the theoretical results. By verifying the assumptions required in Theorems 3.2, 3.9, and 3.10, the example confirms the practical usefulness of the established existence, uniqueness, and UHML stability criteria. Furthermore, the effectiveness of the framework is illustrated through three generalized models: Newton's law of cooling, world population growth, and blood alcohol level dynamics. Despite describing phenomena from different scientific disciplines, all three models are successfully analyzed within the same fractional framework, highlighting the flexibility and broad applicability of the proposed operator. This demonstrates the capability of the (k, ρ, ϕ) -Caputo-FDO formulation to capture diverse memory-dependent processes using a unified mathematical structure. A noteworthy outcome of these examples is the statistical validation performed using observed cooling data. The comparison among the classical, Caputo, k -Caputo, ρ -Caputo, and (k, ρ) -Caputo formulations reveal that the generalized operator consistently produce the lowest values of the L^2 -norm error, MSE, and RMSE.

This study is restricted to time-fractional differential equations. An extension of the proposed framework to space-fractional or space-time fractional problems would require a separate treatment of spatial operators, geometries, and boundary conditions and is left for future investigation. Taken together, the analytical results, numerical investigations, and statistical assessments indicate that the (k, ρ, ϕ) -Caputo-FDO framework serves not only as a mathematical generalization of existing fractional operators but also as a bridge between theoretical fractional modeling and observed physical phenomena. In addition, the computational time required for each fractional definition is evaluated. The computational times of the classical, Caputo, k -Caputo, ρ -Caputo, and (k, ρ) -Caputo formulations are compared under the same computational settings. This comparison is intended to quantify the additional computational cost associated with the fractional and generalized fractional operators. The results are summarized in Table 5. The consistency between the theoretical analysis, numerical approximations, and statistical evaluations demonstrates the effectiveness of the proposed framework in describing memory-dependent processes. In future work, researchers may focus on extending this framework to nonlinear coupled systems, other fractional operators, and parameter estimation based on real observational data.

Author contributions

Weerawat Sudsutad: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing, formal analysis; Jutarat Kongson: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing, formal analysis; Chatthai Thaiprayoon: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing, funding acquisition, formal analysis, supervision.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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