



Research article

Computation of weighted Jensen and Lah-Ribarič inequalities for analogue of $(g, h; m)$ -convexity

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Abstract: We formulate an H -analogue of $(g, h; m)$ -convexity by treating the nonnegative coefficient H as an independent structural variable. Two endpoint representations of a point in an interval are used to establish weighted Lah-Ribarič inequalities for the resulting convex and concave classes. Under concavity or convexity assumptions on H , weighted Jensen's inequality yields normalized endpoint refinements and corresponding reverse estimates. A further extension is obtained when H is supermultiplicative; its natural concave counterpart follows under submultiplicativity. Finite-sum versions produce H -analogues of the Giaccardi, Popoviciu, and Petrović inequalities. We also derive an increasing-functional inequality and an estimate for the associated Jensen gap. The framework contains the $(g, h; m)$ -convex setting and, through linear, power, constant, and product-generated choices of H , recovers or extends a broad collection of known Lah-Ribarič and Jensen-type inequalities. A complementary two-coefficient formulation is also established to cover the general $(g, h; \alpha - m)$ -convex case without identifying $h(1 - t^\alpha)$ with $h((1 - t)^\alpha)$.

Keywords: $(g, h; m)$ -convexity; Lah-Ribarič inequality; Jensen inequality; supermultiplicative functions; integral inequalities

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1. Introduction

Convexity is one of the principal tools for comparing the value of a function at an average point with suitable averages of its values. The classical Jensen inequality provides the fundamental lower estimate associated with a convex function and has numerous consequences throughout the theory of inequalities [1]. A complementary foundational treatment of convex functions and their principal inequalities was given by Roberts and Varberg [2]. Its converse counterpart, usually called the Lah-Ribarič inequality, controls the mean value of a convex function by a linear combination of its values at the endpoints of the underlying interval [3]. Consequently, the Jensen and Lah-Ribarič inequalities yield complementary estimates for the same weighted functional. A modern introduction to convex analysis and its applications, including the central role of Jensen-type reasoning, is provided by Mordukhovich and Nam [4]. Recent developments devoted specifically to Jensen and Lah-Ribarič inequalities have also been collected by Andrić et al., emphasizing refinements, converses, and related functionals [5].

Several generalized notions of convexity have been introduced by replacing the ordinary coefficients t and $1 - t$ with more flexible weight functions. Among them, h -convexity and m -convexity provide unified settings for a number of classical and nonclassical convexity classes.

For clarity, if $f \geq 0$ is defined on an interval, then h -convexity means

$$f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y),$$

whereas m -convexity means

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y),$$

for admissible x, y and $m \in (0, 1]$, $t \in [0, 1]$. Writing $(fg)(x) = f(x)g(x)$, the $(g, h; m)$ -convex class is determined by

$$f(tx + m(1 - t)y) \leq h(t)(fg)(x) + mh(1 - t)(fg)(y).$$

Thus h - and m -convexity arise from the latter condition by the appropriate choices of g , h , and m .

This class is denoted $(h, g; m)$ -convexity by Andrić and Pečarić; it incorporates both an auxiliary positive function g and the deformation parameter m [6]. Hermite-Hadamard and Lah-Ribarič type inequalities for this class were investigated in [7]. More recently, the $(g, h; \alpha - m)$ -convex framework has been used to obtain further weighted integral inequalities by introducing a power parameter into the coefficients [8].

Recent work has continued to broaden the coefficient-based approach to convexity. Breaz et al. established Hadamard- and Fejér-type inequalities for modified h -convex functions, including estimates for products of functions and consequences of superadditive weight functions [9]. Farid and Pečarić extended the same setting through quasi-arithmetic means and obtained corresponding Hermite-Hadamard bounds, thereby showing that the same coefficient structure can be transported to nonlinear mean geometries [10]. Liu et al. introduced strongly $(g, h; \alpha - m)$ -convex functions and derived Riemann-Liouville fractional Hermite-Hadamard inequalities with explicit strong-convexity correction terms [11]. Ali and Bibi recently developed Jensen- and Mercer-type inequalities for h -convex functions by exploiting supermultiplicative and superadditive properties of h , with particular emphasis on refined Jensen-Mercer gaps [12]. These contributions motivate treating the coefficient function as an independent structural object, as is done here through the general function H .

The reason for introducing H is structural rather than merely notational. In the usual $(g, h; m)$ presentation, the same symbol h identifies a particular named class and supplies both complementary coefficients. This can obscure which properties of the coefficient are actually used in a proof and makes composite choices, such as products of two weight mechanisms, cumbersome to track. Treating H as an independent coefficient variable separates the coefficient law from the functions f and g and from the deformation parameter m . The basic endpoint estimate then requires only nonnegativity of H ; the normalized Jensen refinement additionally requires concavity or convexity; and the rescaled estimates require supermultiplicativity or submultiplicativity. The formulation therefore identifies the exact structural assumption responsible for each layer of the result and accommodates linear, power, constant, regularized reciprocal, and product-generated coefficients in one notation. Since h may itself be arbitrary, this abstraction is not claimed to enlarge the one-coefficient class set-theoretically. Its value is to expose the coefficient as a variable of the theory. When the two endpoint coefficients are not complementary evaluations of one function, as in the general $(g, h; \alpha - m)$ case, the separate H_1, H_2 formulation in Remark 3.3 records the genuinely additional structure.

Other recent studies confirm that flexible coefficient functions interact fruitfully with several forms of generalized convexity. Varošanec combined h -convexity with quasi-arithmetic mean transformations and obtained Jensen-, Schur-, and weighted Hermite-Hadamard-type inequalities [13]. Saeed et al. used a center-radius total order to develop Jensen- and Hermite-Hadamard-type estimates for interval-valued (h_1, h_2) -convex functions [14]. Afzal et al. established corresponding Jensen and Hermite-Hadamard inclusions for interval-valued (h_1, h_2) -Godunova-Levin functions, showing how two coefficient functions can govern nonclassical interval bounds [15]. Fahad et al. proved Jensen-Mercer and weighted Hermite-Hadamard-Mercer inequalities for GA- h -convex functions and their subclasses, including versions based on Hadamard fractional integrals [16].

Further extensions emphasize the adaptability of Jensen-type arguments. Nazari Pasari et al. derived generalized integral Jensen inequalities for preinvex functions and recovered a Hermite-Hadamard consequence [17]. Sayyari et al. extended Jensen, Mercer, and Fejér-Hermite-Hadamard inequalities under strong convexity and applied the resulting estimates to entropy [18]. Tariq et al. obtained Hermite-Hadamard-Fejér estimates for m -convex functions through Raina fractional integrals, demonstrating the role of the deformation parameter in weighted integral bounds [19]. Javed et al. recently treated interval-valued super-quadratic mappings controlled by an h -type function and derived Jensen, reverse Jensen, Jensen-Mercer, and Hermite-Hadamard inequalities in classical and fractional settings [20].

Three related contributions in Axioms further clarify the position of the present results. Rodić established generalizations of Jensen-type inequalities, including converse forms connected with the Lah-Ribarič and Edmundson-Lah-Ribarič principles, and discussed applications of the resulting estimates [21]. Bayraktar, Kórus, and Nápoles Valdés combined Jensen-Mercer arguments with non-conformable fractional operators to derive Hermite-Hadamard variants for convex and modified (h, m) -convex functions [22]. Khan, Ivelić Bradanović, and Mahmoud obtained improved Jensen-Mercer bounds for strongly convex functions and applied them to divergence measures, entropy, and Mercer-type means [23]. These works reinforce the relevance of weighted Jensen converses and generalized coefficient functions to the H -analogue studied below.

Recent developments on converse Jensen inequalities are equally pertinent. Vukelić extended the Lah-Ribarič inequality to integral arithmetic means and divided differences and used the resulting

functionals to obtain higher-order convexity and exponential-convexity consequences [24]. Mikić, Dilda Pečarić, and Josip Pečarić derived Edmundson-Lah-Ribarič inequalities for 3-convex functions and positive linear functionals, together with applications to generalized f -divergences and Stolarsky-type means [25]. Sohail et al. obtained improvements of the integral Jensen inequality under convexity assumptions on a third derivative, illustrating how additional smoothness can sharpen the Jensen side of the two-sided comparison [26]. Mukhtar et al. combined Jensen and Lah-Ribarič differences with higher-order identities and applied the resulting bounds to divergence and entropy measures in information theory [27]. Together, these studies show that the Jensen/Lah-Ribarič pair remains a productive basis for both structural generalization and quantitative applications.

Additional related results illustrate several complementary directions of this development. Baloch et al. obtained discrete Jensen-type inequalities for harmonic convex functions and extended them to harmonic h -convex functions [28]. Bosch et al. established Jensen-type inequalities for m -convex functions and applied them to generalized Riemann-Liouville-type integral operators [29]. In an operator setting, Ivelić Bradanović et al. sharpened Jensen-Mercer inequalities and developed applications to quasi-arithmetic means [30]. Ahmed et al. derived Milne-type estimates through Atangana-Baleanu-Katugampola fractional integral operators under convexity assumptions on the absolute value of a second derivative [31]. These studies concern different geometries, operators, and quadrature functionals, but collectively demonstrate how a modified convexity structure can be propagated into Jensen-type, converse, and fractional integral bounds. The present work follows this broad line from a different direction: it keeps the affine argument and encodes the coefficient structure through H , with two endpoint representations producing the Lah-Ribarič estimates developed below.

The present H -framework contains the results of [7] as direct special cases. Indeed, the choice $H = h$ reduces the H -analogue to the $(g, h; m)$ -convex class, denoted $(h, g; m)$ in that reference. In particular, Theorem 2.1 recovers [7], while the corresponding (h, m) -, h -, and m -convex consequences follow by imposing, respectively, $g \equiv 1$; $g \equiv 1$ and $m = 1$; and $g \equiv 1$ with $h(t) = t$.

The H -analogue considered here replaces the particular weight h by a general nonnegative function H . This formulation is useful because the choice of H can encode many familiar coefficient systems while the functions f and g , as well as the parameter m , remain unchanged. Our purpose is to establish the basic Lah-Ribarič inequality for this class and its Jensen-type extension. Two different endpoint representations of an arbitrary point of $[a, b]$ lead to two valid estimates. Taking their minimum gives the sharper convex bound; for the corresponding concave class, the same argument produces a maximum. No concavity, convexity, or multiplicativity assumption on H is needed for the first result. When H is concave, weighted Jensen's inequality yields a further normalized upper bound; the dual statement holds for the concave class when H is convex. Finally, when H is supermultiplicative, normalized coefficients can be replaced by ratios of H evaluated at the corresponding unnormalized endpoint distances. This gives an H -analogue of the supermultiplicative refinement and leads to further power and product specializations. Finite-sum versions yield H -analogues of the Giaccardi, Popoviciu, and Petrović inequalities, while an increasing two-variable function produces a functional refinement and a bound for the corresponding Jensen gap.

Guide to the contribution and hypotheses

The hierarchy of the results should be distinguished carefully. Setting $H = h$ recovers the $(g, h; m)$ -class; adding $g \equiv 1$ gives the (h, m) -class, and adding both $g \equiv 1$ and $m = 1$ gives the h -convex class.

The choices $g \equiv 1$, $H(t) = t$; $g \equiv 1$, $H(t) = t^s$; and $g \equiv 1$, $m = 1$, $H \equiv 1$ recover, respectively, the m -convex, (s, m) -convex, and P -function settings. These recoveries verify consistency and are not presented as new classes. Likewise, whenever a theorem below is identified as the $H = h$ or $\alpha = 1$ form of an earlier result, that identification records a direct specialization.

The contribution of the one-coefficient H -setting is instead a unified theorem scheme in which the coefficient law is treated as an independent variable and the exact additional property of that law is isolated at each stage. Within this scheme, the two endpoint representations yield the pointwise minimum or maximum refinements; product-generated coefficients provide systematic composite realizations; the continuous, discrete, and increasing-functional forms are derived in one notation; and Subsection 2 supplies equality and sharpness criteria. The two-coefficient construction in Remark 3.3 is a genuine extension of Definition 2.1: H_1 and H_2 need not satisfy $H_2(t) = H_1(1 - t)$. It is precisely this construction that covers the general coefficients $h(t^\alpha)$ and $h(1 - t^\alpha)$; the single- H setting covers them only when the complementary compatibility relation holds.

Table 1 summarizes the hypotheses used by the main groups of results. All rows retain the interval, measurability, finiteness, and admissibility conditions stated in the corresponding theorem.

Table 1. Assumptions on the coefficient function in the main result groups.

Result group	Assumptions on H	Purpose of the additional assumption
Basic endpoint inequalities and their direct specializations	$H : J \rightarrow [0, \infty)$	Nonnegativity is sufficient for the two pointwise endpoint bounds; no convexity or multiplicativity of H is used.
Normalized Jensen extensions	Basic assumptions, $W > 0$, and H concave for the convex upper bound or convex for the concave lower bound	Weighted Jensen's inequality replaces averages of H by H evaluated at the normalized mean.
Supermultiplicative extension	Basic assumptions, relevant distances in J , $H(b - a) > 0$, and supermultiplicativity; use submultiplicativity for the reversed concave form	The normalized coefficients are controlled by ratios involving unnormalized endpoint distances.
Discrete, Giaccardi, Popoviciu, and Petrović forms	Basic assumptions with a finite atomic measure; the rescaled discrete theorem also inherits the preceding multiplicative assumptions	The integral argument becomes a finite weighted sum without changing the pointwise endpoint step.
Increasing-functional inequalities	Hypotheses of the normalized endpoint theorem and F increasing in its first variable	Monotonicity transports the normalized endpoint comparison through the two-variable functional.

2. The H -analogue and a Lah-Ribarič inequality

Let $I \subseteq \mathbb{R}$ be an interval, let $J \subseteq \mathbb{R}$ be an interval containing $[0, 1]$, and let $m \in (0, 1]$. Assume that

$$H : J \rightarrow [0, \infty) \quad \text{and} \quad g : I \rightarrow (0, \infty).$$

For convenience, write

$$(fg)(x) := f(x)g(x).$$

Definition 2.1. A nonnegative function $f : I \rightarrow [0, \infty)$ is called $(g, H; m)$ -convex, written

$$f \in \text{Conv}_m(g, H)$$

if

$$f(tx + m(1-t)y) \leq H(t)(fg)(x) + mH(1-t)(fg)(y) \quad (2.1)$$

for all $x, y \in I$ and $t \in [0, 1]$ for which $tx + m(1-t)y \in I$. If the reverse inequality holds in (2.1), then f is called $(g, H; m)$ -concave, written $f \in \text{Conc}_m(g, H)$.

Theorem 2.1. Let $a < b$, let $[a, b] \subseteq I$, and assume that $a/m, b/m \in I$. Let $p, \Omega: [a, b] \rightarrow \mathbb{R}$ be measurable functions such that

$$\Omega(\zeta) \geq 0, \quad p(\zeta) \in [a, b] \quad \text{for almost every } \zeta \in [a, b],$$

and suppose that all the integrals below are finite. For $\zeta \in [a, b]$, define

$$A(\zeta) := (fg)(a)H\left(\frac{b-p(\zeta)}{b-a}\right) + m(fg)\left(\frac{b}{m}\right)H\left(\frac{p(\zeta)-a}{b-a}\right), \quad (2.2)$$

$$B(\zeta) := (fg)(b)H\left(\frac{p(\zeta)-a}{b-a}\right) + m(fg)\left(\frac{a}{m}\right)H\left(\frac{b-p(\zeta)}{b-a}\right). \quad (2.3)$$

If $f \in \text{Conv}_m(g, H)$, then

$$\int_a^b \Omega(\zeta)f(p(\zeta))d\zeta \leq \int_a^b \Omega(\zeta)\min\{A(\zeta), B(\zeta)\}d\zeta \leq \min\left\{\int_a^b \Omega(\zeta)A(\zeta)d\zeta, \int_a^b \Omega(\zeta)B(\zeta)d\zeta\right\}. \quad (2.4)$$

If $f \in \text{Conc}_m(g, H)$, then

$$\int_a^b \Omega(\zeta)f(p(\zeta))d\zeta \geq \int_a^b \Omega(\zeta)\max\{A(\zeta), B(\zeta)\}d\zeta \geq \max\left\{\int_a^b \Omega(\zeta)A(\zeta)d\zeta, \int_a^b \Omega(\zeta)B(\zeta)d\zeta\right\}. \quad (2.5)$$

Proof. Fix $u \in [a, b]$. First take $t = (b-u)/(b-a)$, $x = a$, and $y = b/m$. Then $t \in [0, 1]$ and $tx + m(1-t)y = u$. The defining inequality (2.1) therefore gives

$$f(u) \leq (fg)(a)H\left(\frac{b-u}{b-a}\right) + m(fg)\left(\frac{b}{m}\right)H\left(\frac{u-a}{b-a}\right). \quad (2.6)$$

Next take $t = (u-a)/(b-a)$, $x = b$, and $y = a/m$. This also represents u as $tx + m(1-t)y$, and hence

$$f(u) \leq (fg)(b)H\left(\frac{u-a}{b-a}\right) + m(fg)\left(\frac{a}{m}\right)H\left(\frac{b-u}{b-a}\right). \quad (2.7)$$

It follows from (2.6) and (2.7) that

$$f(u) \leq \min\{A_u, B_u\},$$

where A_u and B_u denote the right-hand sides of (2.6) and (2.7), respectively.

Substituting $u = p(\zeta)$, multiplying by the nonnegative function $\Omega(\zeta)$, and integrating over $[a, b]$ proves the first inequality in (2.4). Moreover,

$$\int_a^b \Omega(\zeta)\min\{A(\zeta), B(\zeta)\}d\zeta \leq \min\left\{\int_a^b \Omega(\zeta)A(\zeta)d\zeta, \int_a^b \Omega(\zeta)B(\zeta)d\zeta\right\}.$$

This is precisely the second inequality in (2.4).

For $f \in \text{Conc}_m(g, H)$, inequalities (2.6) and (2.7) are reversed. Thus $f(u) \geq \max\{A_u, B_u\}$, and the same integration argument proves (2.5). $\square$

Remark 2.1. When $H = h$, Theorem 2.1 reduces to the $(g, h; m)$ -convex Lah-Ribarič inequality. It also agrees with the $\alpha = 1$ case of the $(g, h; \alpha - m)$ -convex result. In particular, choosing $g \equiv 1$, $m = 1$, and $H(t) = t$ recovers the classical endpoint form of the Lah-Ribarič estimate after normalization by

$$\int_a^b \Omega(\zeta) d\zeta.$$

Equality conditions and sharpness

The equality cases follow directly from the two steps in the proof. Let $d\mu(\zeta) = \Omega(\zeta) d\zeta$. In the convex case, equality in the first inequality of (2.4) holds if and only if

$$f(p(\zeta)) = \min\{A(\zeta), B(\zeta)\} \quad \text{for } \mu\text{-almost every } \zeta.$$

Write $I_A = \int A d\mu$ and $I_B = \int B d\mu$. If $I_A < I_B$, equality in the second step holds if and only if $A \leq B$ μ -almost everywhere; if $I_B < I_A$, it holds if and only if $B \leq A$ μ -almost everywhere. When $I_A = I_B$, equality holds if and only if $A = B$ μ -almost everywhere. In the concave case, equality in the first step means $f(p) = \max\{A, B\}$ μ -almost everywhere. For the second step, if $I_A > I_B$, equality holds exactly when $A \geq B$ μ -almost everywhere; if $I_B > I_A$, it holds exactly when $B \geq A$ μ -almost everywhere; and if $I_A = I_B$, it again holds exactly when $A = B$ μ -almost everywhere.

The later results add only the equality requirements associated with their additional proof steps. In the Jensen extensions, equality is also required in the relevant weighted Jensen inequalities for H . Thus, when H is strictly concave (or strictly convex in the reverse case) and the corresponding endpoint multipliers are positive, equality forces p to be constant μ -almost everywhere. More generally, Jensen equality is possible when H is affine on the convex hull of the essential range of $q = (b - p)/(b - a)$ (equivalently, of $r = (p - a)/(b - a)$). In the supermultiplicative results, equality additionally requires $H((b - a)s) = H(b - a)H(s)$ at the active normalized distances; this condition is automatic when H is multiplicative. For the discrete results, “ μ -almost everywhere” is replaced by every index with $p_i > 0$.

These bounds are sharp without further hypotheses. Indeed, for $g \equiv 1$, $m = 1$, $H(t) = t$, and any nonnegative affine function f , both endpoint representations equal $f(u)$. Consequently, every step of the classical integral and finite-sum Lah-Ribarič inequalities is an equality, so their constants cannot be improved in the stated generality.

3. Special cases with respect to H

To state the principal specializations compactly, put

$$q(\zeta) := \frac{b - p(\zeta)}{b - a}, \quad 1 - q(\zeta) = \frac{p(\zeta) - a}{b - a}. \quad (3.1)$$

Thus $q(\zeta) \in [0, 1]$ for almost every $\zeta \in [a, b]$. For a given choice of H , define

$$A_H(\zeta) := (fg)(a)H(q(\zeta)) + m(fg)\left(\frac{b}{m}\right)H(1 - q(\zeta)), \quad (3.2)$$

$$B_H(\zeta) := (fg)(b)H(1 - q(\zeta)) + m(fg)\left(\frac{a}{m}\right)H(q(\zeta)). \quad (3.3)$$

Theorem 2.1 may then be written as

$$\int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \int_a^b \Omega(\zeta) \min\{A_H(\zeta), B_H(\zeta)\} d\zeta \leq \min \left\{ \int_a^b \Omega A_H, \int_a^b \Omega B_H \right\}, \quad (3.4)$$

where the limits and the integration variable in the last two integrals are $[a, b]$ and ζ , respectively. The following choices of H recover important classes.

Remark 3.1 (Special choices of H). *The following choices in (3.4) recover several important classes.*

(i) If $H(t) = h(t)$, Definition 2.1 becomes

$$f(tx + m(1-t)y) \leq h(t)(fg)(x) + mh(1-t)(fg)(y).$$

Thus (3.4) gives the $(g, h; m)$ -convex Lah-Ribarič inequality. Setting $g \equiv 1$ gives the (h, m) -convex case, while setting $g \equiv 1$ and $m = 1$ gives the h -convex case. These classes are defined by the displayed inequality after making the stated substitutions. In particular, after the corresponding notational identification of the endpoint interval, this specialization gives [7].

(ii) For $H(t) = t$, set

$$A_1(\zeta) = (fg)(a)q(\zeta) + m(fg)\left(\frac{b}{m}\right)(1 - q(\zeta)), \quad (3.5)$$

$$B_1(\zeta) = (fg)(b)(1 - q(\zeta)) + m(fg)\left(\frac{a}{m}\right)q(\zeta). \quad (3.6)$$

Substitution of A_1 and B_1 into (3.4) gives the (g, m) -convex inequality, whose defining condition is

$$f(tx + m(1-t)y) \leq t(fg)(x) + m(1-t)(fg)(y).$$

When $g \equiv 1$, this becomes the m -convex case.

(iii) Let $s \in (0, 1]$ and $H(t) = t^s$. Then

$$A_s(\zeta) = (fg)(a)q(\zeta)^s + m(fg)\left(\frac{b}{m}\right)(1 - q(\zeta))^s, \quad (3.7)$$

$$B_s(\zeta) = (fg)(b)(1 - q(\zeta))^s + m(fg)\left(\frac{a}{m}\right)q(\zeta)^s. \quad (3.8)$$

Consequently,

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{A_s(\zeta), B_s(\zeta)\} d\zeta \\ &\leq \min \left\{ \int_a^b \Omega(\zeta) A_s(\zeta) d\zeta, \int_a^b \Omega(\zeta) B_s(\zeta) d\zeta \right\}. \end{aligned} \quad (3.9)$$

For $g \equiv 1$, this is the result for the (s, m) -convex class in the second sense, defined by

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y);$$

setting $m = 1$ gives the s -convex case.

(iv) The constant choice $H(t) = 1$ gives

$$A_H(\zeta) = (fg)(a) + m(fg)\left(\frac{b}{m}\right), \quad B_H(\zeta) = (fg)(b) + m(fg)\left(\frac{a}{m}\right),$$

and hence

$$\int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min\left\{(fg)(a) + m(fg)\left(\frac{b}{m}\right), (fg)(b) + m(fg)\left(\frac{a}{m}\right)\right\} \int_a^b \Omega(\zeta) d\zeta. \quad (3.10)$$

For $g \equiv 1$ and $m = 1$, this is the corresponding estimate for P -functions, namely nonnegative functions satisfying $f(tx + (1-t)y) \leq f(x) + f(y)$.

(v) Let $\varepsilon > 0$ and use the regularized reciprocal coefficient $H_\varepsilon(t) = 1/(t + \varepsilon)$ on $[0, 1]$. This choice satisfies the finiteness requirement in Definition 2.1. Set

$$A_\varepsilon(\zeta) = \frac{(fg)(a)}{q(\zeta) + \varepsilon} + \frac{m(fg)(b/m)}{1 - q(\zeta) + \varepsilon}, \quad (3.11)$$

$$B_\varepsilon(\zeta) = \frac{(fg)(b)}{1 - q(\zeta) + \varepsilon} + \frac{m(fg)(a/m)}{q(\zeta) + \varepsilon}. \quad (3.12)$$

Therefore,

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{A_\varepsilon(\zeta), B_\varepsilon(\zeta)\} d\zeta \\ &\leq \min\left\{\int_a^b \Omega(\zeta) A_\varepsilon(\zeta) d\zeta, \int_a^b \Omega(\zeta) B_\varepsilon(\zeta) d\zeta\right\}. \end{aligned} \quad (3.13)$$

For $g \equiv 1$ and $m = 1$, this is a finite regularization of the Godunova-Levin-type estimate. If $p(\zeta) \in (a, b)$ almost everywhere and an integrable majorant permits passage to the limit, the unregularized expressions are recovered formally as $\varepsilon \downarrow 0$; they are not asserted to be a direct specialization of Definition 2.1.

Corollary 3.1 (Classical Lah-Ribarič inequality). *Under the assumptions of Theorem 2.1, let $g \equiv 1$, $m = 1$, and $H(t) = t$. Then*

$$\int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \frac{f(a)}{b-a} \int_a^b \Omega(\zeta)(b-p(\zeta)) d\zeta + \frac{f(b)}{b-a} \int_a^b \Omega(\zeta)(p(\zeta)-a) d\zeta. \quad (3.14)$$

If

$$W := \int_a^b \Omega(\zeta) d\zeta > 0, \quad \text{and} \quad I_M(\Omega, p) := \frac{\int_a^b \Omega(\zeta) p(\zeta) d\zeta}{W},$$

then (3.14) is equivalently expressed as

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \frac{b - I_M(\Omega, p)}{b-a} f(a) + \frac{I_M(\Omega, p) - a}{b-a} f(b). \quad (3.15)$$

The preceding specializations use a single coefficient law—linear, power, constant, or reciprocal—to control both endpoint contributions. A natural next step is to combine two complementary weight mechanisms. The product $H(t) = h_1(t)h_2(1-t)$ does this while remaining within Definition 2.1; it also makes transparent how the corresponding one-factor classes are recovered. This observation motivates the product-generated construction developed next.

Product-generated choices of H

Let $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$. The choice $H(t) = h_1(t)h_2(1-t)$ gives

$$A_{12}(\zeta) = (fg)(a)h_1(q)h_2(1-q) + m(fg)\left(\frac{b}{m}\right)h_1(1-q)h_2(q), \quad (3.16)$$

$$B_{12}(\zeta) = (fg)(b)h_1(1-q)h_2(q) + m(fg)\left(\frac{a}{m}\right)h_1(q)h_2(1-q), \quad (3.17)$$

where $q = q(\zeta)$.

Corollary 3.2 (Lah-Ribarič inequality for the $(h_1, h_2, g; m)$ -convex class). *Let $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$, and suppose that all the assumptions of Theorem 2.1 are satisfied. Assume that f belongs to the $(h_1, h_2, g; m)$ -convex class; that is,*

$$f(tx + m(1-t)y) \leq h_1(t)h_2(1-t)(fg)(x) + mh_1(1-t)h_2(t)(fg)(y) \quad (3.18)$$

for all admissible $x, y \in I$ and $t \in [0, 1]$. Then

$$\begin{aligned} \int_a^b \Omega(\zeta)f(p(\zeta))d\zeta &\leq \int_a^b \Omega(\zeta)\min\{A_{12}(\zeta), B_{12}(\zeta)\}d\zeta \\ &\leq \min\left\{\int_a^b \Omega(\zeta)A_{12}(\zeta)d\zeta, \int_a^b \Omega(\zeta)B_{12}(\zeta)d\zeta\right\}. \end{aligned} \quad (3.19)$$

If the reverse of (3.18) holds, then

$$\begin{aligned} \int_a^b \Omega(\zeta)f(p(\zeta))d\zeta &\geq \int_a^b \Omega(\zeta)\max\{A_{12}(\zeta), B_{12}(\zeta)\}d\zeta \\ &\geq \max\left\{\int_a^b \Omega(\zeta)A_{12}(\zeta)d\zeta, \int_a^b \Omega(\zeta)B_{12}(\zeta)d\zeta\right\}. \end{aligned} \quad (3.20)$$

Proof. Set $H(t) = h_1(t)h_2(1-t)$ in (2.1). Then $H(1-t) = h_1(1-t)h_2(t)$, and hence (2.1) becomes (3.18). Moreover, $H(q(\zeta)) = h_1(q(\zeta))h_2(1-q(\zeta))$ and $H(1-q(\zeta)) = h_1(1-q(\zeta))h_2(q(\zeta))$. Substituting these identities into (2.4) gives (3.19). The reverse case follows from (2.5). $\square$

Corollary 3.3 (The $(h_1, h_2; m)$ -convex case). *Under the assumptions of Corollary 3.2, let $g \equiv 1$. Define*

$$A_{12}^{(m)}(\zeta) := f(a)h_1(q(\zeta))h_2(1-q(\zeta)) + mf\left(\frac{b}{m}\right)h_1(1-q(\zeta))h_2(q(\zeta)), \quad (3.21)$$

$$B_{12}^{(m)}(\zeta) := f(b)h_1(1-q(\zeta))h_2(q(\zeta)) + mf\left(\frac{a}{m}\right)h_1(q(\zeta))h_2(1-q(\zeta)). \quad (3.22)$$

If f is $(h_1, h_2; m)$ -convex, then

$$\begin{aligned} \int_a^b \Omega(\zeta)f(p(\zeta))d\zeta &\leq \int_a^b \Omega(\zeta)\min\{A_{12}^{(m)}(\zeta), B_{12}^{(m)}(\zeta)\}d\zeta \\ &\leq \min\left\{\int_a^b \Omega(\zeta)A_{12}^{(m)}(\zeta)d\zeta, \int_a^b \Omega(\zeta)B_{12}^{(m)}(\zeta)d\zeta\right\}. \end{aligned} \quad (3.23)$$

For an $(h_1, h_2; m)$ -concave function, the inequalities in (3.23) are reversed and each minimum is replaced by a maximum.

Proof. Set $g \equiv 1$ in Corollary 3.2. □

Corollary 3.4 (The (h_1, h_2, g) -convex case). *Under the assumptions of Corollary 3.2, let $m = 1$. If f is (h_1, h_2, g) -convex, then*

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq (fg)(a) \int_a^b \Omega(\zeta) h_1(q(\zeta)) h_2(1 - q(\zeta)) d\zeta \\ &\quad + (fg)(b) \int_a^b \Omega(\zeta) h_1(1 - q(\zeta)) h_2(q(\zeta)) d\zeta. \end{aligned} \quad (3.24)$$

For an (h_1, h_2, g) -concave function, inequality (3.24) holds in the reverse direction.

Proof. Set $m = 1$ in (3.19). In this case, $a/m = a$ and $b/m = b$, so the two endpoint expressions inside the minimum coincide. □

Corollary 3.5 (The (h_1, h_2) -convex case). *Under the assumptions of Corollary 3.2, let $g \equiv 1$ and $m = 1$. If f is (h_1, h_2) -convex, then*

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq f(a) \int_a^b \Omega(\zeta) h_1(q(\zeta)) h_2(1 - q(\zeta)) d\zeta \\ &\quad + f(b) \int_a^b \Omega(\zeta) h_1(1 - q(\zeta)) h_2(q(\zeta)) d\zeta. \end{aligned} \quad (3.25)$$

For an (h_1, h_2) -concave function, inequality (3.25) is reversed.

Proof. Set $g \equiv 1$ in Corollary 3.4. □

Corollary 3.6 (The symmetric product-weight case). *Under the assumptions of Corollary 3.2, let $h_1 = h_2 = h$. Suppose that*

$$f(tx + m(1 - t)y) \leq h(t)h(1 - t)((fg)(x) + m(fg)(y)). \quad (3.26)$$

Then

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \min \left\{ (fg)(a) + m(fg)\left(\frac{b}{m}\right), (fg)(b) + m(fg)\left(\frac{a}{m}\right) \right\} \\ &\quad \times \int_a^b \Omega(\zeta) h(q(\zeta)) h(1 - q(\zeta)) d\zeta. \end{aligned} \quad (3.27)$$

If the reverse of (3.26) holds, then (3.27) is reversed and the minimum is replaced by a maximum.

Proof. When $h_1 = h_2 = h$, the weight

$$H(t) = h(t)h(1 - t)$$

is symmetric; hence $H(1 - t) = H(t)$. The conclusion follows immediately from Corollary 3.2. □

Corollary 3.7 (Recovery of the $(g, h; m)$ -convex case). *Under the assumptions of Corollary 3.2, let $h_1 = h$ and $h_2 \equiv 1$. Define*

$$A_h(\zeta) := (fg)(a)h(q(\zeta)) + m(fg)\left(\frac{b}{m}\right)h(1 - q(\zeta)), \quad (3.28)$$

$$B_h(\zeta) := (fg)(b)h(1 - q(\zeta)) + m(fg)\left(\frac{a}{m}\right)h(q(\zeta)). \quad (3.29)$$

If f is $(g, h; m)$ -convex, then

$$\begin{aligned} \int_a^b \Omega(\zeta)f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{A_h(\zeta), B_h(\zeta)\} d\zeta \\ &\leq \min \left\{ \int_a^b \Omega(\zeta)A_h(\zeta) d\zeta, \int_a^b \Omega(\zeta)B_h(\zeta) d\zeta \right\}. \end{aligned} \quad (3.30)$$

For a $(g, h; m)$ -concave function, the inequalities in (3.30) are reversed and the minima are replaced by maxima.

Proof. For $h_1 = h$ and $h_2 \equiv 1$, one has $H(t) = h(t)$. Apply Corollary 3.2. $\square$

Remark 3.2 (Recovery of the results of reference [7]). *The results proved in [7] for the classes denoted there by $(h, g; m)$ -convexity and $(h, g; m)$ -concavity are contained in the present H -framework. To see this, take*

$$h_1 = h, \quad h_2 \equiv 1, \quad \text{so that} \quad H(t) = h(t).$$

With this choice, Definition 2.1 becomes the defining inequality used in [7], and Theorem 2.1, equivalently Corollary 3.7, gives [7]. The special results for (h, m) -convex, h -convex, and m -convex functions are then obtained by taking, respectively,

$$g \equiv 1, \quad g \equiv 1, m = 1, \quad g \equiv 1, h(t) = t.$$

The concave counterparts follow by reversing the inequalities and replacing the minima by maxima. Hence every consequence of [7] obtained by these parameter specializations is recovered from the results of the present paper.

Remark 3.3 (Recovery of Theorem 1 and Corollaries 1 and 2 in [8]). *For this remark, consider the following two-coefficient analogue of Definition 2.1: let $H_1, H_2 : [0, 1] \rightarrow [0, \infty)$ and assume that*

$$f(tx + m(1 - t)y) \leq H_1(t)(fg)(x) + mH_2(t)(fg)(y) \quad (3.31)$$

for every admissible $x, y \in I$ and $t \in [0, 1]$. The single- H condition (2.1) is recovered from (3.31) by setting $H_1(t) = H(t)$ and $H_2(t) = H(1 - t)$.

Let

$$q(\zeta) = \frac{b - p(\zeta)}{b - a}, \quad r(\zeta) = 1 - q(\zeta) = \frac{p(\zeta) - a}{b - a}.$$

For an arbitrary $\alpha \in (0, 1]$, introduce the complementary weight pair

$$H_{\alpha,1}(t) := h(t^\alpha), \quad H_{\alpha,2}(t) := h(1 - t^\alpha). \quad (3.32)$$

Set $H_1 = H_{\alpha,1}$ and $H_2 = H_{\alpha,2}$ in (3.31). This is the general $(g, h; \alpha - m)$ -convex condition. Applying it first with $t = q(\zeta)$, $x = a$, and $y = b/m$, and then with $t = r(\zeta)$, $x = b$, and $y = a/m$, gives

$$A_\alpha(\zeta) = (fg)(a)h(q(\zeta)^\alpha) + m(fg)\left(\frac{b}{m}\right)h(1 - q(\zeta)^\alpha), \quad (3.33)$$

$$B_\alpha(\zeta) = (fg)(b)h(r(\zeta)^\alpha) + m(fg)\left(\frac{a}{m}\right)h(1 - r(\zeta)^\alpha). \quad (3.34)$$

Therefore,

$$\begin{aligned} \int_a^b \Omega(\zeta)f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{A_\alpha(\zeta), B_\alpha(\zeta)\} d\zeta \\ &\leq \min\left\{\int_a^b \Omega(\zeta)A_\alpha(\zeta) d\zeta, \int_a^b \Omega(\zeta)B_\alpha(\zeta) d\zeta\right\}. \end{aligned} \quad (3.35)$$

This is precisely [8, Theorem 1]; its concave version follows by reversing the inequalities and replacing the minima by maxima.

To obtain [8, Corollary 1], take $g \equiv 1$, $h(u) = u$. Then (3.33) and (3.34) become

$$A_\alpha(\zeta) = f(a)q(\zeta)^\alpha + mf\left(\frac{b}{m}\right)(1 - q(\zeta)^\alpha), \quad (3.36)$$

$$B_\alpha(\zeta) = f(b)r(\zeta)^\alpha + mf\left(\frac{a}{m}\right)(1 - r(\zeta)^\alpha). \quad (3.37)$$

Substitution of these expressions into (3.35) yields the stated (α, m) -convex inequality.

Finally, impose $\Omega \equiv 1$, $p(\zeta) = \zeta$. Since

$$\int_a^b q(\zeta)^\alpha d\zeta = \int_a^b r(\zeta)^\alpha d\zeta = \frac{b-a}{\alpha+1},$$

and

$$\int_a^b (1 - q(\zeta)^\alpha) d\zeta = \int_a^b (1 - r(\zeta)^\alpha) d\zeta = \frac{\alpha(b-a)}{\alpha+1},$$

one obtains

$$\int_a^b f(\zeta) d\zeta \leq \frac{b-a}{\alpha+1} \min\left\{f(a) + \alpha mf\left(\frac{b}{m}\right), f(b) + \alpha mf\left(\frac{a}{m}\right)\right\}, \quad (3.38)$$

which is [8, Corollary 2].

For $\alpha = 1$, the pair in (3.32) reduces to $H_{1,1}(t) = h(t)$, $H_{1,2}(t) = h(1 - t)$, and hence (3.31) becomes exactly the single- H specialization $H = h$ used in Theorem 2.1. For general α , the complementary pair (3.32) is essential; the substitution $H(t) = h(t^\alpha)$ alone would instead give $H(1 - t) = h((1 - t)^\alpha)$, which is a different coefficient.

More generally, for $\alpha \in (0, 1]$, the power-product choice $H(t) = h_1(t^\alpha)h_2(1 - t^\alpha)$ gives

$$A_{\alpha,12}(\zeta) = (fg)(a)h_1(q^\alpha)h_2(1 - q^\alpha) + m(fg)\left(\frac{b}{m}\right)h_1((1 - q)^\alpha)h_2(1 - (1 - q)^\alpha), \quad (3.39)$$

$$B_{\alpha,12}(\zeta) = (fg)(b)h_1((1-q)^\alpha)h_2(1-(1-q)^\alpha) + m(fg)\left(\frac{a}{m}\right)h_1(q^\alpha)h_2(1-q^\alpha). \quad (3.40)$$

Substitution of (3.39) and (3.40) into (3.4) gives the power-product H -analogue. If $h_1 = h_2 = h$, the coefficients are generated by $H(t) = h(t^\alpha)h(1-t^\alpha)$.

Remark 3.4. Every convex special case above has a concave counterpart. It is obtained by reversing the inequalities and replacing each minimum by the corresponding maximum, exactly as in Theorem 2.1.

The following theorem strengthens the Lah-Ribarič inequalities for $\text{Conv}_m(g, H)$ by applying weighted Jensen's inequality to the coefficient function H .

Theorem 3.1 (Jensen extension of the H -analogue). Assume the hypotheses of Theorem 2.1, and suppose that

$$W := \int_a^b \Omega(\zeta) d\zeta > 0.$$

Define

$$\begin{aligned} I_M(\Omega, p) &:= \frac{1}{W} \int_a^b \Omega(\zeta) p(\zeta) d\zeta, \\ q(\zeta) &:= \frac{b - p(\zeta)}{b - a}, \\ r(\zeta) &:= \frac{p(\zeta) - a}{b - a}. \end{aligned} \quad (3.41)$$

With A and B defined by (2.2) and (2.3), respectively, set

$$\begin{aligned} A_M(\Omega, p) &:= (fg)(a)H\left(\frac{b - I_M(\Omega, p)}{b - a}\right) + m(fg)\left(\frac{b}{m}\right)H\left(\frac{I_M(\Omega, p) - a}{b - a}\right), \\ B_M(\Omega, p) &:= (fg)(b)H\left(\frac{I_M(\Omega, p) - a}{b - a}\right) + m(fg)\left(\frac{a}{m}\right)H\left(\frac{b - I_M(\Omega, p)}{b - a}\right). \end{aligned}$$

If $f \in \text{Conv}_m(g, H)$ and H is concave on $[0, 1]$, then

$$\begin{aligned} \frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \min \left\{ \frac{1}{W} \int_a^b \Omega(\zeta) A(\zeta) d\zeta, \frac{1}{W} \int_a^b \Omega(\zeta) B(\zeta) d\zeta \right\} \\ &\leq \min \{A_M(\Omega, p), B_M(\Omega, p)\}. \end{aligned} \quad (3.42)$$

If $f \in \text{Conc}_m(g, H)$ and H is convex on $[0, 1]$, then both inequalities in (3.42) are reversed and each minimum is replaced by the corresponding maximum.

Proof. Divide the convex inequality (2.4) by W . Since $q(\zeta), r(\zeta) \in [0, 1]$ and $q(\zeta) + r(\zeta) = 1$, this gives the first inequality in (3.42).

Because H is concave, weighted Jensen's inequality yields

$$\frac{1}{W} \int_a^b \Omega(\zeta) H(q(\zeta)) d\zeta \leq H\left(\frac{1}{W} \int_a^b \Omega(\zeta) q(\zeta) d\zeta\right), \quad (3.43)$$

$$\frac{1}{W} \int_a^b \Omega(\zeta) H(r(\zeta)) d\zeta \leq H\left(\frac{1}{W} \int_a^b \Omega(\zeta) r(\zeta) d\zeta\right). \quad (3.44)$$

Moreover,

$$\begin{aligned} \frac{1}{W} \int_a^b \Omega(\zeta) q(\zeta) d\zeta &= \frac{b - I_M(\Omega, p)}{b - a}, \\ \frac{1}{W} \int_a^b \Omega(\zeta) r(\zeta) d\zeta &= \frac{I_M(\Omega, p) - a}{b - a}. \end{aligned} \quad (3.45)$$

Substitution of (3.43)–(3.45) proves the second inequality in (3.42).

If $f \in \text{Conc}_m(g, H)$, Theorem 2.1 holds in the reverse direction with a maximum. If H is convex, weighted Jensen's inequalities (3.43) and (3.44) are also reversed. The asserted concave counterpart follows. $\square$

Remark 3.5. For $H = h$, Theorem 3.1 coincides with the $\alpha = 1$ specialization of Theorem 2 in [8].

Theorem 3.2 (Normalized endpoint H -analogue). Assume the hypotheses of Theorem 2.1, let

$$W = \int_a^b \Omega(\zeta) d\zeta > 0, \quad I_M(\Omega, p) = \frac{1}{W} \int_a^b \Omega(\zeta) p(\zeta) d\zeta.$$

If $f \in \text{Conv}_m(g, H)$ and H is concave on $[0, 1]$, then

$$\begin{aligned} \frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \min\left\{ \right. \\ &\quad (fg)(a)H\left(\frac{b - I_M(\Omega, p)}{b - a}\right) + m(fg)\left(\frac{b}{m}\right)H\left(\frac{I_M(\Omega, p) - a}{b - a}\right), \\ &\quad \left. (fg)(b)H\left(\frac{I_M(\Omega, p) - a}{b - a}\right) + m(fg)\left(\frac{a}{m}\right)H\left(\frac{b - I_M(\Omega, p)}{b - a}\right) \right\}. \end{aligned} \quad (3.46)$$

If $f \in \text{Conc}_m(g, H)$ and H is convex on $[0, 1]$, then (3.46) is reversed and the minimum is replaced by a maximum.

Proof. The asserted convex estimate is the terminal inequality in (3.42). The reverse statement follows from the concave part of Theorem 3.1. $\square$

Remark 3.6. Setting $H = h$ in Theorem 3.2 gives the $\alpha = 1$ specialization of Theorem 3 in [8]. Thus Theorem 3.1 supplies the two-step refinement, while Theorem 3.2 records its direct normalized endpoint consequence.

4. Special cases of Theorems 3.1 and 3.2

To state the consequences of Theorems 3.1 and 3.2 compactly, put

$$\lambda := \frac{I_M(\Omega, p) - a}{b - a}, \quad 1 - \lambda = \frac{b - I_M(\Omega, p)}{b - a}. \quad (4.1)$$

Since $p(\zeta) \in [a, b]$ almost everywhere and $\Omega \geq 0$, one has $I_M(\Omega, p) \in [a, b]$, and hence $\lambda \in [0, 1]$. Define

$$X_H(\lambda) := (fg)(a)H(1 - \lambda) + m(fg)\left(\frac{b}{m}\right)H(\lambda), \quad (4.2)$$

$$Y_H(\lambda) := (fg)(b)H(\lambda) + m(fg)\left(\frac{a}{m}\right)H(1 - \lambda). \quad (4.3)$$

Thus, for concave H and $f \in \text{Conv}_m(g, H)$,

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min\{X_H(\lambda), Y_H(\lambda)\}. \quad (4.4)$$

If H is convex and $f \in \text{Conc}_m(g, H)$, then the inequality in (4.4) is reversed and the minimum is replaced by a maximum.

Corollary 4.1 (The $(g, h; m)$ -convex case). *Assume the hypotheses of Theorem 3.1. Let $H = h$, where h is concave on $[0, 1]$, and suppose that $f \in \text{Conv}_m(g, h)$. Then*

$$\begin{aligned} \frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min \left\{ (fg)(a)h(1 - \lambda) + m(fg)\left(\frac{b}{m}\right)h(\lambda), \right. \\ \left. (fg)(b)h(\lambda) + m(fg)\left(\frac{a}{m}\right)h(1 - \lambda) \right\}. \end{aligned} \quad (4.5)$$

If $f \in \text{Conc}_m(g, h)$ and h is convex, then (4.5) is reversed and the minimum is replaced by a maximum.

Proof. Set $H = h$ in (4.4). The reverse case follows from the concave part of Theorem 3.1. $\square$

Corollary 4.2 (The linear choice and the classical Lah-Ribarič inequality). *Let $H(t) = t$. If $f \in \text{Conv}_m(g, H)$, then*

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min \left\{ (1 - \lambda)(fg)(a) + \lambda m(fg)\left(\frac{b}{m}\right), \lambda(fg)(b) + (1 - \lambda)m(fg)\left(\frac{a}{m}\right) \right\}. \quad (4.6)$$

If $f \in \text{Conc}_m(g, H)$, then (4.6) is reversed and the minimum is replaced by a maximum. In particular, if $g \equiv 1$ and $m = 1$, then both endpoint expressions in (4.6) coincide, and

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq (1 - \lambda)f(a) + \lambda f(b). \quad (4.7)$$

Proof. The function $H(t) = t$ is both concave and convex. Substitution in (4.4) gives (4.6). For $g \equiv 1$ and $m = 1$, the two terms inside the minimum are equal, which proves (4.7). $\square$

Corollary 4.3 (The power choice $H(t) = t^s$). *Let $s > 0$, let $H_s(t) = t^s$, and define*

$$\begin{aligned} X_s &:= (fg)(a)(1 - \lambda)^s + m(fg)\left(\frac{b}{m}\right)\lambda^s, \\ Y_s &:= (fg)(b)\lambda^s + m(fg)\left(\frac{a}{m}\right)(1 - \lambda)^s. \end{aligned}$$

If $0 < s \leq 1$ and $f \in \text{Conv}_m(g, H_s)$, then

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min\{X_s, Y_s\}. \quad (4.8)$$

If $s \geq 1$ and $f \in \text{Conc}_m(g, H_s)$, then

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \geq \max\{X_s, Y_s\}. \quad (4.9)$$

Proof. The function t^s is concave on $[0, 1]$ for $0 < s \leq 1$ and convex there for $s \geq 1$. The conclusions follow from the two parts of Theorem 3.1. $\square$

Corollary 4.4 (The constant choice $H \equiv 1$). *If $H(t) = 1$ and $f \in \text{Conv}_m(g, H)$, then*

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min \left\{ (fg)(a) + m(fg) \left(\frac{b}{m} \right), (fg)(b) + m(fg) \left(\frac{a}{m} \right) \right\}. \quad (4.10)$$

For $f \in \text{Conc}_m(g, H)$, the inequality is reversed and the minimum is replaced by a maximum.

Proof. The constant function $H \equiv 1$ is both concave and convex. Therefore, the result follows directly from (4.4). $\square$

Corollary 4.5 (The product-generated $(h_1, h_2, g; m)$ -convex case). *Let $H_{12}(t) := h_1(t)h_2(1-t)$, $t \in [0, 1]$. Assume that H_{12} is concave and that f belongs to the $(h_1, h_2, g; m)$ -convex class. Then*

$$\begin{aligned} \frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \leq \min & \left\{ (fg)(a)h_1(1-\lambda)h_2(\lambda) + m(fg) \left(\frac{b}{m} \right) h_1(\lambda)h_2(1-\lambda), \right. \\ & \left. (fg)(b)h_1(\lambda)h_2(1-\lambda) + m(fg) \left(\frac{a}{m} \right) h_1(1-\lambda)h_2(\lambda) \right\}. \end{aligned} \quad (4.11)$$

If H_{12} is convex and f belongs to the corresponding $(h_1, h_2, g; m)$ -concave class, then (4.11) is reversed and the minimum is replaced by a maximum.

Proof. For the product choice,

$$H_{12}(\lambda) = h_1(\lambda)h_2(1-\lambda), \quad H_{12}(1-\lambda) = h_1(1-\lambda)h_2(\lambda).$$

Substitution of these identities into (4.4) proves the convex case. The reverse case follows from the convexity of H_{12} . $\square$

Remark 4.1 (Further classes generated by the product choice). *Subject to the same concavity or convexity requirement on $H_{12}(t) = h_1(t)h_2(1-t)$, Corollary 4.5 contains the following cases:*

- (i) $g \equiv 1$ gives the $(h_1, h_2; m)$ -convex or concave class;
- (ii) $m = 1$ gives the (h_1, h_2, g) -convex or concave class;
- (iii) $g \equiv 1$ and $m = 1$ give the (h_1, h_2) -convex or concave class;
- (iv) $h_1 = h_2 = h$ gives the symmetric product coefficient $H(t) = h(t)h(1-t)$;
- (v) $h_2 \equiv 1$ gives $H = h_1$, and hence recovers Corollary 4.1.

Corollary 4.6 (A normalized Hadamard-type specialization). *Assume the hypotheses of Theorem 3.1, let $\Omega \equiv 1$, and let $p(\zeta) = \zeta$. If H is concave and $f \in \text{Conv}_m(g, H)$, then*

$$\frac{1}{b-a} \int_a^b f(\zeta) d\zeta \leq H\left(\frac{1}{2}\right) \min\left\{(fg)(a) + m(fg)\left(\frac{b}{m}\right), (fg)(b) + m(fg)\left(\frac{a}{m}\right)\right\}. \quad (4.12)$$

If H is convex and $f \in \text{Conc}_m(g, H)$, then (4.12) is reversed and the minimum is replaced by a maximum.

Proof. For $\Omega \equiv 1$ and $p(\zeta) = \zeta$, one has $W = b - a$, $I_M(\Omega, p) = (a + b)/2$, and $\lambda = 1/2$. Substitution in (4.4) gives the result. $\square$

Corollary 4.7 (A regularized reciprocal choice). *Let $\varepsilon > 0$ and*

$$H_\varepsilon(t) := \frac{1}{t + \varepsilon}, \quad t \in [0, 1].$$

If $f \in \text{Conc}_m(g, H_\varepsilon)$, then

$$\frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta \geq \max\left\{\frac{(fg)(a)}{1 - \lambda + \varepsilon} + \frac{m(fg)(b/m)}{\lambda + \varepsilon}, \frac{(fg)(b)}{\lambda + \varepsilon} + \frac{m(fg)(a/m)}{1 - \lambda + \varepsilon}\right\}. \quad (4.13)$$

Proof. The function H_ε is convex on $[0, 1]$. Hence the reverse part of Theorem 3.1 applies. $\square$

Remark 4.2. *The regularization in Corollary 4.7 is necessary because the unregularized coefficient $H(t) = 1/t$ is not finite at $t = 0$, whereas Definition 2.1 assumes that H is finite on $[0, 1]$. When $\lambda \in (0, 1)$, the right-hand side of (4.13) has the corresponding formal limit as $\varepsilon \downarrow 0$.*

5. A supermultiplicative H -extension

A nonnegative function $H : J \rightarrow [0, \infty)$ is called supermultiplicative on J if

$$H(xy) \geq H(x)H(y)$$

whenever $x, y, xy \in J$. It is called submultiplicative when the reverse inequality holds.

Theorem 5.1 (Supermultiplicative H -analogue). *Assume the hypotheses of Theorem 2.1, and suppose that $H(b-a) > 0$. Assume also that $b-a$, $b-p(\zeta)$, $p(\zeta)-a$, belong to J for almost every $\zeta \in [a, b]$. Define*

$$C_H(\zeta) := \frac{(fg)(a)H(b-p(\zeta)) + m(fg)(b/m)H(p(\zeta)-a)}{H(b-a)}, \quad (5.1)$$

$$D_H(\zeta) := \frac{(fg)(b)H(p(\zeta)-a) + m(fg)(a/m)H(b-p(\zeta))}{H(b-a)}. \quad (5.2)$$

If $f \in \text{Conv}_m(g, H)$ and H is supermultiplicative on J , then

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{C_H(\zeta), D_H(\zeta)\} d\zeta \\ &\leq \min\left\{\int_a^b \Omega(\zeta) C_H(\zeta) d\zeta, \int_a^b \Omega(\zeta) D_H(\zeta) d\zeta\right\}. \end{aligned} \quad (5.3)$$

If $f \in \text{Conc}_m(g, H)$ and H is submultiplicative on J , then

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\geq \int_a^b \Omega(\zeta) \max\{C_H(\zeta), D_H(\zeta)\} d\zeta \\ &\geq \max \left\{ \int_a^b \Omega(\zeta) C_H(\zeta) d\zeta, \int_a^b \Omega(\zeta) D_H(\zeta) d\zeta \right\}. \end{aligned} \quad (5.4)$$

Proof. Put

$$q(\zeta) = \frac{b - p(\zeta)}{b - a}, \quad r(\zeta) = \frac{p(\zeta) - a}{b - a}.$$

Then $b - p(\zeta) = (b - a)q(\zeta)$, $p(\zeta) - a = (b - a)r(\zeta)$. If H is supermultiplicative, then

$$H(q(\zeta)) \leq \frac{H(b - p(\zeta))}{H(b - a)}, \quad H(r(\zeta)) \leq \frac{H(p(\zeta) - a)}{H(b - a)}. \quad (5.5)$$

Consequently, $A(\zeta) \leq C_H(\zeta)$ and $B(\zeta) \leq D_H(\zeta)$. The first inequality in (2.4) and (5.5) therefore give the first inequality in (5.3). The second follows from

$$\int_a^b \Omega \min\{C_H, D_H\} \leq \min \left\{ \int_a^b \Omega C_H, \int_a^b \Omega D_H \right\}.$$

If H is submultiplicative, both inequalities in (5.5) are reversed. Hence $A(\zeta) \geq C_H(\zeta)$ and $B(\zeta) \geq D_H(\zeta)$. The concave inequality (2.5) now yields (5.4). $\square$

Remark 5.1. For $H = h$, the convex part of Theorem 5.1 reduces to the $\alpha = 1$ specialization of Theorem 4 in [8]. The submultiplicative statement is the corresponding concave extension.

6. Special cases of Theorem 5.1

Corollary 6.1 (Recovery of the h -supermultiplicative case). *Under the hypotheses of Theorem 5.1, set $H = h$, where h is supermultiplicative and $h(b - a) > 0$. Define*

$$\begin{aligned} C_h(\zeta) &:= \frac{(fg)(a)h(b - p(\zeta)) + m(fg)(b/m)h(p(\zeta) - a)}{h(b - a)}, \\ D_h(\zeta) &:= \frac{(fg)(b)h(p(\zeta) - a) + m(fg)(a/m)h(b - p(\zeta))}{h(b - a)}. \end{aligned}$$

If $f \in \text{Conv}_m(g, h)$, then

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{C_h(\zeta), D_h(\zeta)\} d\zeta \\ &\leq \min \left\{ \int_a^b \Omega(\zeta) C_h(\zeta) d\zeta, \int_a^b \Omega(\zeta) D_h(\zeta) d\zeta \right\}. \end{aligned} \quad (6.1)$$

Proof. This is Theorem 5.1 with $H = h$. It is the $\alpha = 1$ specialization of Theorem 4 in [8]. $\square$

Corollary 6.2 (Scaled power coefficients). *Let*

$$H_{c,s}(t) := ct^s, \quad 0 < c \leq 1, \quad s > 0.$$

Assume that the hypotheses of Theorem 5.1 hold for $H_{c,s}$, and put

$$q(\zeta) = \frac{b - p(\zeta)}{b - a}, \quad r(\zeta) = \frac{p(\zeta) - a}{b - a}.$$

Define

$$P_s(\zeta) := (fg)(a)q(\zeta)^s + m(fg)(b/m)r(\zeta)^s, \quad (6.2)$$

$$Q_s(\zeta) := (fg)(b)r(\zeta)^s + m(fg)(a/m)q(\zeta)^s. \quad (6.3)$$

If $f \in \text{Conv}_m(g, H_{c,s})$, then

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{P_s(\zeta), Q_s(\zeta)\} d\zeta \\ &\leq \min \left\{ \int_a^b \Omega(\zeta) P_s(\zeta) d\zeta, \int_a^b \Omega(\zeta) Q_s(\zeta) d\zeta \right\}. \end{aligned} \quad (6.4)$$

Proof. For $x, y \geq 0$,

$$H_{c,s}(xy) = cx^s y^s \geq c^2 x^s y^s = H_{c,s}(x) H_{c,s}(y),$$

so $H_{c,s}$ is supermultiplicative. Moreover, the factor c cancels in the ratios defining C_H and D_H , which gives (6.2)–(6.4). For $c = 1$, the power function is multiplicative and the ratio comparison becomes an equality. $\square$

Corollary 6.3 (Product of supermultiplicative coefficients). *Let*

$$h_1, h_2: J \rightarrow [0, \infty)$$

be supermultiplicative, and assume that $h_1(b - a)h_2(b - a) > 0$. Set $H_\Pi(t) := h_1(t)h_2(t)$. Under the remaining hypotheses of Theorem 5.1, define

$$\begin{aligned} C_\Pi(\zeta) &:= (fg)(a) \frac{h_1(b - p(\zeta))h_2(b - p(\zeta))}{h_1(b - a)h_2(b - a)} + m(fg)(b/m) \frac{h_1(p(\zeta) - a)h_2(p(\zeta) - a)}{h_1(b - a)h_2(b - a)}, \\ D_\Pi(\zeta) &:= (fg)(b) \frac{h_1(p(\zeta) - a)h_2(p(\zeta) - a)}{h_1(b - a)h_2(b - a)} + m(fg)(a/m) \frac{h_1(b - p(\zeta))h_2(b - p(\zeta))}{h_1(b - a)h_2(b - a)}. \end{aligned}$$

If $f \in \text{Conv}_m(g, H_\Pi)$, then

$$\begin{aligned} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta &\leq \int_a^b \Omega(\zeta) \min\{C_\Pi(\zeta), D_\Pi(\zeta)\} d\zeta \\ &\leq \min \left\{ \int_a^b \Omega(\zeta) C_\Pi(\zeta) d\zeta, \int_a^b \Omega(\zeta) D_\Pi(\zeta) d\zeta \right\}. \end{aligned} \quad (6.5)$$

Proof. The product $H_{\Pi} = h_1 h_2$ is supermultiplicative because

$$H_{\Pi}(xy) = h_1(xy)h_2(xy) \geq h_1(x)h_1(y)h_2(x)h_2(y) = H_{\Pi}(x)H_{\Pi}(y).$$

The conclusion follows by substituting $H = H_{\Pi}$ in Theorem 5.1. $\square$

Remark 6.1 (Complementary product and reduced classes). *The complementary product*

$$H_{12}(t) = h_1(t)h_2(1-t)$$

from Corollary 4.5 also yields a special case of Theorem 5.1 whenever it extends to J , is supermultiplicative there, and $H_{12}(b-a) > 0$. In every preceding supermultiplicative result, $g \equiv 1$ gives the corresponding $(H; m)$ -convex case, $m = 1$ gives the (g, H) -convex case, and $g \equiv 1$, $m = 1$ give the H -convex case.

7. Discrete forms and classical consequences

The passage from the continuous results to finite sums is measure-theoretic: Replace the measure $d\mu = \Omega d\zeta$ by the atomic measure $\mu = \sum_{i=1}^n p_i \delta_{x_i}$. Because the endpoint estimates are pointwise, this substitution preserves the proof and converts each integral into a weighted sum. In the terminology used below, the Giaccardi form retains a reference point x_0 together with the weighted aggregate $\bar{x} = \sum_i p_i x_i$; the Popoviciu form is obtained by taking $x_0 = 0$; and the Petrović form further uses unit weights. This provides the logical bridge from the continuous inequalities of the preceding sections to their classical discrete consequences.

Theorem 7.1 (Discrete Lah-Ribarič H -analogue). *Assume the interval hypotheses of Theorem 2.1. Let $n \in \mathbb{N}$, $p_i \geq 0$, and $x_i \in [a, b]$ for $i = 1, \dots, n$. Put*

$$q_i := \frac{b - x_i}{b - a}, \quad r_i := \frac{x_i - a}{b - a},$$

and define

$$A_i^H := (fg)(a)H(q_i) + m(fg)(b/m)H(r_i), \quad (7.1)$$

$$B_i^H := (fg)(b)H(r_i) + m(fg)(a/m)H(q_i). \quad (7.2)$$

If $f \in \text{Conv}_m(g, H)$, then

$$\sum_{i=1}^n p_i f(x_i) \leq \sum_{i=1}^n p_i \min\{A_i^H, B_i^H\} \leq \min \left\{ \sum_{i=1}^n p_i A_i^H, \sum_{i=1}^n p_i B_i^H \right\}. \quad (7.3)$$

If $f \in \text{Conc}_m(g, H)$, then (7.3) is reversed and each minimum is replaced by a maximum.

Proof. Apply the two pointwise endpoint bounds (2.6) and (2.7) to $u = x_i$. Thus $f(x_i) \leq \min\{A_i^H, B_i^H\}$. Multiplication by $p_i \geq 0$ and summation proves the first inequality. The second follows from

$$\sum_{i=1}^n p_i \min\{A_i^H, B_i^H\} \leq \min \left\{ \sum_{i=1}^n p_i A_i^H, \sum_{i=1}^n p_i B_i^H \right\}.$$

The concave case follows from the reversed endpoint bounds. $\square$

Remark 7.1. For $H = h$, Theorem 7.1 reduces to the $\alpha = 1$ case of Theorem 5 in [8]. The choices $H(t) = t^s$, $H \equiv 1$, and $H(t) = h_1(t)h_2(1 - t)$ give, respectively, the power, constant, and product-generated discrete forms.

Theorem 7.2 (Giaccardi H -analogue). Let $I = [0, \infty)$, let $p_i \geq 0$ with $\sum_{i=1}^n p_i > 0$, let $x_0, x_1, \dots, x_n \in I$, and put $\bar{x} := \sum_{i=1}^n p_i x_i$. Assume that $x_0 < \bar{x}$ and $x_i \in [x_0, \bar{x}]$ for $i = 1, \dots, n$. Define

$$u_i := \frac{\bar{x} - x_i}{\bar{x} - x_0}, \quad v_i := \frac{x_i - x_0}{\bar{x} - x_0},$$

and

$$G_i^H := (fg)(x_0)H(u_i) + m(fg)(\bar{x}/m)H(v_i), \quad (7.4)$$

$$K_i^H := (fg)(\bar{x})H(v_i) + m(fg)(x_0/m)H(u_i). \quad (7.5)$$

If $f \in \text{Conv}_m(g, H)$, then

$$\sum_{i=1}^n p_i f(x_i) \leq \sum_{i=1}^n p_i \min\{G_i^H, K_i^H\} \leq \min \left\{ \sum_{i=1}^n p_i G_i^H, \sum_{i=1}^n p_i K_i^H \right\}. \quad (7.6)$$

If $f \in \text{Conc}_m(g, H)$, then (7.6) is reversed and each minimum is replaced by a maximum.

Proof. Apply Theorem 7.1 with $a = x_0$ and $b = \bar{x}$. □

Remark 7.2. The condition $x_i \in [x_0, \bar{x}]$ is equivalent to $(x_i - x_0)(\bar{x} - x_i) \geq 0$ when $x_0 < \bar{x}$. For $H = h$, Theorem 7.2 recovers the $\alpha = 1$ form of Theorem 6 in [8].

Corollary 7.1 (Popoviciu H -analogue). Assume the hypotheses of Theorem 7.2 with $x_0 = 0$, $\bar{x} > 0$, and $0 \leq x_i \leq \bar{x}$. Define

$$U_i^H := (fg)(0)H\left(1 - \frac{x_i}{\bar{x}}\right) + m(fg)(\bar{x}/m)H\left(\frac{x_i}{\bar{x}}\right),$$

$$V_i^H := (fg)(\bar{x})H\left(\frac{x_i}{\bar{x}}\right) + m(fg)(0)H\left(1 - \frac{x_i}{\bar{x}}\right).$$

If $f \in \text{Conv}_m(g, H)$, then

$$\sum_{i=1}^n p_i f(x_i) \leq \sum_{i=1}^n p_i \min\{U_i^H, V_i^H\} \leq \min \left\{ \sum_{i=1}^n p_i U_i^H, \sum_{i=1}^n p_i V_i^H \right\}. \quad (7.7)$$

The concave counterpart is obtained by reversing (7.7) and replacing the minima by maxima.

Proof. Set $x_0 = 0$ in Theorem 7.2. □

Corollary 7.2 (Petrović H -analogue). Let $x_1, \dots, x_n \geq 0$, and put $S := \sum_{i=1}^n x_i > 0$. Define

$$R_i^H := (fg)(0)H\left(\frac{S - x_i}{S}\right) + m(fg)(S/m)H\left(\frac{x_i}{S}\right),$$

$$T_i^H := (fg)(S)H\left(\frac{x_i}{S}\right) + m(fg)(0)H\left(\frac{S - x_i}{S}\right).$$

If $f \in \text{Conv}_m(g, H)$, then

$$\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n \min\{R_i^H, T_i^H\} \leq \min\left\{\sum_{i=1}^n R_i^H, \sum_{i=1}^n T_i^H\right\}. \quad (7.8)$$

The concave counterpart is obtained by reversing (7.8) and replacing the minima by maxima.

Proof. In Theorem 7.2, take $p_i = 1$, $x_0 = 0$, and $\bar{x} = S$. □

Theorem 7.3 (Supermultiplicative discrete H -analogue). Assume the hypotheses of Theorem 7.2. Suppose that $H(\bar{x} - x_0) > 0$. Also assume that $\bar{x} - x_0$, $\bar{x} - x_i$, and $x_i - x_0$ belong to J for $i = 1, \dots, n$. Define

$$\widehat{G}_i^H := \frac{(fg)(x_0)H(\bar{x} - x_i) + m(fg)(\bar{x}/m)H(x_i - x_0)}{H(\bar{x} - x_0)}, \quad (7.9)$$

$$\widehat{K}_i^H := \frac{(fg)(\bar{x})H(x_i - x_0) + m(fg)(x_0/m)H(\bar{x} - x_i)}{H(\bar{x} - x_0)}. \quad (7.10)$$

If $f \in \text{Conv}_m(g, H)$ and H is supermultiplicative on J , then

$$\begin{aligned} \sum_{i=1}^n p_i f(x_i) &\leq \sum_{i=1}^n p_i \min\{G_i^H, K_i^H\} \\ &\leq \sum_{i=1}^n p_i \min\{\widehat{G}_i^H, \widehat{K}_i^H\} \\ &\leq \min\left\{\sum_{i=1}^n p_i \widehat{G}_i^H, \sum_{i=1}^n p_i \widehat{K}_i^H\right\}. \end{aligned} \quad (7.11)$$

If $f \in \text{Conc}_m(g, H)$ and H is submultiplicative on J , then

$$\begin{aligned} \sum_{i=1}^n p_i f(x_i) &\geq \sum_{i=1}^n p_i \max\{G_i^H, K_i^H\} \\ &\geq \sum_{i=1}^n p_i \max\{\widehat{G}_i^H, \widehat{K}_i^H\} \\ &\geq \max\left\{\sum_{i=1}^n p_i \widehat{G}_i^H, \sum_{i=1}^n p_i \widehat{K}_i^H\right\}. \end{aligned} \quad (7.12)$$

Proof. Since $\bar{x} - x_i = (\bar{x} - x_0)u_i$ and $x_i - x_0 = (\bar{x} - x_0)v_i$, supermultiplicativity gives

$$H(u_i) \leq \frac{H(\bar{x} - x_i)}{H(\bar{x} - x_0)}, \quad H(v_i) \leq \frac{H(x_i - x_0)}{H(\bar{x} - x_0)}.$$

Thus $G_i^H \leq \widehat{G}_i^H$ and $K_i^H \leq \widehat{K}_i^H$. Combining these comparisons with (7.6) proves (7.11). For submultiplicative H , all coefficient comparisons reverse, and the concave part of Theorem 7.2 gives (7.12). □

Remark 7.3 (Special cases of Theorem 7.3). For $H = h$, the convex part of Theorem 7.3 reduces to the $\alpha = 1$ case of Theorem 7 in [8]. Taking $x_0 = 0$ gives the supermultiplicative Popoviciu form; taking additionally $p_i = 1$ and $\bar{x} = \sum_{i=1}^n x_i$ gives the supermultiplicative Petrović form. The scaled-power choice $H(t) = ct^s$, $0 < c \leq 1$, $s > 0$, is admissible, and the factor c cancels from $\widehat{G}_i^H$ and $\widehat{K}_i^H$.

8. An increasing-functional H -analogue

For $x \in [a, b]$, introduce the endpoint functions

$$\Phi_H(x) := (fg)(a)H\left(\frac{b-x}{b-a}\right) + m(fg)(b/m)H\left(\frac{x-a}{b-a}\right), \quad (8.1)$$

$$\Psi_H(x) := (fg)(b)H\left(\frac{x-a}{b-a}\right) + m(fg)(a/m)H\left(\frac{b-x}{b-a}\right). \quad (8.2)$$

Theorem 8.1 (Increasing-functional H -analogue). *Assume the hypotheses of Theorem 3.2, and put*

$$M_f(\Omega, p) := \frac{1}{W} \int_a^b \Omega(\zeta) f(p(\zeta)) d\zeta.$$

Let $V \subseteq \mathbb{R}$ be an interval containing $M_f(\Omega, p)$, $f(x)$, $\Phi_H(x)$, $\Psi_H(x)$ ($x \in [a, b]$), and let $F : V \times V \rightarrow \mathbb{R}$ be nondecreasing in its first variable. Assume that the extrema displayed below exist.

If $f \in \text{Conv}_m(g, H)$ and H is concave on $[0, 1]$, then

$$\begin{aligned} & F(M_f(\Omega, p), f(I_M(\Omega, p))) \\ & \leq \min \left\{ F(\Phi_H(I_M(\Omega, p)), f(I_M(\Omega, p))), \right. \\ & \quad \left. F(\Psi_H(I_M(\Omega, p)), f(I_M(\Omega, p))) \right\} \\ & \leq \max_{x \in [a, b]} \min \left\{ F(\Phi_H(x), f(x)), \right. \\ & \quad \left. F(\Psi_H(x), f(x)) \right\}. \end{aligned} \quad (8.3)$$

If $f \in \text{Conc}_m(g, H)$ and H is convex on $[0, 1]$, then

$$\begin{aligned} & F(M_f(\Omega, p), f(I_M(\Omega, p))) \\ & \geq \max \left\{ F(\Phi_H(I_M(\Omega, p)), f(I_M(\Omega, p))), \right. \\ & \quad \left. F(\Psi_H(I_M(\Omega, p)), f(I_M(\Omega, p))) \right\} \\ & \geq \min_{x \in [a, b]} \max \left\{ F(\Phi_H(x), f(x)), \right. \\ & \quad \left. F(\Psi_H(x), f(x)) \right\}. \end{aligned} \quad (8.4)$$

Proof. Theorem 3.2 gives

$$M_f(\Omega, p) \leq \min \{ \Phi_H(I_M(\Omega, p)), \Psi_H(I_M(\Omega, p)) \}.$$

Because F is nondecreasing in its first variable, it follows that

$$F(M_f(\Omega, p), f(I_M(\Omega, p))) \leq \min \{ F(\Phi_H(I_M), f(I_M)), F(\Psi_H(I_M), f(I_M)) \},$$

where $I_M = I_M(\Omega, p)$. Since $I_M \in [a, b]$, the last minimum is bounded above by the maximum over $x \in [a, b]$. This proves (8.3).

In the concave case,

$$M_f(\Omega, p) \geq \max \{ \Phi_H(I_M(\Omega, p)), \Psi_H(I_M(\Omega, p)) \}.$$

The same monotonicity argument, followed by minimization over $[a, b]$, proves (8.4). $\square$

Remark 8.1. With $H = h$, the convex part of Theorem 8.1 reduces to the $\alpha = 1$ case of Theorem 8 in [8]. The concave statement is the corresponding reverse form.

Corollary 8.1 (Jensen-gap estimate). *Under the convex hypotheses of Theorem 8.1,*

$$\begin{aligned} & M_f(\Omega, p) - f(I_M(\Omega, p)) \\ & \leq \min\{\Phi_H(I_M(\Omega, p)) - f(I_M(\Omega, p)), \Psi_H(I_M(\Omega, p)) - f(I_M(\Omega, p))\} \\ & \leq \max_{x \in [a, b]} \min\{\Phi_H(x) - f(x), \Psi_H(x) - f(x)\}. \end{aligned} \quad (8.5)$$

Under the concave hypotheses, the inequalities reverse, the minima become maxima, and the final maximum becomes a minimum.

Proof. Apply Theorem 8.1 with $F(u, v) = u - v$, which is increasing in u . □

Remark 8.2 (Special choices in Theorem 8.1). *The substitutions $H(t) = t$, $H(t) = t^s$, and*

$$H(t) = h_1(t)h_2(1 - t)$$

give the classical, power, and product-generated functional bounds, respectively. In particular, when $H(t) = t$, $g \equiv 1$, and $m = 1$, one has

$$\Phi_H(x) = \Psi_H(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b),$$

so Corollary 8.1 reduces to the classical Lah-Ribarič estimate for the Jensen gap.

9. Conclusions

We have developed weighted Jensen and Lah-Ribarič inequalities for the $(g, H; m)$ -convex and $(g, H; m)$ -concave classes. The use of two endpoint representations produces a pointwise minimum in the convex case and a pointwise maximum in the concave case. Concavity or convexity of H then yields normalized Jensen refinements, while supermultiplicativity or submultiplicativity provides rescaled endpoint estimates. Linear, power, constant, regularized reciprocal, and product-generated coefficient functions recover familiar subclasses, and the two-coefficient construction clarifies the general $(g, h; \alpha - m)$ situation. Atomic measures connect the continuous theory to the Giaccardi, Popoviciu, and Petrović forms, and an increasing two-variable functional gives a final family of comparison inequalities.

The equality analysis shows that the basic estimates are sharp and identifies the additional requirements introduced by the Jensen and multiplicative steps. Possible directions for further work include stability estimates measuring the defect from equality, multivariable and operator versions, fractional-integral analogues, and optimal selection of H for prescribed endpoint data. It would also be useful to determine minimal regularity assumptions under which singular coefficient laws can be obtained as limits of finite regularizations.

Author contributions

Conceptualization: M.A., M.E., G.F., J. P. and J.R. Validation: M.A., M.E., G.F., J. P. and J.R. Formal Analysis: M.A., M.E., G.F., J. P. and J.R. Writing manuscript: M.A., G.F. Editing and Review manuscript: M.A., M.E., G.F., J. P. and J.R. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest.

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