



*Research article***Edge pairings of regular polygons and reassembling objects****Pantaleón D. Romero^{1,*} and Catarina Mendes de Jesus S²**

¹ Departamento de Matemáticas, Física y CC. Tecnológicas, Universidad Cardenal Herrera – CEU, C/ San Bartolomé 55, 46115 Alfara del Patriarca, Valencia, Spain

² Department of Mathematics, ICE, Federal University of Juiz de Fora, Rua José Lourenço Kelmer, s/n Campus Universitário, 36036-900, Juiz de Fora, MG, Brazil

* **Correspondence:** Email: pantaleon.romero@uchceu.es.

Abstract: Reassembling or classifying an object built from a bounded number of congruent pieces often reduces to a combinatorial question: identifying the sides of a regular polygon in pairs so as to recover a closed surface. This procedure, an edge pairing, produces a graph embedded in the resulting surface, whose genus, orientability, vertex count, and edge count are entirely determined by the pairing rule. Graphs of this kind are closely related to those underlying error-correcting and quantum surface codes, where a polygon boundary encodes redundancy, and the graph governs decoding performance; the same combinatorial structure arises whenever a boundary of matching physical pieces must be paired locally, as in computer-assisted object reconstruction. Existing constructions of pairing graphs on orientable surfaces rely on three local operations: extension, contraction, and edge exchange applied to a canonical pairing with a single vertex. This paper extends that framework to closed surfaces of both orientable and nonorientable type and introduces three families of surgery for combining pairing graphs by connected sum. We prove that every graph obtained by any combination of these operations is itself a valid edge-pairing graph, with explicit formulas for the resulting genus, vertex, and edge counts, and we classify canonical pairings for the bitorus, the 4-projective plane, and the tritorus. Building on this closure property, we formulate a greedy pairing algorithm (GPA) for reconstructing an object from polygonlike fragments and prove that every pairing it produces, at every intermediate step, is a valid edge-pairing graph on the surface fixed at initialization.

Keywords: edge pairings; regular polygons; closed surfaces; graph embedding; surgery of graphs; greedy pairing algorithm; quantum surface codes

Mathematics Subject Classification: 05C10, 57M15, 57M20, 94B25

1. Introduction

Reconstructing a physical object from a finite set of fragments (a broken ceramic vessel, an inscribed tablet, a mosaic floor recovered in pieces) is, at its combinatorial core, a matching problem: Every fragment carries boundary edges that must be identified, edge to edge, with the boundary of another fragment, subject to compatibility constraints inherited from the shape of the original object. When the object is naturally divided into a bounded number of roughly congruent pieces, this reconstruction problem can be modeled as an edge pairing of a regular polygon: The outlines of the fragments are represented as the sides of a polygon $\mathcal{P}_n$, and reconstructing the object amounts to identifying these sides in pairs so as to recover a closed surface M_g consistent with the object being reassembled.

A natural computational strategy for this class of problems is *greedy pairing*: At each step, the two boundary edges currently most compatible, according to a local geometric or combinatorial criterion, are identified and removed from further consideration, and the procedure repeats until no admissible pair remains. Greedy pairing is inexpensive to compute and, crucially, produces a valid partial pairing at every intermediate step, a property that is essential in practice, where an automatically generated pairing is typically inspected, corrected, and refined by a human operator before being accepted (see Section 6 below).

Reconstructing an object from its fragments is itself a long-standing computational problem, most often studied under the names of jigsaw puzzle solving and fractured-object reassembly. Foundational geometric matching approaches identify compatible fracture surfaces directly [1]. The combinatorial structure of edge-matching puzzles, and its computational complexity, were formalized by Demaine and Demaine [2]; graph-theoretic formulations closer in spirit to the present work recover a global assembly by imposing loop or connectivity constraints on a graph of candidate matches [3–5], and recent graph-diffusion models pursue the same goal with learned pairwise compatibilities [6]. A recent survey [7] gives a comprehensive account of this literature. None of these approaches, however, addresses the question posed here, on which the soundness of any greedy or heuristic reconstruction procedure (which is purely topological and independent of the specific reconstruction task) rests: Which graphs can arise as valid edge-pairing graphs of a regular polygon on a closed surface, and which elementary local moves preserve the graphs' class validity as they transform, independently of any particular matching heuristic?

Answering this question precisely is what makes a greedy strategy trustworthy; every local move it performs, and every partial result it produces along the way, must itself be a legitimate edge-pairing graph. Section 6 shows how the closure properties established in this paper supply exactly this correctness guarantee for a greedy assembly procedure of this kind.

This same topological question is also relevant in the setting of error-correcting codes, where reliable data transmission in communication systems such as computing and telephony depends on the algebraic and combinatorial structure of the underlying graph [8–10]. Related graph-based constructions appear in quantum error correction: Quantum surface codes derived from semi-regular tessellations [11], polygon-based color code constructions [12], and quantum Tanner codes [13] all rely on graphs embedded in surfaces to strengthen error-correction guarantees on tessellation structures related to, though distinct from, the edge-pairing graphs studied here. Expander codes, built from pseudo-random graphs, achieve strong performance in low-rate error correction [14].

In general topology an edge pairing is defined as a quotient map from a polygon onto a closed

surface, identifying pairs of edges along the polygon boundary; this creates a connected graph on the surface, the regular polygon edge-pairing graph, consisting of the image of the boundary of the polygon together with a complementary, simply connected region. This construction is, in the language of the classical literature on graph embeddings, a one-face or unicellular map: An embedding of a graph in a closed surface whose complement is a single disc. The enumeration of such maps by genus, and their combinatorial and bijective structure, have been studied extensively, both for the orientable case (notably by Harer and Zagier [23], who related it to the Euler characteristic of moduli spaces of curves, and by Chapuy [25], who gave a direct bijective proof of a combinatorial identity for unicellular maps) and for the general (not necessarily maximum-genus) embedding problem, addressed by Xuong [24]. Proposition 5.1 below (Section 5) makes this identification explicit: An edge-pairing graph is precisely a graph whose complement is simply connected, that is, a unicellular map.

The search for new families of edge-pairing graphs has motivated the development of systematic techniques for constructing them. For trivalent graphs associated with the tessellation $\{12g - 6, 3\}$, see [15–17]. Jorgensen and Naatanen [18] showed that a regular 18-gon admits eight edge pairings, corresponding to five nonisomorphic graphs (Figure 16), and Nakamura [19] showed that a regular 30-gon admits 927 pairings on the tritorus, corresponding to 65 distinct trivalent graphs. To reach surfaces of genus greater than three without an exhaustive search, [16] introduced surgery on edge pairings, proving that any graph obtained by finitely many such surgeries between trivalent edge-pairing graphs is again a trivalent edge-pairing graph. In [20], extension and contraction of graphs on a surface were introduced, showing that every edge-pairing graph can be obtained by extending some canonical pairing, that is, a pairing whose graph has a single vertex. Combining surgery with extension on oriented surfaces, [21] derived new families of edge-pairing graphs and introduced edge exchange, which yields new pairing graphs from a known one while keeping the number of vertices and edges fixed.

The present paper extends the results of [21] for orientable surfaces to the general case of closed surfaces, including the nonorientable ones. Beyond this generalization, our main contribution is to show that extension, contraction, edge exchange, and the three families of surgery introduced here can be freely combined (in any order, on both orientable and nonorientable surfaces), and that the resulting graph is always a valid edge-pairing graph on a closed surface whose genus, vertex count and edge count are given by explicit formulas (Theorems 5.1 and 5.2). This closure property is precisely what a greedy pairing algorithm needs to be provably correct at every step, and is what connects the topological theory developed here to the reconstruction problem discussed in Section 6.

This work is organized as follows: Section 2 contains a summary of regular polygon edge-pairing graphs, along with extension, contraction and edge-swapping operations. Section 3 discusses edge pairings of regular polygons and their equivalence. Section 4 introduces canonical edge pairings and classifies them on the bitorus, the 4-projective plane and the tritorus. Section 5 presents the construction of edge pairings by extension, contraction and edge swap, along with an overview of edge pairing surgeries based on [16, 21]. In Section 6, we introduce the greedy pairing algorithm and prove its topological correctness by combining the results of Sections 2–5. Section 7 illustrates these results on two concrete fragment-reconstruction instances. Section 8 provides concluding remarks.

2. Graphs and surfaces

Section 2.1 recalls, for completeness and to affix notation, standard material on graphs embedded in surfaces; Sections 2.2–2.4 generalize the extension, contraction, edge-exchange, and connected-sum operations of [16, 20, 21], previously stated only for oriented surfaces, to the case of arbitrary closed surfaces, and are among the new contributions of this paper.

2.1. Graphs embedded in surfaces

Throughout the paper, $g\mathbb{T}^2$ denotes the closed, orientable surface of genus g (the connected sum of g tori), $g\mathbb{P}^2$ denotes the closed, nonorientable surface of (nonorientable) genus g (the connected sum of g projective planes), and M_g denotes either one generically, the type being clear from context, or being stated explicitly.

The *genus* of a graph G is the smallest number g such that there exists an embedding $\iota : G \rightarrow g\mathbb{T}^2$. If G has genus m , then there is always an embedding $\iota : G \rightarrow g\mathbb{T}^2$ when $m \leq g$ and an embedding $\iota : G \rightarrow g\mathbb{P}^2$ when $m \leq 2g$. In this case, the graph image $\iota(G)$ in M_g will be denoted by $\mathcal{G}(V, A)$ or simply $\mathcal{G}$. Here, V and A denote, respectively, the number of vertices and the number of edges of the graph $\mathcal{G}$; this pair (V, A) is carried in the notation throughout the paper.

The complement of a graph $\mathcal{G}$ in M_g will be denoted by $\mathcal{B}(\mathcal{G}) = M_g \setminus \mathcal{G}$, and the number of connected components of $\mathcal{B}(\mathcal{G})$ will be denoted by F .

Remark. Let $\mathcal{G}(V, A)$ be a connected graph on a closed surface M_g . If $\mathcal{B}(\mathcal{G})$ is a simply connected region, then

- i) The Euler characteristic of the graph G satisfies $\chi(\mathcal{G}) = \begin{cases} 1 - 2g, & \text{for } M_g = g\mathbb{T}^2, \\ 1 - g, & \text{for } M_g = g\mathbb{P}^2. \end{cases}$
- ii) The graph G is a tree type ($\chi(\mathcal{G}) = 1$) if and only if $g = 0$.
- iii) If $V = 1$, then $A = 2g$ if M_g is orientable, and $A = g$ if M_g is nonorientable.

Example 2.1. Figure 1 shows that three graphs on the bitorus $2\mathbb{T}^2$ are planar graphs (genus zero). In (b), there is $V = 5$ vertices, $A = 8$ edges, and $F = 3$ faces, whereas in (a) and (c), there are $V = 6$ vertices, $A = 9$ edges, and $F = 3$ faces. In all cases, one of the regions of $\mathcal{B}(\mathcal{G})$ is not simply connected and has genus 1.

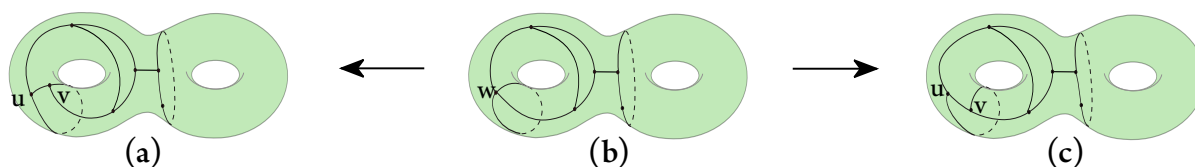


Figure 1. Examples of graphs on the bitorus.

The graphs embedded in the surfaces illustrated in Figure 7 are characterized by their complement being simply connected. In (a), $M_g = \mathbb{S}^2$, and the graph type is $\mathcal{G}(3, 2)$; in (b), $M_g = \mathbb{T}^2$, and the graph type is $\mathcal{G}(1, 2)$; in (c), $M_g = \mathbb{K}^2 = 2\mathbb{P}^2$, and the graph type is $\mathcal{G}(1, 2)$; and in (d), $M_g = \mathbb{P}^2$, and the graph type is $\mathcal{G}(2, 2)$.

Proposition 2.1. If $\mathcal{G} = (V, A)$ is a graph on M_g , then the Euler characteristic of M_g satisfies $\chi(M_g) = \chi(\mathcal{G}) + \sum_{i=1}^F \chi(R_i)$, where R_i ($i = 1, \dots, F$) denotes a region of $\mathcal{B}(\mathcal{G})$.

If each R_i ($i = 1, \dots, F$) is a simply connected region, then $\chi(M_g) = \chi(\mathcal{G}) + F$.

Proof. Suppose that every vertex of $\mathcal{G}$ has degree greater than 2, as a vertex with degree 2 can be disregarded as if it were a point on a larger edge. Removing such a vertex and merging its two incident edges into one decreases both V and A by one, leaving $\chi(\mathcal{G}) = V - A$ unchanged, and it does not alter the regions R_i of $\mathcal{B}(\mathcal{G})$, as these depend only on the position of the edges in the surface and not on the presence of a degree-two vertex along an arc. The Euler characteristic of M_g is given by $\chi(M_g) = \chi(\cup_{i=1}^F R_i)$; the characteristic of the union of two regions R_i, R_j is $\chi(R_i \cup R_j) = \chi(R_i) + \chi(R_j) - \chi(R_i \cap R_j)$, and the Euler characteristic of a region R_i is given by $\chi(R_i) = 2 - 2g_i - k_i$ if R_i is orientable; and $\chi(R_i) = 2 - g_i - k_i$ if R_i is nonorientable, consistent with the two cases distinguished in the Remark of §2.1, where g_i is the genus of R_i , and k_i is the number of boundary components of R_i .

The set $R_i \cap R_j$, for $i \neq j$, is either empty or consists of a set of edges intersecting at a set of vertices. Here, $\{a_i\}_{i=1}^A$ and $\{v_i\}_{i=1}^V$ denote, respectively, the edges and vertices of $\mathcal{G}$. The set $\cup_{i \neq j} (R_i \cap R_j)$ coincides with $\mathcal{G}$ itself, as the graph is exactly the common boundary shared by neighboring regions of $\mathcal{B}(\mathcal{G})$; viewing $\mathcal{G}$ as a one-dimensional closure-finite weak (CW) complex with V vertices and A edges gives $\chi(\cup_{i=1}^A a_i) - \chi(\cup_{i=1}^V v_i) = A - V = -\chi(\mathcal{G})$. Thus, $\chi(R_i \cap R_j) = \chi(\cup_{i \neq j} R_i \cap R_j) - \chi(\cup_{i \neq j} a_i \cap a_j) = \chi(\cup_{i=1}^A a_i) - \chi(\cup_{i=1}^V v_i) = A - V$. From these equalities, it follows that $\chi(M_g) = \chi(\cup_{i=1}^F R_i) - [\chi(\cup_{i \neq j} R_i \cap R_j) - \chi(\cup_{i \neq j} a_i \cap a_j)] = \sum_{i=1}^F \chi(R_i) - [A - V] = \sum_{i=1}^F \chi(R_i) + \chi(\mathcal{G})$. Therefore, $\chi(M_g) = \chi(\mathcal{G}) + \sum_{i=1}^F \chi(R_i)$. The Euler characteristic of a region R_i , simply connected, is $\chi(R_i) = 1$. If all regions R_i in $\mathcal{B}(\mathcal{G})$ are simply connected for $i = 1, \dots, F$, then $\chi(M_g) = \chi(\mathcal{G}) + \sum_{i=1}^F \chi(R_i) = \chi(\mathcal{G}) + F$. $\square$

2.2. Extension and contraction of graphs

In [20], the extension (contraction) of graphs on an oriented surface was introduced with the aim of obtaining edge-pairing graphs from a known graph. These techniques are proven here in the general case of a closed surface M_g , and they will be applied in Section 5.1.

Definition 2.1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be connected graphs on the closed surface M_g . An edge $uv \in \mathcal{G}_2$ connecting vertices u and v is called an extension of vertex $w \in \mathcal{G}_1$ if uv can be obtained by a “stretching” of vertex w over M_g (see Figure 1). In this case, it is said that $\mathcal{G}_2$ is an extension of $\mathcal{G}_1$ or that $\mathcal{G}_1$ is a contraction of $\mathcal{G}_2$ on M_g .

Example 2.2. The graphs (a) and (c) on the bitorus, illustrated in Figure 1, can be obtained from graph (b) by extending the vertex w . The graph (b) can be obtained by contracting the edge uv in the graph (a) or the graph in (c).

Proposition 2.2. If $\mathcal{G}'$ is a graph obtained from the graph $\mathcal{G}$ by the extension or contraction over M_g , then the sets $\mathcal{B}(\mathcal{G})$ and $\mathcal{B}(\mathcal{G}')$ are homeomorphic. Consequently, $\mathcal{B}(\mathcal{G}')$ is simply connected if and only if $\mathcal{B}(\mathcal{G})$ is simply connected.

Proof. Let $\mathcal{G}'$ be the graph obtained by extending the graph $\mathcal{G}$ on a surface M_g , through the stretching of vertex w of $\mathcal{G}$ onto edge uv of $\mathcal{G}'$.

If w is a vertex on the boundaries of regions U and W in $\mathcal{B}(\mathcal{G})$ (as w is a single point, and the extension modifies the boundary only along the single new edge uv , at most two regions of $\mathcal{B}(\mathcal{G})$ — those meeting at w along that edge — can be affected; the remainder of the local structure around w ,

and hence, every other region, is left unchanged), then the new regions U' and W' in $\mathcal{B}(\mathcal{G}')$ have one additional edge and vertex on the boundary compared to U and W , which is the edge uv . Because only the boundary of the region is altered, U' and W' have the same topology as U and W . Thus, the set $\mathcal{B}(\mathcal{G}')$ has the same topology as the set $\mathcal{B}(\mathcal{G})$. Similarly, contracting an edge uv does not alter the regions in the complement, keeping the topology of the set $\mathcal{B}(\mathcal{G})$ invariant. $\square$

2.3. Edge swapping in graphs

In [21], the edge-swapping technique for a matching graph on an oriented surface was introduced. This technique will be extended to an arbitrary graph on closed surfaces.

Let $\mathcal{G}$ be a connected graph on the surface M_g , and let λ be a path in $\mathcal{G}$ with endpoints at vertices u and v , both with degree greater than 3. Suppose there exists an edge a incident to u , which can be “dragged” along λ across M_g to vertex v such that a does not intersect any other edge of $\mathcal{G}$, as locally illustrated in Figure 2. This action keeps the subgraph $\mathcal{G}\{a\}$ invariant while maintaining the invariance of the numbers V and A .

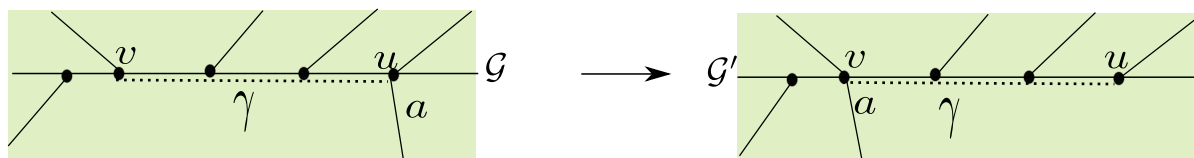


Figure 2. Local example of edge exchange in a graph.

Definition 2.2. A graph $\mathcal{G}'(V, A)$ is said to be obtained from $\mathcal{G}(V, A)$ by edge swaps if it is possible to move, as described, one endpoint of an edge from vertex u to another vertex v along a path λ in $\mathcal{G}(V, A)$ without intersecting any point of $\mathcal{G}(V, A)$.

Proposition 2.3. If $\mathcal{G}'$ is a graph obtained from graph $\mathcal{G}$ by swapping edges over M_g , then the sets $\mathcal{B}(\mathcal{G})$ and $\mathcal{B}(\mathcal{G}')$ are homeomorphic. Consequently, $\mathcal{B}(\mathcal{G}')$ is simply connected if and only if $\mathcal{B}(\mathcal{G})$ is simply connected.

Proof. Let U and W in $\mathcal{B}(\mathcal{G})$ be regions whose boundaries contain the edge uv_1 , connected to vertex v_1 of $\mathcal{G}$. In M_g , the edge uv_2 is, by the hypothesis of Definition 2.2, chosen disjoint from every other edge of the original graph $\mathcal{G}$; the substitution of edge uv_1 by this edge uv_2 , linking to vertex v_2 , solely alters the count of edges and vertices on the boundaries of the new regions U' and W' relative to U and W . This does not modify the topology of U and W . Replacing the boundary segment uv_1 by uv_2 amounts to sliding an arc of ∂U while preserving the topology of its interior; the number of sides on the boundary changes, but no handle is added or removed, and no new gluing is introduced, so the genus and number of boundary components of U (and hence its topological type as a surface with boundary) are unaffected. Hence, U' and W' maintain the same topology as U and W , respectively. Because the edge substitution does not intersect any other edges, the complement $\mathcal{B}(\mathcal{G}')$ retains the same topology as $\mathcal{B}(\mathcal{G})$. $\square$

2.4. Connected sum of graphs and of surfaces

This section uses the terms *surgery* and *connected sum* at two related but distinct levels: first, between closed surfaces (removing a disc from each and gluing along the resulting boundary circles),

and second, between the graphs embedded in those surfaces (Definition 2.3 below). In both cases, *surgery* denotes the operation in the general case, which may occur on a single connected component or between two connected components, whereas *connected sum* denotes the particular case in which the surgery occurs between two disjoint connected components, resulting in a single connected component.

A surgery between two surfaces M_{g_1} and M_{g_2} can be realized by removing a disc D_i from M_{g_i} ($i = 1, 2$) and then identifying the boundary of the closure of $M_{g_1} \setminus D_1$ with the boundary of the closure of $M_{g_2} \setminus D_2$. If M_{g_1} and M_{g_2} are disjoint surfaces, this surgery is called the connected sum of the surfaces and is denoted by $M_{g_1} \# M_{g_2}$. Then, the surface may hold the following properties, depending on context:

- i) $M_{g_1} \# M_{g_2}$ is orientable if and only if both M_{g_1} and M_{g_2} are orientable. In this case, the genus g is given by $g = g_1 + g_2$.
- ii) $M_{g_1} \# M_{g_2}$ is nonorientable with genus $g = 2g_1 + g_2$ if M_{g_1} is orientable, and M_{g_2} is nonorientable.
- iii) If both surfaces are nonorientable, then $g = g_1 + g_2$.

Definition 2.3. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two graphs on the surface M_g . It is called *surgery between the graphs* $\mathcal{G}_1$ and $\mathcal{G}_2$, the connection of the graphs by the edges a_1 of $\mathcal{G}_1$ and a_2 of $\mathcal{G}_2$ in one of the three ways, as illustrated in Figure 3:

- S_1 : One end of a line segment l is identified with a point on a_1 , and the other end of l is identified with a point on a_2 , dividing the edge a_i ($i = 1, 2$) into two new edges separated by the vertex $u = a_1 \cap l$ and $v = a_2 \cap l$.
- S_2 : The line segment l is identified with an arc of a_1 and a_2 , dividing the edge a_i ($i = 1, 2$) into two new edges separated by the edge l connected to the vertices u and v .
- S_3 : At a point P on the surface M_g , it identifies points u_1 from a_1 and u_2 from a_2 , generating a vertex w in the new graph, where u_i divides a_i ($i = 1, 2$) into two new edges.

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint connected graphs, then the surgery S_ν ($\nu = 1, 2, 3$) is called the *connected sum of graphs*, denoted by $\mathcal{G}_1 \oplus_{S_\nu} \mathcal{G}_2$; we also write $\mathcal{G}_1 \oplus_\nu \mathcal{G}_2$ for this graph when the type of surgery S_ν is clear from context.

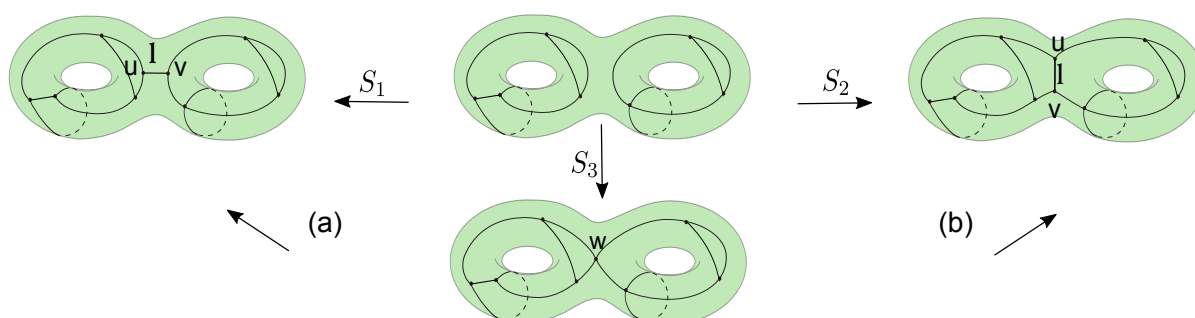


Figure 3. Examples of connected sums of graphs on the bitorus.

Figure 4 further illustrates the three surgeries S_1 , S_2 and S_3 of Definition 2.3 applied between two arbitrary graphs, independently of the connected-sum case shown in Figure 3.

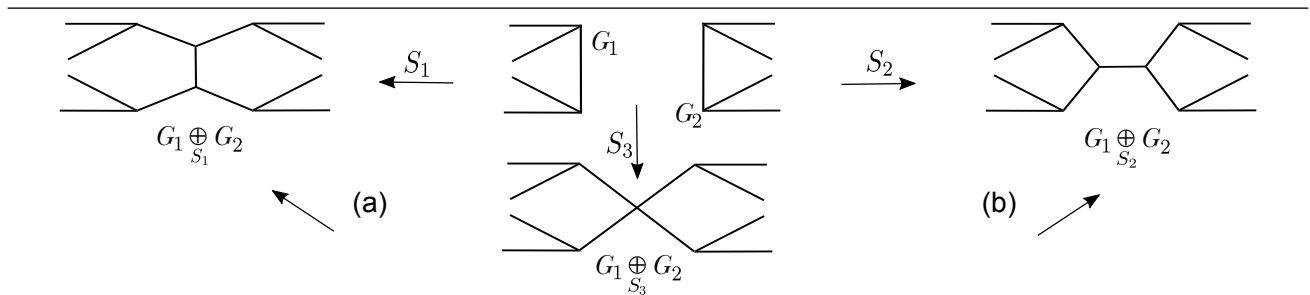


Figure 4. Examples of surgeries between two graphs.

Remark. Let $\mathcal{G}(V, A)$ denote the graph obtained from the connected sum $\mathcal{G}_1(V_1, A_1) \oplus_{S_\nu} \mathcal{G}_2(V_2, A_2)$ ($\nu = 1, 2, 3$). Then:

- 1) $V = V_1 + V_2 + 2$, and $A = A_1 + A_2 + 3$ for $\nu = 1, 2$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are trivalent graphs, then $\mathcal{G}$ is a trivalent graph.
- 2) If $\nu = 3$, then $V = V_1 + V_2 + 1$, and $A = A_1 + A_2 + 2$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are quadrivalent graphs, then $\mathcal{G}$ is a quadrivalent graph.
- 3) $\mathcal{G}_1 \oplus_{S_1} \mathcal{G}_2$ and $\mathcal{G}_1 \oplus_{S_2} \mathcal{G}_2$ can be obtained by extending the vertex w in $\mathcal{G}_1 \oplus_{S_3} \mathcal{G}_2$.

Example 2.3. Figure 3 illustrates the connected sum of graphs $\mathcal{G}_1(4, 6)$ and $\mathcal{G}_2(3, 5)$ on a bitorus by surgeries S_ν for $\nu = 1, 2, 3$, where in (a) and (b), the graphs $\mathcal{G}(9, 14)$ are not isomorphic but can be obtained by extending the vertex w in the graph $\mathcal{G}(8, 13) = \mathcal{G}_1 \oplus_{S_3} \mathcal{G}_2$.

Remark. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be connected graphs on M_g . The region $\mathcal{B}(\mathcal{G}_1 \oplus_\nu \mathcal{G}_2)$ ($\nu = 1, 2, 3$) is simply connected if and only if $\mathcal{B}(\mathcal{G}_1 \cup \mathcal{G}_2)$ is homeomorphic to a cylinder, as the edge l in the connected sums S_1 and S_2 , or the vertex w in S_3 , would split the region into two if it were simply connected.

Definition 2.4. Let $\mathcal{G}_i$ be a connected graph on the surface M_{g_i} for $i = 1, 2$. We define the connected sum of the pairs $(M_{g_1}, \mathcal{G}_1)$ and $(M_{g_2}, \mathcal{G}_2)$ as the connected sum of the surfaces $M_{g_1} \# M_{g_2}$ followed by the connected sum of the graphs $\mathcal{G}_1 \oplus_\nu \mathcal{G}_2$ ($\nu = 1, 2, 3$).

Example 2.4. Figure 5 illustrates the connected sums S_1 and S_2 between two pairs of type $(\mathbb{T}^2, \mathcal{G}(2, 3))$, resulting from the pair $(2\mathbb{T}^2, \mathcal{G}(6, 9))$, and the surgery S_3 between two pairs of type $(\mathbb{T}^2, \mathcal{G}(1, 2))$, resulting from the pair $(2\mathbb{T}^2, \mathcal{G}(3, 6))$.

Figure 6 illustrates the connected sums S_1 and S_2 between the pairs $(\mathbb{T}^2, \mathcal{G}(2, 3))$ and $(2\mathbb{P}^2, \mathcal{G}(2, 3))$, resulting in the pair $(4\mathbb{P}^2, \mathcal{G}(6, 9))$, and the surgery S_3 between the pairs $(\mathbb{T}^2, \mathcal{G}(1, 2))$ and $(2\mathbb{P}^2, \mathcal{G}(1, 2))$, resulting in the pair $(4\mathbb{P}^2, \mathcal{G}(3, 6))$.

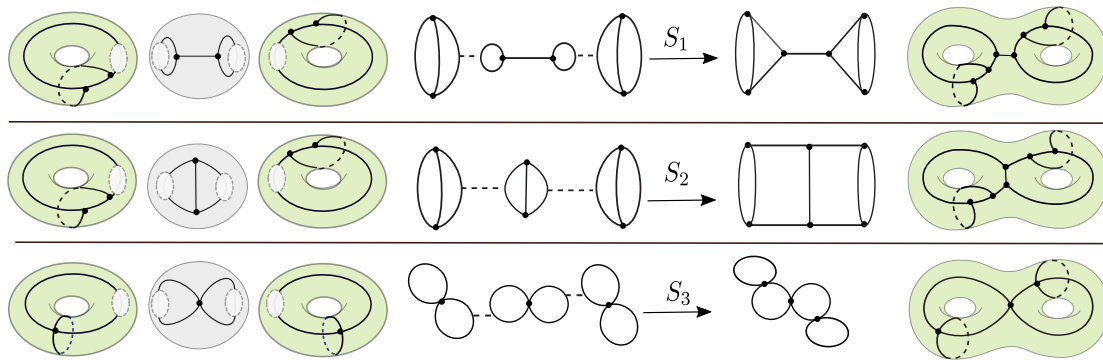


Figure 5. Examples of surgeries between two tori.

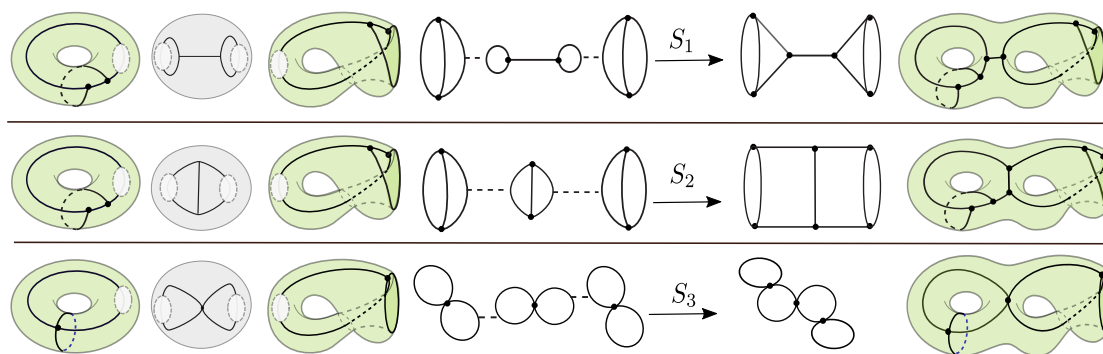


Figure 6. Examples of surgeries between a torus and a Klein bottle.

Proposition 2.4. *If $\mathcal{G}_i$ is a connected graph on M_{g_i} , and $\mathcal{B}_i(\mathcal{G}_i)$ is a simply connected region for $i = 1, 2$, then $\mathcal{B}(\mathcal{G}_1 \oplus_v \mathcal{G}_2)$ ($v = 1, 2, 3$) is a simply connected region on $M_{g_1} \# M_{g_2}$.*

Proof. To realize the connected sum of surfaces M_{g_1} and M_{g_2} , a disk D_i is removed from $\mathcal{B}_i$ ($i = 1, 2$) in order to obtain the regions $\mathcal{B}'_1 = \mathcal{B}_1 \setminus D_1$ and $\mathcal{B}'_2 = \mathcal{B}_2 \setminus D_2$, both homeomorphic to a cylinder. Then, the boundaries of $\mathcal{B}'_1$ and $\mathcal{B}'_2$ are identified to determine the surface $M_{g_1} \# M_{g_2}$, leading to a region $\mathcal{B}'$. The complement of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is homeomorphic to a cylinder. In the connected sum of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, they are joined by an edge or a point, opening one side of the cylindrical region. Formally speaking, a cylinder $C = S^1 \times [0, 1]$ has two boundary components and $\chi(C) = 0$; connecting by an edge a point of an arc on one boundary component to a point of an arc on the other corresponds exactly to the connection of two graphs, joining two degree-2 vertices by an edge. This is topologically equivalent to cutting the cylinder along an arc joining its two boundary circles, which merges the two boundary components into one and yields a disc D^2 with $\chi = 1$ and a single boundary component. A disc is simply connected, yielding the simply connected region $\mathcal{B}(\mathcal{G}_1 \oplus_v \mathcal{G}_2)$. $\square$

3. Pairings of edges of regular polygons

A regular polygon $\mathcal{P}_n$ with n sides, referred to as edges, forms the boundary of the polygon denoted by $\partial\mathcal{P}_n$. The boundary includes vertices, which are intersections between pairs of edges. As formalized in Definition 3.1 below, a quotient map identifies the sides of the polygon in pairs, gluing them via an orientation-respecting homeomorphism to recover a closed surface M_g . For the purposes of this work,

we consider $n = 2A$, where $A > 1$ so that the n sides of $\partial\mathcal{P}_n$ can always be grouped into A pairs to be so identified. Each such pair of edges is represented in the polygon diagram by a line segment joining their midpoints.

Definition 3.1. Let $q : \mathcal{P}_{2A} \rightarrow M_g$ be a quotient map (injective inside the polygon $\mathcal{P}_{2A}$) that maps each pair of edges (a_i, a_j) ($i, j \in \{1, \dots, n\}$) of $\partial\mathcal{P}_{2A}$ onto a curve arc α_s ($s \in \{1, \dots, A\}$) on the closed surface M_g and maps sets of vertices of $\partial\mathcal{P}_{2A}$ onto points in M_g . In this context, q is defined as the edge pairing of the regular polygon $\mathcal{P}_{2A}$.

The image of $\partial\mathcal{P}_{2A}$ in M_g will be denoted by $\mathcal{G}_q(V, A) = q(\partial\mathcal{P}_{2A})$, or simply $\mathcal{G}$, and is called the edge-pairing graph associated to q .

The set consisting of A line segments in $\mathcal{P}_{2A}$, connected pairwise by edges identified by q , is called the **edge-pairing diagram** and is denoted by $\mathcal{D}(g, A)$.

An edge pairing q is said to be k -regular if $\mathcal{G}$ is a k -regular graph.

The edges of the polygon should be oriented according to their identifications. This is crucial to determine whether M_g is an oriented or nonoriented surface.

Example 3.1. Figure 7 illustrates the possible edge pairings of the polygon $\mathcal{P}_4$ on oriented and nonoriented surfaces, with their pairing diagrams: (a) The nonregular pairing on $M_0 = \mathbb{S}^2$; (b) the 4-regular pairing on $M_1 = \mathbb{T}^2$ (orientable); (c) the 4-regular pairing on $M_2 = 2\mathbb{P}^2 = \mathbb{K}^2$; (d) the 2-regular pairing on $M_1 = \mathbb{P}^2$ (nonorientable).

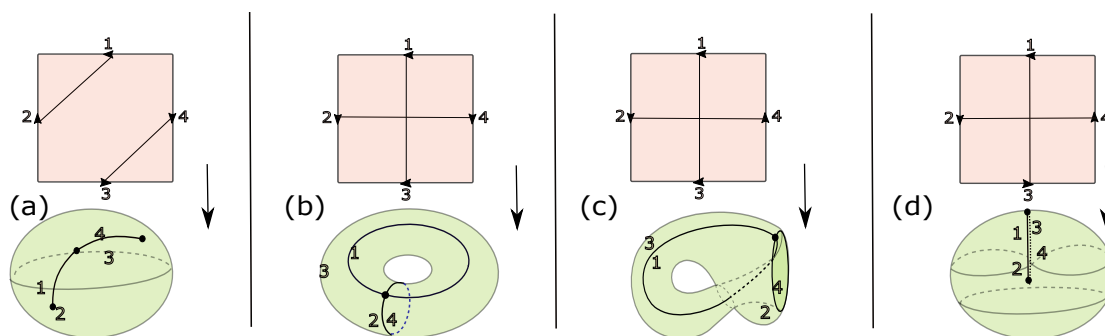


Figure 7. Possible pairings of $\mathcal{P}_4$.

Definition 3.2. Two pairings $q_1, q_2 : \mathcal{P}_n \rightarrow M_g$ are equivalent if their diagrams $\mathcal{D}_1(g, A)$ and $\mathcal{D}_2(g, A)$, corresponding to their respective pairings q_1 and q_2 , can be brought onto each other (respecting orientations on edges) through some rotations and reflections in the plane.

Example 3.2. Figure 7 shows that the diagrams of the pairings (b) $\mathcal{D}(1, 2)$ on the torus, and (c) $\mathcal{D}(2, 2)$ on the Klein bottle are identical except for the orientation of the edges of the polygon, distinguishing the two surfaces. The same applies to the polygons in Figure 8, distinguished by edge orientations: (a) and (b) are equivalent pairings on the bitorus, with diagram $\mathcal{D}(2, 4)$; (c) and (d) are equivalent pairings on the 4-projective plane, with diagram $\mathcal{D}(4, 4)$.

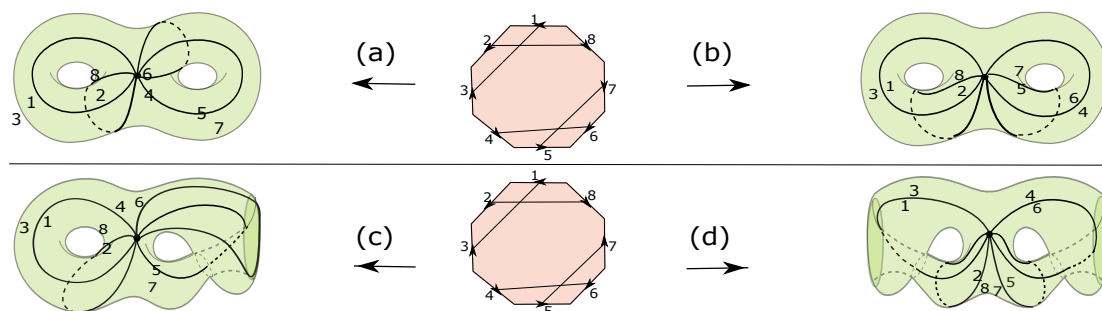


Figure 8. Examples of equivalent canonical pairings.

Definition 3.3. A graph $G(V, A)$ is said to be an edge-pairing graph if there exists an embedding $\iota : G \rightarrow M_g$ such that $q(\partial\mathcal{P}_{2A}) = \iota(G) = \mathcal{G}$, where $q : \mathcal{P}_{2A} \rightarrow M_g$ is some edge pairing.

Proposition 3.1. Let $\mathcal{G}(V, A)$ be a k -regular edge-pairing graph $q : \mathcal{P}_n \rightarrow M_g$, where n is even and necessarily $k \neq 2$, as the denominator $k - 2$ appears in every formula below. Then,

- a) $V = \frac{2(2g-1)}{k-2}$, $A = \frac{k(2g-1)}{k-2}$ and $n = \frac{k(4g-2)}{k-2}$ if $g > 0$ and $M_g = g\mathbb{T}^2$ (see [20]).
- b) $V = \frac{2(g-1)}{k-2}$, $A = \frac{k(g-1)}{k-2}$ and $n = \frac{k(2g-2)}{k-2}$ if $g > 1$ and $M_g = g\mathbb{P}^2$.

Proof. If $\mathcal{G}(V, A)$ is a k -regular pairing graph of the regular polygon $\mathcal{P}_n$ on M_g , where n is even, then $2A = kV$. Here, $F = 1$, as $\mathcal{G} = q(\partial\mathcal{P}_{2A})$ is the image of the boundary of the polygon, so its complement $M_g \setminus \mathcal{G}$ is exactly the image of the interior of the polygon (a single simply connected region). If M_g is oriented, then $2 - 2g = V - A + 1$. If M_g is nonoriented, then $2 - g = V - A + 1$. From these equalities, the above statements follow. $\square$

Corollary 3.1. Let $\mathcal{G}(V, A)$ be an edge-pairing graph on M_g .

- a) If $\mathcal{G}$ is 3-regular, then
- i) $V = 4g - 2$, $A = 6g - 3$, and $n = 12g - 6$ if $M_g = g\mathbb{T}^2$ ($g > 0$);
 - ii) $V = 2g - 2$, $A = 3g - 3$, and $n = 6g - 6$ if $M_g = g\mathbb{P}^2$ ($g > 1$ and even; see the remark following this corollary on the origin of this parity restriction).
- b) If $\mathcal{G}$ is 4-regular, then
- i) $V = 2g - 1$, $A = 4g - 2$, and $n = 8g - 4$ if $M_g = g\mathbb{T}^2$ ($g > 0$);
 - ii) $V = g - 1$, $A = 2g - 2$, and $n = 4g - 4$ if $M_g = g\mathbb{P}^2$ ($g > 1$ and even).

Remark. The parity restriction on g in the nonorientable case reflects the scope of the constructive method of Section 5.2, which builds such graphs from an even-genus, nonorientable seed by connected sum with orientable pieces (each such surgery adding an even increment $2g_1$ to the genus by Proposition 5.2) rather than a topological obstruction to k -regular pairing graphs on odd-genus, nonorientable surfaces; establishing or ruling out the odd-genus case is left open.

4. Canonical edge pairings

Let $\mathcal{G}(V, A)$ denote the graph associated with $q : \mathcal{P}_{2A} \rightarrow M_g$, the mapping of edge pairings. On M_g , it is possible to select a simply connected region B_q that contains all vertices of $\mathcal{G}$ such that B_q

$\cap \mathcal{G}$ remains connected, and the complement $M_g \setminus B_q$ contains $2g$ arcs for oriented M_g and g arcs for nonoriented M_g . This region B_q will be called the center of edge pairings.

Definition 4.1. An edge pairing $q : \mathcal{P}_{2A} \rightarrow M_g$ is called a canonical edge pairing if the graph $\mathcal{G} = q(\mathcal{P}_{2A})$ contains a unique vertex.

Definition 4.2. The canonical pairing center $C_q(g, A)$ associated to $q : \mathcal{P}_{2A} \rightarrow M_g$ is defined as follows: If the pairing center B_q has a unique vertex v with degree $2A = 4g$, then there exists a disk D centered on v with $4g$ radial segments dividing D into equal parts (see Figure 9), homeomorphic to B_q . The set D together with its $4g$ radial segments is referred to as the canonical pairing center associated with q .

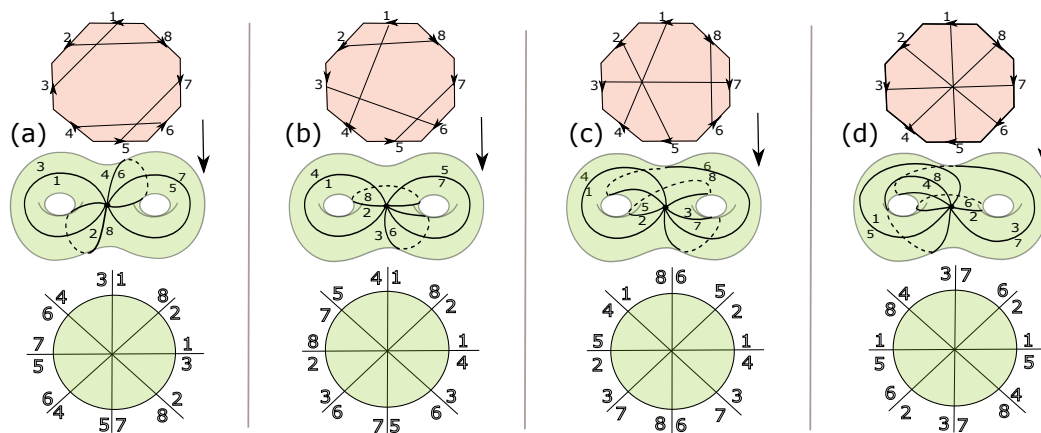


Figure 9. Canonical nonequivalent pairs in the bitorus $\mathbb{T}^2$.

Example 4.1. In Figure 7, the pairings of $\mathcal{P}_4$ in (b) $\mathcal{D}(1, 2)$ and (c) $\mathcal{D}(2, 2)$ are canonical, with the same diagram but distinguished by edge orientations. The pairings are noncanonical in (a) $\mathcal{D}(0, 2)$ and (d) $\mathcal{D}(1, 2)$, as they involve multiple vertices.

Figure 9 illustrates the four possible centers of canonical pairings on the bitorus, and Figure 10 illustrates four centers of canonical pairings on the 4-projective plane.

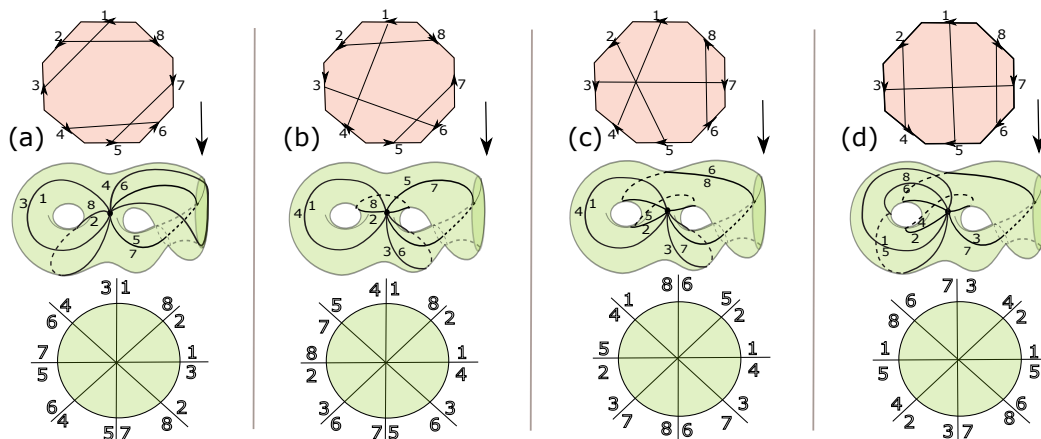


Figure 10. Example of canonical pairings on the 4-projective plane.

The canonical pairing center $C_q(g, A)$ can be represented by a sequence of numbers, denoted by $S(g, A)$, corresponding to the $2A$ sides of the polygon, labeled as $l_1, \dots, l_{2A}$. The numbers are separated by commas and semicolons as follows: A comma separating the pair (l_m, l_n) indicates that l_m and l_n are edges of $\mathcal{P}_{2A}$ identified by q ; dots separate two consecutive edges of $\mathcal{P}_{2A}$; that is, $l_i.l_{i+1}$ indicates that l_i and l_{i+1} are consecutive edges.

Remark. If $S(g, A)$ is a sequence associated with the canonical pairing center $C(g, A)$ of $\mathcal{P}_{2A}$ with edges $\{l_1, \dots, l_{2A}\}$, then the following hold for $S(g, A)$:

- Contains pairs (l_m, l_n) and (l_n, l_m) , $m, n \in \{1, \dots, 2A - 1\}$ (see Table 1).
- Always contains pairs of the type $l_i.l_{i+1}$ ($i = 1, \dots, 2A - 1$) if M_g is oriented (see Table 1).
- May contain pairs of the type $l_{i+1}.l_i$ if M_g is nonoriented (see Figure 10(d)).
- Does not contain the pair (l_{2A}, l_1) , which closes the cycle of the sequence.
- Does not contain pairs of the type (l_i, l_{i+1}) and (l_{2A}, l_1) .

Table 1. Canonical pairing sequences in the bitorus and the 4-projective space.

(a)	1.2, 8.1, 3.4, 6.7, 5.6, 4.5, 7.8, 2.3	(a)	1.2, 8.1, 3.4, 6.7, 5.6, 4.5, 7.8, 2.3
(b)	1.2, 8.1, 4.5, 7.8, 2.3, 6.7, 5.6, 3.4	(b)	1.2, 8.1, 4.5, 7.8, 2.3, 6.7, 5.6, 3.4
(c)	1.2, 5.6, 8.1, 4.5, 2.3, 7.8, 6.7, 3.4	(c)	1.2, 5.6, 8.1, 4.5, 2.3, 7.8, 6.7, 3.4
(d)	1.2, 6.7, 3.4, 8.1, 5.6, 2.3, 7.8, 4.5	(d)	1.2, 4.3, 7.6, 8.1, 5.4, 2.3, 7.8, 6.5

Definition 4.3. Two sequences $S_1(g, A)$ and $S_2(g, A)$ on a surface M_g are equivalent if there exists a bijective correspondence between pairs (l_m, l_n) in $S_1(g, A)$ and $S_2(g, A)$ for $m, n \in \{1, \dots, 2A - 1\}$, preserving the adjacency relation (in the sense of consecutive edges $l_i.l_{i+1}$ defined above) between pairs of the form $l_i.l_{i+1}$ ($i = 1, \dots, 2A - 1$) in both sequences.

Example 4.2. The centers of the canonical pairings of $\mathcal{P}_4$ in Figure 7 are $C(1, 2)$ in (b) and $C(2, 2)$ in (c). Both share the sequence 1.2, 4.1, 3.4, 2.3 but cannot be compared, as they lie on different surfaces.

Figure 11 illustrates the centers corresponding to possible diagrams of canonical pairings of $\mathcal{P}_8$ on the bitorus. Note that there are only four nonequivalent classes of these pairings, represented in Figure 9, with their sequences in Table 1 on the left.

Figure 10 illustrates four nonequivalent canonical pairings of $\mathcal{P}_8$ in the 4-projective plane, whose sequences are in Table 1 on the right. Note that only center (d) in Figure 9 differs between the bitorus and the 4-projective plane.

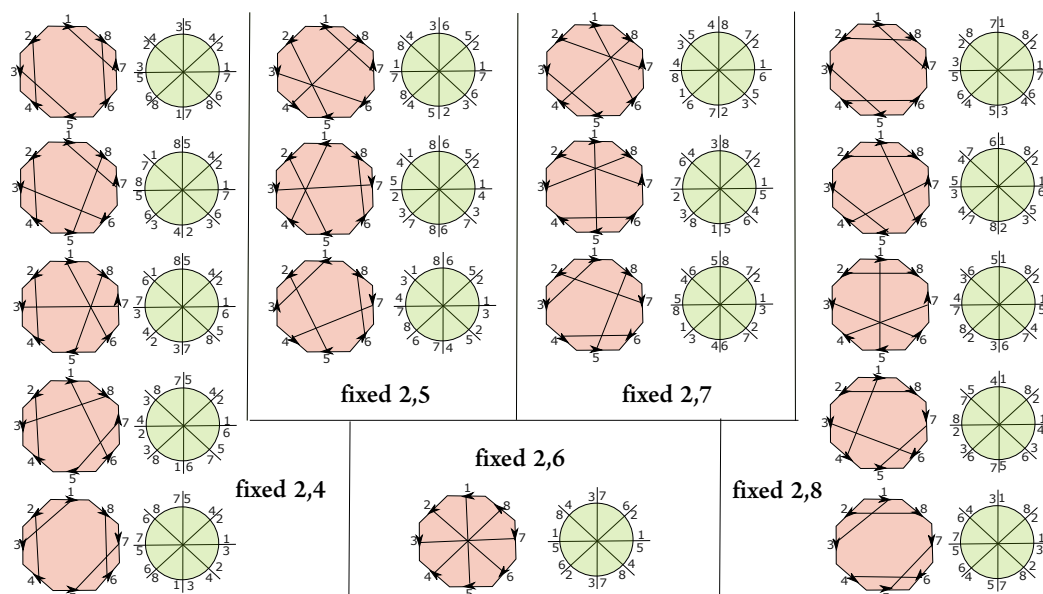
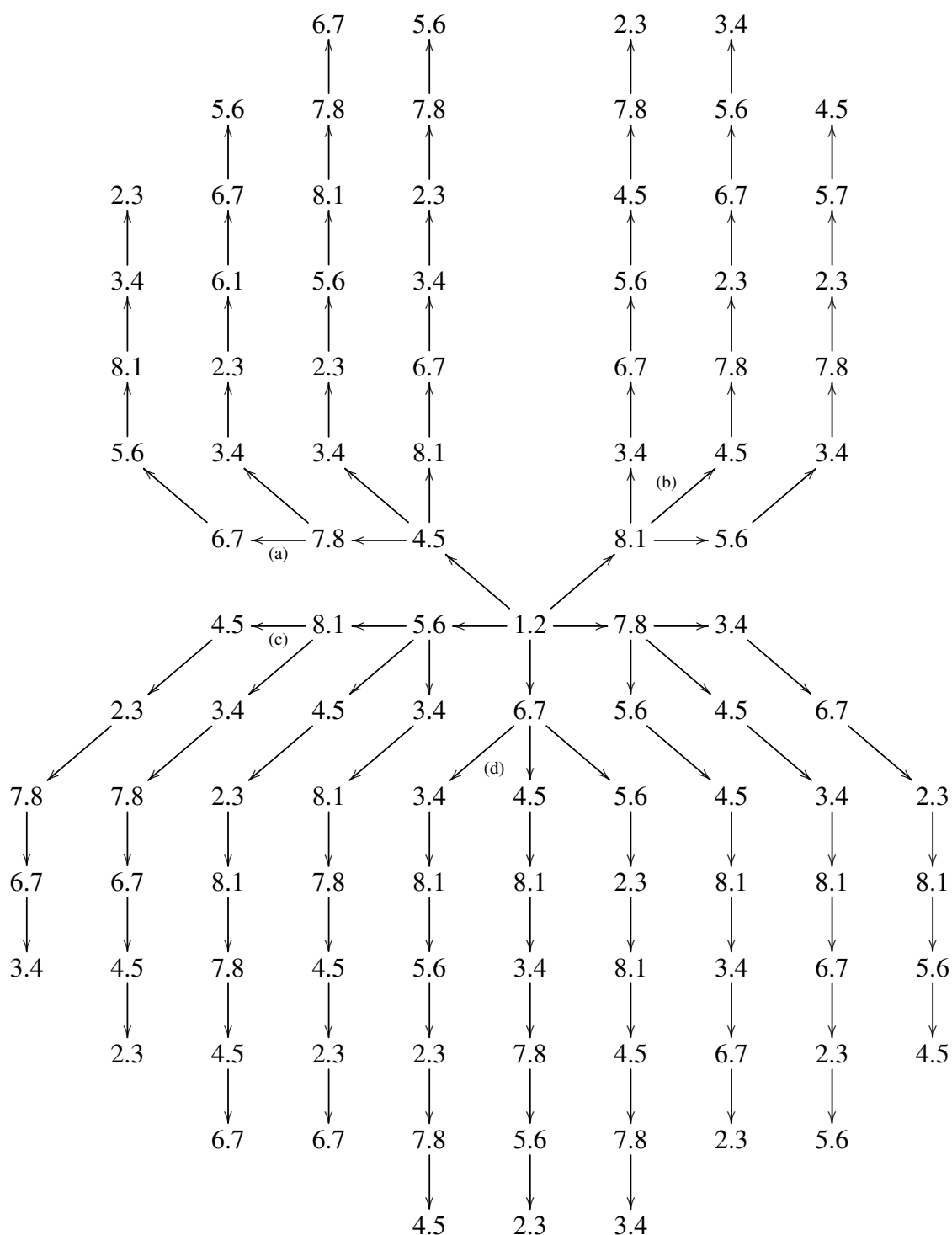


Figure 11. Possible diagrams of canonical pairings on the bitorus.

Proposition 4.1. *There exist four canonical matchings on the bitorus.*

Proof. Table 2 illustrates all possible sequences associated with the canonical pairings of $\mathcal{P}_8$, starting with sides 1.2 (fixing the pair 1.2 as the starting point is a normalizing convention: By Definition 4.3, sequences related by a cyclic rotation of the edge labels of the polygon are equivalent, so every sequence can be rotated to begin with the pair containing Edge 1 without loss of generality), corresponding to the pairing centers illustrated in Figure 11. Among these sequences, there are only four nonequivalent sequences, which are presented in Table 1.



9

(i) and (l), and (j) and (p). The sequences corresponding to these diagrams are found in Table 3. Let us remark that Entry (a) in the table corresponds to Diagram (a) in Figure 13.

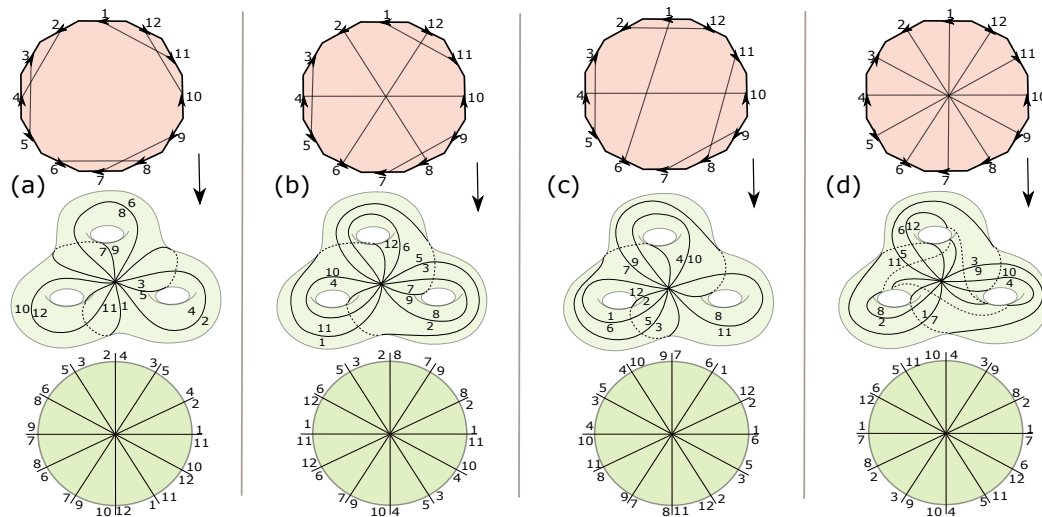


Figure 12. Examples of canonical pairings in the tritorus.

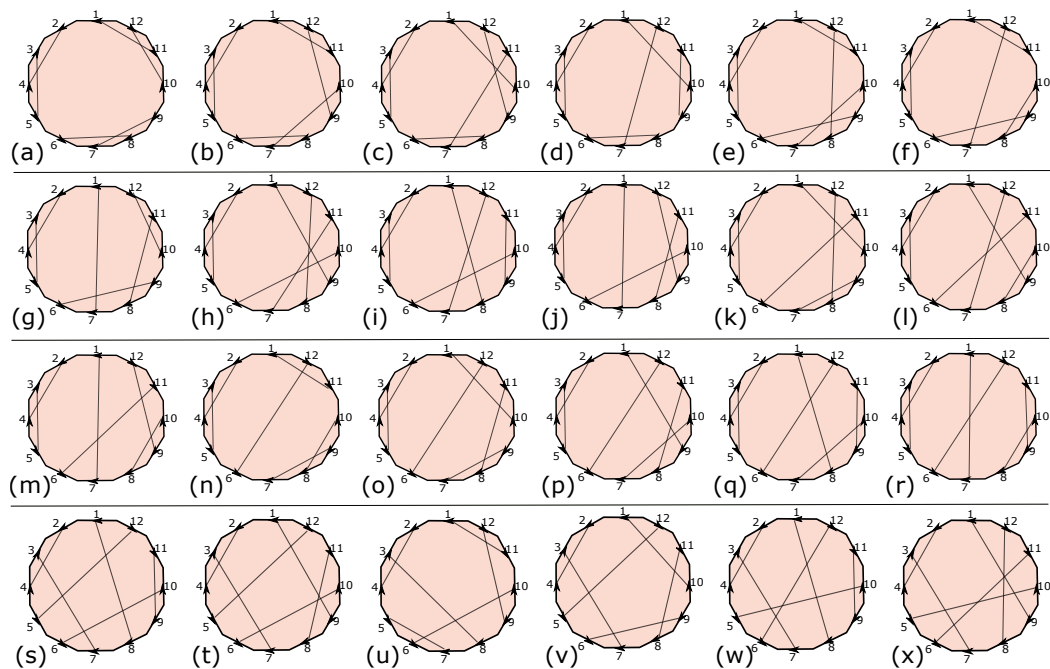


Figure 13. Examples of canonical pairings diagrams on tritorus.

Remark. Table 3 displays all the possible pairing sequences of $\mathcal{P}_{12}$ in the tritorus. Only four of them are nonequivalent. In the case of the pairings of $\mathcal{P}_{12}$ in the tritorus, with the graph $\mathcal{G}(1, 6)$, we do not intend to show the exact number of all possible canonical pairings here. From Figure 13 and Example 4.3, it can be asserted that the number of these pairings exceeds 19. It is possible to construct the sequences as in Table 3 or use computational resources to verify this without much manual effort.

Table 3. Canonical pairings sequences in $3\mathbb{T}^2$ fixed at 2, 4 and 5, 3.

(a)	1.2, 4.5, 3.4, 2.3, 5.6, 8.9, 7.8, 6.7, 9.10, 12.1, 11.12, 10.11, 1
(b)	1.2, 4.5, 3.4, 2.3, 5.6, 8.9, 12.1, 11.12, 9.10, 7.8, 6.7, 10.11, 1
(c)	1.2, 4.5, 3.4, 2.3, 5.6, 8.9, 12.1, 10.11, 7.8, 6.7, 11.12, 9.10, 1
(d)	1.2, 4.5, 3.4, 2.3, 5.6, 8.9, 11.12, 7.8, 6.7, 12.1, 10.11, 9.10, 1
(e)	1.2, 4.5, 3.4, 2.3, 5.6, 9.10, 7.8, 12.1, 11.12, 8.9, 6.7, 10.11, 1
(f)	1.2, 4.5, 3.4, 2.3, 5.6, 9.10, 8.9, 6.7, 12.1, 11.12, 7.8, 10.11, 1
(g)	1.2, 4.5, 3.4, 2.3, 5.6, 9.10, 12.1, 7.8, 11.12, 10.11, 8.9, 6.7, 1
(h)	1.2, 4.5, 3.4, 2.3, 5.6, 10.11, 7.8, 12.1, 9.10, 6.7, 11.12, 8.9, 1
(i)	1.2, 4.5, 3.4, 2.3, 5.6, 10.11, 9.10, 6.7, 12.1, 8.9, 11.12, 7.8, 1
(j)	1.2, 4.5, 3.4, 2.3, 5.6, 10.11, 8.9, 12.1, 7.8, 11.12, 9.10, 6.7, 1
(k)	1.2, 4.5, 3.4, 2.3, 5.6, 11.12, 8.9, 7.8, 12.1, 10.11, 6.7, 9.10, 1
(l)	1.2, 4.5, 3.4, 2.3, 5.6, 11.12, 7.8, 10.11, 6.7, 12.1, 9.10, 8.9, 1
(m)	1.2, 4.5, 3.4, 2.3, 5.6, 11.12, 9.10, 8.9, 12.1, 7.8, 10.11, 6.7, 1
(n)	1.2, 4.5, 3.4, 2.3, 5.6, 12.1, 11.12, 6.7, 9.10, 8.9, 7.8, 10.11, 1
(o)	1.2, 4.5, 3.4, 2.3, 5.6, 12.1, 10.11, 8.9, 7.8, 11.12, 6.7, 9.10, 1
(p)	1.2, 4.5, 3.4, 2.3, 5.6, 12.1, 9.10, 7.8, 11.12, 6.7, 10.11, 8.9, 1
(q)	1.2, 4.5, 3.4, 2.3, 5.6, 12.1, 8.9, 11.12, 6.7, 10.11, 9.10, 7.8, 1
(r)	1.2, 4.5, 3.4, 2.3, 5.6, 12.1, 7.8, 10.11, 9.10, 8.9, 11.12, 6.7, 1
(s)	1.2, 4.5, 12.1, 8.9, 11.12, 5.6, 10.11, 9.10, 6.7, 3.4, 2.3, 7.8, 1
(t)	1.2, 4.5, 12.1, 9.10, 6.7, 3.4, 2.3, 7.8, 11.12, 5.6, 10.11, 8.9, 1
(u)	1.2, 4.5, 7.8, 3.4, 2.3, 8.9, 12.1, 11.12, 9.10, 6.7, 5.6, 10.11, 1
(v)	1.2, 4.5, 12.1, 10.11, 8.9, 6.7, 3.4, 2.3, 7.8, 11.12, 5.6, 9.10, 1
(w)	1.2, 4.5, 10.11, 9.10, 5.6, 12.1, 8.9, 11.12, 6.7, 3.4, 2.3, 7.8, 1
(x)	1.2, 4.5, 10.11, 6.7, 3.4, 2.3, 7.8, 12.1, 9.10, 5.6, 11.12, 8.9, 1

5. Construction of edge pairings

Jorgensen and Naatanen [18] determined the five possible trivalent graphs of type $\mathcal{G}(6, 9)$, associated with the eight nonequivalent pairings of edges on the bitorus, from the polygon $\mathcal{P}_{18}$. The graphs and edge-pairing diagrams are illustrated in Figure 16.

Nakamura [19] showed that there exist 65 graphs of trivalent edge pairings over the tritorus. In [20], it was proven that all these graphs can be obtained through the extension of canonical pairings on the tritorus. Moreover, graph extension, edge swapping, and surgeries between pairings are important tools for determining families of graphs associated with pairings of $\mathcal{P}_{2A}$ for $A > 3$.

5.1. Extension, contraction, and edge swap

The extension, contraction, and exchange of edges in graphs on surfaces were introduced in [20, 21] for the case of edge-pairing graphs and generalized for arbitrary graphs on closed surfaces in Section 2 (see Definitions 2.1 and 2.2). These techniques assist in the determination of new edge pairs on a surface M_g , starting from other already known pairs, such as the canonical pairs.

In [20], it was proved that every graph obtained by the extension or contraction of a graph with edge

pairs on a closed, oriented surface is also a graph with edge pairs. This result can be extended to the case of closed surfaces as an immediate consequence of Proposition 2.2 and Proposition 2.3.

Proposition 5.1. *Let $(M_g, \mathcal{G}(V, A))$ be a pair associated with some edge pairings of a polygon $\mathcal{P}_{2A}$. If $\mathcal{G}_1(V_1, A_1)$ is a graph obtained from $\mathcal{G}(V, A)$ by extensions, contractions, or edge swaps, then $\mathcal{G}_1(V_1, A_1)$ is a pairing graph of a regular polygon $\mathcal{P}_{2A_1}$ on M_g .*

Proof. By Definition 3.3, $\mathcal{G}(V, A)$ is the edge-pairing graph of a regular $2A$ -gon on M_g if and only if $\mathcal{B}(\mathcal{G})$ is a single simply connected region, as q is injective on the interior of $\mathcal{P}_{2A}$, and $M_g \setminus \mathcal{G} = q(\text{int } \mathcal{P}_{2A})$ is then exactly the image of an open disc. By hypothesis, $\mathcal{B}(\mathcal{G})$ is simply connected. If $\mathcal{G}_1$ is obtained from $\mathcal{G}$ by extension, contraction, or edge swaps, Propositions 2.2 and 2.3 show that $\mathcal{B}(\mathcal{G})$ and $\mathcal{B}(\mathcal{G}_1)$ are homeomorphic; hence, $\mathcal{B}(\mathcal{G}_1)$ is also simply connected, and therefore, $\mathcal{G}_1(V_1, A_1)$ is the edge-pairing graph of some regular polygon $\mathcal{P}_{2A_1}$ on M_g , where $2A_1$ is the edge count of $\mathcal{G}_1$. $\square$

Example 5.1. Figure 14 illustrates three examples of graph extensions on the bitorus, with the three possible 5-regular graphs of type $\mathcal{G}(2, 5)$. Corresponding to these graphs are the twelve diagrams of 5-regular pairings of the polygon $\mathcal{P}_{10}$ on the bitorus, as illustrated in Figure 15. Diagram A1 corresponds to graph (a) (each vertex has two loops and one edge); diagrams B1, C1, D1, D2, D3 and A3 correspond to graph (b) (each vertex has five edges); and diagrams A2, B2, B3, C2 and C3 correspond to graph (c) (each vertex has one loop and three edges).

These diagrams are obtained by extending the graphs of the four canonical pairings shown in Figure 9. It should be noted that the diagrams in each set $\{D1, D2, D3, C1, A3, A2, B2\}$ and $\{C2, B3, C3\}$ are equivalent, resulting in at least six equivalence classes of these pairings.

Figure 16 illustrates the eight diagrams of pairings of the polygon $\mathcal{P}_{18}$, along with the centers of edge pairings associated with the five graphs of type $\mathcal{G}(6, 9)$ on the bitorus [18]. Note that the unordered numbering in the polygon corresponds to the new edges obtained from the extensions of the 5-regular graphs illustrated in Figure 15.

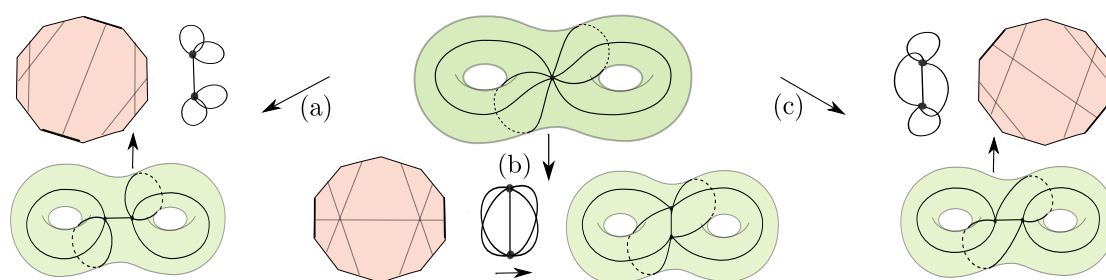


Figure 14. Examples of extension on the bitorus.

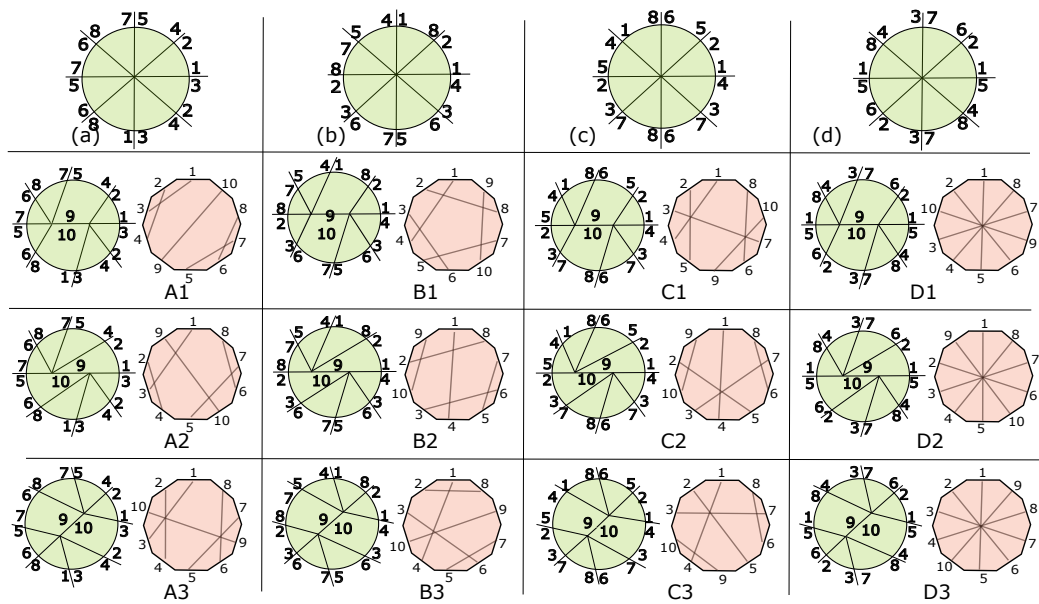


Figure 15. Centers and pairings diagrams of $\mathcal{P}_{10}$ on the bitorus.

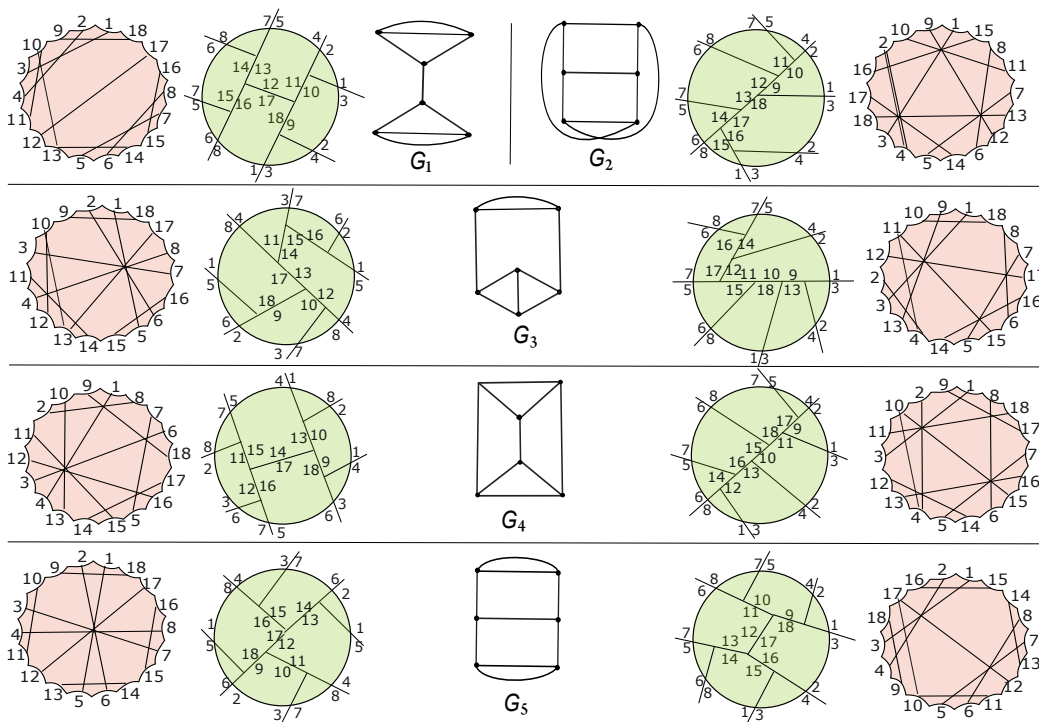


Figure 16. 3-regular pairing graphs of $\mathcal{P}_{18}$ determined in [18].

5.2. Pairing surgeries

In [16], two trivalent pairing surgeries, S_1 and S_2 , were introduced in Section 2.4, which add vertices of degree 3 to generate families of trivalent pairings from previously known ones. In [21], the surgery S_3 , which adds vertices of degree 4, was introduced, allowing the creation of families of tetravalent pairings from previously known ones. The surgery S_3 followed by extensions of the vertex w generates

the surgeries S_1 and S_2 , as illustrated in Figure 4. The results obtained in [21] can be extended to the general case of closed surfaces as an immediate consequence of Proposition 2.4.

Proposition 5.2. *Let $(M_{g_1}, \mathcal{G}_1(V_1, A_1))$ and $(M_{g_2}, \mathcal{G}_2(V_2, A_2))$ be pairings of edges of the respective polygons $\mathcal{P}_{2A_1}$ and $\mathcal{P}_{2A_2}$. Then, $(M_g, \mathcal{G}_1 \oplus_v \mathcal{G}_2)$ is a pair associated with some pairing of edges of the polygon $\mathcal{P}_{2(A_1+A_2+3)}$ for $v = 1, 2$ or the polygon $\mathcal{P}_{2(A_1+A_2+2)}$ for $v = 3$, where $g = g_1 + g_2$ if M_{g_1} and M_{g_2} are both oriented or both nonoriented, and $g = 2g_1 + g_2$ if only M_{g_1} is oriented.*

Corollary 5.1. *If $\mathcal{G}_1(V_1, A_1)$ and $\mathcal{G}_2(V_2, A_2)$ are graphs of pairings of edges, then the possible graphs resulting from surgeries S_v ($v = 1, 2, 3$) between the two graphs are also graphs of pairings of edges. Furthermore, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are trivalent graphs, then $\mathcal{G}_1 \oplus_{S_v} \mathcal{G}_2$ for $v = 1, 2$ is also a trivalent graph; moreover, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are quadrivalent graphs, then $\mathcal{G}_1 \oplus_{S_3} \mathcal{G}_2$ is also a quadrivalent graph.*

Example 5.2. Figure 5 illustrates the connected sum of two pairing graphs on the torus, resulting in pairing graphs on the bitorus. The trivalent graphs of type $\mathcal{G}(6, 9)$ associated with the pairings of $\mathcal{P}_{18}$ result from surgeries S_1 and S_2 between the two pairings of $\mathcal{P}_6$ with graphs $\mathcal{G}(2, 3)$. The quadrivalent graph of type $\mathcal{G}(3, 6)$ associated with the pairings of $\mathcal{P}_{12}$ results from surgery S_3 between two pairings of $\mathcal{P}_4$ with graphs $\mathcal{G}(1, 2)$.

Similarly, Figure 6 illustrates the connected sum of two pairing graphs, one on the torus and the other on the Klein bottle, resulting in pairing graphs on the 4-projective plane. With surgeries S_1 and S_2 between the two pairings of $\mathcal{P}_6$ with graphs $\mathcal{G}(2, 3)$ (on the torus and on the Klein bottle), we obtain pairings of $\mathcal{P}_{18}$ with graphs of type $\mathcal{G}(6, 9)$. Surgery S_3 between two pairings of $\mathcal{P}_4$ with graphs $\mathcal{G}(1, 2)$ generates the pairing of $\mathcal{P}_{12}$ with the quadrivalent graph type $\mathcal{G}(3, 6)$.

Figure 17 illustrates three examples of trivalent graphs obtained by connected sums of the graph $\mathcal{G}(2, 3)$: in (a) and (b) by three surgeries S_1 and in (c) by three surgeries S_2 . Various other families can be obtained by just one of the surgeries.

Figures 18(a) and 18(b) illustrate two examples of quadrivalent pairings on the bitorus associated with the graph $\mathcal{G}(3, 6)$, where each vertex is connected by four different edges. These pairings can be obtained by edge swaps of the pairing obtained by S_3 in Figure 5. In (c), the pairing can be obtained by surgery S_3 between two type (a) pairings.

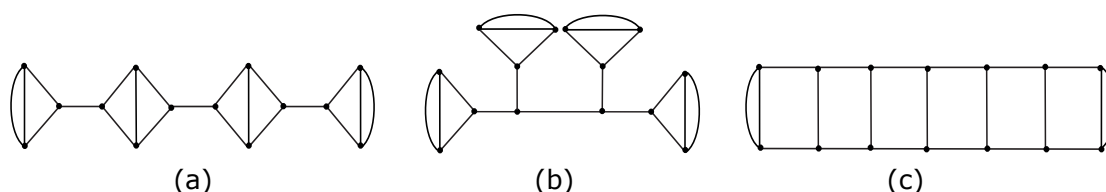


Figure 17. Examples of surgeries S_1 and S_2 among four graphs $\mathcal{G}(2, 3)$.

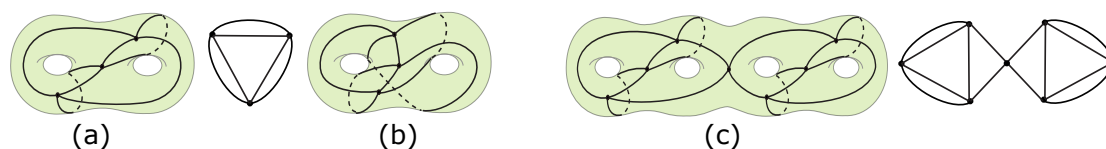


Figure 18. Examples of surgeries S_3 between two tori.

Figure 19 illustrates examples of surgeries S_1 and S_2 of the graph $G(2, 3)$ with the graphs $G_1(6, 9)$ and $G_5(6, 9)$ from Figure 16. Note that the pairs of graphs (a)-1 and (b)-1, (a)-4 and (a)-6, and (a)-3 and

(b)-5 are isomorphic. Furthermore, any other surgeries involving other edges of $G_1(6, 9)$ and $G_5(6, 9)$ will result in a graph isomorphic to one of these graphs.

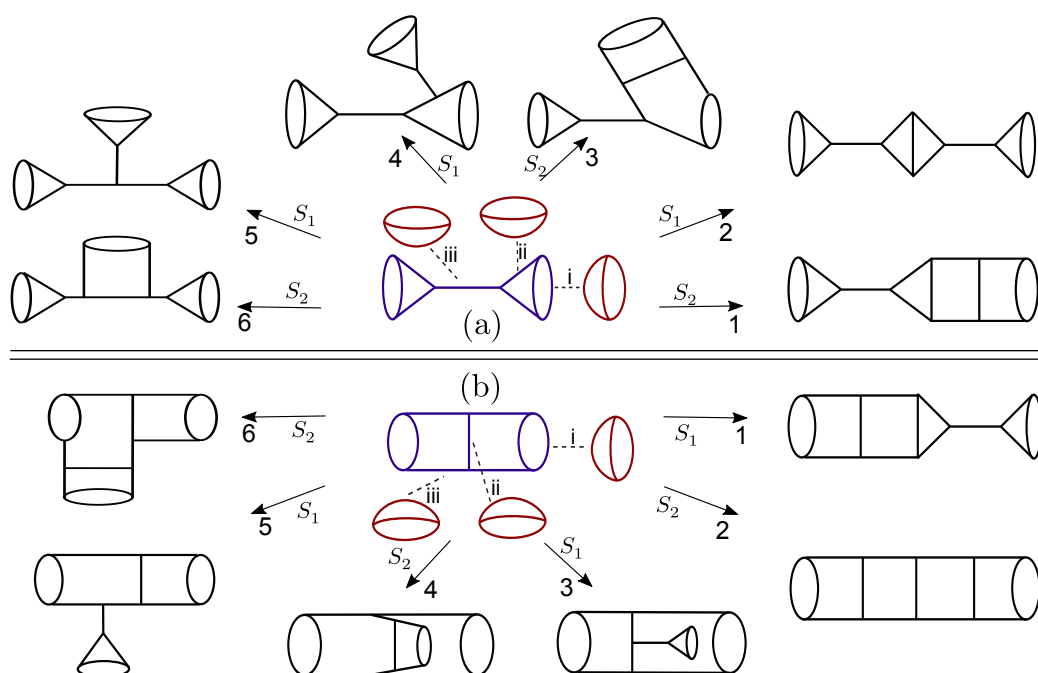


Figure 19. Possible surgeries between $\mathcal{G}_1(6, 9)$ and $\mathcal{G}_1(2, 3)$.

For surgeries between $G(2, 3)$ and graphs like $G_2(6, 9)$, $G_3(6, 9)$, and $G_4(6, 9)$ from Figure 16, there are four edges that can result in nonisomorphic graphs. By the same principle applied explicitly to $G_1(6, 9)$ and $G_5(6, 9)$ via Figure 19, the three edges of $G(2, 3)$ combine with the automorphism-inequivalent edges of the larger graph to produce all nonisomorphic outcomes, each of $G_2(6, 9)$, $G_3(6, 9)$, and $G_4(6, 9)$ has four edges out of the overall nine it possesses that are pairwise nonequivalent under its automorphism group. No analog of Figure 19 is included for these three graphs, so this count should be read as a structural upper bound rather than an exhaustively illustrated one. Therefore, an upper bound on the number of nonisomorphic trivalent graphs of type $\mathcal{G}(10, 15)$ that can be obtained by these two surgeries between $G(2, 3)$ and the five graphs of type $\mathcal{G}(6, 9)$ is $2 \times 3 \times 4 + 2 \times 2 \times 3 = 24 + 12 = 36$ (the first term: two surgery types $\times$ three graphs $G_2, G_3, G_4 \times$ four nonequivalent edges per graph; the second term: two surgery types $\times$ two graphs $G_1, G_5 \times$ three nonequivalent edges per graph, that is, the edges of $G(2, 3)$ itself, per Figure 19). Swapping edges on the tritorus allows for obtaining the other nonisomorphic graphs presented by Nakamura in [19].

Let $(M_{g_1} \# M_{g_2}, \mathcal{G}_1 \oplus \mathcal{G}_2)$, $(\nu = 1, 2, 3)$ be the pair resulting from the surgery between two pairings associated with $(M_{g_1}, \mathcal{G}_1(V_1, A_1))$ and $(M_{g_2}, \mathcal{G}_2(V_2, A_2))$ (see the example in Figure 20). In the connected sum of surfaces with graphs, the addition of diagrams naturally involves the sum of diagrams; the edges of the polygon sides involved in the surgery can decompose into two or three, depending on the surgery. Diagrams will not receive as much attention here, as the primary focus lies on the graphs.

Example 5.3. Consider $(\mathbb{T}^2 \# \mathbb{T}^2, \mathcal{G}_1 \oplus \mathcal{G}_2)$, $(\nu = 1, 2, 3)$, the resulting pair from surgeries between two tori.

- i) The sum of two pairings diagrams of $\mathcal{P}_6$ on the torus, resulting in $(2\mathbb{T}^2, \mathcal{G}_1(2, 3) \oplus_{\nu} \mathcal{G}_2(2, 3))$ ($\nu = 1, 2$) (see Figure 20), can occur in one of two forms as described below. Denote the edges $1, \dots, 6$ as the sides of $\mathcal{P}_6$ corresponding to graph $\mathcal{G}_1$ and edges $7, \dots, 12$ corresponding to $\mathcal{G}_2$. We choose those that are identified on the torus, where the surgery occurs, as $(1, 4)$ and $(7, 10)$. Then, the surgery is as follows:

S_1 : A vertex is added on each edge of pairs $(1, 4)$ and $(7, 10)$, edge $1 \rightarrow 14.1'$, which is identified with the pair at $4 \rightarrow 4'.13$, and the pair obtained at $7 \rightarrow 7'.16$ is identified with the pair at $10 \rightarrow 10'.15$ such that the sequence of edges in $\mathcal{P}_{18}$, clockwise, is $1'.2.3.4'.13.5.6.14.17.7'.8.9.10'.15.11.12.16.18$, where 17 is identified with 18.

S_2 : Two vertices are added on each edge of pairs $(1, 4)$ and $(7, 10)$, edge $1 \rightarrow 14.17'.1'$, which is identified with the trio at $4 \rightarrow 4'.17.13$, and the trio obtained at $7 \rightarrow 16.18'.7'$ is identified with the trio at $10 \rightarrow 10'.18.15$ such that the sequence of edges in $\mathcal{P}_{18}$, clockwise, is $1'.2.3.4'.17.13.5.6.14.7'.8.9.10'.18.15.11.12.16$.

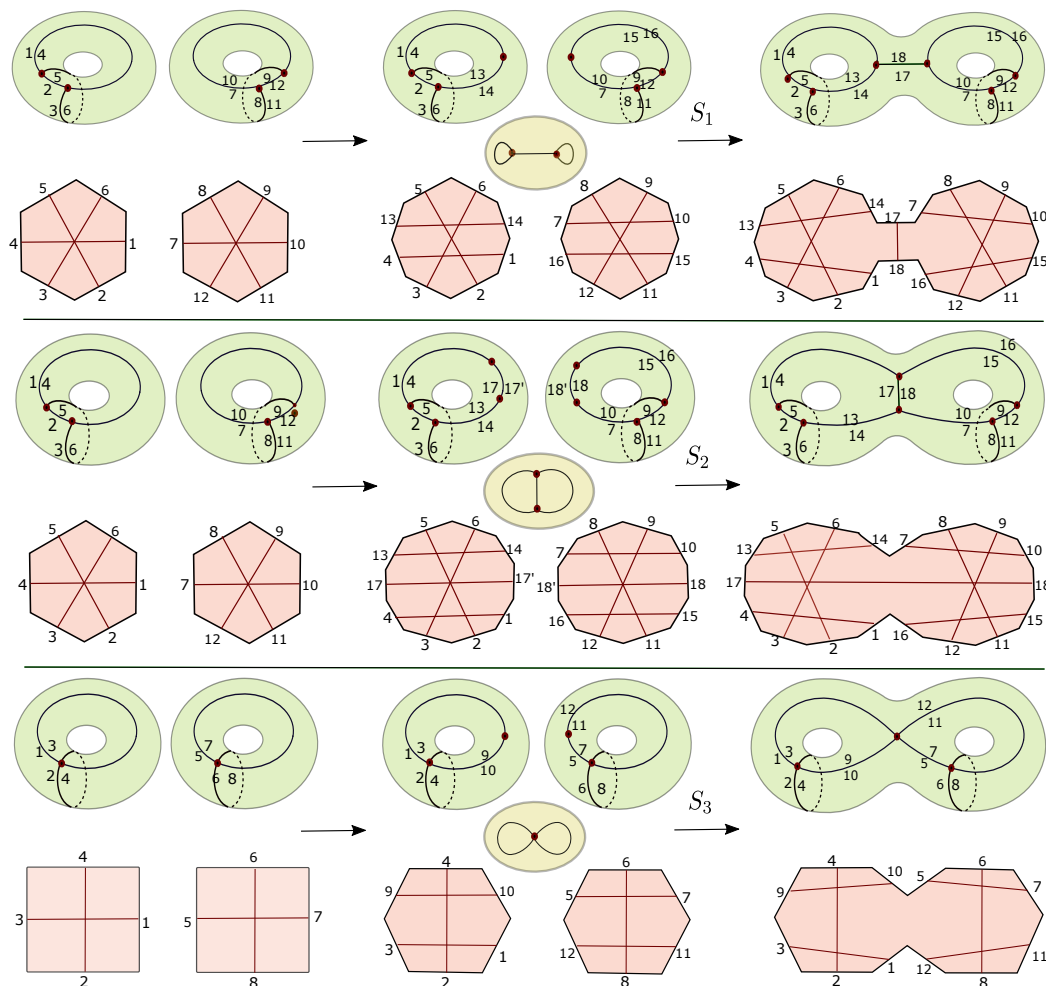


Figure 20. Examples of surgeries between diagrams of two tori.

- ii) Now, we observe the sum of two pairing diagrams of $\mathcal{P}_4$ on the torus, resulting in $(2\mathbb{T}^2, \mathcal{G}_1(1, 2) \oplus_{S_3} \mathcal{G}_2(1, 2))$. Denote the edges $1, \dots, 4$ as the sides of $\mathcal{P}_4$ corresponding to graph $\mathcal{G}_1$ and edges

5, $\dots$, 8 corresponding to $\mathcal{G}_2$. We choose those that are identified on the torus, where the surgery occurs, as (1, 3) and (5, 7). The surgery is as follows:

S_3 : A vertex is added on each edge of pairs (1,3) and (5,7), edge $1 \rightarrow 10.1'$, which is identified with the pair at $3 \rightarrow 3'.9$, and the pair obtained at $5 \rightarrow 12.5'$ is identified with the pair at $7 \rightarrow 7'.11$ such that the sequence of edges in $\mathcal{P}_{12}$, clockwise, is $1'.2.3'.9.4.10.5'.6.7'.11.8.12$.

Remark. *The number of pairing graphs obtained through surgeries can be large. However, there are still numerous graphs that cannot be generated by such methods. Edge swaps may produce pairing graphs from those obtained via surgeries, which will correspond to graphs that can be obtained through extension.*

Determining the number of pairing graphs for a given polygon reduces to counting isomorphism classes of such graphs. The exchange of edges on graphs on a surface allows us to obtain new graphs, with a fixed number of vertices and edges, from a given graph. Extending and contracting graphs on a surface allows one to determine the possible pairing graphs starting from the canonical pairing (see [20]). However, this can require exhaustive work, depending on the size of g . Surgeries can significantly contribute, especially when aiming to obtain pairings for surfaces with large genus g . In such cases, combining surgeries with edge exchanges is quite efficient for determining new pairings. The following results are consequences of Propositions 2.2, 2.3, and 5.2.

Theorem 5.1. *If $\mathcal{G}$ is a graph obtained from other graphs by a combination of edge exchanges, extensions, contractions of graphs on surfaces, and surgeries between pairing graphs, then $\mathcal{G}$ is a pairing graph.*

Proposition 5.3. *Let $\mathcal{G}_1(V_1, A_1)$ and $\mathcal{G}_2(V_2, A_2)$ be pairing graphs on the respective closed surfaces M_{g_1} and M_{g_2} . Then,*

- 1) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are trivalent graphs, the resulting graph from $\mathcal{G}_1 \oplus_{S_3} \mathcal{G}_2$ by extending the vertex added by the 3-sum connective S_3 is a trivalent pairing graph of the polygon $\mathcal{P}_{2(A_1+A_2+3)}$.
- 2) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are quadrivalent graphs, the resulting graph from $\mathcal{G}_1 \oplus_{S_\nu} \mathcal{G}_2$ by contracting the two vertices added by the ν -sum connective S_ν , for $\nu = 1, 2$ is a quadrivalent pairing graph of the polygon $\mathcal{P}_{2(A_1+A_2+2)}$.

Theorem 5.2. *Let $\{(M_{g_i}, \mathcal{G}_i(V_i, A_i))\}_{i=1}^k$ denote a collection of edge pairings of respective regular polygons $\{\mathcal{P}_{n_i}\}_{i=1}^k$. Every pair $(M_g, \mathcal{G}(V, A))$ resulting from the combination of $k - 1$ connected sums of type S_ν ($\nu = 1, 2, 3$) between these pairings of edges is also an edge pairing on some polygon $\mathcal{P}_n$, where*

- 1) $g = 2 \sum_{i=1}^j g_i + \sum_{i=j+1}^k g_i$ if only the first j surfaces are orientable;
- 2) $n = \sum_{i=1}^k n_i + 4j + 6(k - 1 - j)$ if only the first j of the $k - 1$ surgeries are of type $\nu = 3$;
- 3) $V = \sum_{i=1}^k V_i + j + 2(k - 1 - j)$ if only the first j of the $k - 1$ surgeries are of type $\nu = 3$;
- 4) $A = \sum_{i=1}^k A_i + 2j + 3(k - 1 - j)$ if only the first j of the $k - 1$ surgeries are of type $\nu = 3$.

6. Greedy pairing algorithm (GPA) for canonical pairing

The results of Sections 3–5 were stated for their own topological interest, but they also directly answer the correctness question that any greedy pairing algorithm (GPA) for reconstructing an object from regular-polygon-like fragments must face.

In a GPA-based reconstruction pipeline, each detected fragment is treated as one side of a polygon $\mathcal{P}_n$, and the algorithm builds a pairing $q : \mathcal{P}_n \rightarrow M_g$ incrementally: Starting from an initial matching, naturally taken to be a canonical pairing $C_q(g, A)$, the single-vertex pairing graph of Definition 4.1 with a unique vertex of degree $2A$, in the sense of Section 4, as it is the simplest edge-pairing graph with which to seed the procedure, the algorithm then greedily refines the matching by locally modifying the current pairing graph, accepting a modification whenever it improves some compatibility score between the reassembled edges. Read topologically, every such local modification is one of the four operations studied in this paper: an edge exchange (Section 2.3) when two existing pairs are recombined, an extension or contraction (Section 2.2) when a vertex of the working graph is split or merged, or a surgery S_1 , S_2 , or S_3 (Section 2.4) when two separately paired components of the reconstruction are joined.

6.1. The greedy pairing algorithm

We now state the greedy pairing algorithm (GPA) formally, in the notation of Sections 2–5.2. The input is a set of n fragments, represented as the n edges of a polygon $\mathcal{P}_n$ together with a compatibility function w that assigns a nonnegative real score to every pair of edges, and the output is an edge-pairing graph $\mathcal{G}(V, A)$ on a closed surface M_g .

The graph $\mathcal{G}$ returned by Algorithm 1 is the complete, autonomously produced pairing. Only once the algorithm has terminated is $\mathcal{G}$ presented to a human operator for inspection, correction, and refinement, a step required in practice in artifact-restoration workflows, where a specialist validates the reconstruction before it is accepted. Proposition 6.1 guarantees that $\mathcal{G}$, and every graph that preceded it during the run, is a valid edge-pairing graph regardless of any such subsequent human adjustment.

Every operation the algorithm is allowed to perform, listed in the set $\mathcal{O}$, is precisely one of the four local operations studied in Sections 2.2–2.4 of this paper: extension; contraction; edge exchange; and the three families of surgery S_1 , S_2 , S_3 . The algorithm never modifies $\mathcal{G}$ through any other means. The surgeries S_1, S_2, S_3 used here, are the graph-level operations of Definition 2.3 applied to two components of the single working graph $\mathcal{G}$ that already coexist within the fixed surface M_{g_0} (as opposed to the surface-level connected sum of Definition 2.4, which combines two *different* surfaces $M_{g_1} \neq M_{g_2}$ and is used instead in the classification results of §5.2). Consequently, none of the four operations in $\mathcal{O}$ alters the ambient surface: By Proposition 2.2, extension and contraction preserve the ambient surface M_{g_0} fixed at Step 1, and S_1, S_2, S_3 in this graph-level sense leave M_{g_0} unchanged by construction. Definition 2.3 joins $\mathcal{G}_1$ and $\mathcal{G}_2$ using a segment or point l already lying in the common surface M_g without removing any disc from, or gluing anything to, the surface itself (unlike Definition 2.4, whose connected sum $M_{g_1} \# M_{g_2}$ genuinely enlarges the surface and is reserved for combining pairings originally defined on *different* surfaces). Every $\mathcal{G}_i$ produced by the algorithm therefore remains an edge-pairing graph on the same surface M_{g_0} with which it was initialized, growing from the n_0 edges of the canonical seed (the canonical pairing $C_q(g_0, A_0)$ of Step 1, which we call the canonical seed) toward the n edges of the input polygon $\mathcal{P}_n$. This invariance is what allows the topological correctness of every intermediate and final output of Algorithm 1 to be established as a direct consequence of the results already proved in this paper rather than as a separate claim requiring independent verification.

Algorithm 1 Greedy pairing algorithm (GPA)

Require: Polygon $\mathcal{P}_n$ with edges $\{l_1, \dots, l_n\}$; compatibility function w ; an initial subpolygon size $n_0 \leq n$ such that a canonical pairing $C_q(g_0, A_0)$ exists with $2A_0 = n_0$ on the ambient surface M_{g_0} fixed for the reconstruction; local operation set $\mathcal{O} = \{S_1, S_2, S_3, \text{extension, contraction, edge exchange}\}$, subject to the invariant that whenever extension is selected as o , it is applied together with its inverse contraction (Proposition 2.2) within the same iteration of the while-loop below before the next edge pair is selected; extension and contraction are thus used only as transient internal restructuring moves and never persist, unreversed, between successive observed graphs $\mathcal{G}_{t-1}$ and $\mathcal{G}_t$

Ensure: An edge-pairing graph $\mathcal{G}(V, A)$ on the surface M_{g_0} made available on termination for inspection and adjustment by a human operator

- 1: $\mathcal{G} \leftarrow$ the canonical pairing $C_q(g_0, A_0)$ of a subpolygon $\mathcal{P}_{n_0} \subseteq \mathcal{P}_n$, as in Definition of Section 4
- 2: **while** there exists an unpaired edge of $\mathcal{P}_n$ or an admissible local improvement remains **do**
- 3: Select the operation $o \in \mathcal{O}$ and edge pair (l_i, l_j) that maximize the increase in w
- 4: Apply o to $\mathcal{G}$, obtaining a new graph $\mathcal{G}'$
- 5: **if** $\mathcal{G}'$ improves the total compatibility score w **then**
- 6: $\mathcal{G} \leftarrow \mathcal{G}'$
- 7: **end if**
- 8: **end while**
- 9: **return** $\mathcal{G}$

6.2. Correctness of the algorithm

Proposition 6.1 (Topological correctness of GPA). *Let $\mathcal{G}_0$ be the canonical pairing $C_q(g_0, A_0)$ of the subpolygon $\mathcal{P}_{n_0}$ with which GPA is initialized, and let $\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_\kappa$ denote the sequence of graphs produced by the algorithm, where $\mathcal{G}_t$ is obtained from $\mathcal{G}_{t-1}$ by one operation in $\mathcal{O}$. Then, for every $t = 0, 1, \dots, \kappa$, the graph $\mathcal{G}_t$ is an edge-pairing graph of some polygon $\mathcal{P}_{n_t}$ ($n_0 \leq n_t \leq n$) on the fixed surface M_{g_0} (not merely on some closed surface M_{g_t}). We write κ for the total number of iterations of Algorithm 1, reserving k for the number of initial pairings combined in Theorem 5.2.*

Proof. The proof proceeds by induction on t . The base case, $t = 0$, holds because $\mathcal{G}_0$ is, by construction, the canonical pairing $C_q(g_0, A_0)$ of $\mathcal{P}_{n_0}$ in the sense of Section 4, and every canonical pairing is an edge-pairing graph.

For the inductive step, suppose that $\mathcal{G}_{t-1}$ is an edge-pairing graph on the surface M_{g_0} (inductive hypothesis). By construction of GPA, the graph $\mathcal{G}_t$ is obtained from $\mathcal{G}_{t-1}$ by applying a single operation $o \in \mathcal{O}$, that is, an edge exchange, an extension, a contraction, or a surgery of type S_1, S_2 , or S_3 . If o is an extension or contraction, Proposition 2.2 shows the ambient surface is unchanged. If o is an edge exchange, Proposition 2.3 likewise leaves the surface unchanged. If o is a surgery S_1, S_2 , or S_3 , it is applied in the graph-level sense of Definition 2.3 (joining two components that already coexist within the single surface M_{g_0} via a segment or point already lying in M_{g_0} without removing a disc from or gluing anything to the surface), so the ambient surface is unchanged in this case, too. Each of these operations, applied to a graph that is itself an edge-pairing graph, again produces an edge-pairing graph: This is precisely the content of Theorem 5.1, which states that any graph obtained from edge-pairing graphs by a combination of edge exchanges, extensions, contractions of graphs on surfaces,

and surgeries between pairing graphs is itself a pairing graph. Applying this result with the pairing graph $\mathcal{G}_{t-1}$, it follows that $\mathcal{G}_t$ is an edge-pairing graph on M_{g_0} .

By induction, $\mathcal{G}_t$ is an edge-pairing graph on M_{g_0} for every $t = 0, 1, \dots, \kappa$, which proves the topological part of the proposition.

It remains to establish $n_0 \leq n_t \leq n$. By the invariant imposed on Algorithm 1, extension is always paired with its inverse contraction within the same iteration, so the net change in A between the observed graphs $\mathcal{G}_{t-1}$ and $\mathcal{G}_t$ is due only to edge exchange (which leaves A unchanged, by Proposition 2.3) or to a surgery S_1 , S_2 , or S_3 (which increases A by 3 or 2, respectively, by Remark of §2.4). In either case $A_t \geq A_{t-1}$, so by induction, $A_t \geq A_0$ for every t , giving $n_t = 2A_t \geq 2A_0 = n_0$. For the upper bound, every surgery performed by Algorithm 1 consumes edges exclusively from the finite pool $\{l_1, \dots, l_n\}$ of input fragments (Step 3 selects (l_i, l_j) from this pool, and the while-loop condition halts once no unpaired edge of $\mathcal{P}_n$ remains), so $\mathcal{G}_t$ never identifies more than n polygon sides in total, giving $n_t \leq n$. $\square$

Proposition 6.1 does not merely certify that the final output $\mathcal{G}_\kappa$ of GPA is a valid edge-pairing graph; it certifies that *every intermediate graph* produced during the execution of the algorithm has this property. This distinction is significant in a reconstruction setting, where fragments are typically supplied incrementally (as further pieces are excavated, photographed, or identified), so GPA may be run repeatedly on a growing input set rather than once on a fixed one. Proposition 6.1 guarantees that the pairing returned at the end of every such run, however partial the fragment set available at that time, is always a legitimate edge-pairing graph, never an intermediate structure whose topological status is undefined, and so may safely be presented to the human operator for review at that stage.

Corollary 6.1. *Let $\mathcal{G}_\kappa$ be the output of Algorithm 1, obtained from the canonical pairing $\mathcal{G}_0(V_0, A_0)$ on M_{g_0} after κ total iterations, of which $\sigma \leq \kappa$ are surgeries (j of type S_3 and $\sigma - j$ of type S_1 or S_2), interleaved with extensions, contractions, and edge exchanges. Then, $g_\kappa = g_0$, and $V_\kappa = V_0 + j + 2(\sigma - j)$, $A_\kappa = A_0 + 2j + 3(\sigma - j)$.*

Proof. By Proposition 6.1, $\mathcal{G}_\kappa$ is an edge-pairing graph on M_{g_0} itself, so $g_\kappa = g_0$; no genus formula is needed. Extensions, contractions, and edge exchanges modify vertex- and edge-count by fixed known increments, already quantified in Propositions 2.2 and 2.3, but by the invariant of Algorithm 1 (extension always paired with its inverse contraction within the same iteration), their net contribution to (V_κ, A_κ) relative to (V_0, A_0) is zero. Each of the σ surgeries is a graph-level operation in the sense of Definition 2.3, contributing $(\Delta V, \Delta A) = (1, 2)$ if of type S_3 or $(2, 3)$ if of type S_1 or S_2 , by the Remark following that definition. Summing these contributions over the j surgeries of type S_3 and $\sigma - j$ of type S_1, S_2 gives $V_\kappa = V_0 + j + 2(\sigma - j)$ and $A_\kappa = A_0 + 2j + 3(\sigma - j)$. $\square$

Corollary 6.1 allows an implementation of GPA to predict, prior to committing to a proposed surgery, the resulting vertex and edge counts (V_κ, A_κ) (the genus and orientability class are, by Proposition 6.1, guaranteed to remain those of M_{g_0} throughout) and to reject any candidate operation whose resulting counts are inconsistent with the fragment budget expected of the object being reconstructed.

We have to point out that Proposition 6.1 and Corollary 6.1 above already connect the closure theorem of this paper, Theorem 5.1, and the local vertex/edge-count formulas underlying it (Propositions 2.2, 2.3, and the Remark of §2.4), directly to GPA. Theorem 5.2, which concerns the

surface-changing connected sum of Definition 2.4, is not invoked by GPA itself, as the surgeries GPA performs act within the single fixed surface M_{g_0} in the graph-level sense of Definition 2.3. A third result of this paper serves the algorithm in the same way: The classification of canonical pairings on the bitorus and the tritorus, given in Section 4 and summarized in Tables 1 and 3 supplies a small, complete library of initial states with which GPA may be seeded, for objects with a correspondingly small number of fragments. This library removes the need to search for a valid canonical pairing from first principles in these low-complexity cases.

A purely heuristic greedy matching procedure offers no a priori guarantee that its intermediate or final pairings are structurally consistent; verifying this after the fact for each run would require an ad hoc topological check. Grounding GPA in the results of this paper removes that need: Consistency is guaranteed by construction for every combination of the four local operations on every closed surface. This also clarifies which local moves are safe to expose to a human operator correcting the completed reconstruction once Algorithm 1 has terminated (Sections 2.3–2.4 give an exhaustive list) and which combinations of moves are equivalent via the equivalence classes discussed in Section 3 and therefore need not be explored separately by the algorithm.

It is worth stressing explicitly what Proposition 6.1 does and does not guarantee. The proposition certifies that every graph $\mathcal{G}_t$ produced by Algorithm 1 is topologically valid; it says nothing about the value of w attained at termination. Because Algorithm 1 accepts, at each step, only the locally best-compatible admissible operation, it is a hill-climbing procedure on the space of valid edge-pairing graphs, and as such, it may terminate at a local maximum of w rather than at a global one. Guaranteeing global optimality would require either an exhaustive search over the (generally large) space of valid pairings reachable from the canonical seed or a specific acceptance rule with a proven approximation guarantee, neither of which is claimed here. We note that this greedy character is combinatorial rather than continuous: Algorithm 1 does not follow a gradient of w in any differentiable sense, nor does it construct conjugate search directions as in continuous optimization methods such as gradient descent or the conjugate gradient method; at each step, it simply selects, among a finite set of admissible discrete operations, the one yielding the largest increase in w .

Two directions follow naturally from this correspondence. First, the closure property of Theorem 5.1 concerns topological validity, not optimality: Relating it to an approximation guarantee for a specific greedy acceptance rule (for instance, one that accepts the locally best-compatible edge pair at each step) would connect the combinatorics of this paper to the approximation-theoretic analysis of greedy algorithms. Second, the canonical-pairing classifications of Section 4 are currently complete only for small genus (bitorus, 4-projective plane, tritorus); extending them, even partially, to the higher-genus, nonorientable surfaces that arise when GPA is applied to objects with a larger number of fragments would directly enlarge the library of admissible initial states available to the algorithm.

7. Canonical pairings for fragment reconstruction

The results of Sections 2–5.2 guarantee via Proposition 6.1 that every graph produced by Algorithm 1 is a valid edge-pairing graph $\mathcal{G}_q(V, A)$ in the sense of Definition 3.3 on the surface fixed at initialization.

For expository purposes, the pairing obtained in each instance is also described through a fragment-adjacency graph $G(V, E)$: the graph with one vertex for each of the m fragments involved and one edge

for each identified pair among them. This graph is used here purely as a descriptive device, to display the matching on the fragments themselves, and it is not the object $\mathcal{G}_q(V, A)$ studied in Sections 2–5.2: Its vertex set indexes fragments, whereas the vertex set of $\mathcal{G}_q(V, A)$ indexes polygon vertices identified under the quotient map q , and the two coincide only under a correspondence that is not assumed here. No result of Sections 2–5.2 is stated for, or applies to, $G(V, E)$; in particular, we make no claim about its Euler characteristic, as it plays no role in the guarantee of Proposition 6.1, which concerns $\mathcal{G}_q(V, A)$ alone. All values of V and A reported below for $G(V, E)$ are labeled as such to avoid any conflation with the edge-pairing graph $\mathcal{G}_q(V, A)$.

Let w denote the compatibility weight required as input by Algorithm 1: a function assigning to each candidate pair of fragments a score in $[0, 1]$, nonnegative and bounded, as its statement requires. The specific geometric quantities from which w is built in the instances below are an implementation choice and are not fixed by the theory of Sections 2–5.2; what the correctness of Algorithm 1 requires of w is only that it be a well-defined function of this kind, independently of its particular construction.

7.1. Plate: Eight-fragment reconstruction

For the first instance, $m = 8$ fragments were detected and their contact boundaries were identified with the $n = 20$ sides of the polygon $\mathcal{P}_{20}$, giving $A = 10$ candidate edges. The ambient surface is the tritorus $M_3 = 3\mathbb{T}^2$: By Corollary 3.1(b)(i), a 4-regular pairing graph on $g\mathbb{T}^2$ satisfies $A = 4g - 2$, so $g = 3$ is precisely the genus determined by $A = 10$. Algorithm 1 constructs the edge-pairing graph $\mathcal{G}_k(V, 10)$ on $3\mathbb{T}^2$ directly, matching all $A = 10$ candidate edges detected among the fragments through the local operations of Sections 2.2–2.4, in accordance with Proposition 6.1. We denote by $\mathcal{G}_k(V, A)$ the pairing graph returned by Algorithm 1 at termination, and by $G(V, E)$ the fragment-adjacency graph reported alongside it (Section 6, notational conventions).

The canonical pairing shown on the left of Figure 21 is the single-vertex pairing $C_q(3, 6)$ of Section 4, constructed directly from the pair $(g, A_0) = (3, 6)$ fixed at initialization, without reference to the geometry of the detected fragments: By Definition 4.2, it has a single vertex of degree $2A_0 = 6$. The optimized pairing, shown on the right, is the graph $\mathcal{G}_k(V, 10)$ obtained from $C_q(3, 6)$ by the sequence of local operations of Sections 2.2–2.4, each accepted only when it improves the compatibility score w defined above. Because extensions and contractions change the number of vertices and edges of the working graph (Proposition 2.2), the accepted operations reported below extend $\mathcal{G}_k$ from the six edges of the canonical seed to the $A = 10$ candidate edges detected among the eight fragments, and the ambient surface $3\mathbb{T}^2$ fixed at initialization remains unchanged. By Proposition 6.1, $\mathcal{G}_k$ is, independently of its final vertex and edge count, a valid edge-pairing graph on the surface fixed at initialization.

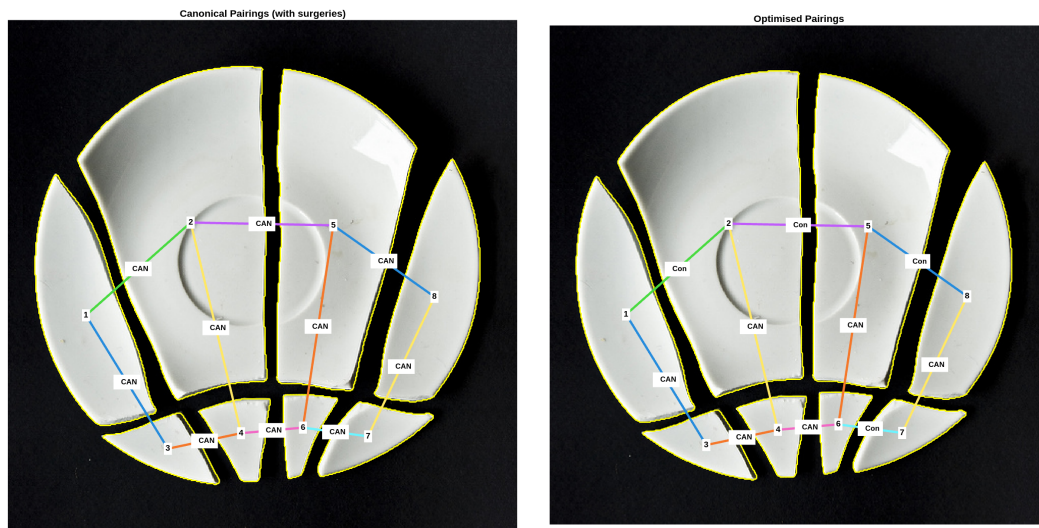
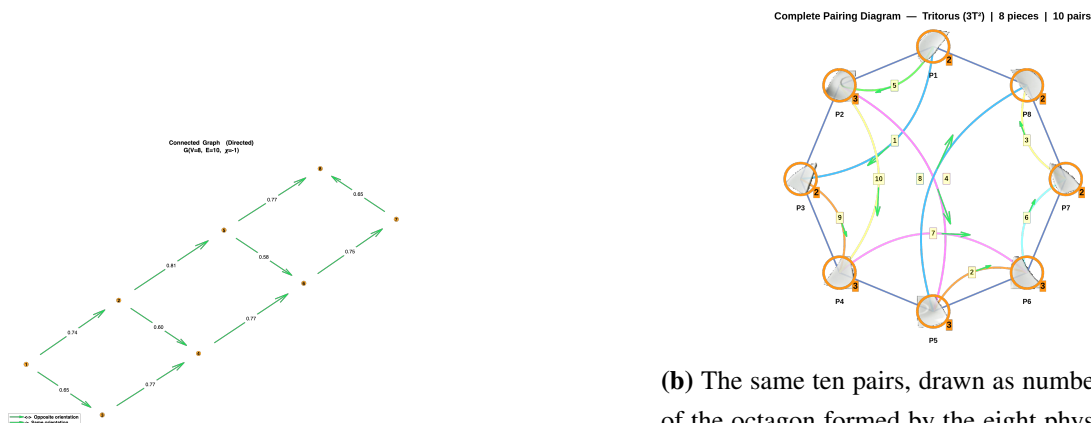


Figure 21. Canonical pairing $C_q(3, 6)$ on the tritorus $3\mathbb{T}^2$ (left) against the pairing graph $\mathcal{G}_k(V, 10)$ returned by Algorithm 1 after the extension and surgery stages (right), both drawn on the eight detected fragments of the plate.



(a) Fragment-adjacency graph $G(V = 8, E = 10)$: one node per detected fragment, one edge per accepted pair, with pairwise compatibility values w labeled on each edge.

(b) The same ten pairs, drawn as numbered chords of the octagon formed by the eight physical pieces $P_1, \dots, P_8$; ring color indicates the number of pairs incident to each piece (orange: two or more; green: exactly one), and arrows indicate edge orientation.

Figure 22. Two representations of the fragment-adjacency graph $G(V, E)$ underlying the optimized pairing, drawn on the surface $M_3 = 3\mathbb{T}^2$ fixed at initialization. (a): As a directed graph. (b): As chords of a polygon formed by the eight detected pieces.

The fragment-adjacency graph shown in Figure 22 has $V = 8$ nodes and $E = 10$ edges. Its Euler characteristic, as a one-dimensional CW-complex, is $\chi(G) = V - E$ (independently of the number C of connected components), giving $\chi(G) = 8 - 10 = -2$; the graph is connected ($C = 1$) in this instance. The value of this example is a property of the fragment-adjacency graph $G(V, E)$ and must not be confused with the Euler characteristic of the edge-pairing graph $\mathcal{G}_q(V, A)$ itself: The Remark of Section 2.1(i), $\chi(\mathcal{G}_q) = 1 - 2g$, applies to $\mathcal{G}_q$ under the hypothesis that $\mathcal{B}(\mathcal{G}_q)$ is simply connected, and its vertex count V counts identified polygon vertices, not physical fragments. The two quantities coincide

only when every fragment corresponds to exactly one vertex of $\mathcal{G}_q$, a correspondence not established here. Consequently, the value $\chi(G) = -2$ reported for the fragment-adjacency graph should not be read as an independent verification of Theorem 5.2 or Corollary 6.1, which concern $\mathcal{G}_q(V, A)$; it documents the connectivity of the fragment matching itself.

The average degree of $G(V, E)$ is $2E/V = 20/8 = 2.5$, confirming that $G(V, E)$ is not a regular graph; this is expected, as G records fragment adjacency rather than the regular structure of a canonical edge-pairing graph, and no claim of regularity is made for it.

The surgery stage of Algorithm 1 did not alter which fragments were matched after the extension stage that produced $\mathcal{G}_\kappa$: The fragment pairing recorded in $G(V, E)$ coincides with the one recorded immediately after extension. This is confirmed by the aggregate comparison of mean normalized distance and mean compatibility between the two pairings, which reports a relative difference of 0%.

Figure 23 displays the ten identified pairs directly on the photographed fragments together with the compatibility value w obtained for each pair, ranging from $w = 0.58$ (pair P_5 – P_6) to $w = 0.81$ (pair P_2 – P_5). Each of the ten chords shown corresponds to a physical adjacency between two real fragments and to the numerical compatibility that Algorithm 1 evaluated when deciding not to alter it. The figure also shows pieces P_2 , P_4 , P_5 , and P_6 incident to two or more pairs, consistent with the nonregular degree sequence of $G(V, E)$ already noted above.

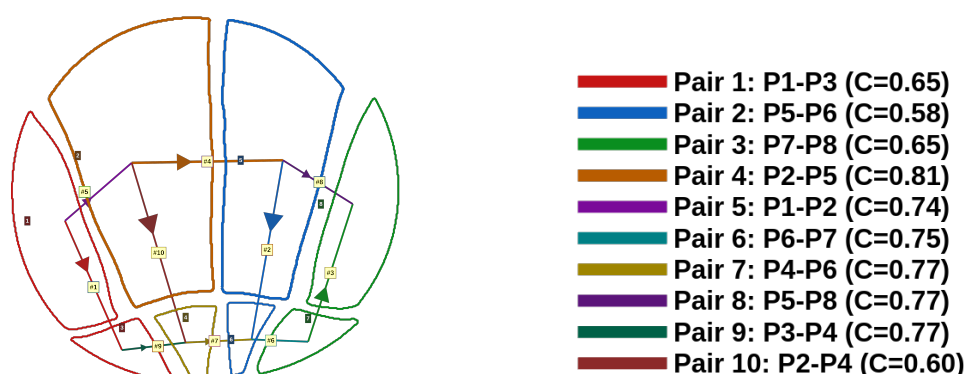


Figure 23. The ten identified pairs, drawn directly on the eight detected fragments of the plate, each pair shown in a distinct color with an arrow indicating edge orientation together with the compatibility value w (labeled C in the figure, identical to the compatibility value w used in the text) for every pair, as defined in the notational conventions above.

Running Algorithm 1 to completion on this instance did not alter the fragment pairing established after the extension stage: No further operation improved the compatibility of the matching, and the algorithm converged to the pairing shown in Figure 23. Proposition 6.1 guarantees that every graph in the sequence $\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_\kappa$ actually produced by Algorithm 1 (that is, the edge-pairing graph obtained after each accepted operation) remained a valid edge-pairing graph on the surface fixed at initialization, independently of the nonregular structure of the fragment-adjacency graph $G(V, E)$ used to report and visualize the matching itself.

This convergence is consistent with, rather than contrary to, the hill-climbing nature of Algorithm 1 discussed in Section 6.2: Once no single admissible local operation improves w , the algorithm has reached a local maximum of the compatibility score, and Proposition 6.1 guarantees that this local maximum is itself a valid edge-pairing graph without asserting that it is the global maximum attainable

on this instance.

7.2. Reconstruction of an archaeological artifact

The second instance corresponds to a set of terracotta fragments of a Greek vase, dated 500–490 BCE, from which twenty-seven fragments were detected (Figure 24).



Figure 24. Collection of ancient Greek terracotta fragments, illustrating a broken ceramic vessel prior to reconstruction. Source: [22], Image ID 2P8KYHE (accessed 2025). Used under license.

Twenty-eight pairs were identified among the twenty-seven detected fragments (Figure 25), each labeled with its compatibility value w , ranging from $w = 0.62$ (pairs P_{15} – P_{16} and P_{24} – P_{27}) to $w = 0.83$ (pair P_1 – P_8). Each of the twenty-seven detected pieces appears in at least one pair.

The edge-pairing graph identified for the matched fragments (Figure 26) is embedded on the closed orientable surface of genus $g = 7$, denoted $7\mathbb{T}^2$. Figure 27 redraws the same matched fragments as a connected chain ordered by piece compatibility, complementing the polygon-chord view of Figure 25. Algorithm 1 was initialized with the canonical pairing $C_q(7, 14)$: By Definition 4.2, this canonical seed has a single vertex of degree $2A_0 = 14$, consistent with the Remark of Section 2.1(iii) for an oriented surface of genus $g = 7$. As in the first instance, the graph obtained by GPA after the application of the local operations coincided, for twenty-six of the twenty-eight pairs, with the target 4-regular graph of $A = 4g - 2 = 26$ edges predicted by Corollary 3.1 for $g = 7$; the remaining two pairs were produced by local operations beyond this canonical target, bringing the final edge count to $A = 28$. This again instantiates Proposition 6.1: Every pair, whether coinciding with the 4-regular target or produced by a local operation, is part of a valid edge-pairing graph on $7\mathbb{T}^2$.

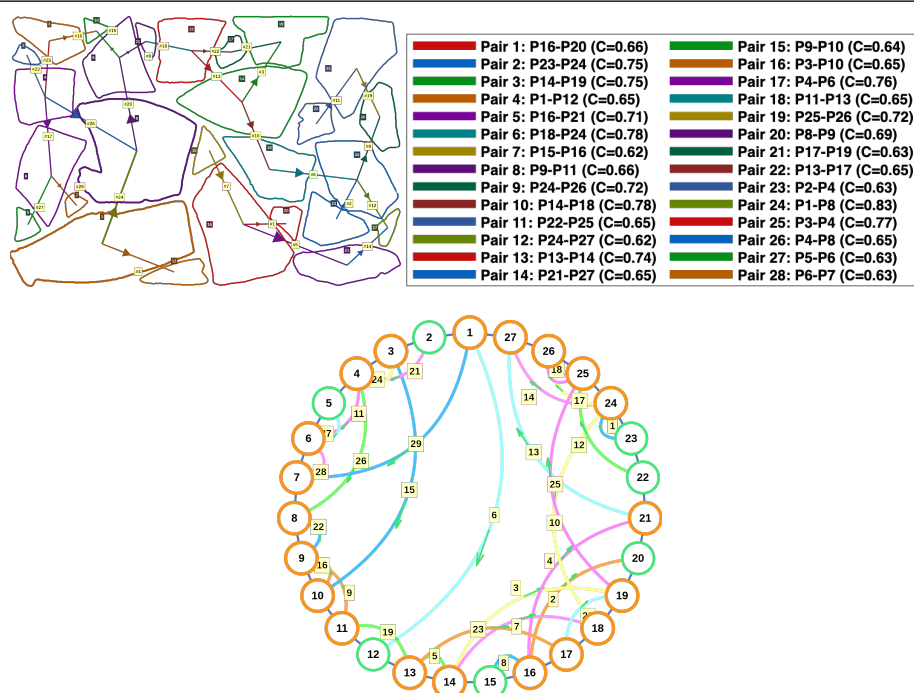


Figure 25. Left: Legend giving, for each of the twenty-eight identified pairs, the two pieces matched and the compatibility value w (labeled C in the figure, identical to the compatibility value w used in the text). Above: Pairs drawn as chords of the canonical polygon, with pieces arranged in cyclic order; orange rings denote a piece incident to two or more pairs, green rings a piece incident to exactly one pair.



Figure 26. Canonical pairing $C_q(7, 14)$ on the heptatorus $7T^2$ (left) against the pairing graph $G_\kappa(V, 28)$ returned by GPA after the surgery stage (right), both drawn on the twenty-seven detected fragments of the vase. Pairs are labeled CAN where the surgery stage left the canonical identification unchanged, CON where a contraction was applied, and S1/S2/BRIDGE where the corresponding surgery of Section 2.4 was applied.

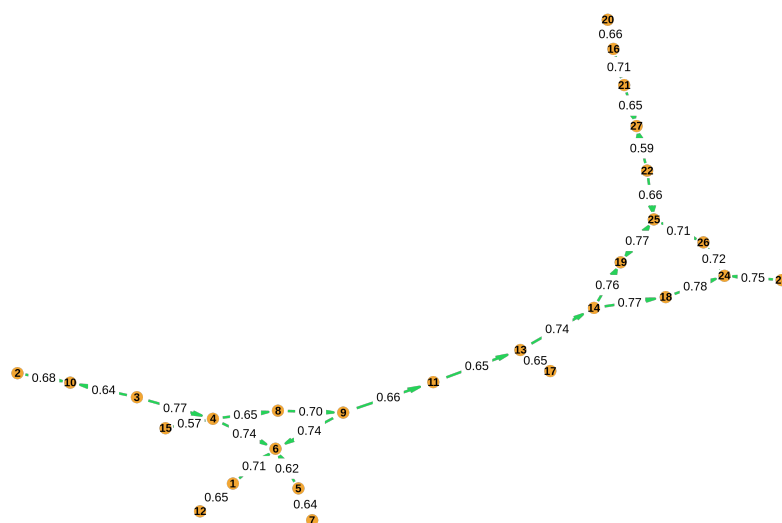


Figure 27. Connected graph of the matched fragments, drawn as a chain ordered by piece compatibility, with the compatibility value w labeled on each edge.

Across both instances, the majority of the final edge-pairing graph is consistent with the 4-regular formula $A = 4g - 2$ of Corollary 3.1: exactly $A = 10$ for the tritorus ($g = 3$) in the first instance and $A = 26$ of the $A = 28$ total for the heptatorus ($g = 7$) in the present instance, the latter two edges arising from local operations that extend the graph beyond the canonical 4-regular target. In both cases, the canonical seed used to initialize GPA satisfies $A_0 = 2g$ (Remark of Section 2.1(iii)), and the final edge count is reached by the extension and surgery operations accepted during the run, not by the canonical seed itself. The coincidence between the canonical pairing diagram and a substantial majority of the optimized pairing graph, observed in both instances, provides evidence that the operations of extension, contraction, edge exchange, and surgery described in this paper, although guaranteed by Theorem 5.1 to remain within the class of edge-pairing graphs whenever they are applied, are not necessarily required to depart extensively from the canonical pairing in order to produce a topologically admissible reconstruction.

8. Conclusions

This paper set out to answer a purely topological question with direct practical consequences: Which graphs can arise as edge-pairing graphs of a regular polygon on a closed surface, and by which elementary local moves can one such graph be transformed into another without leaving that class? We extended the framework of extension, contraction, and edge exchange, previously established for orientable surfaces, to the general case of closed surfaces, including nonorientable ones, and introduced three families of surgery for combining pairing graphs by connected sum. Theorems 5.1 and 5.2 show that these four operations can be freely combined, in any order and on any closed surface, while remaining within the class of valid edge-pairing graphs, with explicit formulas for the resulting genus, vertex count, and edge count. We further classified canonical pairings on the bitorus, the 4-projective plane, and the tritorus (Section 4), supplying a small library of initial states for low-complexity surfaces.

This closure property answers the motivating question directly: It is precisely what allows a

greedy reconstruction procedure to be provably correct at every intermediate step, not only at termination. Building on it, we formulated the greedy pairing algorithm (GPA, Section 6) and proved, in Proposition 6.1, that every graph it produces (at every step of its execution, not merely its final output) is a valid edge-pairing graph on the surface fixed at initialization. Corollary 6.1 further shows that the genus, vertex count, and edge count of the algorithm output can be predicted in advance from the sequence of surgeries performed, allowing an implementation to reject candidate operations inconsistent with the expected topological type of the object being reconstructed.

The two fragment-reconstruction instances of Section 7 illustrate these results on concrete data. In both cases, the graph returned by GPA departed only moderately from its canonical seed: For the eight-fragment plate, all ten edges of the final graph were reached by extending a six-edge canonical seed on the tritorus; for the twenty-seven-fragment vase, twenty-six of the twenty-eight identified pairs coincided with the four-regular target predicted by Corollary 3.1 for the heptatorus, with only two pairs requiring operations beyond this target. This coincidence between the canonical pairing and a substantial majority of the optimized pairing, observed independently in both instances, suggests that the operations studied here are not merely a theoretical safeguard but a practically economical route from a canonical seed to a topologically admissible reconstruction.

Two directions for future work follow naturally from this correspondence. First, the closure property established here concerns topological validity, not optimality; relating it to an approximation guarantee for a specific greedy acceptance rule would connect the combinatorics developed in this paper to the approximation-theoretic analysis of greedy algorithms. Second, the canonical-pairing classifications of Section 4 are currently complete only for small genus (the bitorus, the 4-projective plane, and the tritorus); extending them, even partially, to higher-genus and nonorientable surfaces would directly enlarge the library of admissible initial states available to GPA for objects assembled from a larger number of fragments, and it would strengthen the connection noted in the Introduction between the edge-pairing graphs studied here and the tessellation structures underlying quantum surface codes.

Author contributions

Catarina Mendes de Jesus S: Conceptualization, Methodology, Formal analysis, Investigation, Writing—original draft, Writing—review and editing; Pantaleón D. Romero: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing—original draft, Writing—review and editing, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Pantaleón D. Romero was supported by Universidad CEU Cardenal Herrera under grant number INDI23/17 and ESI Chair Group.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. Q.-X. Huang, S. Flöry, N. Gelfand, M. Hofer, H. Pottmann, Reassembling fractured objects by geometric matching, *ACM T. Graphic.*, **25** (2006), 569–578. <https://doi.org/10.1145/1141911.1141925>
2. E. D. Demaine, M. L. Demaine, Jigsaw puzzles, edge matching, and polyomino packing: Connections and complexity, *Graph. Combinator.*, **23** (2007), 195–208. <https://doi.org/10.1007/s00373-007-0713-4>
3. K. Son, J. Hays, D. B. Cooper, Solving square jigsaw puzzle by hierarchical loop constraints, *IEEE T. Pattern Anal.*, **41** (2019), 2222–2235. <https://doi.org/10.1109/TPAMI.2018.2857776>
4. V. Huroyan, G. Lerman, H.-T. Wu, Solving jigsaw puzzles by the graph connection Laplacian, *SIAM J. Imaging Sci.*, **13** (2020), 1717–1753. <https://doi.org/10.1137/19M1290760>
5. C. Y. Le, X. Li, JigsawNet: Shredded image reassembly using convolutional neural network and loop-based composition, *IEEE T. Image Process.*, **28** (2019), 4000–4015. <https://doi.org/10.1109/TIP.2019.2903298>
6. G. Scarpellini, S. Fiorini, F. Giuliani, P. Moreiro, A. Del Bue, DiffAssemble: A unified graph-diffusion model for 2D and 3D reassembly, In: *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR)*, 2024, 28098–28108.
7. J. Lu, Y. Q. Liang, H. J. Han, J. C. Hua, J. F. Jiang, X. Li, et al., A survey on computational solutions for reconstructing complete objects by reassembling their fractured parts, *Computer Graph. Forum*, **44** (2025), e70081. <https://doi.org/10.1111/cgf.70081>
8. R. G. Cavalcante, H. Lazari, J. D. de Lima, R. Palazzo, A new approach to the design of digital communication systems, In: *Algebraic coding theory and information theory*, Providence: American Mathematical Society, 2005, 145–177. <https://doi.org/10.1090/dimacs/068/10>
9. J. D. de Lima, R. Palazzo, Embedding discrete memoryless channels on compact and minimal surfaces, *Proceedings of the IEEE Information Theory Workshop*, Bangalore, India, 2002, 183–186. <https://doi.org/10.1109/ITW.2002.1115448>
10. E. B. Silva, M. Firer, S. R. Costa, R. Palazzo, Signal constellations in the hyperbolic plane: A proposal for new communication systems, *J. Franklin I.*, **343** (2006), 69–82. <https://doi.org/10.1016/j.jfranklin.2005.09.001>
11. E. B. Da Silva, E. M. Brizola, W. S. Soares, D. F. Copatti, New quantum surface codes from semi-regular tessellations, *Quantum. Inf. Process.*, **22** (2023), 398. <https://doi.org/10.1007/s11128-023-04147-2>
12. E. B. Da Silva, W. S. Soares, Construction of color codes from polygons, *J. Phys. Commun.*, **2** (2018), 095011. <https://doi.org/10.1088/2399-6528/aad062>
13. A. Leverrier, G. Zémor, Decoding quantum Tanner codes, *IEEE T. Inform. Theory*, **69** (2023), 5100–5115. <https://doi.org/10.1109/TIT.2023.3267945>

14. G. Zémor, On expander codes, *IEEE T. Inform. Theory*, **47** (2001), 835–837. <https://doi.org/10.1109/18.910593>
15. C. D. Albuquerque, R. Palazzo, E. B. Silva, Topological quantum codes on compact surfaces with genus $g \geq 2$, *J. Math. Phys.*, **50** (2009), 023513. <https://doi.org/10.1063/1.3081056>
16. M. B. Faria, C. Mendes de Jesus, P. D. R. Sanchez, Surgeries of pairing of edges associated to trivalent graphs, *Bull. Braz. Math. Soc. New Series*, **47** (2016), 1085–1095. <https://doi.org/10.1007/s00574-016-0205-3>
17. M. B. Faria, R. Palazzo Jr., Emparelhamentos generalizados associados a tesselação $\{12g - 6, 3\}$, *Trends in Computational and Applied Mathematics*, **11** (2010), 59–67. <https://doi.org/10.5540/tema.2010.011.01.0059>
18. T. Jorgensen, M. Naatänen, Surfaces of genus 2: generic fundamental polygons, *Q. J. Math.*, **33** (1982), 451–461. <https://doi.org/10.1093/qmath/33.4.451>
19. J. Chadzyński, G. Nakamura, Generic fundamental polygons for surfaces of genus three, *Kodai Math. J.*, **27** (2004), 88–104. <https://doi.org/10.2996/kmj/1085143791>
20. C. M. de Jesus, P. D. Romero, Graphs and closed surfaces associated with a pairing of edges for regular polygons, *Bull. Braz. Math. Soc. New Series*, **51** (2020), 527–541. <https://doi.org/10.1007/s00574-019-00163-y>
21. C. M. de Jesus, Grafos associados aos emparelhamentos de arestas de polígonos regulares, *Trends in Computational and Applied Mathematics*, **23** (2022), 335–347. <https://doi.org/10.5540/tcam.2022.023.02.00335>
22. V. Bucys, Alamy Stock Photo, Background of ancient classical Greek terracotta fragments. Collection of antique ceramics, pieces of broken vase, amphora, jug and jar, Image ID: 2P8KYHE, 2023. Available from: <https://www.alamy.com/background-of-ancient-classical-greek-terracotta-fragments-collection-of-antique-ceramics-pieces-of-broken-vase-amphora-jug-and-jar-image538877402.html>.
23. J. Harer, D. Zagier, The Euler characteristic of the moduli space of curves, *Invent. Math.*, **85** (1986), 457–485. <https://doi.org/10.1007/BF01390325>
24. N. H. Xuong, How to determine the maximum genus of a graph, *J. Comb. Theory B*, **26** (1979), 217–225. [https://doi.org/10.1016/0095-8956\(79\)90058-3](https://doi.org/10.1016/0095-8956(79)90058-3)
25. G. Chapuy, A new combinatorial identity for unicellular maps, via a direct bijective approach, *Adv. Appl. Math.*, **47** (2011), 874–893. <https://doi.org/10.1016/j.aam.2011.04.004>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)