



Research article

Variation comparison between the F -distribution and the normal distribution

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Abstract: We investigated the variation comparison between the F -distribution and the normal distribution. Let X_{d_1,d_2} be an F -random variable with numerator and denominator degrees of freedom as d_1 and d_2 , respectively. We proved that, for $d_1 \in \{1, 2, 3, 4\}$ and $5 \leq d_2 \in \mathbb{N}$, $P\{|X_{d_1,d_2} - E[X_{d_1,d_2}]| \leq \sqrt{\text{Var}(X_{d_1,d_2})}\} > P\{|Z| \leq 1\}$, where Z is a standard normal random variable. Moreover, for each fixed $d_1 \in \{1, 2, 3, 4\}$, $\inf_{d_2 \geq 5} P\{|X_{d_1,d_2} - E[X_{d_1,d_2}]| \leq \sqrt{\text{Var}(X_{d_1,d_2})}\} = P\{|\chi_{d_1}^2 - E[\chi_{d_1}^2]| \leq \sqrt{\text{Var}(\chi_{d_1}^2)}\}$, where $\chi_{d_1}^2$ is a $\chi^2(d_1)$ -random variable. These four structurally representative cases capture the main features of the problem, while the case $d_1 \geq 5$ remains open.

Keywords: variation comparison inequality; F -distribution; normal distribution; χ^2 -distribution; infinitely divisible distribution

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1. Introduction

In [6, 7], we initiated the study of the variation comparison between familiar distributions and the normal distribution. Let Z be a standard normal random variable and $\{X_\tau\}$ be a family of random variables with the same type of distributions and different parameters τ . We discovered that for most familiar infinitely divisible continuous distributions listed in Sato's book [5] (cf. [5, p. 46 and 47]), the following inequality holds:

$$P\{|X_\tau - E[X_\tau]| \leq \sqrt{\text{Var}(X_\tau)}\} > P\{|Z| \leq 1\} \approx 0.6827.$$

Further, for a large number of infinitely divisible continuous distributions including the gamma, log-normal, student's t , and inverse Gaussian distributions, we have

$$\inf_{\tau} P\{|X_{\tau} - E[X_{\tau}]| \leq \sqrt{\text{Var}(X_{\tau})}\} = P\{|Z| \leq 1\}.$$

The F -distribution is a basic and important probability distribution, which has many statistical applications including comparing the variability of two population samples, comparing the means of two or more groups, and testing the validity of a multiple regression equation. Up to now, the variation comparison between the F -distribution and the normal distribution remains unknown. In this paper, we consider the following conjecture.

Conjecture 1.1. *Let $d_1 \in \mathbb{N}$, $5 \leq d_2 \in \mathbb{N}$, and X_{d_1, d_2} be an F -random variable with parameters d_1 and d_2 . Then,*

$$P\{|X_{d_1, d_2} - E[X_{d_1, d_2}]| \leq \sqrt{\text{Var}(X_{d_1, d_2})}\} > P\{|Z| \leq 1\} \approx 0.6827$$

and

$$\inf_{d_1 \in \mathbb{N}, d_2 \geq 5} P\{|X_{d_1, d_2} - E[X_{d_1, d_2}]| \leq \sqrt{\text{Var}(X_{d_1, d_2})}\} = P\{|Z| \leq 1\}.$$

Note that many two-parameter infinitely divisible distributions have been considered in [6, 7]. However, one of the apparent parameters in those distributions can be eliminated by location or scale invariance. Hence, the essential comparisons reduce to one-parameter inequalities. The case of the F -random variable is fundamentally distinct: As a ratio of two independent normalized χ^2 -random variables, both degrees of freedom persist after standardization. Thus, the problem addressed in this paper demands a genuinely two-parameter variation comparison, distinguishing it from the one-parameter frameworks established in [6, 7].

We have been able to resolve Conjecture 1.1 for the case where d_1 is small and d_2 is arbitrary. More precisely, we have the following result.

Theorem 1.2. *Let $d_1 \in \{1, 2, 3, 4\}$, $5 \leq d_2 \in \mathbb{N}$, X_{d_1, d_2} be an F -random variable with parameters d_1 and d_2 , and $\chi_{d_1}^2$ be a $\chi^2(d_1)$ -random variable. Then,*

$$P\{|X_{d_1, d_2} - E[X_{d_1, d_2}]| \leq \sqrt{\text{Var}(X_{d_1, d_2})}\} > P\{|Z| \leq 1\} \approx 0.6827$$

and

$$\inf_{d_2 \geq 5} P\{|X_{d_1, d_2} - E[X_{d_1, d_2}]| \leq \sqrt{\text{Var}(X_{d_1, d_2})}\} = P\{|\chi_{d_1}^2 - E[\chi_{d_1}^2]| \leq \sqrt{\text{Var}(\chi_{d_1}^2)}\}.$$

Note that, by [6, Theorem 1.2], we have

$$\inf_{d_1 \in \mathbb{N}} P\{|\chi_{d_1}^2 - E[\chi_{d_1}^2]| \leq \sqrt{\text{Var}(\chi_{d_1}^2)}\} = P\{|Z| \leq 1\}.$$

Remark 1.3. *The values $d_1 = 1, 2, 3, 4$ are not just the first four cases of Conjecture 1.1. They are the minimal representatives of the different cases that arise in the proof. First, when $d_1 = 1, 2$, the lower endpoint $E[X_{d_1, d_2}] - \sqrt{\text{Var}(X_{d_1, d_2})}$ is non-positive. Then, the required probability is only determined by the upper endpoint. When $d_1 = 3, 4$, the lower endpoint becomes positive and hence the comparison becomes two-sided and both endpoints must be controlled. Second, for even d_1 , the beta parameter $\frac{d_1}{2}$ is an integer and then the relevant beta integrands arising in the proof have finite*

polynomial expansions. For odd d_1 , the beta parameter is a half-integer and then a square-root factor remains. The values $d_1 = 1, 2, 3, 4$ represent the odd one-sided, even one-sided, odd two-sided, and even two-sided regimes, respectively. Thus, Theorem 1.2 resolves some of the most representative cases that occur in Conjecture 1.1.

Beyond providing a partial solution to Conjecture 1.1, Theorem 1.2 gives a uniform finite-sample concentration bound for F -distributions with $d_1 \in \{1, 2, 3, 4\}$ and $d_2 \geq 5$. It shows that more than 68.27% of the probability mass for F -distributions lies within one standard deviation of the mean. Since F -distributions arise in tests concerning population variances, analysis of variance, and regression, this result provides a rigorous reference point for understanding the concentration of the corresponding F -statistics. Moreover, Theorem 1.2 identifies the limiting χ^2 -distribution as the sharp lower benchmark. The recurrence method developed in this paper may also be useful for studying related probability inequalities for other beta-type and ratio distributions.

Numerical examples for

$$F_{d_1, d_2} := P\{|X_{d_1, d_2} - E[X_{d_1, d_2}]| \leq \sqrt{\text{Var}(X_{d_1, d_2})}\}$$

are provided in Table 1 to illustrate Conjecture 1.1 and Theorem 1.2 (cf. (2.1) and the Mathematica code in the Supplementary Material).

Table 1. Some numerical values of F_{d_1, d_2} .

$d_1 \backslash d_2$	5	6	7	10	1000	100,000
1	0.9472136398	0.9302063994	0.9203979875	0.9056443691	0.8799791819	0.8797638312
2	0.9435474805	0.9245008973	0.9132938058	0.8960789690	0.8649357488	0.8646674235
3	0.9423311144	0.9225198424	0.9107495178	0.8924660105	0.7673768557	0.7659852731
4	0.9417676183	0.9215728753	0.9095054501	0.8906250000	0.7387636790	0.7375311593
5	0.9414580781	0.9210402197	0.9087930817	0.8822866278	0.7247684612	0.7236081865
6	0.9412690533	0.9207088210	0.9083433939	0.8741491264	0.7164484664	0.7155296962
10	0.9409503774	0.9201344424	0.9075462812	0.8584654666	0.7017835855	0.7007354844
1000	0.9407217788	0.9196986640	0.9066106258	0.8451487351	0.6839040183	0.6828606791
100,000	0.9407217490	0.9196986029	0.9066103937	0.8451464241	0.6838226866	0.6827015911

Our original motivation of studying $\inf_{\tau} P\{|X_{\tau} - E[X_{\tau}]| \leq \sqrt{\text{Var}(X_{\tau})}\}$ comes from the following conjecture.

Tomaszewski's Conjecture: Let $X = \sum_{i=1}^n a_i X_i$, where $\sum_{i=1}^n a_i^2 = 1$ and $\{X_i, i \geq 1\}$ is a sequence of independent $\{-1, 1\}$ -valued symmetric random variables. Then, $P\{|X| \leq 1\} \geq 1/2$.

Tomaszewski's conjecture has many applications in probability theory, geometric analysis, optimization, and computer science. Keller and Klein [3] completely solved Tomaszewski's conjecture.

Before ending this introduction, we would like to mention some other relevant work. Zhang et al. [8] considered the measure concentration of general infinitely divisible distributions. Let $\mathcal{I}$ be the set of all infinitely divisible random variables with finite second moments, $\mathcal{I}_0 = \{X \in \mathcal{I} : \text{Var}(X) > 0\}$, $P_{\mathcal{I}} = \inf_{X \in \mathcal{I}} P\{|X - E[X]| \leq \sqrt{\text{Var}(X)}\}$, and $P_{\mathcal{I}_0} = \inf_{X \in \mathcal{I}_0} P\{|X - E[X]| < \sqrt{\text{Var}(X)}\}$. They proved that $P_{\mathcal{I}} \geq P_{\mathcal{I}_0} > 0$ and obtained the exact values of $\inf_{X \in \mathcal{J}} P\{|X - E[X]| \leq \sqrt{\text{Var}(X)}\}$ and $\inf_{X \in \mathcal{J}} P\{|X - E[X]| < \sqrt{\text{Var}(X)}\}$.

$\sqrt{\text{Var}(X)}$ for the cases where $\mathcal{J}$ is the set of all geometric random variables, symmetric geometric random variables, Poisson random variables, and symmetric Poisson random variables, respectively. Hu et al. [1] investigated the anti-concentration function $(0, \infty) \ni y \rightarrow \inf_{\tau} P\{|X_{\tau} - E[X_{\tau}]| \geq y \sqrt{\text{Var}(X_{\tau})}\}$. They showed that the anti-concentration function is not identically zero for certain familiar families of distributions, including the uniform distribution, exponential distribution, non-degenerate Gaussian distribution, and student's t -distribution.

The remainder of this paper is organized as follows. First, we make some preparation in Section 2 and discuss different cases of the variation comparison inequality for the F -distribution. Then, in Sections 3 and 4, we give the proof of Theorem 1.2 for cases $d_1 = 1, 2$ and $d_1 = 3, 4$, respectively. Finally, we make some remarks on the case $d_1 \geq 5$ in Section 5. In Sections 3 and 4, we use Mathematica to perform fast and accurate numerical and symbolic computations. We emphasize that all proofs in this paper remain completely rigorous and easily verifiable. The corresponding Mathematica code and output are provided as Supplemental Material.

2. Preliminary

Let X_{d_1, d_2} be an F -random variable with numerator and denominator degrees of freedom as d_1 and d_2 , respectively. We have (cf. [2, 4])

$$E[X_{d_1, d_2}] = \frac{d_2}{d_2 - 2}, \quad \text{Var}(X_{d_1, d_2}) = \frac{2d_2^2(d_1 + d_2 - 2)}{d_1(d_2 - 2)^2(d_2 - 4)},$$

and

$$P\{X_{d_1, d_2} \leq x\} = I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right), \quad x > 0,$$

where I is the regularized incomplete beta function:

$$I_x(a, b) = \frac{B(x; a, b)}{B(a, b)} = \frac{\int_0^x t^{a-1}(1-t)^{b-1} dt}{B(a, b)}.$$

Note that X_{d_1, d_2} has a finite variance only if $d_2 > 4$. Then, we get:

$$\begin{aligned} F_{d_1, d_2} &= P\{|X_{d_1, d_2} - E[X_{d_1, d_2}]| \leq \sqrt{\text{Var}(X_{d_1, d_2})}\} \\ &= I_{\frac{d_1}{d_1 + (d_2 - 2)\left(1 + \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}}\right)^{-1}}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right) - I_{\frac{d_1}{d_1 + (d_2 - 2)\left(\max\left\{0, 1 - \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}}\right\}\right)^{-1}}}\left(\frac{d_1}{2}, \frac{d_2}{2}\right). \end{aligned} \quad (2.1)$$

Hereafter, we adopt the convention that $0^{-1} = \infty$ and $\frac{1}{\infty} = 0$.

Note that X_{d_1, d_2} has the following expression:

$$X_{d_1, d_2} = \frac{\chi_{d_1}^2/d_1}{\chi_{d_2}^2/d_2},$$

where $\chi_{d_1}^2$ and $\chi_{d_2}^2$ are independent χ^2 -random variables with respective degrees of freedom as d_1 and d_2 . By Slutsky's theorem, we know that X_{d_1, d_2} converges to $\chi_{d_1}^2/d_1$ in distribution as $d_2 \rightarrow \infty$ for any fixed $d_1 \in \mathbb{N}$. Then by [6, Theorem 1.2], to complete the proof of Theorem 1.2, it suffices to show that

$$F_{d_1, d_2+2} < F_{d_1, d_2}, \quad \forall d_2 \geq 5. \quad (2.2)$$

To simplify notation, we define

$$A := \frac{d_1}{d_1 + d_2 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right)^{-1}}, \quad B := \frac{d_1}{d_1 + (d_2 - 2) \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right)^{-1}},$$

$$C := \frac{d_1}{d_1 + d_2 \left(\max \left\{0, 1 - \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right\}\right)^{-1}}, \quad D := \frac{d_1}{d_1 + (d_2 - 2) \left(\max \left\{0, 1 - \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right\}\right)^{-1}}.$$

By the relation $I_x(a, b+1) = I_x(a, b) + \frac{x^a(1-x)^b}{bB(a, b)}$, we get:

$$\begin{aligned} (2.2) \text{ holds } &\Leftrightarrow I_A\left(\frac{d_1}{2}, \frac{d_2}{2}\right) + \frac{A^{\frac{d_1}{2}}(1-A)^{\frac{d_2}{2}}}{\frac{d_2}{2}B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} - I_C\left(\frac{d_1}{2}, \frac{d_2}{2}\right) - \frac{C^{\frac{d_1}{2}}(1-C)^{\frac{d_2}{2}}}{\frac{d_2}{2}B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} \\ &< I_B\left(\frac{d_1}{2}, \frac{d_2}{2}\right) - I_D\left(\frac{d_1}{2}, \frac{d_2}{2}\right) \\ &\Leftrightarrow 2A^{\frac{d_1}{2}}(1-A)^{\frac{d_2}{2}} + d_2 \int_C^D t^{\frac{d_1}{2}-1}(1-t)^{\frac{d_2}{2}-1} dt \\ &< d_2 \int_A^B t^{\frac{d_1}{2}-1}(1-t)^{\frac{d_2}{2}-1} dt + 2C^{\frac{d_1}{2}}(1-C)^{\frac{d_2}{2}}. \end{aligned} \quad (2.3)$$

The reduction in (2.2) is the key conceptual step of this paper. Instead of directly comparing the F -distribution with the normal distribution, we compare two consecutive denominator degrees of freedom. Further, we convert the comparison between F_{d_1, d_2+2} and F_{d_1, d_2} into the inequality (2.3), which involves the endpoints A , B , C , and D . Once the strict monotonicity in d_2 is established, the limit $d_2 \rightarrow \infty$ identifies the infimum with the corresponding probability for $\chi^2(d_1)$. Hence, we can apply [6, Theorem 1.2]. This recurrence and limit argument constitutes the principal methodological advance beyond the techniques used in [6, 7].

Note that $A < B$ since

$$\frac{d_1}{d_1 + d_2 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right)^{-1}}$$

is a strictly decreasing function of d_2 for any fixed d_1 . We have

$$\begin{aligned} C > 0 &\Leftrightarrow 1 - \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} > 0, \\ D > 0 &\Leftrightarrow 1 - \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} > 0, \\ D > 0 &\Rightarrow C > 0. \end{aligned} \quad (2.4)$$

We now classify the possible configurations of the endpoints C and D . Since $D > 0$ implies $C > 0$, there are exactly three mutually exclusive cases:

Case 1:

$$C = 0 \Leftrightarrow 1 - \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} \leq 0 \Leftrightarrow \begin{cases} d_2 \geq 5, & \text{if } d_1 = 1 \text{ or } 2, \\ 5 \leq d_2 \leq 4 + \frac{8}{d_1 - 2}, & \text{if } 3 \leq d_1 \leq 10. \end{cases}$$

Case 2:

$$\begin{aligned} C > 0 \text{ and } D = 0 &\Leftrightarrow 1 - \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} > 0 \text{ and } 1 - \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}} \leq 0 \\ &\Leftrightarrow \begin{cases} 4 + \frac{8}{d_1 - 2} < d_2 \leq 6 + \frac{8}{d_1 - 2}, & \text{if } 3 \leq d_1 \leq 10, \\ d_2 = 5 \text{ or } 6, & \text{if } d_1 \geq 11. \end{cases} \end{aligned}$$

Case 3:

$$D > 0 \Leftrightarrow 1 - \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}} \geq 0 \Leftrightarrow d_1 \geq 3 \text{ and } d_2 > 6 + \frac{8}{d_1 - 2}. \quad (2.5)$$

These cases determine whether the integral $\int_C^D t^{\frac{d_1}{2}-1}(1-t)^{\frac{d_2}{2}-1}dt$ in (2.3) vanishes, has a favorable sign, or requires a separate estimate.

It is worth noting that $C > D$ is possible. Under the assumption that $D > 0$, we have

$$\begin{aligned} D > C &\Leftrightarrow d_2 \left\{ 1 - \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} \right\}^{-1} > (d_2 - 2) \left\{ 1 - \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}} \right\}^{-1} \\ &\Leftrightarrow d_2 \left\{ 1 - \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}} \right\} > (d_2 - 2) \left\{ 1 - \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} \right\} \\ &\Leftrightarrow 2 + (d_2 - 2) \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} > d_2 \sqrt{\frac{2(d_1 + d_2 - 2)}{d_1(d_2 - 4)}} \\ &\Leftrightarrow 4 + \frac{2(d_1 + d_2)(d_2 - 2)}{d_1} + 4(d_2 - 2) \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} > \frac{2d_2^2(d_1 + d_2 - 2)}{d_1(d_2 - 4)} \\ &\Leftrightarrow 4(d_2 - 2) \sqrt{\frac{2(d_1 + d_2)}{d_1(d_2 - 2)}} > \frac{8d_2(d_1 + d_2 - 2)}{d_1(d_2 - 4)} \\ &\Leftrightarrow (d_2 - 4) \sqrt{d_1(d_2 - 2)(d_1 + d_2)} > \sqrt{2}d_2(d_1 + d_2 - 2) \\ &\Leftrightarrow d_1(d_2 - 2)(d_1 + d_2)(d_2 - 4)^2 > 2d_2^2(d_1 + d_2 - 2)^2. \end{aligned} \quad (2.6)$$

If $d_1 \geq 3$, then we can show that (2.6) holds without the extra assumption that $D > 0$. That is, we have that

$$D > C \Leftrightarrow d_1(d_2 - 2)(d_1 + d_2)(d_2 - 4)^2 > 2d_2^2(d_1 + d_2 - 2)^2, \quad \forall d_1 \geq 3. \quad (2.7)$$

In fact, suppose the right-hand side of (2.7) holds. By (2.4), we get:

$$D > 0 \Leftrightarrow d_1^2 d_2^2 (d_2 - 4)^2 > 4d_2^2 (d_1 + d_2 - 2)^2$$

$$\begin{aligned}
&\Leftrightarrow d_1^2 d_2^2 (d_2 - 4)^2 > 2d_1 (d_2 - 2)(d_1 + d_2)(d_2 - 4)^2 \\
&\Leftrightarrow d_1 d_2^2 > 2(d_2 - 2)(d_1 + d_2) \\
&\Leftrightarrow (d_1 - 2)d_2^2 + 4d_1 + 4d_2 > 2d_1 d_2 \\
&\Leftrightarrow (d_1 - 2)d_2 + 4 > 2d_1 \\
&\Leftrightarrow (d_1 - 2)5 + 4 > 2d_1 \\
&\Leftrightarrow d_1 > 2.
\end{aligned}$$

The distinction between $C \geq D$ and $C < D$ is important because the integral $\int_C^D t^{\frac{d_1}{2}-1} (1-t)^{\frac{d_2}{2}-1} dt$ is signed. If $C \geq D$, this integral is non-positive and the contribution favors the desired inequality. If $D > C > 0$, the integral is positive and must be controlled separately. Equation (2.7) gives an explicit characterization for this more difficult case.

Before ending this section, we would like to point out that the verifications of inequality (2.3) for different cases lead to case-specific algebraic inequalities. However, these inequalities all serve the same underlying purpose. The separate calculations for $d_1 = 1, 2, 3, 4$ arise because the beta density factor $t^{\frac{d_1}{2}-1}$ has different integer or half-integer forms and the lower endpoint is absent in some parameter ranges. Therefore, the symbolic computations given in the following sections should be viewed as certifications of a common endpoint comparison mechanism rather than as four unrelated calculations.

3. Proof of (2.2) for the cases where $d_1 = 1, 2$

For these two cases, $C = D = 0$. By (2.3), it suffices to prove the following inequality:

$$2A^{\frac{d_1}{2}} (1-A)^{\frac{d_2}{2}} < d_2 \int_A^B t^{\frac{d_1}{2}-1} (1-t)^{\frac{d_2}{2}-1} dt. \quad (3.1)$$

By virtue of the transformation $t = Aw$, we get:

$$\begin{aligned}
&(3.1) \text{ holds} \\
&\Leftrightarrow 2(1-A) < d_2 \int_1^{\frac{B}{A}} w^{\frac{d_1}{2}-1} \left(1 - \frac{A}{1-A}(w-1)\right)^{\frac{d_2}{2}-1} dw \\
&\Leftrightarrow 2(1-A) < d_2 \int_0^{\frac{B-A}{A}} (1+y)^{\frac{d_1}{2}-1} \left(1 - \frac{A}{1-A}y\right)^{\frac{d_2}{2}-1} dy \\
&\Leftrightarrow 2A(1-A) < d_2(B-A) \int_0^1 \left(1 + \frac{B-A}{A}x\right)^{\frac{d_1}{2}-1} \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}-1} dx. \quad (3.2)
\end{aligned}$$

3.1. The case where $d_1 = 2$

Suppose $d_1 = 2$. By (3.2), we get:

$$\begin{aligned}
(3.1) \text{ holds} &\Leftrightarrow 2A(1-A) < 2(1-A) \left[1 - \left(\frac{1-B}{1-A}\right)^{\frac{d_2}{2}}\right] \\
&\Leftrightarrow (1-B)^{\frac{d_2}{2}} < (1-A)^{\frac{d_2}{2}+1}.
\end{aligned}$$

We have

$$\begin{aligned}
 & (1-B)^{\frac{d_2}{2}} < (1-A)^{\frac{d_2}{2}+1} \\
 \Leftrightarrow & \left[\frac{d_2-2}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) + (d_2-2)} \right]^{\frac{d_2}{2}} < \left[\frac{d_2}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right) + d_2} \right]^{\frac{d_2}{2}+1} \\
 \Leftrightarrow & \frac{d_2}{2} \ln \left[\frac{d_2-2}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) + (d_2-2)} \right] < \left(\frac{d_2}{2} + 1 \right) \ln \left[\frac{d_2}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right) + d_2} \right] \\
 \Leftrightarrow & \frac{d_2}{2} \ln \left[1 + \frac{d_1}{d_2-2} \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) \right] > \left(\frac{d_2}{2} + 1 \right) \ln \left[1 + \frac{d_1}{d_2} \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right) \right]. \quad (3.3)
 \end{aligned}$$

For $x = 1, 2, 3$ and $3 \leq y \in \mathbb{R}$, define

$$H_x(y) := \left(\frac{y}{2} + 1 \right) \ln \left[1 + \frac{x}{y} \left(1 + \sqrt{\frac{2(x+y)}{x(y-2)}} \right) \right]. \quad (3.4)$$

We will prove (3.3) by showing that H_2 is a decreasing function for $y \in \{3, 4, 5\}$ and $H'_2(y) < 0$ for $5 \leq y \in \mathbb{R}$. A routine calculation in Mathematica yields the following decreasing values:

$$H_2(3) = 2.87436, \quad H_2(4) = 2.58363, \quad H_2(5) = 2.44523.$$

We have

$$H'_x(y) = \frac{1}{2} \ln \left[1 + \frac{x}{y} \left(1 + \sqrt{\frac{2(x+y)}{x(y-2)}} \right) \right] + \left(\frac{y}{2} + 1 \right) \left\{ \ln \left[1 + \frac{x}{y} \left(1 + \sqrt{\frac{2(x+y)}{x(y-2)}} \right) \right] \right\}'. \quad (3.5)$$

Now let $x = 2$. Define

$$p_2(y) := \frac{2}{y} \left(1 + \sqrt{\frac{2+y}{y-2}} \right).$$

We have

$$\begin{aligned}
 H'_2(y) & \leq \frac{1}{2} \left(p_2(y) - \frac{p_2^2(y)}{2} + \frac{p_2^3(y)}{3} \right) + \left(\frac{y}{2} + 1 \right) \left\{ \ln \left[1 + \frac{2}{y} \left(1 + \sqrt{\frac{2+y}{y-2}} \right) \right] \right\}' \\
 & := L_2(y).
 \end{aligned}$$

Following an exact symbolic computation in Mathematica, we obtain that for $y \geq 5$,

$$\begin{aligned}
 & -3y^3(2+y) \left(2+y+2\sqrt{\frac{2+y}{y-2}} \right) L_2(y) \\
 & = 2 \left(\frac{2+y}{y-2} \right)^{\frac{3}{2}} \left[32\sqrt{\frac{2+y}{y-2}} - 4y \left(9+5\sqrt{\frac{2+y}{y-2}} \right) + 2y^2 \left(2 - \sqrt{\frac{2+y}{y-2}} \right) + 3y^3 \right]
 \end{aligned}$$

$$\begin{aligned}
&> 2 \left(\frac{2+y}{y-2} \right)^{\frac{3}{2}} y \left(39 - 20 \sqrt{\frac{2+y}{y-2}} \right) \\
&= \frac{1}{20} \left(\frac{2+y}{y-2} \right)^{\frac{3}{2}} y \left(1.95 - \sqrt{1 + \frac{4}{y-2}} \right) \\
&\geq \frac{1}{20} \left(\frac{2+y}{y-2} \right)^{\frac{3}{2}} y \left(1.95 - \sqrt{1 + \frac{4}{5-2}} \right) \\
&> 0.42247 \cdot \frac{1}{20} \left(\frac{2+y}{y-2} \right)^{\frac{3}{2}} y.
\end{aligned}$$

Then, $H'_2(y) < 0$ for $5 \leq y \in \mathbb{R}$. Thus, $(1-B)^{\frac{d_2}{2}} < (1-A)^{\frac{d_2}{2}+1}$ by (3.3). Therefore, (3.1) holds for the case where $d_1 = 2$.

To support the rigorous monotonicity analysis above, we provide the graph of $H_2(y)$ for $y \geq 3$ in Figure 1.

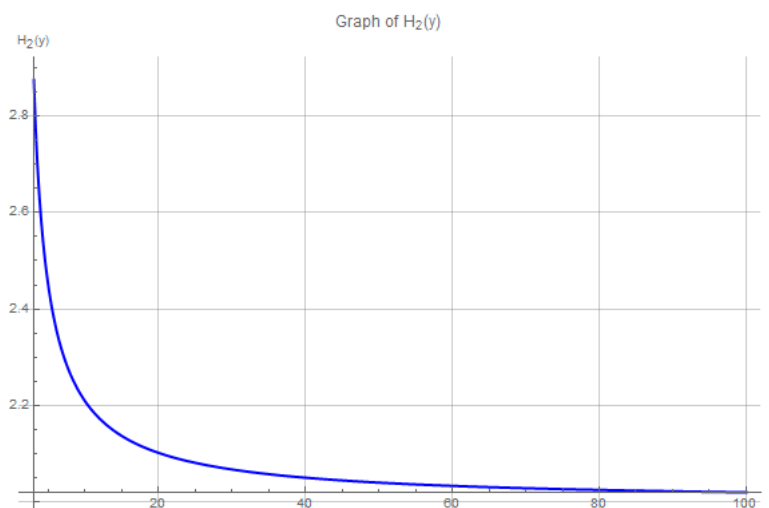


Figure 1. Function $H_2(y)$ for $y \geq 3$.

3.2. The case where $d_1 = 1$

Suppose $d_1 = 1$. By (3.2), we get:

(3.1) holds

$$\begin{aligned}
&\Leftrightarrow 2A(1-A) < d_2(B-A) \int_0^1 \left(1 + \frac{-\frac{1}{2}(B-A)}{A}x \right) \left(1 - \frac{B-A}{1-A}x \right)^{\frac{d_2}{2}-1} dx \\
&\Leftrightarrow 2A(1-A) < d_2(B-A) \left\{ \frac{3A-1}{2A} \int_0^1 \left(1 - \frac{B-A}{1-A}x \right)^{\frac{d_2}{2}-1} dx + \frac{1-A}{2A} \int_0^1 \left(1 - \frac{B-A}{1-A}x \right)^{\frac{d_2}{2}} dx \right\} \\
&\Leftrightarrow 2A(1-A) < \frac{(3A-1)(1-A)}{A} \left[1 - \left(\frac{1-B}{1-A} \right)^{\frac{d_2}{2}} \right] + \frac{d_2(1-A)^2}{(d_2+2)A} \left[1 - \left(\frac{1-B}{1-A} \right)^{\frac{d_2}{2}+1} \right]
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow 2A^2 < \frac{2(d_2+3)A-2}{d_2+2} - \frac{3(d_2+2)A-2-d_2B}{d_2+2} \left(\frac{1-B}{1-A} \right)^{\frac{d_2}{2}} \\
&\Leftrightarrow [3(d_2+2)A-2-d_2B](1-B)^{\frac{d_2}{2}} < 2[(d_2+2)A-1](1-A)^{\frac{d_2}{2}+1}.
\end{aligned} \tag{3.6}$$

Below, we analyze the coefficients of inequality (3.6). First, we have

$$(d_2+2)A > 1 \Leftrightarrow d_1(d_2+2) > d_1+d_2 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right)^{-1},$$

which holds obviously. Second, we have

$$\begin{aligned}
&3(d_2+2)A-2-d_2B > 0 \\
&\Leftrightarrow 3(d_2+2)A > 2+d_2B \\
&\Leftrightarrow \frac{3d_1(d_2+2) \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right)}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right) + d_2} > \frac{d_1(d_2+2) \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) + 2(d_2-2)}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) + (d_2-2)} \\
&\Leftrightarrow 2d_1^2(d_2+2) \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right) \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) \\
&\quad + d_1(d_2-2)(3d_2+4) \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}} \right) \\
&> d_1d_2(d_2+2) \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}} \right) + 2d_2(d_2-2) \\
&\Leftrightarrow 2(d_2-2) + (3d_2^2-4) \sqrt{\frac{2(1+d_2)}{d_2-2}} + 2(d_2+2) \left(1 + \sqrt{\frac{2(1+d_2)}{d_2-2}} \right) \sqrt{\frac{2(d_2-1)}{d_2-4}} \\
&> d_2(d_2+2) \sqrt{\frac{2(d_2-1)}{d_2-4}} \\
&\Leftrightarrow d_2(3d_2+2) \sqrt{\frac{2(1+d_2)}{d_2-2}} > d_2(d_2+2) \sqrt{\frac{2(d_2-1)}{d_2-4}} \\
&\Leftrightarrow (3d_2+2)^2(1+d_2)(d_2-4) > (d_2+2)^2(d_2-1)(d_2-2) \\
&\Leftrightarrow 2.4^2(d_2-4) > d_2-2 \\
&\Leftrightarrow 5(d_2-4) > d_2-2 \\
&\Leftrightarrow 4d_2 > 18,
\end{aligned}$$

which holds for $d_2 \geq 5$. Third, we have

$$\begin{aligned}
&2[(d_2+2)A-1] > 3(d_2+2)A-2-d_2B \\
&\Leftrightarrow d_2B > (d_2+2)A
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \frac{d_1 d_2 \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right)}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right) + (d_2 - 2)} > \frac{d_1(d_2 + 2) \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right)}{d_1 \left(1 + \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right) + d_2} \\
&\Leftrightarrow \frac{d_2 \left(1 + \sqrt{\frac{2(d_2-1)}{d_2-4}}\right)}{\left(1 + \sqrt{\frac{2(d_2-1)}{d_2-4}}\right) + (d_2 - 2)} > \frac{(d_2 + 2) \left(1 + \sqrt{\frac{2(1+d_2)}{d_2-2}}\right)}{\left(1 + \sqrt{\frac{2(1+d_2)}{d_2-2}}\right) + d_2}.
\end{aligned} \tag{3.7}$$

Define

$$K(y) := \frac{y \left(1 + \sqrt{\frac{2(y-1)}{y-4}}\right)}{\left(1 + \sqrt{\frac{2(y-1)}{y-4}}\right) + (y - 2)}, \quad 5 \leq y \in \mathbb{R}.$$

Through straightforward algebraic reduction assisted by Mathematica, we get:

$$\begin{aligned}
&-2(y-4)^2 \sqrt{\frac{y-1}{y-4}} \left(y-1 + \sqrt{\frac{2(y-1)}{y-4}}\right)^2 K'(y) \\
&= \left(3\sqrt{2} - 2\sqrt{\frac{y-1}{y-4}}\right)y^2 - \left(6\sqrt{2} - 4\sqrt{\frac{y-1}{y-4}}\right)y + 16\sqrt{\frac{y-1}{y-4}} \\
&> \left(9\sqrt{2} - 6\sqrt{\frac{y-1}{y-4}}\right)y,
\end{aligned}$$

which is positive for $y \geq 5$. Then, $K'(y) < 0$ for $y \geq 5$. Thus, by (3.7), we obtain that

$$2[(d_2 + 2)A - 1] > 3(d_2 + 2)A - 2 - d_2 B.$$

Hence, by (3.6), we conclude that

$$(3.1) \text{ holds } \Leftrightarrow (1 - B)^{\frac{d_2}{2}} < (1 - A)^{\frac{d_2}{2}+1}.$$

Let $H_x(y)$ be defined by (3.4). Note that (3.3) and (3.5) hold. Now let $x = 1$. We have

$$H'_1(y) = \frac{1}{2} \ln \left[1 + \frac{1}{y} \left(1 + \sqrt{\frac{2(1+y)}{y-2}} \right) \right] + \left(\frac{y}{2} + 1 \right) \left\{ \ln \left[1 + \frac{1}{y} \left(1 + \sqrt{\frac{2(1+y)}{y-2}} \right) \right] \right\}'.$$

Define

$$p_1(y) := \frac{1}{y} \left(1 + \sqrt{\frac{2(1+y)}{y-2}} \right).$$

We have

$$H'_1(y) \leq \frac{1}{2} \left(p_1(y) - \frac{p_1^2(y)}{2} + \frac{p_1^3(y)}{3} \right) + \left(\frac{y}{2} + 1 \right) \left\{ \ln \left[1 + \frac{1}{y} \left(1 + \sqrt{\frac{2(1+y)}{y-2}} \right) \right] \right\}'$$

$$:= L_1(y).$$

Following an exact symbolic computation in Mathematica, we obtain that for $y \geq 3$,

$$\begin{aligned} & -12y^3(y-2)^2 \sqrt{\frac{1+y}{y-2}} \left(1+y + \sqrt{\frac{2(1+y)}{y-2}} \right) L_1(y) \\ = & 32 \sqrt{\frac{1+y}{y-2}} - 4y \left(7\sqrt{2} - 2 \sqrt{\frac{1+y}{y-2}} \right) - y^2 \left(35\sqrt{2} - 4 \sqrt{\frac{1+y}{y-2}} \right) \\ & + y^3 \left(17\sqrt{2} - 23 \sqrt{\frac{1+y}{y-2}} \right) + 3y^4 \left(5\sqrt{2} + \sqrt{\frac{1+y}{y-2}} \right) \\ > & \sqrt{2}y(-28 - 35y + 16y^2) + y^3 \left(46\sqrt{2} - 23 \sqrt{\frac{1+y}{y-2}} \right) \\ > & 0. \end{aligned}$$

Then, $H'_1(y) < 0$ for $3 \leq y \in \mathbb{R}$. Thus, $(1-B)^{\frac{d_2}{2}} < (1-A)^{\frac{d_2}{2}+1}$ by (3.3). Therefore, (3.1) holds for the case where $d_1 = 1$.

To support the rigorous monotonicity analysis above, we provide the graph of $H_1(y)$ for $y \geq 3$ in Figure 2.

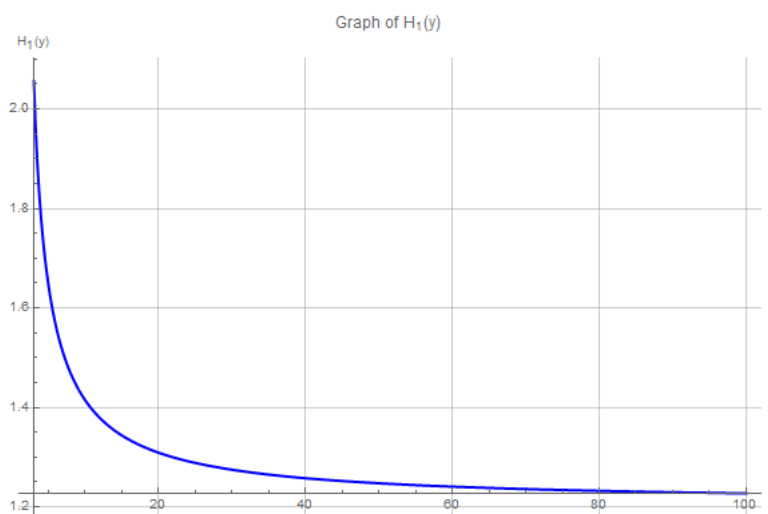


Figure 2. Function $H_1(y)$ for $y \geq 3$.

4. Proof of (2.2) for the cases where $d_1 = 3, 4$

By (2.3), it suffices to prove the following two inequalities:

$$2A^{\frac{d_1}{2}}(1-A)^{\frac{d_2}{2}} < d_2 \int_A^B t^{\frac{d_1}{2}-1}(1-t)^{\frac{d_2}{2}-1} dt, \quad (4.1)$$

$$2C^{\frac{d_1}{2}}(1-C)^{\frac{d_2}{2}} > d_2 \int_C^D t^{\frac{d_1}{2}-1}(1-t)^{\frac{d_2}{2}-1} dt. \quad (4.2)$$

4.1. The case where $d_1 = 4$

4.1.1. Proof of (4.1)

By (3.2), we get:

$$\begin{aligned}
 & (4.1) \text{ holds} \\
 \Leftrightarrow & 2A(1-A) < d_2(B-A) \int_0^1 \left(1 + \frac{B-A}{A}x\right) \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}-1} dx \\
 \Leftrightarrow & 2A(1-A) < d_2(B-A) \left\{ \frac{1}{A} \int_0^1 \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}-1} dx - \frac{1-A}{A} \int_0^1 \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}} dx \right\} \\
 \Leftrightarrow & 2A(1-A) < \left\{ \frac{2(1-A)}{A} \left[1 - \left(\frac{1-B}{1-A} \right)^{\frac{d_2}{2}} \right] - \frac{2d_2(1-A)^2}{(d_2+2)A} \left[1 - \left(\frac{1-B}{1-A} \right)^{\frac{d_2}{2}+1} \right] \right\} \\
 \Leftrightarrow & (d_2+2)A^2 < \left\{ (d_2A+2) - (d_2B+2) \left(\frac{1-B}{1-A} \right)^{\frac{d_2}{2}} \right\} \\
 \Leftrightarrow & (d_2+2)A^2(1-A)^{\frac{d_2}{2}} < \left\{ (d_2A+2)(1-A)^{\frac{d_2}{2}} - (d_2B+2)(1-B)^{\frac{d_2}{2}} \right\} \\
 \Leftrightarrow & (d_2B+2)(1-B)^{\frac{d_2}{2}} < [(d_2+2)A+2](1-A)^{\frac{d_2}{2}+1} \\
 \Leftrightarrow & \ln(d_2B+2) + \frac{d_2}{2} \ln(1-B) < \ln[(d_2+2)A+2] + \left(\frac{d_2}{2} + 1 \right) \ln(1-A) \\
 \Leftrightarrow & \ln \left[\frac{4(d_2+2) \left(1 + \sqrt{\frac{2+d_2}{2(d_2-4)}} \right) + 2(d_2-2)}{4 \left(1 + \sqrt{\frac{2+d_2}{2(d_2-4)}} \right) + (d_2-2)} \right] - \frac{d_2}{2} \ln \left[1 + \frac{4}{d_2-2} \left(1 + \sqrt{\frac{2+d_2}{2(d_2-4)}} \right) \right] \\
 & < \ln \left[\frac{4(d_2+4) \left(1 + \sqrt{\frac{4+d_2}{2(d_2-2)}} \right) + 2d_2}{4 \left(1 + \sqrt{\frac{4+d_2}{2(d_2-2)}} \right) + d_2} \right] - \frac{d_2+2}{2} \ln \left[1 + \frac{4}{d_2} \left(1 + \sqrt{\frac{4+d_2}{2(d_2-2)}} \right) \right]. \quad (4.3)
 \end{aligned}$$

For $3 \leq y \in \mathbb{R}$, define

$$\begin{aligned}
 H_4(y) &:= -\frac{y+2}{2} \ln \left[1 + \frac{4}{y} \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) \right] + \ln \left[\frac{4(y+4) \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + 2y}{4 \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + y} \right] \\
 &= \frac{y+2}{2} \ln y - \frac{y+4}{2} \ln \left[4 \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + y \right] + \ln \left[4(y+4) \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + 2y \right].
 \end{aligned}$$

We will prove (4.3) by showing that H_4 is an increasing function for $y \in \{3, 4, \dots, 12\}$ and $H'_4(y) > 0$ for $12 \leq y \in \mathbb{R}$. A routine calculation in Mathematica yields the following decreasing values:

$$\begin{aligned}
 H_4(3) &= -2.15017, \quad H_4(4) = -1.85244, \quad H_4(5) = -1.70932, \quad H_4(6) = -1.62225, \\
 H_4(7) &= -1.563, \quad H_4(8) = -1.51983, \quad H_4(9) = -1.48688, \quad H_4(10) = -1.46088,
 \end{aligned}$$

$$H_4(11) = -1.43981, \quad H_4(12) = -1.42238.$$

We have

$$\begin{aligned} H'_4(y) &= \frac{1}{2} \ln y + \frac{y+2}{2y} \\ &\quad - \frac{1}{2} \ln \left[4 \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + y \right] - \frac{y+4}{2} \left\{ \ln \left[4 \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + y \right] \right\}' \\ &\quad + \left\{ \ln \left[4(y+4) \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + 2y \right] \right\}' \\ &= -\frac{1}{2} \ln \left[1 + \frac{4}{y} \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) \right] + \frac{y+2}{2y} \\ &\quad - \frac{y+4}{2} \left\{ \ln \left[4 \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + y \right] \right\}' + \left\{ \ln \left[4(y+4) \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + 2y \right] \right\}'. \end{aligned}$$

Define

$$p_4(y) := \frac{4}{y} \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right).$$

We have

$$\begin{aligned} -H'_4(y) &\leq \frac{1}{2} \left(p_4(y) - \frac{p_4^2(y)}{2} + \frac{p_4^3(y)}{3} \right) - \frac{y+2}{2y} \\ &\quad + \frac{y+4}{2} \left\{ \ln \left[4 \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + y \right] \right\}' - \left\{ \ln \left[4(y+4) \left(1 + \sqrt{\frac{4+y}{2(y-2)}} \right) + 2y \right] \right\}' \\ &:= L_4(y). \end{aligned}$$

Following an exact symbolic computation in Mathematica, we get:

$$\begin{aligned} &-3y^3(y-2)^2 \left(4 + y + 2 \sqrt{\frac{2(4+y)}{y-2}} \right) \left[8 + 4 \sqrt{\frac{2(4+y)}{y-2}} + y \left(3 + \sqrt{\frac{2(4+y)}{y-2}} \right) \right] L_4(y) \\ &= 4 \sqrt{\frac{4+y}{y-2}} \left[(2048 + 1536y - 1040y^2 - 368y^3 + 10y^4 + 3y^5) \sqrt{2} \right. \\ &\quad \left. + (-2048 + 3072y - 112y^2 - 484y^3 + 8y^4 + 3y^5) \sqrt{\frac{4+y}{y-2}} \right]. \end{aligned}$$

Define

$$\begin{aligned} T_1(y) &:= 2048 + 1536y - 1040y^2 - 368y^3 + 10y^4 + 3y^5, \\ T_2(y) &:= -2048 + 3072y - 112y^2 - 484y^3 + 8y^4 + 3y^5. \end{aligned}$$

We have

$$\begin{aligned}T_1(12+r) &= 188672 + 197760r + 46192r^2 + 4432r^3 + 190r^4 + 3r^5, \\T_2(12+r) &= 94720 + 157632r + 41216r^2 + 4220r^3 + 188r^4 + 3r^5.\end{aligned}$$

Then, $T_1(y), T_2(y) > 0$ for $12 \leq y \in \mathbb{R}$. Thus, $H'_4(y) > 0$ for $12 \leq y \in \mathbb{R}$. Therefore, (4.3) holds for the case where $d_1 = 4$.

To support the rigorous monotonicity analysis above, we provide the graph of $H_4(y)$ for $y \geq 3$ in Figure 3.

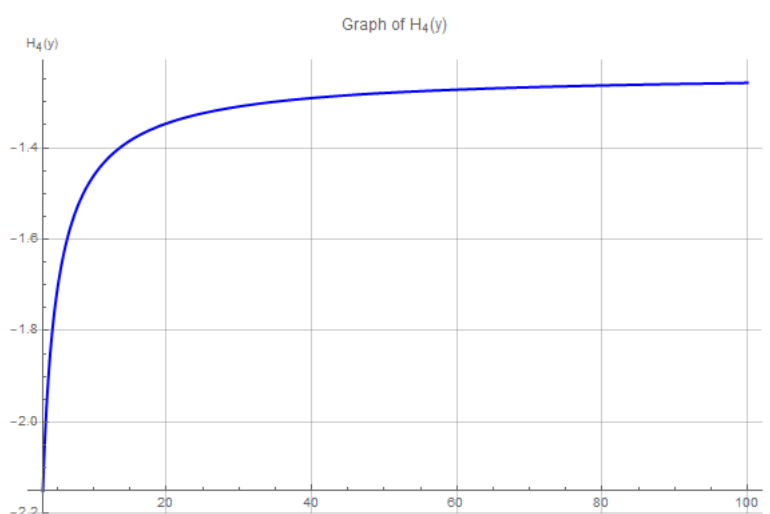


Figure 3. Function $H_4(y)$ for $y \geq 3$.

4.1.2. Proof of (4.2)

We assume without loss of generality that $D > C > 0$. By (2.5), we get $D > 0 \Leftrightarrow d_2 \geq 11$. Further, by (2.7), we get:

$$\begin{aligned}D > C > 0 &\Leftrightarrow 2(d_2 - 2)(d_2 + 4)(d_2 - 4)^2 > d_2^2(d_2 + 2)^2 \\&\Leftrightarrow -256 + 192d_2 - 20d_2^2 - 16d_2^3 + d_2^4 > 0.\end{aligned}$$

Define

$$P_4(d_2) := -256 + 192d_2 - 20d_2^2 - 16d_2^3 + d_2^4.$$

We have

$$\begin{aligned}P_4(11) &= -7219, \quad P_4(12) = -7744, \quad P_4(13) = -7731, \\P_4(14) &= -6976, \quad P_4(15) = -5251, \quad P_4(16) = -2304, \\P_4(17+r) &= 2141 + 5292r + 898r^2 + 52r^3 + r^4.\end{aligned}$$

Then,

$$D > C > 0 \Leftrightarrow d_2 \geq 17.$$

Similar to (4.3), we get:

(4.2) holds

$$\begin{aligned}
 &\Leftrightarrow \ln(d_2 D + 2) + \frac{d_2}{2} \ln(1 - D) > \ln[(d_2 + 2)C + 2] + \left(\frac{d_2}{2} + 1\right) \ln(1 - C) \\
 &\Leftrightarrow \ln \left[\frac{4(d_2 + 2) \left(1 - \sqrt{\frac{2+d_2}{2(d_2-4)}}\right) + 2(d_2 - 2)}{4 \left(1 - \sqrt{\frac{2+d_2}{2(d_2-4)}}\right) + (d_2 - 2)} \right] - \frac{d_2}{2} \ln \left[1 + \frac{4}{d_2 - 2} \left(1 - \sqrt{\frac{2+d_2}{2(d_2-4)}}\right) \right] \\
 &> \ln \left[\frac{4(d_2 + 4) \left(1 - \sqrt{\frac{4+d_2}{2(d_2-2)}}\right) + 2d_2}{4 \left(1 - \sqrt{\frac{4+d_2}{2(d_2-2)}}\right) + d_2} \right] - \frac{d_2 + 2}{2} \ln \left[1 + \frac{4}{d_2} \left(1 - \sqrt{\frac{4+d_2}{2(d_2-2)}}\right) \right]. \quad (4.4)
 \end{aligned}$$

For $15 \leq y \in \mathbb{R}$, define

$$\begin{aligned}
 R_4(y) &:= -\frac{y+2}{2} \ln \left[1 + \frac{4}{y} \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) \right] + \ln \left[\frac{4(y+4) \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + 2y}{4 \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + y} \right] \\
 &= \frac{y+2}{2} \ln y - \frac{y+4}{2} \ln \left[4 \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + y \right] + \ln \left[4(y+4) \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + 2y \right].
 \end{aligned}$$

We will prove (4.4) by showing that $R'_4(y) < 0$ for $15 \leq y \in \mathbb{R}$.

We have

$$\begin{aligned}
 R'_4(y) &= \frac{1}{2} \ln y + \frac{y+2}{2y} \\
 &\quad - \frac{1}{2} \ln \left[4 \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + y \right] - \frac{y+4}{2} \left\{ \ln \left[4 \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + y \right] \right\}' \\
 &\quad + \left\{ \ln \left[4(y+4) \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + 2y \right] \right\}' \\
 &= -\frac{1}{2} \ln \left[1 + \frac{4}{y} \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) \right] + \frac{y+2}{2y} \\
 &\quad - \frac{y+4}{2} \left\{ \ln \left[4 \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + y \right] \right\}' + \left\{ \ln \left[4(y+4) \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right) + 2y \right] \right\}'.
 \end{aligned}$$

Define

$$r(y) := \frac{4}{y} \left(1 - \sqrt{\frac{4+y}{2(y-2)}}\right).$$

We have

$$-R'_4(y) \geq \frac{1}{2} \left(r(y) - \frac{r^2(y)}{2} \right) - \frac{y+2}{2y}$$

$$\begin{aligned}
& + \frac{y+4}{2} \left\{ \ln \left[4 \left(1 - \sqrt{\frac{4+y}{2(y-2)}} \right) + y \right] \right\}' - \left\{ \ln \left[4(y+4) \left(1 - \sqrt{\frac{4+y}{2(y-2)}} \right) + 2y \right] \right\}' \\
& := Q_4(y).
\end{aligned}$$

Through straightforward algebraic reduction assisted by Mathematica, we obtain that for $y \geq 15$,

$$\begin{aligned}
& y^2(4+y) \left(4+y-2\sqrt{\frac{2(4+y)}{y-2}} \right) \left[8-4\sqrt{\frac{2(4+y)}{y-2}} + y \left(3 - \sqrt{\frac{2(4+y)}{y-2}} \right) \right] Q_4(y) \\
& = 4 \left(\frac{4+y}{y-2} \right)^{\frac{3}{2}} \left[128 \sqrt{\frac{4+y}{y-2}} + y \left(88\sqrt{2} - 40 \sqrt{\frac{4+y}{y-2}} \right) + y^2 \left(26\sqrt{2} - 34 \sqrt{\frac{4+y}{y-2}} \right) \right. \\
& \quad \left. + y^3 \left(\sqrt{2} - \sqrt{\frac{4+y}{y-2}} \right) \right].
\end{aligned}$$

Note that the first term in the square brackets above is decreasing in y , whereas the remaining three terms are increasing in y . Then, we have that for $y \geq 15$,

$$\begin{aligned}
& y^2(4+y) \left(4+y-2\sqrt{\frac{2(4+y)}{y-2}} \right) \left[8-4\sqrt{\frac{2(4+y)}{y-2}} + y \left(3 - \sqrt{\frac{2(4+y)}{y-2}} \right) \right] Q_4(y) \\
& > 4 \left(\frac{4+y}{y-2} \right)^{\frac{3}{2}} \min \left\{ \min_{y \geq 17} \left[y^2 \left(43\sqrt{2} - 51 \sqrt{\frac{4+y}{y-2}} \right) \right], 128 \sqrt{\frac{4+17}{17-2}} + 15 \left(88\sqrt{2} - 40 \sqrt{\frac{4+15}{15-2}} \right) \right. \\
& \quad \left. + 15^2 \left(26\sqrt{2} - 34 \sqrt{\frac{4+15}{15-2}} \right) + 15^3 \left(\sqrt{2} - \sqrt{\frac{4+15}{15-2}} \right) \right\} \\
& > \frac{4}{51} \left(\frac{4+y}{y-2} \right)^{\frac{3}{2}} \min \left\{ \min_{y \geq 17} \left[y^2 \left(1.19 - \sqrt{1 + \frac{6}{y-2}} \right) \right], 19.81 \right\} \\
& \geq \frac{4}{51} \left(\frac{4+y}{y-2} \right)^{\frac{3}{2}} \min \left\{ 17^2 \left(1.19 - \sqrt{1 + \frac{6}{17-2}} \right), 19.81 \right\} \\
& > 1.96 \cdot \frac{4}{51} \left(\frac{4+y}{y-2} \right)^{\frac{3}{2}}.
\end{aligned}$$

Then, $R'_4(y) < 0$ for $15 \leq y \in \mathbb{R}$. Therefore, (4.4) holds for the case where $d_1 = 4$.

To support the rigorous monotonicity analysis above, we provide the graph of $R_4(y)$ for $y \geq 15$ in Figure 4.

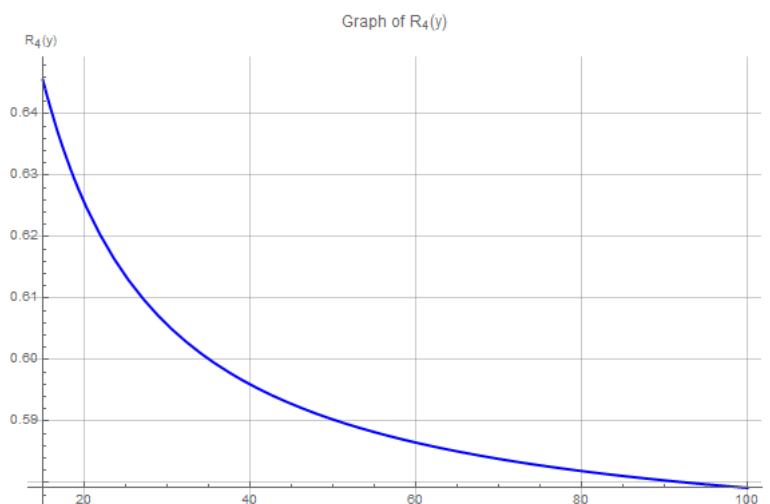


Figure 4. Function $R_4(y)$ for $y \geq 15$.

4.2. The case where $d_1 = 3$

4.2.1. Proof of (4.1)

By (3.2), we get:

$$\begin{aligned}
 (4.1) \text{ holds} &\Leftrightarrow 2A(1-A) < d_2(B-A) \int_0^1 \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}-1} dx \\
 &\Leftrightarrow 2A(1-A) < 2(1-A) \left[1 - \left(\frac{1-B}{1-A}\right)^{\frac{d_2}{2}}\right] \\
 &\Leftrightarrow (1-B)^{\frac{d_2}{2}} < (1-A)^{\frac{d_2}{2}+1}.
 \end{aligned} \tag{4.5}$$

Let $H_x(y)$ be defined by (3.4). Note that (3.3) and (3.5) hold. Now let $x = 3$. We have

$$H_3(y) = \left(\frac{y}{2} + 1\right) \ln \left[1 + \frac{3}{y} \left(1 + \sqrt{\frac{2(3+y)}{3(y-2)}}\right)\right].$$

We will prove (4.5) by showing that H_3 is a decreasing function for $y \in \{3, 4, \dots, 12\}$ and $H'_3(y) < 0$ for $12 \leq y \in \mathbb{R}$. A routine calculation in Mathematica yields the following decreasing values:

$$\begin{aligned}
 H_3(3) &= 3.46574, \quad H_3(4) = 3.18962, \quad H_3(5) = 3.06414, \quad H_3(6) = 2.99125, \\
 H_3(7) &= 2.94353, \quad H_3(8) = 2.9099, \quad H_3(9) = 2.88497, \quad H_3(10) = 2.86578, \\
 H_3(11) &= 2.85058, \quad H_3(12) = 2.83826.
 \end{aligned}$$

We have

$$H'_3(y) = \frac{1}{2} \ln \left[1 + \frac{3}{y} \left(1 + \sqrt{\frac{2(3+y)}{3(y-2)}}\right)\right] + \left(\frac{y}{2} + 1\right) \left\{ \ln \left[1 + \frac{3}{y} \left(1 + \sqrt{\frac{2(3+y)}{3(y-2)}}\right)\right] \right\}'.$$

Define

$$p_3(y) := \frac{3}{y} \left(1 + \sqrt{\frac{2(3+y)}{3(y-2)}} \right).$$

We have

$$\begin{aligned} H'_3(y) &\leq \frac{1}{2} \left(p_3(y) - \frac{p_3^2(y)}{2} + \frac{p_3^3(y)}{3} \right) + \left(\frac{y}{2} + 1 \right) \left\{ \ln \left[1 + \frac{3}{y} \left(1 + \sqrt{\frac{2(3+y)}{3(y-2)}} \right) \right] \right\}' \\ &:= L_3(y). \end{aligned}$$

Through straightforward algebraic reduction assisted by Mathematica and the fact that

$$\sqrt{6} > 1.5 \sqrt{\frac{3+y}{y-2}}, \quad y \geq 12,$$

we obtain that for $y \geq 12$,

$$\begin{aligned} &-4y^3(y-2)^2 \sqrt{\frac{3+y}{y-2}} \left(3+y + \sqrt{\frac{6(3+y)}{y-2}} \right) L_3(y) \\ &= 864 \sqrt{\frac{3+y}{y-2}} - 36y \left(11\sqrt{6} + 6\sqrt{\frac{3+y}{y-2}} \right) - 3y^2 \left(29\sqrt{6} + 88\sqrt{\frac{3+y}{y-2}} \right) \\ &\quad + y^3 \left(19\sqrt{6} + 9\sqrt{\frac{3+y}{y-2}} \right) + 3y^4 \left(\sqrt{6} - \sqrt{\frac{3+y}{y-2}} \right) \\ &> y \left[(-396 - 87y + 19y^2) \sqrt{6} + (-216 - 264y + 9y^2 + 1.5y^3) \sqrt{\frac{3+y}{y-2}} \right]. \end{aligned}$$

Define

$$\begin{aligned} U_1(y) &:= -396 - 87y + 19y^2, \\ U_2(y) &:= 2(-216 - 264y + 9y^2 + 1.5y^3). \end{aligned}$$

We have

$$\begin{aligned} U_1(12+r) &= 1296 + 369r + 19r^2, \\ U_2(12+r) &= 1008 + 1200r + 126r^2 + 3r^3. \end{aligned}$$

Then, $U_1(y), U_2(y) > 0$ for $12 \leq y \in \mathbb{R}$. Thus, $H'_3(y) < 0$ for $12 \leq y \in \mathbb{R}$. Therefore, (4.5) holds for the case where $d_1 = 3$.

To support the rigorous monotonicity analysis above, we provide the graph of $H_3(y)$ for $y \geq 3$ in Figure 5.

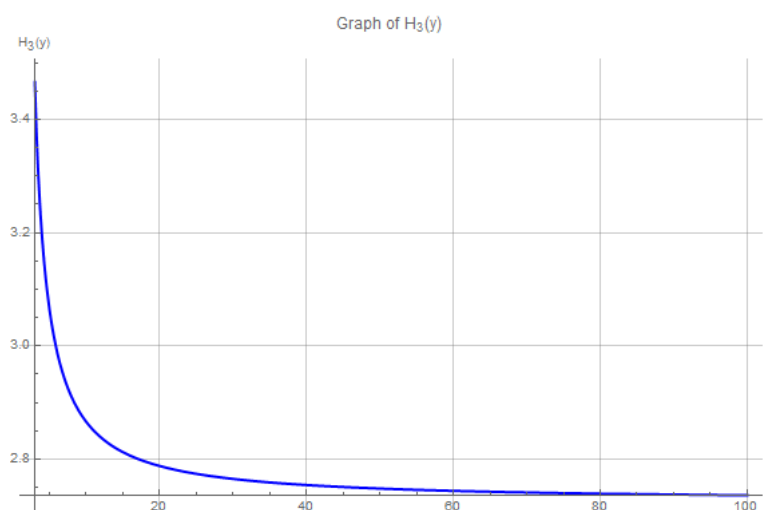


Figure 5. Function $H_3(y)$ for $y \geq 3$.

4.2.2. Proof of (4.2)

We assume without loss of generality that $D > C > 0$. By (2.5), we get $D > 0 \Leftrightarrow d_2 \geq 15$. Further, by (2.7), we get:

$$\begin{aligned} D > C > 0 &\Leftrightarrow 3(d_2 - 2)(d_2 + 3)(d_2 - 4)^2 > 2d_2^2(d_2 + 1)^2 \\ &\Leftrightarrow -288 + 192d_2 + 4d_2^2 - 25d_2^3 + d_2^4 > 0. \end{aligned}$$

Define

$$P_3(d_2) := -288 + 192d_2 + 4d_2^2 - 25d_2^3 + d_2^4.$$

We have

$$\begin{aligned} P_3(15) &= -30258, \quad P_3(16) = -33056, \quad P_3(17) = -35172, \quad P_3(18) = -36360, \\ P_3(19) &= -36350, \quad P_3(20) = -34848, \quad P_3(21) = -31536, \quad P_3(22) = -26072, \\ P_3(23) &= -18090, \quad P_3(24) = -7200, \\ P(25 + r) &= 7012 + 16017r + 1879r^2 + 75r^3 + r^4. \end{aligned}$$

Then,

$$D > C > 0 \Leftrightarrow d_2 \geq 25.$$

Similar to (3.2), we get:

$$(4.2) \text{ holds} \Leftrightarrow 2C(1 - C) > d_2(D - C) \int_0^1 \left(1 + \frac{D - C}{C}x\right)^{0.5} \left(1 - \frac{D - C}{1 - C}x\right)^{\frac{d_2}{2} - 1} dx.$$

Then,

$$(4.2) \text{ holds} \Leftrightarrow 2C(1 - C) > d_2(D - C) \int_0^1 \left(1 + \frac{D - C}{2C}x\right) \left(1 - \frac{D - C}{1 - C}x\right)^{\frac{d_2}{2} - 1} dx. \quad (4.6)$$

We have

$$\begin{aligned}
& 2C(1-C) > d_2(D-C) \int_0^1 \left(1 + \frac{D-C}{2C}x\right) \left(1 - \frac{D-C}{1-C}x\right)^{\frac{d_2}{2}-1} dx \\
\Leftrightarrow & 2C(1-C) > d_2(D-C) \left\{ \frac{1+C}{2C} \int_0^1 \left(1 - \frac{D-C}{1-C}x\right)^{\frac{d_2}{2}-1} dx - \frac{1-C}{2C} \int_0^1 \left(1 - \frac{D-C}{1-C}x\right)^{\frac{d_2}{2}} dx \right\} \\
\Leftrightarrow & 2C(1-C) > \frac{1-C^2}{C} \left[1 - \left(\frac{1-D}{1-C}\right)^{\frac{d_2}{2}} \right] - \frac{d_2(1-C)^2}{(d_2+2)C} \left[1 - \left(\frac{1-D}{1-C}\right)^{\frac{d_2}{2}+1} \right] \\
\Leftrightarrow & 2C^2 > \frac{2(d_2+1)C+2}{d_2+2} - \frac{2(1+C)+d_2(C+D)}{d_2+2} \left(\frac{1-D}{1-C}\right)^{\frac{d_2}{2}} \\
\Leftrightarrow & [2(1+C)+d_2(C+D)](1-D)^{\frac{d_2}{2}} > 2[(d_2+2)C+1](1-C)^{\frac{d_2}{2}+1} \\
\Leftrightarrow & \ln[2(1+C)+d_2(C+D)] + \frac{d_2}{2} \ln(1-D) > \ln\{2[(d_2+2)C+1]\} + \left(\frac{d_2}{2}+1\right) \ln(1-C) \\
\Leftrightarrow & \ln\left(1 + \frac{C}{1-C}\right) > \ln\left(1 + \frac{(d_2+2)C-d_2D}{2(1+C)+d_2(C+D)}\right) + \frac{d_2}{2} \ln\left(1 + \frac{D-C}{1-D}\right),
\end{aligned}$$

which together with (4.6) implies that

$$(4.2) \text{ holds} \Leftrightarrow \ln\left(1 + \frac{C}{1-C}\right) > \ln\left(1 + \frac{(d_2+2)C-d_2D}{2(1+C)+d_2(C+D)}\right) + \frac{d_2}{2} \ln\left(1 + \frac{D-C}{1-D}\right). \quad (4.7)$$

Note that

$$\begin{aligned}
& (d_2+2)C > d_2D \\
\Leftrightarrow & \frac{d_2+2}{d_1+d_2\left(1 - \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right)^{-1}} > \frac{d_2}{d_1+(d_2-2)\left(1 - \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right)^{-1}} \\
\Leftrightarrow & 2d_1\left(1 - \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right)\left(1 - \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right) + (d_2^2-4)\left(1 - \sqrt{\frac{2(d_1+d_2)}{d_1(d_2-2)}}\right) \\
& > d_2^2\left(1 - \sqrt{\frac{2(d_1+d_2-2)}{d_1(d_2-4)}}\right) \\
\Leftrightarrow & \left(d_2^2-6+6\sqrt{\frac{2(3+d_2)}{3(d_2-2)}}\right)\sqrt{\frac{2(d_2+1)}{3(d_2-4)}}+2 > (d_2^2+2)\sqrt{\frac{2(3+d_2)}{3(d_2-2)}} \\
\Leftrightarrow & (d_2^2-6+2\sqrt{6})\sqrt{\frac{2(d_2+1)}{3(d_2-4)}} > (d_2^2+2)\sqrt{\frac{2(3+d_2)}{3(d_2-2)}} \\
\Leftrightarrow & (d_2^2-2)\sqrt{\frac{2(d_2+1)}{3(d_2-4)}} > (d_2^2+2)\sqrt{\frac{2(3+d_2)}{3(d_2-2)}} \\
\Leftrightarrow & (d_2^2-2)^2(d_2+1)(d_2-2) > (d_2^2+2)^2(3+d_2)(d_2-4)
\end{aligned}$$

$$\Leftrightarrow 2(20 + 28d_2^2 + 4d_2^3 + d_2^4) > 0.$$

Then, $(d_2 + 2)C > d_2D$. Since $\ln(1 + x) > x - \frac{x^2}{2}$ and $\ln(1 + x) < x$ for any $x > 0$, we get:

$$\begin{aligned} \ln\left(1 + \frac{C}{1-C}\right) &> \ln\left(1 + \frac{(d_2 + 2)C - d_2D}{2(1+C) + d_2(C+D)}\right) + \frac{d_2}{2} \ln\left(1 + \frac{D-C}{1-D}\right) \\ \Leftrightarrow \frac{C}{1-C} \left(1 - \frac{C}{2(1-C)}\right) &> \frac{(d_2 + 2)C - d_2D}{2(1+C) + d_2(C+D)} + \frac{d_2(D-C)}{2(1-D)}, \end{aligned}$$

which together with (4.7) implies that

$$(4.2) \text{ holds } \Leftrightarrow \frac{C}{1-C} \left(1 - \frac{C}{2(1-C)}\right) > \frac{(d_2 + 2)C - d_2D}{2(1+C) + d_2(C+D)} + \frac{d_2(D-C)}{2(1-D)}. \quad (4.8)$$

For $5 \leq y \in \mathbb{R}$, define

$$\begin{aligned} C(y) &:= \frac{3}{3 + y \left(1 - \sqrt{\frac{2(3+y)}{3(y-2)}}\right)^{-1}}, \quad D(y) := \frac{3}{3 + (y-2) \left(1 - \sqrt{\frac{2(y+1)}{3(y-4)}}\right)^{-1}}, \\ V(y) &:= \frac{C(y)}{1-C(y)} \left(1 - \frac{C(y)}{2(1-C(y))}\right) - \frac{(y+2)C(y) - yD(y)}{2(1+C(y)) + y(C(y) + D(y))} - \frac{y(D(y) - C(y))}{2(1-D(y))}, \end{aligned}$$

and

$$s_1(y) := \sqrt{\frac{y+1}{y-4}}, \quad s_2(y) := \sqrt{\frac{y+3}{y-2}}.$$

Through straightforward algebraic reduction assisted by Mathematica, we get:

$$V(y) = \frac{G_1(y)}{G_2(y)},$$

where

$$\begin{aligned} G_1(y) = & -3 \left\{ y^8 [2\sqrt{6}s_1(y) - 2\sqrt{6}s_2(y)] \right. \\ & + y^7 [-4 - 8\sqrt{6}s_1(y) + 8\sqrt{6}s_2(y) + 4s_1(y)s_2(y)] \\ & + y^6 [-16 - 25\sqrt{6}s_1(y) + 23\sqrt{6}s_2(y) - 20s_1(y)s_2(y)] \\ & + y^5 [-392 + 77\sqrt{6}s_1(y) + 125\sqrt{6}s_2(y) - 70s_1(y)s_2(y)] \\ & + y^4 [-500 + 410\sqrt{6}s_1(y) - 364\sqrt{6}s_2(y) - 304s_1(y)s_2(y)] \\ & + 4y^3 [1134 - 108\sqrt{6}s_1(y) - 457\sqrt{6}s_2(y) + 654s_1(y)s_2(y)] \\ & + 16y^2 [435 - 237\sqrt{6}s_1(y) + 160\sqrt{6}s_2(y) + 192s_1(y)s_2(y)] \\ & + 192y [-54 + 11\sqrt{6}s_1(y) - 66s_1(y)s_2(y)] \\ & \left. + 6912 [-1 + \sqrt{6}s_1(y)] \right\} \end{aligned} \quad (4.9)$$

and

$$\begin{aligned}
 G_2(y) = & 2(-4+y)(-2+y)^2 y^2 \left[3+y-\sqrt{6}s_2(y) \right] \\
 & \cdot \left\{ y^2 \left[-8+\sqrt{6}s_1(y)+\sqrt{6}s_2(y) \right] + 4 \left[-3+3\sqrt{6}s_1(y)+\sqrt{6}s_2(y)-6s_1(y)s_2(y) \right] \right. \\
 & \left. + 2y \left[-13+4\sqrt{6}s_1(y)+4\sqrt{6}s_2(y)-6s_1(y)s_2(y) \right] \right\}.
 \end{aligned}$$

In what follows, we will show that both $G_1(y)$ and $G_2(y)$ are negative for $y \in \{25, 26, \dots\}$. Once this is proved, we deduce that $V(y)$ is a positive function of $y \in \{25, 26, \dots\}$ and then the proof of (4.2) is complete by (4.8).

First, we show that $G_1(y)$ is negative for $y \in \{25, 26, \dots\}$. A routine calculation in Mathematica yields

$$\begin{aligned}
 G_1(25) &= -1.46179 \cdot 10^9, \quad G_1(26) = -1.91825 \cdot 10^9, \quad G_1(27) = -2.48438 \cdot 10^9, \\
 G_1(28) &= -3.17992 \cdot 10^9, \quad G_1(29) = -4.02709 \cdot 10^9, \quad G_1(30) = -5.05084 \cdot 10^9, \\
 G_1(31) &= -6.27904 \cdot 10^9, \quad G_1(32) = -7.74269 \cdot 10^9, \quad G_1(33) = -9.47617 \cdot 10^9.
 \end{aligned}$$

Then, we need only show that $G_1(y)$ is negative for $y \in \{34, 35, \dots\}$.

We consider expression (4.9) of $G_1(y)$. Note that

$$1 < s_2(y) < s_1(y) \leq \sqrt{\frac{26}{21}} \approx 1.112697, \quad 25 \leq y \in \mathbb{R}.$$

Then, for $25 \leq y \in \mathbb{R}$, we have

$$6912 \left[-1 + \sqrt{6}s_1(y) \right] > 0,$$

$$\begin{aligned}
 & y^5 \left[-392 + 77\sqrt{6}s_1(y) + 125\sqrt{6}s_2(y) - 70s_1(y)s_2(y) \right] \\
 & + 192y \left[-54 + 11\sqrt{6}s_1(y) - 66s_1(y)s_2(y) \right] \\
 & > y \left\{ 25^4 \left[-392 + 202\sqrt{6} - 70(1.113)^2 \right] + 192 \left[-54 + 11\sqrt{6} - 66(1.113)^2 \right] \right\} \\
 & \approx (6.26157 \cdot 10^6)y > 0,
 \end{aligned}$$

$$\begin{aligned}
 & 16y^2 \left[435 - 237\sqrt{6}s_1(y) + 160\sqrt{6}s_2(y) + 192s_1(y)s_2(y) \right] \\
 & > 16y^2 \left[435 - 237\sqrt{6}(1.113) + 160\sqrt{6} + 192 \right] \approx (16y^2)(372.7895) > 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & 4y^3 \left[1134 - 108\sqrt{6}s_1(y) - 457\sqrt{6}s_2(y) + 654s_1(y)s_2(y) \right] \\
 & > 4y^3 \left[1134 - 565\sqrt{6}(1.113) + 654 \right] \approx (4y^3)(247.6506) > 0.
 \end{aligned}$$

We now consider the other terms in expression (4.9) of $G_1(y)$. Note that

$$(y-5)s_1(y)s_2(y) - y > (y-5)[s_2(y)]^2 - y = -\frac{15}{y-2},$$

and, by setting $y = 30 + r$ with $r \geq 0$, we get:

$$\begin{aligned} & \sqrt{y-4}(2y^2 - 8y - 23) > 1.93 \sqrt{y+1}(y^2 - 6y + 8) \\ \Leftrightarrow & -23543936 - 8238032y + 6438956y^2 - 489960y^3 - 70261y^4 + 2751y^5 > 0 \\ \Leftrightarrow & 2233345504 + 2608569328r + 325703156r^2 + 15837720r^3 + 342389r^4 + 2751r^5 > 0. \end{aligned}$$

Then, we obtain that for $34 \leq y \in \mathbb{R}$,

$$\begin{aligned} & y^8 [2\sqrt{6}s_1(y) - 2\sqrt{6}s_2(y)] + y^7 [-4 - 8\sqrt{6}s_1(y) + 8\sqrt{6}s_2(y) + 4s_1(y)s_2(y)] \\ & + y^6 [-16 - 25\sqrt{6}s_1(y) + 23\sqrt{6}s_2(y) - 20s_1(y)s_2(y)] \\ & + y^4 [-500 + 410\sqrt{6}s_1(y) - 364\sqrt{6}s_2(y) - 304s_1(y)s_2(y)] \\ = & \sqrt{6}y^6 [s_1(y) - s_2(y)](2y^2 - 8y - 23) - y^6 \{16 + 2\sqrt{6}s_1(y) - 4[(y-5)s_1(y)s_2(y) - y]\} \\ & + y^4 [-500 + 410\sqrt{6}s_1(y) - 364\sqrt{6}s_2(y) - 304s_1(y)s_2(y)] \\ > & \frac{10\sqrt{6}y^6(2y^2 - 8y - 23)}{[s_1(y) + s_2(y)](y^2 - 6y + 8)} - y^6 \left[16 + 2\sqrt{6}(1.113) + \frac{60}{y-2}\right] \\ & - y^4 [500 - 410\sqrt{6} + 364(1.113) + 304(1.113)^2] \\ > & \frac{5\sqrt{6}y^6\sqrt{y-4}(2y^2 - 8y - 23)}{\sqrt{y+1}(y^2 - 6y + 8)} - y^6 \left[16 + 2\sqrt{6}(1.113) + \frac{60}{y-2}\right] - 278y^4 \\ > & y^6 \left[5\sqrt{6} \cdot 1.93 - 16 - 2\sqrt{6}(1.113) - \frac{60}{y-2} - \frac{278}{y^2}\right] \\ \geq & y^6 \left[5\sqrt{6} \cdot 1.93 - 16 - 2\sqrt{6}(1.113) - \frac{60}{34-2} - \frac{278}{34^2}\right] \\ \approx & 0.0695274. \end{aligned}$$

Thus, $G_1(y) < 0$ for $y \in \{25, 26, \dots\}$.

Finally, we show that $G_2(y)$ is negative for $y \in \{25, 26, \dots\}$. For $25 \leq y \in \mathbb{R}$, define

$$y = 25 + x, \quad u = \sqrt{\frac{26+x}{21+x}}, \quad v = \sqrt{\frac{28+x}{23+x}}.$$

We have that

$$1 < v < u \leq \sqrt{\frac{26}{21}} \approx 1.112697.$$

Following an exact symbolic computation in Mathematica, we get:

$$G_2(y) = 2(25+x)^2(483 + 44x + x^2)$$

$$\begin{aligned}
& \cdot \left\{ -3785600 + 548100 \sqrt{6}u + 664102 \sqrt{6}v - 324162uv + x^4 \left[-8 + \sqrt{6}u + \sqrt{6}v \right] \right. \\
& \quad + x(-577824 + 80699 \sqrt{6}u + 95091 \sqrt{6}v - 37278uv) \\
& \quad + x^2(-33056 + 4451 \sqrt{6}u + 5041 \sqrt{6}v - 1422uv) \\
& \quad \left. + x^3(-840 + 109 \sqrt{6}u + 117 \sqrt{6}v - 18uv) \right\} \\
& < 2(25 + x)^2(483 + 44x + x^2) \\
& \cdot \left\{ -3785600 + 548100 \sqrt{6}(1.113) + 664102 \sqrt{6}(1.113) - 324162 \right. \\
& \quad + x^4 \left[-8 + \sqrt{6}(1.113) + \sqrt{6}(1.113) \right] \\
& \quad + x(-577824 + 80699 \sqrt{6}(1.113) + 95091 \sqrt{6}(1.113) - 37278) \\
& \quad + x^2(-33056 + 4451 \sqrt{6}(1.113) + 5041 \sqrt{6}(1.113) - 1422) \\
& \quad \left. + x^3(-840 + 109 \sqrt{6}(1.113) + 117 \sqrt{6}(1.113) - 18) \right\} \\
& < 0.
\end{aligned}$$

Therefore, the proof is complete.

5. Reduction and remaining difficulties for $d_1 \geq 5$

Theorem 1.2 provides a partial solution to Conjecture 1.1. It establishes the proposed comparison in four structurally representative cases. In this section, we will show that the same framework continues to apply when $d_1 \geq 5$, although the resulting estimates become increasingly complicated as d_1 grows. We believe that our method offers a possible route to the remaining cases. The arguments presented here do not by themselves settle the problem for $d_1 \geq 5$. Rather, they reduce it to explicit inequalities that are uniform in both d_1 and d_2 .

5.1. The case where $d_1 = 2(k + 1)$, $2 \leq k \in \mathbb{N}$

In this subsection, we assume that $d_1 \geq 6$ and d_1 is an even number. By (2.3), to complete the proof of (2.2), it suffices to prove inequalities (4.1) and (4.2).

By (2.3), we get:

$$\begin{aligned}
& (4.1) \text{ holds} \\
& \Leftrightarrow I_A\left(\frac{d_1}{2}, \frac{d_2+2}{2}\right) < I_B\left(\frac{d_1}{2}, \frac{d_2}{2}\right) \\
& \Leftrightarrow \frac{\int_0^A t^{\frac{d_1}{2}-1} (1-t)^{\frac{d_2+2}{2}-1} dt}{B\left(\frac{d_1}{2}, \frac{d_2+2}{2}\right)} < \frac{\int_0^B t^{\frac{d_1}{2}-1} (1-t)^{\frac{d_2}{2}-1} dt}{B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} \\
& \Leftrightarrow \frac{\int_{1-A}^1 (1-t)^{\frac{d_1}{2}-1} t^{\frac{d_2+2}{2}-1} dt}{B\left(\frac{d_1}{2}, \frac{d_2+2}{2}\right)} < \frac{\int_{1-B}^1 (1-t)^{\frac{d_1}{2}-1} t^{\frac{d_2}{2}-1} dt}{B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)}
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \frac{\int_0^{1-A} (1-t)^{\frac{d_1}{2}-1} t^{\frac{d_2+2}{2}-1} dt}{B\left(\frac{d_1}{2}, \frac{d_2+2}{2}\right)} > \frac{\int_0^{1-B} (1-t)^{\frac{d_1}{2}-1} t^{\frac{d_2}{2}-1} dt}{B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)} \\
&\Leftrightarrow (d_1 + d_2) \int_0^{1-A} (1-t)^{\frac{d_1}{2}-1} t^{\frac{d_2+2}{2}-1} dt > d_2 \int_0^{1-B} (1-t)^{\frac{d_1}{2}-1} t^{\frac{d_2}{2}-1} dt \\
&\Leftrightarrow (d_1 + d_2) \int_0^{1-A} \sum_{n=0}^{\frac{d_1}{2}-1} (-1)^n \binom{\frac{d_1}{2}-1}{n} t^{n+\frac{d_2+2}{2}-1} dt > d_2 \int_0^{1-B} \sum_{n=0}^{\frac{d_1}{2}-1} (-1)^n \binom{\frac{d_1}{2}-1}{n} t^{n+\frac{d_2}{2}-1} dt \\
&\Leftrightarrow (d_1 + d_2)(1-A)^{\frac{d_2+2}{2}} \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(A-1)^n}{2n+d_2+2} > d_2(1-B)^{\frac{d_2}{2}} \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(B-1)^n}{2n+d_2} \\
&\Leftrightarrow (1-B)^{\frac{d_2}{2}} \left\{ \prod_{n=0}^{\frac{d_1}{2}-1} (2n+d_2) \right\} \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(B-1)^n}{2n+d_2} \\
&< (1-A)^{\frac{d_2+2}{2}} \left\{ \prod_{n=0}^{\frac{d_1}{2}-1} (2n+d_2+2) \right\} \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(A-1)^n}{2n+d_2+2}. \tag{5.1}
\end{aligned}$$

For $d_1 \in \mathbb{N}$ and $3 \leq y \in \mathbb{R}$, define

$$\begin{aligned}
J_{d_1}(y) &:= -\frac{y+2}{2} \ln \left[1 + \frac{d_1}{y} \left(1 + \sqrt{\frac{2(d_1+y)}{d_1(y-2)}} \right) \right] + \sum_{n=0}^{\frac{d_1}{2}-1} \ln(2n+y+2) \\
&\quad + \ln \left\{ \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(-1)^n}{(2n+y+2) \left[1 + \frac{d_1}{y} \left(1 + \sqrt{\frac{2(d_1+y)}{d_1(y-2)}} \right) \right]^n} \right\}.
\end{aligned}$$

To complete the proof of (4.1), we may try to show that J_{d_1} is a strictly increasing function through proving that $J'_{d_1}(y) > 0$.

We assume without loss of generality that $D > C > 0$. By (2.7), this assumption is equivalent to

$$d_1(d_2-2)(d_1+d_2)(d_2-4)^2 > 2d_2^2(d_1+d_2-2)^2.$$

Similar to (5.1), we can show that

$$\begin{aligned}
(4.2) \text{ holds } &\Leftrightarrow (1-D)^{\frac{d_2}{2}} \left\{ \prod_{n=0}^{\frac{d_1}{2}-1} (2n+d_2) \right\} \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(D-1)^n}{2n+d_2} \\
&> (1-C)^{\frac{d_2+2}{2}} \left\{ \prod_{n=0}^{\frac{d_1}{2}-1} (2n+d_2+2) \right\} \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(C-1)^n}{2n+d_2+2}.
\end{aligned}$$

For $d_1 \in \mathbb{N}$ and $3 \leq y \in \mathbb{R}$, define

$$K_{d_1}(y) := -\frac{y+2}{2} \ln \left[1 + \frac{d_1}{y} \left(1 - \sqrt{\frac{2(d_1+y)}{d_1(y-2)}} \right) \right] + \sum_{n=0}^{\frac{d_1}{2}-1} \ln(2n+y+2)$$

$$+ \ln \left\{ \sum_{n=0}^{\frac{d_1}{2}-1} \binom{\frac{d_1}{2}-1}{n} \frac{(-1)^n}{(2n+y+2) \left[1 + \frac{d_1}{y} \left(1 - \sqrt{\frac{2(d_1+y)}{d_1(y-2)}} \right) \right]^n} \right\}.$$

To complete the proof of (4.2), we may try to show that K_{d_1} is a strictly decreasing function through proving that $K'_{d_1}(y) < 0$.

5.2. The case where $d_1 = 2k + 1$, $2 \leq k \in \mathbb{N}$

In this subsection, we assume that $d_1 \geq 5$ and d_1 is an odd number. By (2.3), to complete the proof of (2.2), it suffices to prove (4.1) and (4.2).

For $x > 0$, denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x and

$$(x)_{\langle n \rangle} := x(x-1) \cdots (x-n+1).$$

Similar to (3.2), we get:

$$\begin{aligned} & (4.1) \text{ holds} \\ \Leftrightarrow & 2A(1-A) < d_2(B-A) \int_0^1 \left(1 + \frac{B-A}{A}x\right)^{\frac{d_1}{2}-1} \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}-1} dx \\ \Leftrightarrow & 2A(1-A) < d_2(B-A) \sum_{n=0}^{\lfloor \frac{d_1}{2}-1 \rfloor} \frac{\left(\frac{d_1}{2}-1\right)_{\langle n \rangle} \left(\frac{B-A}{A}\right)^n}{n!} \int_0^1 x^n \left(1 - \frac{B-A}{1-A}x\right)^{\frac{d_2}{2}-1} dx \\ \Leftrightarrow & 2A < d_2 \sum_{n=0}^{\lfloor \frac{d_1}{2}-1 \rfloor} \frac{\left(\frac{d_1}{2}-1\right)_{\langle n \rangle} \left(\frac{1-A}{A}\right)^n}{n!} \sum_{l=0}^n \frac{(-1)^l}{\frac{d_2}{2}+l} \binom{n}{l} \left[1 - \left(\frac{1-B}{1-A}\right)^{\frac{d_2}{2}+l} \right]. \end{aligned} \quad (5.2)$$

We assume without loss of generality that $D > C > 0$. By (2.7), this assumption is equivalent to

$$d_1(d_2-2)(d_1+d_2)(d_2-4)^2 > 2d_2^2(d_1+d_2-2)^2.$$

For $x > 0$, denote by $\lceil x \rceil$ the smallest integer greater than or equal to x . Similar to (5.2), we get:

$$(4.2) \text{ holds} \Leftrightarrow 2C > d_2 \sum_{n=0}^{\lceil \frac{d_1}{2}-1 \rceil} \frac{\left(\frac{d_1}{2}-1\right)_{\langle n \rangle} \left(\frac{1-C}{C}\right)^n}{n!} \sum_{l=0}^n \frac{(-1)^l}{\frac{d_2}{2}+l} \binom{n}{l} \left[1 - \left(\frac{1-D}{1-C}\right)^{\frac{d_2}{2}+l} \right]. \quad (5.3)$$

Conjecture 1.1 can be resolved if we are able to establish inequalities (5.2) and (5.3) with two variables $d_1 = 2k + 1$, $2 \leq k \in \mathbb{N}$, and $d_2 \in \mathbb{N}$.

Note that the floor and ceiling functions in (5.2) and (5.3) reflect the half-integer exponent $\frac{d_1}{2} - 1$ occurring when d_1 is odd. Unlike the even case, the corresponding binomial expansion does not terminate. Thus, we must use finite lower and upper truncations and control the remainders. We expect the general strategy to remain the same as in the even case. However, the required uniform remainder estimates are technically more delicate.

Author contributions

Ze-Chun Hu suggested the problems and discussed with Ping Sun and Wei Sun. Ping Sun and Wei Sun wrote the main manuscript text and prepared Table 1 and Figures 1–5. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

References

1. Z. C. Hu, R. Song, Y. Tan, On the anti-concentration functions of some familiar families of distributions, *Math. Theory Appl.*, **44** (2024), 1–15.
2. N. L. Johnson, S. Kotz, N. Balakrishnan, *Continuous univariate distributions*, Wiley, **2** (1995).
3. N. Keller, O. Klein, Proof of Tomaszewski's conjecture on randomly signed sums, *Adv. Math.*, **407** (2022), 108558. <https://doi.org/10.1016/j.aim.2022.108558>
4. K. Krishnamoorthy, *Handbook of statistical distributions with applications*, 2 Eds., New York: Chapman and Hall/CRC, 2016. <https://doi.org/10.1201/b19191>
5. K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, 1999.
6. P. Sun, Z. C. Hu, W. Sun, The infimum values of two probability functions for the Gamma distribution, *J. Inequal. Appl.*, **2024** (2024), 5. <https://doi.org/10.1186/s13660-024-03081-w>
7. P. Sun, Z. C. Hu, W. Sun, Variation comparison between infinitely divisible distributions and the normal distribution, *Stat. Pap.*, **65** (2024), 4405–4429. <https://doi.org/10.1007/s00362-024-01561-1>
8. J. Zhang, Z. C. Hu, W. Sun, On the measure concentration of infinitely divisible distributions, *Acta Math. Sci.*, **45** (2025), 473–492. <https://doi.org/10.1007/s10473-025-0211-x>

Supplementary material: Mathematica code and output

§1. Values of F_{d_1, d_2} :

```
(* Definition of F_{d1, d2} *)
FValue[d1_Integer, d2_Integer] := Module[{s, xUpper, xLower},
s = N[Sqrt[2(d1 + d2 - 2)/(d1(d2 - 4))], 80];
xUpper = N[d1/(d1 + (d2 - 2)/(1 + s)), 80];
(* If the maximum in the definition is zero, xLower=0 *)
xLower = If[2(d1 + d2 - 2) ≥ d1(d2 - 4), 0, N[d1/(d1 + (d2 - 2)/(1 - s)), 80]];
Block[{$MaxExtraPrecision = 10000},
N[BetaRegularized[xUpper, d1/2, d2/2] - BetaRegularized[xLower, d1/2, d2/2], 30]];
(* Requested values of d1 and d2 *)
d1Values = {1, 2, 3, 4, 5, 6, 10, 1000, 100000};
d2Values = {5, 6, 7, 10, 1000, 100000};
(* Compute all values *)
values = Table[FValue[d1, d2], {d1, d1Values}, {d2, d2Values}];
(* Display ten digits after the decimal point *)
formattedValues = Map[NumberForm[#, {12, 10}, NumberPadding → {"", "0"}]&, values, {2}];
(* Add row and column headings *)
resultTable = Prepend[MapThread[Prepend, {formattedValues, d1Values}],
Prepend[d2Values, "d1\nd2"]];
(* Create the title and display the completed table *)
Column[{Style[Row[{"Values of", Subscript[F, Row[{Subscript[d, 1], ", ", Subscript[d, 2]}]]], 15, Bold],
Grid[resultTable, Frame → All, Alignment → Center,
Background → {None, {LightBlue, {White, LightGray}}}, ItemStyle → Directive[FontSize → 12],
Spacings → {1.5, 1}], Alignment → Center, Spacings → 1]
```

§3.1. Some numerical values of $H_2(y)$:

```
H2[y_] := (y/2+1)*Log[1+2/y*(1+(2*(2+y)/2/(y-2))^(1/2))];
Table[N[H2[i]], {i, 3, 5}]
{2.87436, 2.58363, 2.44523}
```

Symbolic computation of $L_2(y)$:

```
p2=2/y*(1+(2*(2+y)/2/(y-2))^(1/2));
g2=1/2*(p2-p2^2/2+p2^3/3);
L2=g2+(y/2+1)*D[Log[1+2/y*(1+(2*(2+y)/2/(y-2))^(1/2))],y]
```

Simplify[L2]

$$\frac{2\left(\frac{2+y}{-2+y}\right)^{3/2}\left(-3y^3-32\sqrt{\frac{2+y}{-2+y}}+2y^2\left(-2+\sqrt{\frac{2+y}{-2+y}}\right)+4y\left(9+5\sqrt{\frac{2+y}{-2+y}}\right)\right)}{3y^3(2+y)\left(2+y+2\sqrt{\frac{2+y}{-2+y}}\right)}$$

Graph of $H_2(y)$ for $y \geq 3$:

```
h[x_, y_] := (y/2 + 1)Log[1 + (x/y)(1 + Sqrt[2(x + y)/(x(y - 2))])];
```

```
Plot[h[2, y], {y, 3, 100}, PlotRange → All,
AxesLabel → {y, Row[{H2, "(y)"}]},
PlotLabel → Row[{"Graph of ", H2, "(y)"}],
GridLines → Automatic, PlotStyle → {Blue, Thick},
ImageSize → Large]
```

§3.2 Symbolic computation of $K'(y)$:

```
K = y*(1+(2*(y-1)/(y-4))^(1/2)) / (1+(2*(y-1)/(y-4))^(1/2)+y-2);
```

```
K1 = D[K, y];
```

```
Simplify[K1]
```

$$\frac{-16\sqrt{\frac{-1+y}{-4+y}} + \left(6\sqrt{2}-4\sqrt{\frac{-1+y}{-4+y}}\right)y + \left(-3\sqrt{2}+2\sqrt{\frac{-1+y}{-4+y}}\right)y^2}{2(-4+y)^2\sqrt{\frac{-1+y}{-4+y}}\left(-1+\sqrt{2}\sqrt{\frac{-1+y}{-4+y}}+y\right)^2}$$

Symbolic computation of $L_1(y)$:

```
p1 = 1/y*(1+(2*(1+y)/(y-2))^(1/2));
```

```
g1 = 1/2*(p1-p1^2/2+p1^3/3);
```

```
L1 = g1+(y/2+1)*D[Log[1+1/y*(1+(2*(1+y)/(y-2))^(1/2))],y];
```

```
Simplify[L1]
```

$$\frac{-32\sqrt{\frac{1+y}{-2+y}}+y^2\left(35\sqrt{2}-4\sqrt{\frac{1+y}{-2+y}}\right)+4y\left(7\sqrt{2}-2\sqrt{\frac{1+y}{-2+y}}\right)-3y^4\left(5\sqrt{2}+\sqrt{\frac{1+y}{-2+y}}\right)+y^3\left(-17\sqrt{2}+23\sqrt{\frac{1+y}{-2+y}}\right)}{12(y-2)^2y^3\sqrt{\frac{1+y}{-2+y}}\left(1+y+\sqrt{2}\sqrt{\frac{1+y}{-2+y}}\right)}$$

Graph of $H_1(y)$ for $y \geq 3$:

```
h[x_, y_] := (y/2 + 1) * Log[1 + (x/y) * (1 + Sqrt[2(x + y)/(x(y - 2))])];
```

```
Plot[h[1, y], {y, 3, 100}, PlotRange → All,
```

```
AxesLabel → {y, Row[{H1, "(y)"}]},
```

```
PlotLabel → Row[{"Graph of ", H1, "(y)"}],
```

```
GridLines → Automatic, PlotStyle → {Blue, Thick},
```

```
ImageSize → Large]
```

§4.1.1 Some numerical values of $H_4(y)$:

```
H4[y_] := (y/2+1)*Log[y] - ((y+4)/2)*Log[4*(1+((4+y)/2/(y-2))^(1/2))+y]
+ Log[4*(y+4)*(1+((4+y)/2/(y-2))^(1/2))+2*y];
```

```
Table[N[H4[i]], {i, 3, 12}]
```

```
{-2.15017, -1.85244, -1.70932, -1.62225, -1.563, -1.51983, -1.48688, -1.46088, -1.43981, -1.42238}
```

Symbolic computation of $L_4(y)$:

```
p4 = 4/y*(1+((4+y)/2/(y-2))^(1/2));
```

```
g4 = 1/2*(p4-p4^2/2+p4^3/3);
```

```
L4 = g4-((y+2)/2/y)+(y+4)/2*D[tLog[4*(1+((4+y)/2/(y-2))^(1/2))+y],y]-
```

```
D[Log[4*(y+4)*(1+((4+y)/2/(y-2))^(1/2))+2*y],y];
```

```
Simplify[L4]
```

$$\frac{4\sqrt{\frac{4+y}{-2+y}}\left(2048\left(\sqrt{2}-\sqrt{\frac{4+y}{-2+y}}\right)+3y^5\left(\sqrt{2}+\sqrt{\frac{4+y}{-2+y}}\right)+1536y\left(\sqrt{2}+2\sqrt{\frac{4+y}{-2+y}}\right)+2y^4\left(5\sqrt{2}+4\sqrt{\frac{4+y}{-2+y}}\right)-16y^2\left(65\sqrt{2}+7\sqrt{\frac{4+y}{-2+y}}\right)-4y^3\left(92\sqrt{2}+121\sqrt{\frac{4+y}{-2+y}}\right)\right)}{3(-2+y)^2y^3\left(4+y+2\sqrt{2}\sqrt{\frac{4+y}{-2+y}}\right)\left(8+4\sqrt{2}\sqrt{\frac{4+y}{-2+y}}+y\left(3+\sqrt{2}\sqrt{\frac{4+y}{-2+y}}\right)\right)}$$

Graph of $H_4(y)$ for $y \geq 3$:

```
h4[y_] := ((y + 2)/2)Log[y]
- ((y + 4)/2) * Log[4(1 + Sqrt[(y + 4)/(2(y - 2))]) + y]
+ Log[4(y + 4)(1 + Sqrt[(y + 4)/(2(y - 2))]) + 2y];
Plot[h4[y], {y, 3, 100}, PlotRange -> All,
AxesLabel -> {y, Row[{H4, "(y)"}]},
PlotLabel -> Row[{"Graph of ", H4, "(y)"}],
GridLines -> Automatic, PlotStyle -> {Blue, Thick},
ImageSize -> Large]
```

§4.1.2 Symbolic computation of $Q_4(y)$:

```
r = 4/y*(1-((4+y)/2/(y-2))^(1/2));
x4 = 1/2*(r-r^2/2);
Q4 = x4-((y+2)/2/y)+(y+4)/2*D[Log[4*(1-((4+y)/2/(y-2))^(1/2))+y],y]-
D[Log[4*(y+4)*(1-((4+y)/2/(y-2))^(1/2))+2*y],y];
Simplify[Q4]
```

$$-\frac{4\left(\frac{4+y}{-2+y}\right)^{3/2}\left(-128\sqrt{\frac{4+y}{-2+y}}+y^3\left(-\sqrt{2}+\sqrt{\frac{4+y}{-2+y}}\right)+y^2\left(-26\sqrt{2}+34\sqrt{\frac{4+y}{-2+y}}\right)+y\left(-88\sqrt{2}+40\sqrt{\frac{4+y}{-2+y}}\right)\right)}{y^2(4+y)\left(-4-y+2\sqrt{2}\sqrt{\frac{4+y}{-2+y}}\right)\left(-8+4\sqrt{2}\sqrt{\frac{4+y}{-2+y}}+y\left(-3+\sqrt{2}\sqrt{\frac{4+y}{-2+y}}\right)\right)}$$

Graph of $R_4(y)$ for $y \geq 15$:

```
R4[y_] := ((y + 2)/2)Log[y]
- ((y + 4)/2) * Log[4(1 - Sqrt[(y + 4)/(2(y - 2))]) + y]
+ Log[4(y + 4)(1 - Sqrt[(y + 4)/(2(y - 2))]) + 2y];
Plot[R4[y], {y, 15, 100}, PlotRange -> All,
AxesLabel -> {y, Row[{R4, "(y)"}]},
PlotLabel -> Row[{"Graph of ", R4, "(y)"}],
GridLines -> Automatic, PlotStyle -> {Red, Thick},
ImageSize -> Large]
```

§4.2.1 Some numerical values of $H_3(y)$:

```
H3[y_] := (y/2+1)*Log[1+3/y*(1+(2*(3+y)/3/(y-2))^(1/2))];
Table[N[H3[i]], {i, 3, 12}]
{3.46574, 3.18962, 3.06414, 2.99125, 2.94353, 2.9099, 2.88497, 2.86578, 2.85058, 2.83826}
```

Symbolic computation of $L_3(y)$:

```
p3 = 3/y*(1+(2*(3+y)/3/(y-2))^(1/2));
g3 = 1/2*(p3-p3^2/2+p3^3/3);
L3 = g3+(y+2)/2*D[Log[1+3/y*(1+(2*(3+y)/3/(y-2))^(1/2))],y];
Simplify[L3]
```

$$\frac{-864 \sqrt{\frac{3+y}{-2+y}} - 3y^4 \left(\sqrt{6} - \sqrt{\frac{3+y}{-2+y}} \right) + 36y \left(11 \sqrt{6} + 6 \sqrt{\frac{3+y}{-2+y}} \right) - y^3 \left(19 \sqrt{6} + 9 \sqrt{\frac{3+y}{-2+y}} \right) + 3y^2 \left(29 \sqrt{6} + 88 \sqrt{\frac{3+y}{-2+y}} \right)}{4(y-2)^2 y^3 \sqrt{\frac{3+y}{-2+y}} \left(3+y + \sqrt{6} \sqrt{\frac{3+y}{-2+y}} \right)}$$

Graph of $H_3(y)$ for $y \geq 3$:

$h[x_-, y_-] := (y/2 + 1) * \text{Log}[1 + (x/y) * (1 + \text{Sqrt}[2(x+y)/(x(y-2))])]$;

$\text{Plot}[h[3, y], \{y, 3, 100\}, \text{PlotRange} \rightarrow \text{All},$

$\text{AxesLabel} \rightarrow \{y, \text{Row}[\{H_3, "(y)'\}]\},$

$\text{PlotLabel} \rightarrow \text{Row}[\{"Graph of ", H_3, "(y)'\}"],$

$\text{GridLines} \rightarrow \text{Automatic}, \text{PlotStyle} \rightarrow \{\text{Blue}, \text{Thick}\},$

$\text{ImageSize} \rightarrow \text{Large}]$

§4.2.2 Symbolic computation of $V(y)$, $G_1(y)$, $G_2(y)$ and some numerical values of $G_1(y)$:

$\text{AA} = d1 * ((1 + (2 * (d1 + y) / d1 / (y - 2))^{(1/2)})) / (d1 * ((1 + (2 * (d1 + y) / d1 / (y - 2))^{(1/2)})) + y);$

$\text{BB} = d1 * (1 + (2 * (d1 + y - 2) / d1 / (y - 4))^{(1/2)}) / (d1 * (1 + (2 * (d1 + y - 2) / d1 / (y - 4))^{(1/2)}) + (y - 2));$

$\text{CC} = d1 * ((1 - (2 * (d1 + y) / d1 / (y - 2))^{(1/2)})) / (d1 * ((1 - (2 * (d1 + y) / d1 / (y - 2))^{(1/2)})) + y);$

$\text{DD} = d1 * (1 - (2 * (d1 + y - 2) / d1 / (y - 4))^{(1/2)}) / (d1 * (1 - (2 * (d1 + y - 2) / d1 / (y - 4))^{(1/2)}) + (y - 2));$

$d1 = 3;$

$\text{V} = \text{CC} / (1 - \text{CC}) * (1 - \text{CC} / (1 - \text{CC}) / 2) - ((y + 2) * \text{CC} - y * \text{DD}) / (2 * (1 + \text{CC}) + y * (\text{CC} + \text{DD})) -$

$y * (\text{DD} - \text{CC}) / (1 - \text{DD}) / 2;$

$\text{cleanV} = \text{Simplify}[\text{Expand}[\text{Simplify}[\text{V}]]];$

$\text{G1}[y_-] := \text{Evaluate}[\text{Numerator}[\text{cleanV}]];$

$\text{G2}[y_-] := \text{Evaluate}[\text{Denominator}[\text{cleanV}]];$

$\text{G1}[y]$

$\text{G2}[y]$

$\text{Table}[\text{N}[\text{G1}[i]], \{i, 25, 33\}]$

$$\begin{aligned} G_1(y) = & 3 \left(-6912 \left(-1 + \sqrt{6} \sqrt{\frac{1+y}{-4+y}} \right) + 2 \sqrt{6} y^8 \left(-\sqrt{\frac{1+y}{-4+y}} + \sqrt{\frac{3+y}{-2+y}} \right) \right. \\ & + 4y^3 \left(-1134 + 108 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} + 457 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} - 654 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \\ & + 16y^2 \left(-435 + 237 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} - 160 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} - 192 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \\ & - 192y \left(-54 + 11 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} - 66 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \\ & + y^7 \left(4 + 8 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} - 8 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} - 4 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \\ & + y^6 \left(16 + 25 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} - 23 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} + 20 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \\ & \left. + y^5 \left(392 - 77 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} - 125 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} + 70 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \right) \end{aligned}$$

$$+ y^4 \left(500 - 410 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} + 364 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} + 304 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right)$$

$$\begin{aligned} G_2(y) = & 2(-4+y)(-2+y)^2 y^2 \left(3+y - \sqrt{6} \sqrt{\frac{3+y}{-2+y}} \right) \\ & \times \left(y^2 \left(-8 + \sqrt{6} \sqrt{\frac{1+y}{-4+y}} + \sqrt{6} \sqrt{\frac{3+y}{-2+y}} \right) \right. \\ & + 4 \left(-3 + 3 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} + \sqrt{6} \sqrt{\frac{3+y}{-2+y}} - 6 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \\ & \left. + 2y \left(-13 + 4 \sqrt{6} \sqrt{\frac{1+y}{-4+y}} + 4 \sqrt{6} \sqrt{\frac{3+y}{-2+y}} - 6 \sqrt{\frac{1+y}{-4+y}} \sqrt{\frac{3+y}{-2+y}} \right) \right) \end{aligned}$$

$$\{-1.46179 \times 10^9, -1.91825 \times 10^9, -2.48438 \times 10^9, -3.17992 \times 10^9, -4.02709 \times 10^9, \\ -5.05084 \times 10^9, -6.27904 \times 10^9, -7.74269 \times 10^9, -9.47617 \times 10^9\}$$

Symbolic computation of $G_2(y)$:

Simplify $\left[\text{Expand} \left[\text{Simplify} \left[\text{Simplify} [\text{Expand}[G_2[y]]] /. \left\{ y \rightarrow 25 + x, \sqrt{\frac{1+y}{-4+y}} \rightarrow u, \sqrt{\frac{3+y}{-2+y}} \rightarrow v \right\} \right] \right] \right]$

$$\begin{aligned} & 2(25+x)^2 (483 + 44x + x^2) \left(-8(26+x)^2 (700 + 53x + x^2) \right. \\ & \quad + \sqrt{6}v (664102 + 95091x + 5041x^2 + 117x^3 + x^4) \\ & \quad \left. + u (783 + 56x + x^2) (-18v(23+x) + \sqrt{6} (700 + 53x + x^2)) \right) \end{aligned}$$

$$\begin{aligned} T[x_-] := & -8(26+x)^2 (700 + 53x + x^2) + \sqrt{6}v (664102 + 95091x + 5041x^2 + 117x^3 + x^4) \\ & + u (783 + 56x + x^2) (-18v(23+x) + \sqrt{6} (700 + 53x + x^2)); \end{aligned}$$

Collect[Expand[T[x]], x]//TraditionalForm

$$\begin{aligned} & x^4 (\sqrt{6}u + \sqrt{6}v - 8) + x^3 (-18uv + 109 \sqrt{6}u + 117 \sqrt{6}v - 840) \\ & + x^2 (-1422uv + 4451 \sqrt{6}u + 5041 \sqrt{6}v - 33056) \\ & + x (-37278uv + 80699 \sqrt{6}u + 95091 \sqrt{6}v - 577824) \\ & - 324162uv + 548100 \sqrt{6}u + 664102 \sqrt{6}v - 3785600. \end{aligned}$$



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