



*Research article***Attribute reduction approaches based on Boolean reasoning and hypergraph for inconsistent fuzzy formal decision contexts****Jing Zhang^{1,*}, Zengtai Gong^{2,*} and Huai Yu³**¹ School of Mathematical Sciences, Northwest Minzu University, Lanzhou 730030, China² College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China³ Human Resources Department, Northwest Minzu University, Lanzhou 730030, China*** Correspondence:** Email: jingzhangnwnu@163.com, zt-gong@163.com.

Abstract: Attribute reduction in fuzzy formal concept analysis (FFCA) constitutes a core research issue of FFCA, which denotes the process of eliminating redundant attributes while preserving the specific structural properties of the concept lattice. Nevertheless, most existing attribute reduction methods work well merely for consistent fuzzy formal decision contexts (CFFDCs), yet fail to produce valid outcomes for inconsistent fuzzy formal decision contexts (IFFDCs). Motivated by this limitation, this paper concentrates on the research of IFFDCs. First, drawing on the upper and lower approximation operators of rough sets, this paper defines the assignment function, distribution function, maximum distribution function, upper approximation sum function, as well as their corresponding reductions, and further explores the internal relationships among these diverse reductions. Second, reduction approaches based on Boolean reasoning and hypergraph minimal transversals are investigated, and the consistency of results yielded by these two approaches is theoretically proven. On the basis of the above theoretical derivations, the algorithms corresponding to each reduction theory are developed. Numerical examples are further utilized to verify the validity and correctness of the proposed reduction theories and algorithms.

Keywords: inconsistent fuzzy formal decision contexts; attribute reduction; rough set; Boolean reasoning; hypergraph minimal transversal

Mathematics Subject Classification: 03E72, 68T30

1. Introduction

Formal concept analysis (FCA) [1], as a key theoretical tool in knowledge representation and rule mining, has been widely applied in knowledge engineering by using concept lattices to characterise hierarchical relationships between extent and intent. In practice, however, with the continuous growth

of high-dimensional and complex datasets, the original concept lattice suffers from structural redundancy and computational complexity, which severely limit its practical application. Attribute reduction, a key research area within FCA, enables the elimination of redundant attributes whilst preserving core knowledge structures and semantic information. This simplifies the construction of concept lattices and reduces algorithmic complexity, thereby providing a theoretical foundation for efficient knowledge discovery and playing a significant role in refining the FCA theoretical framework. To date, this method has been successfully applied in scenarios such as machine learning [2], knowledge spaces [3], concept cognitive learning [4], information retrieval [5], and recommendation systems [6], offering an effective technical approach for processing high-dimensional data across multiple disciplines.

Attribute reduction is one of the main research areas in FCA. It can be divided into two categories: reduction without decision attributes and reduction with decision attributes, with the latter further classified into consistent and inconsistent subcategories based on consistency. Zhang [7] was the first to propose an attribute reduction method based on Boolean reasoning that ensures the structural invariance of the concept lattice, which was a pioneering work. Building upon this, Wu [8] introduced granular computing into the concept lattice, presented the concept of object granules, and investigated an attribute reduction method based on Boolean reasoning that ensures the structural invariance of object granules; this reduction method is referred to as granular reduction. On this basis, Chen [9] observed that, in data implementation, methods based on Boolean reasoning occupied a significant amount of memory and were time-consuming. He therefore proposed a granularity reduction algorithm based on hypergraph minimal transversals. In [9], the relationship between granularity reduction and the minimal transversal of its derived graph was first discussed, their equivalence was verified, and a new heuristic algorithm was constructed.

Given that fuzzy relations better capture the binary relationships between objects and attributes in the real world, Brusco presented fuzzy formal concept analysis (FFCA) by combining fuzzy sets [10] with formal concepts [11,12]. Furthermore, Yahia [13] and Krajčič [14] introduced the “one-sided fuzzy concept” and studied several of its properties. Thereafter, a number of scholars have combined formal concepts with various fuzzy set models, introducing a range of fuzzy formal concept models, such as L-fuzzy concept lattices [15, 16], intuitionistic L-fuzzy [17], weighted fuzzy concepts [18].

Noting that Shao et al. built upon the work of [8] and incorporated fuzzy concept lattice theory, defined fuzzy object granules, fuzzy granular reduction, and fuzzy consistent sets, and proposed a granular reduction algorithm based on Boolean reasoning [19]. As this algorithm consumed excessive memory and took too long to run during data experiments, Gong and Zhang proposed a new heuristic algorithm based on graph vertex cover theory for fuzzy formal contexts (FFCs) [20]. This algorithm is not only feasible but also highly efficient. It is worth noting that, in this paper, the authors proposed a data processing method with better interpretability, which ingeniously transforms the binary relationships between objects and attributes into fuzzy relationships.

In recent years, Fan et al. proposed a reduction method for symmetric formal contexts driven by the concept of incremental pseudo-equivalence [21]. The proposed method preserves the binary relation of the formal context. Experimental results demonstrated that pseudo-equivalent concept reduction outperforms conventional equivalent concept reduction in terms of redundancy, relevance, and computational time. Gong et al. developed a fuzzy-hypergraph-based visualization method within the framework of rough set theory, aiming to simplify the processes of attribute identification and

reduction in FFCs [22]. Niu et al. proposed several reduction methods for Boolean formal contexts, including reductions that preserve the structure of the concept lattice, the extents of meet-irreducible elements, the extents of join-irreducible elements, and column-vector granular concepts, respectively [23]. Hu et al. improved the applicability of rule-based attribute reduction algorithms by introducing a method to simplify the discernibility matrix, and further proposed a heuristic approach for computing attributes using the ordered attributes method [24]. Moreover, To avoid the limitations of attribute reduction, Zhang et al. proposed attribute combination reduction, which leverages the strengths of both attribute reduction and concept reduction while keeping object intents unchanged [25].

Almost all fuzzy formal decision contexts (CFFDCs), as well as FFCs, have been studied simultaneously, as demonstrated by researchers such as Zhang [7], Wu [8], Shao [19], Chen [9], Lin [26], Gong [20] and others. However, in practice, there are still situations that do not involve CFFDCs, and the reduction methods mentioned above prove ineffective in such cases. Consequently, a small number of scholars have begun to investigate inconsistent formal decision contexts (IFDCs) or inconsistent fuzzy formal decision contexts (IFFDCs).

In response to the above issues, Li et al. proposed an attribute reduction method based on rough set inclusion [27]. Subsequently, drawing on information entropy theory, Huang et al. introduced a new granular reduction algorithm [28]. However, both methods are limited to classical formal contexts, and no research has yet been conducted on FFDCs. Consequently, this paper focuses on exploring the theory of attribute reduction for IFFDCs.

Existing studies, whether based on fuzzy rough sets, graph theory, or information entropy, have mainly focused on CFFDCs, i.e., existing attribute reduction methods are no longer applicable to IFFDCs. Consequently, one of our main motivations is to propose an effective attribute reduction method tailored to such IFFDCs. Furthermore, given that rough sets require no prior knowledge and can characterise fuzzy boundaries and quantify attribute importance through upper and lower approximations, they offer a theoretically rigorous approach well suited to fuzzy data. Graph theory, in turn, can transform attribute dependencies into graph optimisation problems, intuitively modelling redundancy relationships and enhancing reduction efficiency. The integration of these two approaches balances the handling of fuzzy uncertainty with structural optimisation, offering significant advantages in attribute reduction for fuzzy information systems. Therefore, this paper takes IFFDCs as its research subject and proposes an attribute reduction method based on rough set inclusion theory and hypergraph minimal transversals.

The contributions of this paper are mainly as follows:

(1) The assignment function, distribution function, maximum distribution function, and upper approximation sum function based on the inclusion degree of rough set in IFFDCs were first introduced;

(2) A novel heuristic algorithm based on hypergraph minimal transversal in IFFDCs was proposed.

It should be emphasized that the reduction methods based on Boolean reasoning and minimal transversal of hypergraph are not two independently constructed heuristic algorithms. Rather, they are two different computational realizations of the same attribute reduction criterion established through the equivalence-class function. Therefore, their theoretical validity originates from the common reduction framework, while Boolean reasoning and hypergraph minimal transversal provide alternative mechanisms for computing the corresponding reducts.

The remainder of this paper is organised as follows. Section 2 primarily introduces the terminology, concepts and fundamental theories required for this paper. The theory and algorithms of assignment reduction are examined in Section 3. Section 4 discusses the theory and algorithms of distribution reduction. The theory and algorithms of maximum distribution reduction are examined in Section 5. Finally, in Section 6 we summarise this work and offer recommendations for future research. The research framework of the paper see Figure 1.

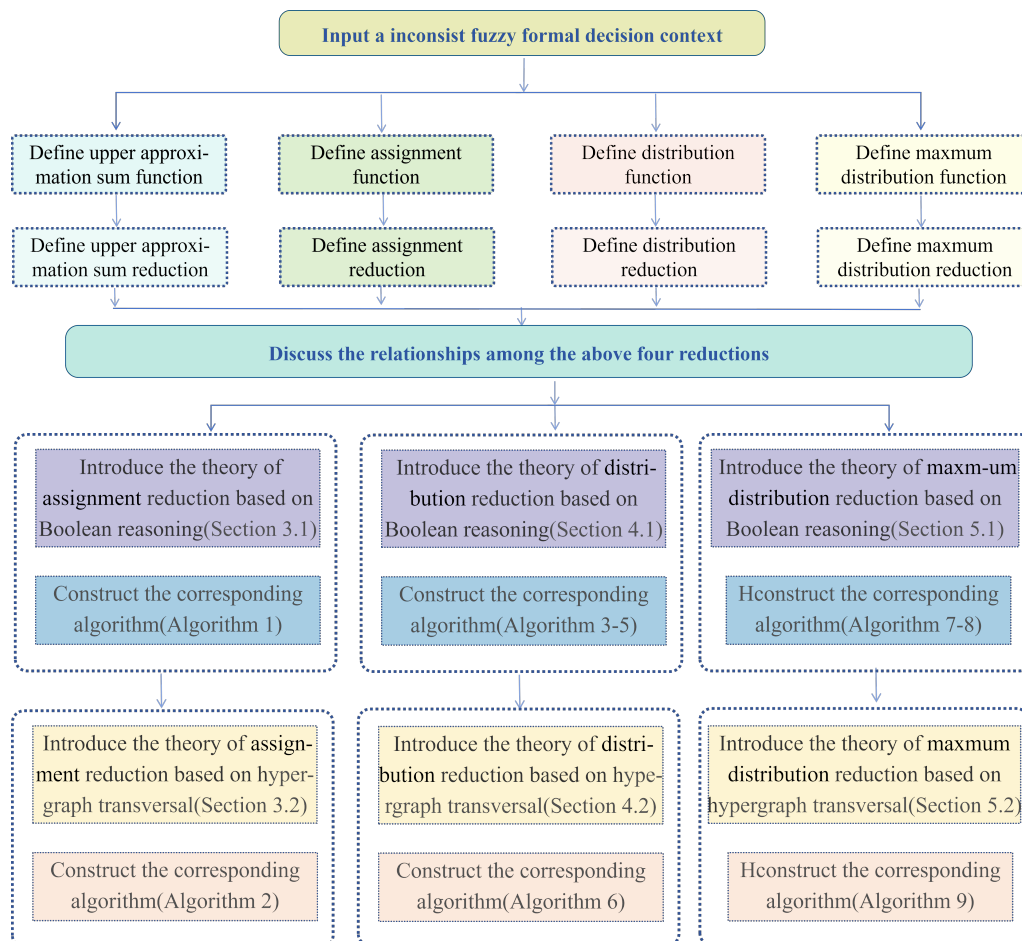


Figure 1. The research framework of the paper.

2. Preliminaries

2.1. Fuzzy formal concept

The paper focuses on a finite and nonempty universe O . For the terminology and operation symbols used, please refer to Table 1. For more detailed information, please refer to [20].

Table 1. Terminology in the paper.

Notation/operator	Description
O	the finite and nonempty universe
$P(O)$	the power set of all subsets of O
$\mathcal{F}(O)$	the power set of fuzzy subsets of O
$\tilde{\pi}$	a one-side fuzzy formal concept
$\tilde{M}(x, a)$	the fuzzy binary relationship between object x and conditional attribute a
$\tilde{N}(x, d)$	the fuzzy binary relationship between object x and decision attribute d
$\mathcal{H} = (O, C, \tilde{M})$	a fuzzy formal context, O is the object set, C is the conditional attribute set
$\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$	a fuzzy formal decision context, D is the decision attribute set, $C \cap D = \emptyset$
$\tilde{L}(O, C, \tilde{M})$	the concept lattice induced by $(O, C, \tilde{M})$
τ_C^*	the distribution function on O with respect to a attribute subset C
λ_C^*	the maximum distribution function on O with respect to a attribute subset C
ζ_C^*	the assignment function on O with respect to a attribute subset C
ξ_C^*	the an upper approximation sum function on the domain O with respect to C
χ_A	the characteristic function of $A \subseteq C$ ($\chi_A(a) = 1$ if $a \in A$, otherwise is 0)
$\kappa_A(X)$	the intent of X under A (equals $\kappa_C(X) \cap \chi_A$)
$\iota_C(\tilde{K})$	the extent of $\tilde{K}$ under C (equals $\{x \in O \forall a \in C, \tilde{K}(a) \leq \tilde{M}(x, a)\}$)
R_C	the fuzzy relation of objects and attribute subset C (equals $\{(x, y) \in O \times O \tilde{M}(x, a) \leq \tilde{M}(y, a), \forall a \in C\}$)
$[x]_{R_C}$	object granule of x (equals $\{y \in O (x, y) \in R_C\} = \{y \in O \tilde{M}(x, a) \leq \tilde{M}(y, a) \forall a \in C\}$)
$\tilde{O}$	a fuzzy set
$\tilde{O}_1 \cup \tilde{O}_2$	the sup of $\tilde{O}_1(x)$ and $\tilde{O}_2(x)$, equals $\tilde{O}_1(x) \vee \tilde{O}_2(x)$
$\tilde{O}_1 \cap \tilde{O}_2$	the inf of $\tilde{O}_1(x)$ and $\tilde{O}_2(x)$, equals $\tilde{O}_1(x) \wedge \tilde{O}_2(x)$

Krajčí [14] and Yahia [13, 19] defined two operations κ_C and ι_C between $P(O)$ and $\mathcal{F}(C)$ in an FFC $\mathcal{H} = (O, C, \tilde{M})$, which form a Galois connection.

Definition 2.1. [13, 19] Let $\mathcal{H} = (O, C, \tilde{M})$ be an FFC. For $X \in P(O)$, $\tilde{K} \in \mathcal{F}(C)$, the operators $\kappa_C : P(O) \rightarrow \mathcal{F}(C)$ and $\iota_C : \mathcal{F}(C) \rightarrow P(O)$ are defined respectively, as follows:

$$\kappa_C(X)(a) = \bigwedge_{x \in X} \tilde{M}(x, a), \quad a \in C, \quad (2.1)$$

$$\iota_C(\tilde{K}) = \{x \in O | \forall a \in C, \tilde{K}(a) \leq \tilde{M}(x, a)\}. \quad (2.2)$$

Additionally, the following properties are given [19]:

- (1) $O_1 \subseteq O_2 \Rightarrow \kappa_C(O_2) \subseteq \kappa_C(O_1)$, $\tilde{K}_1 \subseteq \tilde{K}_2 \Rightarrow \iota_C(\tilde{K}_2) \subseteq \iota_C(\tilde{K}_1)$, ($O_1, O_2 \in P(O)$, $\tilde{K}_1, \tilde{K}_2 \in \mathcal{F}(C)$);
- (2) $O_i \subseteq \iota_C \circ \kappa_C(O_i)$, $\tilde{K} \subseteq \kappa_C \circ \iota_C(\tilde{K})$;
- (3) $\kappa_C(O_i) = \kappa_C \circ \iota_C \circ \kappa_C(O_i)$, $\iota_C(\tilde{K}) = \iota_C \circ \kappa_C \circ \iota_C(\tilde{K})$, $O_i \in P(O)$;
- (4)

$$\kappa_C \left(\bigcup_{i \in J} O_i \right) = \bigcap_{i \in J} \kappa_C(O_i), \quad (2.3)$$

$$\iota_C \left(\bigcup_{i \in J} \tilde{K}_i \right) = \bigcap_{i \in J} \iota_C(\tilde{K}_i), \quad (2.4)$$

$\tilde{K}_i \in \mathcal{F}(C)$, where $i \in J$ (J is an index set). Based on the above properties, it can be deduced that κ and ι form a Galois connection, i.e., $\kappa_C(X) = \tilde{K}$ and $\iota_C(\tilde{K}) = X$. $\tilde{\pi} = (X, \tilde{K})$ is called a one-sided fuzzy formal concept, where X is the extent of $\tilde{\pi}$ and $\tilde{K}$ is the intent of $\tilde{\pi}$. All concepts, ordered by a partial order, form a lattice known as a concept lattice. This partial order is defined as $(X_1, \tilde{K}_1) \leq (X_2, \tilde{K}_2)$ if and only if $X_1 \subseteq X_2$, or equivalently, $\tilde{K}_2 \subseteq \tilde{K}_1$.

Example 2.1. Consider a given FFC $\mathcal{H} = (O, C, \tilde{M})$, where $O = \{x_1, x_2, x_3, x_4, x_5\}$, $C = \{a_1, a_2, a_3, a_4, a_5\}$, which is shown in Table 2. Table 3 lists all the one-sided fuzzy concepts generated by $\mathcal{H}$. And, the Hasse diagram of the generated concept lattice $\tilde{L}(O, C, \tilde{M})$ is shown in Figure 2.

Table 2. A formal fuzzy context $\mathcal{H}$.

$\tilde{M}$	a_1	a_2	a_3	a_4	a_5
x_1	0.7	0.7	0.8	0.7	0.6
x_2	0.8	0.7	1	0.7	1
x_3	0.9	0.9	0.8	0.3	0.8
x_4	0.9	0.9	0.7	0.3	0.7
x_5	0.8	0.7	1	0.3	0.8

Table 3. The one-sided fuzzy concepts derived from $\mathcal{H}$.

Concepts	(object set, attribute set)
$\tilde{\pi}_1$	$(\{x_1, x_2, x_3, x_4, x_5\}, \{\frac{a_1}{0.7}, \frac{a_2}{0.7}, \frac{a_3}{0.1}, \frac{a_4}{0.3}, \frac{a_5}{0.6}\})$
$\tilde{\pi}_2$	$(\{x_1, x_2, x_3, x_4\}, \{\frac{a_1}{0.7}, \frac{a_2}{0.7}, \frac{a_3}{0.7}, \frac{a_4}{0.3}, \frac{a_5}{0.6}\})$
$\tilde{\pi}_3$	$(\{x_2, x_3, x_4, x_5\}, \{\frac{a_1}{0.8}, \frac{a_2}{0.7}, \frac{a_3}{0.1}, \frac{a_4}{0.3}, \frac{a_5}{0.7}\})$
$\tilde{\pi}_4$	$(\{x_1, x_2, x_3\}, \{\frac{a_1}{0.7}, \frac{a_2}{0.7}, \frac{a_3}{0.8}, \frac{a_4}{0.3}, \frac{a_5}{0.6}\})$
$\tilde{\pi}_5$	$(\{x_2, x_3, x_4\}, \{\frac{a_1}{0.8}, \frac{a_2}{0.7}, \frac{a_3}{0.7}, \frac{a_4}{0.3}, \frac{a_5}{0.7}\})$
$\tilde{\pi}_6$	$(\{x_2, x_3, x_5\}, \{\frac{a_1}{0.8}, \frac{a_2}{0.7}, \frac{a_3}{0.1}, \frac{a_4}{0.3}, \frac{a_5}{0.8}\})$
$\tilde{\pi}_7$	$(\{x_1, x_2\}, \{\frac{a_1}{0.7}, \frac{a_2}{0.7}, \frac{a_3}{0.8}, \frac{a_4}{0.7}, \frac{a_5}{0.6}\})$
$\tilde{\pi}_8$	$(\{x_2, x_3\}, \{\frac{a_1}{0.8}, \frac{a_2}{0.7}, \frac{a_3}{0.8}, \frac{a_4}{0.3}, \frac{a_5}{0.8}\})$
$\tilde{\pi}_9$	$(\{x_3, x_4\}, \{\frac{a_1}{0.9}, \frac{a_2}{0.9}, \frac{a_3}{0.7}, \frac{a_4}{0.3}, \frac{a_5}{0.7}\})$
$\tilde{\pi}_{10}$	$(\{x_2\}, \{\frac{a_1}{0.8}, \frac{a_2}{0.7}, \frac{a_3}{1}, \frac{a_4}{0.7}, \frac{a_5}{1}\})$
$\tilde{\pi}_{11}$	$(\{x_3\}, \{\frac{a_1}{0.9}, \frac{a_2}{0.9}, \frac{a_3}{0.8}, \frac{a_4}{0.3}, \frac{a_5}{0.8}\})$
$\tilde{\pi}_{12}$	$(\emptyset, \{\frac{a_1}{1}, \frac{a_2}{1}, \frac{a_3}{1}, \frac{a_4}{1}, \frac{a_5}{1}\})$

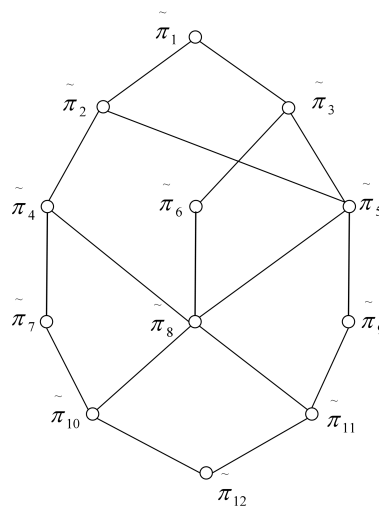


Figure 2. The Hasse diagram of $\tilde{L}(O, C, \tilde{M})$.

For any $A \subseteq C$, $\mathcal{H}_A = (O, A, \tilde{M}_A)$ is called a sub context of $\mathcal{H}$, where $\tilde{M}_A = \tilde{M} \cap (O \times A)$ [19]. Then, the operators $\kappa_A: P(O) \rightarrow \mathcal{F}(A)$ and $\iota_A: \mathcal{F}(A) \rightarrow P(O)$ are defined respectively as follows:

$$\kappa_A(X)(a) = \bigwedge_{x \in X} \tilde{M}(x, a), \quad \forall a \in A, \quad (2.5)$$

$$\iota_A(\tilde{K}) = \{x \in O \mid \forall a \in A, \tilde{K}(a) \leq \tilde{M}(x, a)\}, \quad (2.6)$$

where $X \in P(O)$ and $\tilde{B} \in \mathcal{F}(A)$. χ_A is called the characteristic function, which is defined as $\chi_A(a) = 1$ for $a \in A$, otherwise is 0 for any $A \subseteq C$. Moreover, the following properties holds:

- (1) $\kappa_A(X) = \kappa_C(X) \cap \chi_A$,
- (2) $\iota_C \circ \kappa_A(X) \subseteq \iota_A \circ \kappa_A(X)$.

For abbreviation, we let $\kappa_C(x)$ stand for $\kappa_C(\{x\})$.

2.2. Object granule

The purpose of granular reduction in an FFC is to ensure that the object granule is invariant, so below we start with the definition and some properties of the object granule.

Definition 2.2. [19] Let $\mathcal{H} = (O, C, \tilde{M})$ be an FFC, $R_C = \{(x, y) \in O \times O \mid \tilde{M}(x, a) \leq \tilde{M}(y, a), \forall a \in C\}$ be an ordered relation on object set. The granule of $x \in O$ generated based on R_C is defined as:

$$[x]_{R_C} : O \rightarrow O, \quad [x]_{R_C} = \{y \in O \mid (x, y) \in R_C\}, \quad (2.7)$$

$$[x]_{R_C} = \{y \in O \mid \tilde{M}(x, a) \leq \tilde{M}(y, a), \forall a \in C\}. \quad (2.8)$$

Lemma 2.1. [19] Let $\mathcal{H} = (O, C, \tilde{M})$ be an FFC, $A \subseteq C$ and $x \in O$, then $([x]_{R_A}, \kappa_A([x]_{R_A}))$ is a one-sided fuzzy formal concept of $(O, A, \tilde{M}_A)$ and

$$[x]_{R_A} = \iota_A \circ \kappa_A(x). \quad (2.9)$$

Definition 2.3. [19] Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, where $(O, C, \tilde{M})$ and $(O, D, \tilde{N})$ are FFCs, $C \cap D = \emptyset$. Furthermore, $\mathcal{Q}$ is said to be consistent if $\kappa_C \circ \iota_C(x) \subseteq \kappa_D \circ \iota_D(x)$ for all $x \in O$. Otherwise, it is said to be inconsistent.

Example 2.2. Table 4 lists an FFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$, where $O = \{x_1, x_2, x_3, x_4, x_5\}$, $C = \{a_1, a_2, a_3, a_4, a_5\}$ and $D = \{d_1, d_2, d_3\}$.

Table 4. A fuzzy formal decision context $\mathcal{Q}$.

$\tilde{M}/\tilde{N}$	a_1	a_2	a_3	a_4	a_5	d_1	d_2	d_3
x_1	0.7	0.7	0.8	0.7	0.6	0.8	0.7	0.7
x_2	0.8	0.7	1	0.7	1	0.8	0.8	0.7
x_3	0.9	0.9	0.8	0.3	0.8	0.9	0.6	1
x_4	0.9	0.9	0.7	0.3	0.7	1	0.6	0.8
x_5	0.8	0.7	1	0.3	0.8	0.8	0.6	0.7

By Definition 2.2 and Lemma 2.1, we have

$$\begin{aligned}
 [x_1]_{R_C} &= \iota_C \circ \kappa_C(x_1) = \{x_1, x_2\}, \\
 [x_2]_{R_C} &= \iota_C \circ \kappa_C(x_2) = \{x_2\}, \\
 [x_3]_{R_C} &= \iota_C \circ \kappa_C(x_3) = \{x_3\}, \\
 [x_4]_{R_C} &= \iota_C \circ \kappa_C(x_4) = \{x_3, x_4\}, \\
 [x_5]_{R_C} &= \iota_C \circ \kappa_C(x_5) = \{x_2, x_5\}, \\
 [x_1]_{R_D} &= \iota_D \circ \kappa_D(x_1) = \{x_1, x_2\}, \\
 [x_2]_{R_D} &= \iota_D \circ \kappa_D(x_2) = \{x_2\}, \\
 [x_3]_{R_D} &= \iota_D \circ \kappa_D(x_3) = \{x_3\}, \\
 [x_4]_{R_D} &= \iota_D \circ \kappa_D(x_4) = \{x_4\}, \\
 [x_5]_{R_D} &= \iota_D \circ \kappa_D(x_5) = \{x_1, x_2, x_3, x_4, x_5\}.
 \end{aligned}$$

Furthermore, for a discourse $O = \{x_1, x_2, \dots, x_m\}$, and $X \subseteq O$, the characteristic function φ_X is a m -dimensional row vector φ_X , where

$$\varphi_X = \begin{cases} 1, & x_i \in X, \\ 0, & x_i \notin X. \end{cases} \quad (2.10)$$

Example 2.3. (Continue from Example 2.2) From Example 2.2, we have $[x_1]_{R_C} = \{x_1, x_2\}$, $[x_2]_{R_C} = \{x_2\}$, $[x_3]_{R_C} = \{x_3\}$, $[x_4]_{R_C} = \{x_3, x_4\}$, $[x_5]_{R_C} = \{x_2, x_5\}$. Based on the definition of the characteristic function φ_X , we easily derive $\varphi([x_1]_{R_C}) = \{1, 1, 0, 0, 0\}$, $\varphi([x_2]_{R_C}) = \{0, 1, 0, 0, 0\}$, $\varphi([x_3]_{R_C}) = \{0, 0, 1, 0, 0\}$, $\varphi([x_4]_{R_C}) = \{0, 0, 1, 1, 0\}$, $\varphi([x_5]_{R_C}) = \{0, 1, 0, 0, 1\}$. According Definition 2.3, we know $\mathcal{Q}$ is an IFFDC.

2.3. Definitions, properties of functions and reducts and relationships between different reducts

Definition 2.4. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, $A \subseteq C$, $X \subseteq O$,

$$R_-^A : P(O) \rightarrow P(O), \quad R_-^A(X) = \{x \in O \mid [x]_{R_A} \subseteq X\}, \quad (2.11)$$

$$R_A^- : P(O) \rightarrow P(O), \quad R_A^-(X) = \{x \in O \mid [x]_{R_A} \cap X \neq \emptyset\}, \quad (2.12)$$

called the set of granular upper approximations and the set of granular lower approximations of X about A .

Definition 2.5. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, where $O = \{x_1, x_2, \dots, x_m\}$, and let $O/R_D = \{[x_1]_{R_D}, [x_2]_{R_D}, \dots, [x_s]_{R_D}\}$ be the set of distinct equivalence classes induced by the decision attributes, where $s = |O/R_D|$, $A \subseteq C$. For any $x \in O$, the following functions are defined

$$\zeta_A^* : O \rightarrow P(O/R_D), \quad \zeta_A^*(x) = \{[x_i]_{R_D} \in O/R_D \mid [x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset, x_i \in O\}, \quad (2.13)$$

$$\tau_A^* : O \rightarrow [0, 1]^s, \quad \tau_A^*(x) = \left(\frac{|[x]_{R_A} \cap [x_1]_{R_D}|}{|O|}, \frac{|[x]_{R_A} \cap [x_2]_{R_D}|}{|O|}, \dots, \frac{|[x]_{R_A} \cap [x_m]_{R_D}|}{|O|} \right), \quad (2.14)$$

$$\lambda_A^* : O \rightarrow [0, 1], \quad \lambda_A^*(x) = \max \left(\frac{|[x]_{R_A} \cap [x_1]_{R_D}|}{|O|}, \frac{|[x]_{R_A} \cap [x_2]_{R_D}|}{|O|}, \dots, \frac{|[x]_{R_A} \cap [x_m]_{R_D}|}{|O|} \right), \quad (2.15)$$

$$\xi_A^* : O \rightarrow [0, m], \quad \xi_A^* = \frac{\sum_{i=1}^m |R_A^-([x_i]_{R_D})|}{|O|}. \quad (2.16)$$

ζ_A^* is termed the assignment function on O with respect to a attribute subset A , τ_A^* is termed the distribution function on O with respect to a attribute subset A , λ_A^* is termed the maximum distribution function on O with respect to a attribute subset A , ξ_A^* is the an upper approximation sum function on the domain O with respect to A .

Definition 2.6. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, where $O = \{x_1, x_2, \dots, x_m\}$ and $A \subseteq C$.

(1) For all $x \in O$, it holds that $\zeta_A^*(x) = \zeta_C^*(x)$, then A is termed a granular assignment consistent set (GACS) of $\mathcal{Q}$. If A is a GACS of $\mathcal{Q}$, and an arbitrary proper subset of A is no longer a GACS of $\mathcal{Q}$, then A is termed the granular assignment reduct (GAR) of $\mathcal{Q}$.

(2) For all $x \in O$, it holds that $\tau_A^*(x) = \tau_C^*(x)$, then A is termed a granular distribution consistent set (GDACS) of $\mathcal{Q}$. If A is a GDACS of $\mathcal{Q}$, and an arbitrary proper subset of A is no longer a GDACS of $\mathcal{Q}$, then A is termed the granular distribution reduct (GDR) of $\mathcal{Q}$.

(3) For all $x \in O$, it holds that $\lambda_A^*(x) = \lambda_C^*(x)$, then A is termed a granular maximum distribution consistent set (GMDCS) of $\mathcal{Q}$. If A is a GMDCS of $\mathcal{Q}$, and an arbitrary proper subset of A is no longer a GMDCS of $\mathcal{Q}$, then A is termed the granular maximum distribution reduct (GMDR) of $\mathcal{Q}$.

(4) For all $x \in O$, it holds that $\xi_A^* = \xi_C^*$, then A is termed a granular upper approximation sum consistent set (GUACS) of $\mathcal{Q}$. If A is a GUACS of $\mathcal{Q}$, and an arbitrary proper subset of A is no longer a GUACS of $\mathcal{Q}$, then A is termed the granular upper approximation sum reduct (GUAR) of $\mathcal{Q}$.

It is easy to verify that the GDR, GMDR, GAR and GUAR are all identical for CFFDCs.

Remark 2.1. GACS is an attribute set that preserves possible decision class unchanged for any object, GDACS is an attribute set that preserves the membership degree of objects in each decision class unchanged, GMDCS is an attribute set that preserves the maximum distribution decision class unchanged for each object, and GUACS is an attribute set that preserves the upper approximation sum unchanged of any decision class.

Next, we discuss the relationship between these four kinds of reducts in IFFDCs. Before proceeding further, we first prove the following properties for the four functions given in Definition 2.5.

Property 2.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, $A \subseteq C$. For each $x \in O$, the following conclusions hold,

- (1) $R_C^-([x_i]_{R_D}) \subseteq R_A^-([x_i]_{R_D})$;
- (2) $\tau_C^*(x) \leq \tau_A^*(x)$;
- (3) $\lambda_C^*(x) \leq \lambda_A^*(x)$;
- (4) $\zeta_C^*(x) \subseteq \zeta_A^*(x)$;
- (5) $1 \leq \xi_A^* \leq |O|$;
- (6) $\xi_A^* = 1 \Leftrightarrow R_A^-([x_i]_{R_D}) = [x_i]_{R_D}, \forall i \leq |O|$; $\xi_A^* = |O| \Leftrightarrow R_A^-([x_i]_{R_D}) = O, \forall i \leq |O|$;
- (7) $\xi_C^* \leq \xi_A^*, \forall i \leq |O|$.

Where $\tau_C^*(x) \leq \tau_A^*(x)$ denotes that each component corresponding to the two vectors has a corresponding magnitude relationship between them.

Proof. (1) Since $A \subseteq C$, we have

$$\begin{aligned} [x]_{R_C} &= \{y \in O \mid \tilde{M}(x, a) \leq \tilde{M}(y, a) \forall a \in C\} \\ &\subseteq \{y \in O \mid \tilde{M}(x, a) \leq \tilde{M}(y, a) \forall a \in A\} \\ &= [x]_{R_A}, \end{aligned} \quad (2.17)$$

i.e., $[x]_{R_C} \subseteq [x]_{R_A}$.

Moreover, since

$$\begin{aligned} R_C^-([x_i]_{R_D}) &= \{x \in O \mid [x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset\} \\ &\subseteq \{x \in O \mid [x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset\} \\ &= R_A^-([x_i]_{R_D}), \end{aligned} \quad (2.18)$$

we have $R_C^-([x_i]_{R_D}) \subseteq R_A^-([x_i]_{R_D})$.

(2) From the proof of Property (1), we have $[x]_{R_C} \subseteq [x]_{R_A}$ for any $x \in O$. Further, from Definition 2.5, it follows that $\tau_C^*(x) \leq \tau_A^*(x)$.

(3) From the proof of Property (2), we have

$$\frac{|[x]_{R_C} \cap [x_i]_{R_D}|}{|O|} \leq \frac{|[x]_{R_A} \cap [x_i]_{R_D}|}{|O|}. \quad (2.19)$$

Thus, it follows that

$$\max \left\{ \frac{|[x]_{R_C} \cap [x_i]_{R_D}|}{|O|} \right\} \leq \max \left\{ \frac{|[x]_{R_A} \cap [x_i]_{R_D}|}{|O|} \right\}, \quad (2.20)$$

i.e., $\tau_C^*(x) \leq \tau_A^*(x)$.

(4) From the proof of Property (1), we have $[x]_{R_C} \subseteq [x]_{R_A}$ for any $x \in O$. Thus, it follows that $\{[x_i]_{R_D} \mid [x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset\} \subseteq \{[x_i]_{R_D} \mid [x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset\}$, i.e., $\zeta_C^*(x) \subseteq \zeta_A^*(x)$.

(5) For any $x_i \in O$, there is $[x_i] \subseteq [x]_{R_A} \cap [x_i]_{R_D} \subseteq \{x_1, x_2, \dots, x_m\} = O$. Moreover, by Definition 2.4, we have

$$|O| \leq \sum_{i=1}^m |R_A^-([x_i]_{R_D})| \leq m \cdot |O|, \quad m = |O|. \quad (2.21)$$

Thus, it follows that $1 \leq \xi_A^* \leq |O|$.

(6) If $\xi_A^* = 1$, there is

$$\sum_{i=1}^m |R_A^-([x_i]_{R_D})| = |O|, \quad (2.22)$$

i.e., $[x]_{R_A} \cap [x_i]_{R_D} = [x_i]_{R_D} = R_A^-([x_i]_{R_D})$.

If $\xi_A^* = |O|$, there is

$$\sum_{i=1}^m |R_A^-([x_i]_{R_D})| = m \cdot |O|, \quad (2.23)$$

i.e., $[x]_{R_A} \cap [x_i]_{R_D} = \{x_1, x_2, \dots, x_m\} = O$.

(7) By Property (1) and Definition 2.5, we have $\xi_C^* \leq \xi_A^*, \forall i \leq |O|$. $\square$

Lemma 2.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, $A \subseteq C$. A is a GACS of $\mathcal{Q}$ iff for $\forall x_i, x_j \in O$, when $\zeta_C^*(x_j) \not\subseteq \zeta_C^*(x_i)$, there is $[x_j]_{R_A} \not\subseteq [x_i]_{R_A}$.

Proof. ($\Rightarrow$) If $[x_j]_{R_A} \subseteq [x_i]_{R_A}$, then we have $\zeta_A^*(x_j) \subseteq \zeta_A^*(x_i)$, and since A is a GACS of $\mathcal{Q}$, we have $\zeta_C^*(x_j) = \zeta_A^*(x_j)$, and $\zeta_C^*(x_i) = \zeta_A^*(x_i)$, so there is $\zeta_C^*(x_j) \subseteq \zeta_C^*(x_i)$, which contradicts $\zeta_C^*(x_j) \not\subseteq \zeta_C^*(x_i)$. Thus, there is $[x_j]_{R_A} \not\subseteq [x_i]_{R_A}$.

($\Leftarrow$) To prove that A is a GACS of $\mathcal{Q}$, it is sufficient to show that for any $x \in O$, there is $\zeta_C^*(x) = \zeta_A^*(x)$, which is known as $\zeta_C^*(x) \subseteq \zeta_A^*(x) (\forall x \in O)$, so it is sufficient to show that $\zeta_A^*(x) \subseteq \zeta_C^*(x)$ for all $x \in O$ holds. Take any $[x_j]_{R_D} \in \zeta_A^*(x)$, then $[x_j]_{R_D} \cap [x]_{R_A} \neq \emptyset$. Let $x_i \in [x_j]_{R_D} \cap [x]_{R_A}$, then we have $x_i \in [x_j]_{R_D}$ and $x_i \in [x]_{R_A}$. From $x_i \in [x]_{R_A}$ it follows that $[x_i]_{R_A} \cap [x_j]_{R_D} \neq \emptyset$, hence, $[x_j]_{R_D} \in \zeta_C^*(x_i) \subseteq \zeta_C^*(x)$. From the arbitrariness of $[x_j]_{R_D}$, we have $\zeta_A^*(x) \subseteq \zeta_C^*(x) (\forall x \in O)$. Thus, it follows that A is a GACS of $\mathcal{Q}$. $\square$

Theorem 2.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, $A \subseteq C$. If A is a GDACS of $\mathcal{Q}$, then A is a GMDCS of $\mathcal{Q}$.

Proof. Since A is a GDACS of $\mathcal{Q}$, there is $\tau_A^*(x) = \tau_C^*(x)$. By the definition of GDACS, and since equality of vectors implies componentwise equality, for all $x_i \in O$ we have

$$\frac{|[x]_{R_A} \cap [x_i]_{R_D}|}{|O|} = \frac{|[x]_{R_C} \cap [x_i]_{R_D}|}{|O|}, \quad (2.24)$$

which further implies that

$$\max_{1 \leq i \leq m} \frac{|[x]_{R_A} \cap [x_i]_{R_D}|}{|O|} = \max_{1 \leq i \leq m} \frac{|[x]_{R_C} \cap [x_i]_{R_D}|}{|O|}, \quad m = |O|. \quad (2.25)$$

By the definition of $\lambda_A^*(x)$, we have $\lambda_A^*(x) = \lambda_C^*(x)$. Consequently, A is a GMDCS of $\mathcal{Q}$. $\square$

Theorem 2.4. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC and let $A \subseteq C$. If A is a GDACS of $\mathcal{Q}$, then A is a GACS of $\mathcal{Q}$.

Proof. Since A is a GDSCS of $\mathcal{Q}$, by the definition of GDSCS, this implies that $\tau_A^*(x) = \tau_C^*(x)$. And according to the definition of $\tau_A^*(x)$, we have $|[x]_{R_A} \cap [x_i]_{R_D}| = |[x]_{R_C} \cap [x_i]_{R_D}|$ for any $[x_i]_{R_D}$. Proof by the following two cases.

(1) Assume that $[x_i]_{R_D} \in \zeta_A^*(x)$. By the Definition 2.5, $[x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset$, which implies $|[x]_{R_A} \cap [x_i]_{R_D}| > 0$. Using the equality above, we obtain $|[x]_{R_C} \cap [x_i]_{R_D}| > 0$ and hence $[x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset$. By Definition 2.5, it is known that $[x_i]_{R_D} \in \zeta_C^*(x)$, hence $\zeta_A^*(x) \subseteq \zeta_C^*(x)$.

(2) Assume that $[x_i]_{R_D} \in \zeta_C^*(x)$. By the Definition 2.5, we have $[x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset$. Since $[x]_{R_C} \subseteq [x]_{R_A}$, then we obtain $[x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset$. By Definition 2.5, it is known that $[x_i]_{R_D} \in \zeta_A^*(x)$, hence $\zeta_C^*(x) \subseteq \zeta_A^*(x)$.

Therefore $\zeta_C^*(x) = \zeta_A^*(x)$, for all $x \in O$. By the definition of a GACS, A is a GACS of $\mathcal{Q}$. $\square$

Theorem 2.5. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC and let $A \subseteq C$. Then A is a GACS of $\mathcal{Q}$ if and only if A is a GUACS of $\mathcal{Q}$.

Proof. We prove the two implications separately.

($\Rightarrow$) Since A is a GACS of $\mathcal{Q}$, by the definition of GACS, we have $\zeta_C^*(x) = \zeta_A^*(x)$ for any $x \in O$. That is, there is $\{[x_i]_{R_D} | [x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset, 1 \leq i \leq m\} = \{[x_i]_{R_D} | [x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset, 1 \leq i \leq m\}$. By the definition of $R_A^-([x_i]_{R_D})$, there is $R_A^-([x_i]_{R_D}) = R_C^-([x_i]_{R_D})$ for any $[x_i]_{R_D}$. Otherwise, if there exists $[x_i]_{R_D}$ such that $R_A^-([x_i]_{R_D}) \neq R_C^-([x_i]_{R_D})$, then there exists $x \in O$ such that $[x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset$, but $[x]_{R_A} \cap [x_i]_{R_D} = \emptyset$, which is impossible. Or, there exists $x \in O$ such that $[x]_{R_C} \cap [x_i]_{R_D} = \emptyset$, but $[x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset$, which contradicts $\zeta_C^*(x) = \zeta_A^*(x)$, hence $R_A^-([x_i]_{R_D}) = R_C^-([x_i]_{R_D})$. Therefore $\xi_A^* = \xi_C^*$. By the definition of GUACS, A is a GUACS of $\mathcal{Q}$.

($\Leftarrow$) Since A is a GUACS of $\mathcal{Q}$, by the definition of GUACS, we have $\xi_A^* = \xi_C^*$, i.e.,

$$\frac{\sum_{i=1}^m |R_A^-([x_i]_{R_D})|}{|O|} = \frac{\sum_{i=1}^m |R_C^-([x_i]_{R_D})|}{U}. \quad (2.26)$$

Assuming $\zeta_C^*(x) \neq \zeta_A^*(x)$, we have $\{[x_i]_{R_D} | [x]_{R_A} \cap [x_i]_{R_D} \neq \emptyset, 1 \leq i \leq m\} \neq \{[x_i]_{R_D} | [x]_{R_C} \cap [x_i]_{R_D} \neq \emptyset, 1 \leq i \leq m\}$, that is there exists $[x_r]_{R_D}$ such that $[x]_{R_D} \cap [x_r]_{R_D} = \emptyset$, while $[x]_{R_C} \cap [x_r]_{R_D} \neq \emptyset$, which is impossible. Or, there exists $[x_r]_{R_D}$ such that $[x]_{R_C} \cap [x_r]_{R_D} = \emptyset$, while $[x]_{R_A} \cap [x_r]_{R_D} \neq \emptyset$, hence there is $|\{x \in O | [x]_{R_C} \cap [x_r]_{R_D} = \emptyset\}| < |\{x \in O | [x]_{R_A} \cap [x_r]_{R_D} = \emptyset\}|$, we have $\xi_C^* < \xi_A^*$, which contradicts $\xi_C^* = \xi_A^*$. Therefore $\zeta_C^*(x) = \zeta_A^*(x)$. By the definition of GACS A is a GACS of $\mathcal{Q}$. $\square$

To provide a clearer and more intuitive view of the interrelationships among different reducts, we constructed the diagram shown in Figure 3.

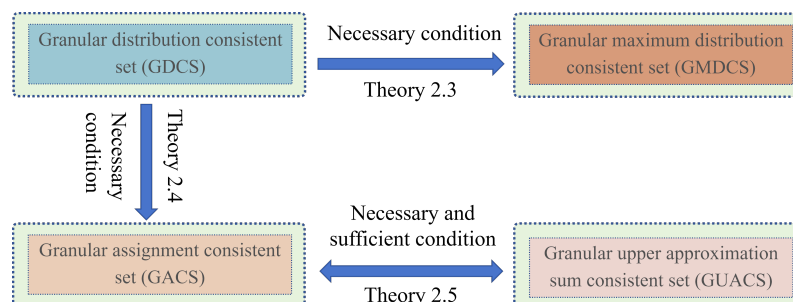


Figure 3. The relationship between GMDCS, GDSCS, GACS and GUACS.

As can be seen from the above analysis, the GACS and GUACS are equivalent for IFFDCs, therefore, we will focus solely on the GACS in this paper.

3. Assignment reduction of IFFDCs

In this section, two granular assignment reduction theories are introduced, one based on rough sets and the other on hypergraph theory.

3.1. Boolean reasoning representation of granular assignment reduction

Definition 3.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, for any $x_i, x_j \in O$, we denote

$$D_{\zeta}^C(x_i, x_j) = \begin{cases} \{a \in C | \tilde{M}(x_i, a) > \tilde{M}(x_j, a)\}, & \zeta_C^*(x_j) \not\subseteq \zeta_C^*(x_i), \\ \emptyset, & \text{others.} \end{cases} \quad (3.1)$$

$D_{\zeta}^C(x_i, x_j)$ is referred to as the granular assignment difference attribute set. And $\mathcal{D}_{\zeta} = \{D_{\zeta}^C(x_i, x_j) | x_i, x_j \in O\}$ is referred to as the granular assignment difference matrices, and written $\mathcal{D}_{\zeta}^0 = \{D_{\zeta}^A(x_i, x_j) | D_{\zeta}^A(x_i, x_j) \neq \emptyset\}$. $f_{\zeta}(\mathcal{Q}) = \bigwedge \{\forall a | a \in D_{\zeta}^C(x_i, x_j), x_i, x_j \in O\}$ is referred to as the granular assignment difference formula.

Theorem 3.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, $A \subseteq C$. A is a GACS of $\mathcal{Q}$ iff $B \cap D_{\zeta}^C(x_i, x_j) \neq \emptyset$ for any $D_{\zeta}^C(x_i, x_j) \in \mathcal{D}_{\zeta}^0$.

Proof. ($\Rightarrow$) When $D_{\zeta}^C(x_i, x_j) \neq \emptyset$, according to Definition 3.1. that there must be $\zeta_C^*(x_j) \not\subseteq \zeta_C^*(x_i)$. Since A is the GACS of $\mathcal{Q}$, there is $[x_j]_{R_A} \not\subseteq [x_i]_{R_A}$. Therefore, we discuss the following 3 cases:

(1) If $[x_i]_{R_A} \not\subseteq [x_j]_{R_A}$. Since $[x_i]_{R_A} \not\subseteq [x_j]_{R_A}$, there exists at least one x_m satisfying $x_m \in [x_j]_{R_A}$, but $x_m \notin [x_i]_{R_A}$, and hence there is $a \in A$, such that $\tilde{M}(x_j, a) \leq \tilde{M}(x_m, a)$, and $\tilde{M}(x_i, a) \not\leq \tilde{M}(x_m, a)$. Thus, there exists $a \in A \subseteq C$, which satisfies $\tilde{M}(x_i, a) > \tilde{M}(x_m, a)$, and which further yields $a \in D_{\zeta}^C(x_i, x_j)$, that is, $A \cap D_{\zeta}^C(x_i, x_j) \neq \emptyset$.

(2) If $[x_i]_{R_A} \cap [x_j]_{R_A} = \emptyset$. Since $[x_i]_{R_A} \cap [x_j]_{R_A} = \emptyset$, it is clear that there exists $a \in A$ such that $a \in D_{\zeta}^A(x_i, x_j)$. Otherwise, if for any $a \in A$, there is $a \notin D_{\zeta}^A(x_i, x_j)$. Thus there is $\{a \in A | \tilde{M}(x_i, a) \leq \tilde{M}(x_j, a)\} \subseteq A$, and then there is $[x_j]_{R_A} \subseteq [x_i]_{R_A}$, and so we get $[x_i]_{R_A} \cap [x_j]_{R_A} = [x_j]_{R_A} \neq \emptyset$, which is contradictory to $[x_i]_{R_A} \cap [x_j]_{R_A} = \emptyset$, hence, $A \cap D_{\zeta}^C(x_i, x_j) \neq \emptyset$.

(3) If $[x_i]_{R_A} \cap [x_j]_{R_A} \subset [x_i]_{R_A}$ and $[x_i]_{R_A} \cap [x_j]_{R_A} \subset [x_j]_{R_A}$. The proof is analogous to that of case (1).

($\Leftarrow$) Since $D_{\zeta}^C(x_i, x_j) \neq \emptyset$, there is $\zeta_C^*(x_j) \not\subseteq \zeta_C^*(x_i)$, and since $A \cap D_{\zeta}^C(x_i, x_j) \neq \emptyset$, there is $a \in A \cap D_{\zeta}^C(x_i, x_j) \neq \emptyset$, hence, there exists $a \in A \cap D_{\zeta}^C(x_i, x_j)$, i.e. $a \in \{b \in C | \tilde{M}(x_i, b) > \tilde{M}(x_j, b)\} \cap A$, i.e. $a \in \{b \in A | \tilde{M}(x_j, b)\} \cap A$, which means $a \in \{b \in A | \tilde{M}(x_i, b) > \tilde{M}(x_j, b)\}$, i.e. $\tilde{M}(x_i, a) > \tilde{M}(x_j, a)$. That is, $\tilde{M}(x_i, a) > \tilde{M}(x_j, a)$, and further, there is $\tilde{M}(x_i, a) \not\leq \tilde{M}(x_j, a)$, i.e., there is $[x_j]_{R_A} \not\subseteq [x_i]_{R_A}$. Thus, by Lemma 2.2, A is a GACS of $\mathcal{Q}$. $\square$

For the IFFDCs $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$, the minimal disjunctive normal form for the granular distribution difference function of $\mathcal{Q}$ is

$$f_{\zeta}(\mathcal{Q}) = \bigvee_m^p \left(\bigwedge_{n=1}^q a_n \right), \quad (3.2)$$

where $a_n \in \mathcal{D}_\zeta^0$, written $A_m = \{a_n | n = 1, 2, \dots, q\}$.

Theorem 3.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, $A \subseteq C$, $R_\zeta(\mathcal{Q})$ is the set consisting of all GARs of $\mathcal{Q}$, then there is $\{A_m(m = 1, 2, \dots, p)\} = R_\zeta(\mathcal{Q})$.

Proof. Denote $\Omega = \{A_m(m = 1, 2, \dots, p)\}$. Prove first that $R_\zeta(\mathcal{Q}) \subseteq \Omega$. For any $R \in R_\zeta(\mathcal{Q})$, it follows from Theorem 3.1 that $R \cap D_\zeta^C(x, y) \neq \emptyset$ ($\forall D_\zeta^C(x, y) \in \mathcal{D}_\zeta^0$), i.e., $R \in \Omega$, so $R_\zeta(\mathcal{Q}) \subseteq \Omega$.

Next, we prove $\Omega \subseteq R_\zeta(\mathcal{Q})$. For any $D_\zeta^C(x, y) \in \mathcal{D}_\zeta^0$, there is $A \cap D_\zeta^C(x, y) \neq \emptyset$ ($\forall A \in \Omega$). According to Theorem 3.1, we know A is a GACS of $\mathcal{Q}$. Removing any attribute from A , the set is denoted A_1 , and hence there exists $x_i, x_j \in O$ such that $A_1 \cap D_\zeta^C(x_i, x_j) = \emptyset$, i.e., A_1 is not a GACS of $\mathcal{Q}$, and by the arbitrariness of A_1 , A is a GAR of $\mathcal{Q}$, which means that $A \in R_\zeta(\mathcal{Q})$, and hence $\Omega \subseteq R_\zeta(\mathcal{Q})$.

In summary, $\{A_m(m = 1, 2, \dots, p)\} = R_\zeta(\mathcal{Q})$. $\square$

Example 3.1. Continue From Example 2.2, it can be seen that $\mathcal{Q}$ represented in Table 4 is an IFFDC. According to Definition 2.5, we can obtain the granular assignment function of $x_i \in O$ hereinafter presented:

$$\zeta_C^*(x_1) = \zeta_C^*(x_2) = \zeta_C^*(x_5) = \{[x_1]_{R_D}, [x_2]_{R_D}, [x_5]_{R_D}\}, \quad (3.3)$$

$$\zeta_C^*(x_3) = \{[x_3]_{R_D}, [x_5]_{R_D}\}, \quad (3.4)$$

$$\zeta_C^*(x_4) = \{[x_3]_{R_D}, [x_4]_{R_D}, [x_5]_{R_D}\}. \quad (3.5)$$

The granular assignment difference matrix for $\mathcal{Q}$ is shown in Table 5.

Table 5. The granular assignment difference matrix for $\mathcal{Q}$.

$D_\zeta^C(x, y)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	a_4	a_3a_4	$\emptyset$
x_2	$\emptyset$	$\emptyset$	$a_3a_4a_5$	$a_3a_4a_5$	$\emptyset$
x_3	$a_1a_2a_5$	a_1a_2	$\emptyset$	a_3a_5	a_1a_2
x_4	$a_1a_2a_5$	a_1a_2	$\emptyset$	$\emptyset$	a_1a_2
x_5	$\emptyset$	$\emptyset$	a_3	a_3a_5	$\emptyset$

The granular assignment difference function for $\mathcal{Q}$ is

$$\begin{aligned} f_\zeta(\mathcal{Q}) &= a_4 \wedge (a_3 \vee a_4) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_5) \wedge (a_1 \vee a_2) \wedge (a_3 \vee a_5) \wedge a_3 \\ &= (a_1 \wedge a_3 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \end{aligned} \quad (3.6)$$

Let $A_1 = \{a_1, a_3, a_4\}$ and $A_2 = \{a_2, a_3, a_4\}$. It is easy to verify that for any $x_i \in O$, there is $\zeta_{A_j}^*(x_i) = \zeta_C^*(x_i)$, $j = 1, 2$. Therefore, A_1 and A_2 are two GACSs of $\mathcal{Q}$. And no any proper subset of A_1 and A_2 is a GACS of $\mathcal{Q}$. Therefore, A_1 and A_2 are GARs of $\mathcal{Q}$.

Considering the complexity of calculating the assignment function for large data, the following is a discussion of the matrix operation method for calculating the assignment function.

Before that, the definitions of the granular assignment characteristic function and granular assignment matrix are given.

Definition 3.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, where $O = \{x_1, x_2, \dots, x_m\}$, $X \subseteq O$. The granular assignment characteristic function of $\zeta_C^*(X)$ is defined by the m -dimensional row vector $\varphi(\zeta_C^*(X))(x)$, where

$$\varphi(\zeta_C^*(X))(x_i) = \begin{cases} 1, & [x_i]_{R_D} \in X, \\ 0, & [x_i]_{R_D} \notin X. \end{cases} \quad (3.7)$$

Definition 3.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, M_ζ is a $|O| \times |O|$ Boolean matrix. Then M_ζ is said to be the granular assignment matrix of $\mathcal{Q}$, if $M_\zeta(i, :)$ is the granular assignment characteristic function of the object x_i , i.e.,

$$M_\zeta(i, :) = \varphi(\zeta_C(x_i)), \quad (3.8)$$

where $M_\zeta(i, :)$ denotes the row vector of the i -th row of matrix M_ζ .

Example 3.2. (Continue from Example 2.3) According to Definitions 2.4, the granular assignment characteristic function of the object x_i in Example 2.3 is as follows:

$$\varphi(\zeta_C^*(x_1)) = \{1, 1, 0, 0, 1\}, \quad (3.9)$$

$$\varphi(\zeta_C^*(x_2)) = \{1, 1, 0, 0, 1\}, \quad (3.10)$$

$$\varphi(\zeta_C^*(x_3)) = \{0, 0, 1, 0, 1\}, \quad (3.11)$$

$$\varphi(\zeta_C^*(x_4)) = \{0, 0, 1, 1, 1\}, \quad (3.12)$$

$$\varphi(\zeta_C^*(x_5)) = \{1, 1, 0, 0, 1\}. \quad (3.13)$$

The granular assignment matrix is

$$M_\zeta = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5}. \quad (3.14)$$

Definition 3.4. Let $A = (a(i, k))_{m \times n}$, $B = (b(j, k))_{q \times n}$ be two Boolean matrices, define the following operations:

$$c(i, j) = a(i, :) * b(j, :) = \begin{cases} 1, & a(i, k) = b(j, k) \neq 0, \exists k \in \{1, 2, \dots, n\}, \\ 0, & \text{others,} \end{cases} \quad (3.15)$$

and $A * B = (c(i, j))_{m \times q}$.

The new matrix obtained after the operation $*$ is also a Boolean matrix.

Example 3.3. (Continue from Example 3.2) For the given Boolean matrices $A_{3 \times 4}$ and $B_{5 \times 4}$, the result after operation by $*$ is as follows.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}_{3 \times 4}, \quad (3.16)$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{5 \times 4}, \quad (3.17)$$

$$C = A * B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}_{3 \times 4} * \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{5 \times 4} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}_{3 \times 5}. \quad (3.18)$$

The element corresponding to c_{ij} is obtained by comparing the i -th row of A with the j -th row of B . For example, c_{22} is determined by comparing $A(2, :)$ with $B(2, :)$. Since $a_{22} = b_{22}$, according to the definition of the $*$ operator, we obtain $c_{22} = 1$.

Theorem 3.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an FFDC, where $|O| = m$, if M_C and M_D are the object granule and decision granule of $\mathcal{Q}$, then $M_\zeta = M_C * M_D$.

Proof. For any $x_i \in O$, there is

$$\zeta_C^*(x_i) = \{[x_j]_{R_D} | [x_i]_{R_C} \cap [x_j]_{R_D} \neq \emptyset, j \in \{1, 2, \dots, m\}\} \quad (3.19)$$

$$= \{[x_j]_{R_D} | M_C(i, k) = M_D(j, k), \exists k \in \{1, 2, \dots, m\}\} \quad (3.20)$$

$$= \{[x_j]_{R_D} | M_C(i, :) * M_D(j, :) = 1\} \quad (3.21)$$

$$= \{[x_j]_{R_D} | M_\zeta(i, j) = 1\}. \quad (3.22)$$

Therefore, the i -th row vector of M_ζ is exactly the characteristic function of $\zeta_C(x_i)$. By Definition 3.1, it follows that M_ζ is the granular assignment matrix of $\mathcal{Q}$. $\square$

Example 3.4. (Continue from Example 2.2) From Example 2.2, the object granule matrix M_C and the decision granule matrix M_D generated by $\mathcal{Q}$ in Table 4 are shown below, and according to Definition 3.4, the granular assignment matrix M_ζ can be obtained after $*$ on M_C and M_D , as shown below

$$M_C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5}, \quad M_D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{5 \times 5}, \quad (3.23)$$

$$M_\zeta = M_C * M_D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5} * \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{5 \times 5} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5}. \quad (3.24)$$

M_ζ is the granular assignment matrix of $\mathcal{Q}$, and $M_\zeta(i, :)$ is the characteristic function of $\zeta_C(x_i)$, which agrees with the result of Example 3.2.

The above results establish the theoretical foundation for attribute reduction based on Boolean reasoning. Specifically, the reduction conditions established in Theorems 3.1 and 3.2 can be represented by the corresponding minimal disjunctive normal form, whose solutions determine the attribute subsets satisfying the reduction criterion. Based on these theoretical results, we therefore develop **Algorithm 1** to compute all GARs of an IFFDC.

Algorithm 1. Computing the GAR of IFFDC.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$;

Output: $r_{\zeta}(\mathcal{Q})$ // the GAR of $\mathcal{Q}$.

- 1: Calculate the object granularity matrix M_C , and the decision granular matrix M_D of $\mathcal{Q}$;
 - 2: Calculate the granular assignment matrix M_{ζ} ;
 - 3: Calculate the granular assignment difference matrix $\mathcal{D}_{\zeta}$;
 - 4: Calculate $f_{\zeta}(\mathcal{Q}) = \bigwedge_{(x,y) \in O \times O} \bigvee D_{\zeta}(x,y)$;
 - 5: Calculate $f_{\zeta}(\mathcal{Q}) = \bigvee_{i=1}^t (\bigwedge_{j=1}^{s_i} a_j)$;
 - 6: Let $N_k = \{a_j | j \leq s_i\}$, $r_{\zeta}(\mathcal{Q}) = \{N_k | k \leq t\}$;
 - 7: Return $r_{\zeta}(\mathcal{Q})$.
-

3.2. Hypergraph representation of granular assignment reduction

Definition 3.5. [29] A hypergraph is defined as a pair $G = (V, E)$, where V is a finite set of vertices and E is a family of subsets of V . The members of E are called hyperedges (or simply edges). A transversal of G is a set $M \subset V$ such that $M \cap e \neq \emptyset$ for every edge $e \in E$. Equivalently, a transversal is a vertex set that intersects every edge. A transversal M is said to be minimal if no proper subset of M is a transversal.

All minimal transversals of $G_{\zeta}(\mathcal{Q})$ are denoted by $\mathcal{T}G_{\zeta}(\mathcal{Q})$, and $\mathcal{R}_{\zeta}(\mathcal{Q})$ denotes all GARs of $\mathcal{Q}$. Without special explanation, the following graph all represent hypergraph.

Definition 3.6. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, $\mathcal{D}_{\zeta}$ is the granular assignment difference matrix of $\mathcal{Q}$. Noting that $V = C$, $E = \{e \in \mathcal{D}_{\zeta}^0 : e \neq \emptyset\}$, the graph $G_{\zeta}(\mathcal{Q}) = (V, E)$ is referred to as the granular assignment induced graph derived from $\mathcal{Q}$.

Lemma 3.4. [30] Let $G = (V, E)$ be a graph (hypergraph), then a vertex subset $V' \subseteq V$ is a minimal vertex cover (minimal transversal) of G if and only if $\bigwedge_{v_i \in V'} v_i'$ is a prime implicant of the Boolean function f_G .

Theorem 3.5. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC. If $G_{\zeta}(\mathcal{Q}) = (V, E)$ is the granular assignment induced graph of $\mathcal{Q}$, then $\mathcal{T}G_{\zeta}(\mathcal{Q}) = \mathcal{R}_{\zeta}(\mathcal{Q})$.

Proof. Evidently, by Definition 3.6, each edge in the granular assignment induced graph G_{ζ} of $\mathcal{Q}$ represents a set of granular assignment difference attribute of $\mathcal{Q}$. In essence, the non-zero entries of the granular assignment difference matrix in $\mathcal{Q}$ are equivalent to the edges in G_{ζ} . Therefore, by Theorems 3.2 and Lemma 3.4, it is easy to see that $f_{G_{\zeta}} = f_{\zeta}(\mathcal{Q})$, that is, $\mathcal{T}G_{\zeta}(\mathcal{Q}) = \mathcal{R}_{\zeta}(\mathcal{Q})$. $\square$

Example 3.5. (Continue from Example 3.4) Table 5 shows the granular assignment difference matrix for $\mathcal{Q}$ in Example 3.4. From Definition 3.5, we obtain its corresponding granular assignment induced

graph $G_{\zeta}(\mathcal{Q}) = (V, E)$, where $V = \{a_1, a_2, a_3, a_4, a_5\}$, $E = \{e_1 = (a_3), e_2 = (a_4), e_3 = (a_3, a_4), e_4 = (a_3, a_5), e_5 = (a_3, a_4, a_5), e_6 = (a_1, a_2), e_7 = (a_1, a_2, a_5)\}$, see Figure 4 for details.

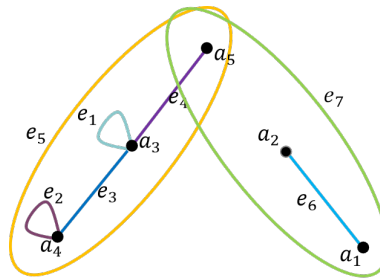


Figure 4. The granular assignment induced graph $G_{\zeta}(\mathcal{Q})$ of $\mathcal{Q}$.

By Lemma 3.4 it follows that all minimal transversals of the graph $G_{\zeta}(\mathcal{Q})$ can be obtained by the following equation

$$\begin{aligned} f_{G_{\zeta}}(\mathcal{Q}) &= a_4 \wedge (a_3 \vee a_4) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_5) \wedge (a_1 \vee a_2) \wedge (a_3 \vee a_5) \wedge a_3 \\ &= (a_1 \wedge a_3 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \end{aligned} \quad (3.25)$$

By Theorem 3.5, it is clear that the minimal transversal of the graph $G_{\zeta}(\mathcal{Q})$ coincides with the GAR of $\mathcal{Q}$.

The preceding theoretical results further establish the correspondence between attribute reducts and minimal transversals of the granular assignment induction graph. This correspondence enables the theoretical reduction problem to be transformed into a hypergraph minimal-transversal problem. Based on this relationship, **Algorithm 2** is developed to efficiently obtain one GAR of an IFFDC.

Algorithm 2. Computing the minimal transversal algorithm for $G_{\zeta}(\mathcal{Q})$.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$;

Output: $r_{\zeta}(\mathcal{Q})$./the minimal transversal of $G_{\zeta}(\mathcal{Q})$.

- 1: Generate the granular assignment induction graph $G_{\zeta}(\mathcal{Q})$ of $\mathcal{Q}$;
 - 2: Generate the incidence matrix $MG_{\zeta}(\mathcal{Q})$ for graph $G_{\zeta}(\mathcal{Q})$;
 - 3: Put all the vertices with ring into $r_{\zeta}(\mathcal{Q})$;
 - 4: For any vertex v in $r_{\zeta}(\mathcal{Q})$, remove all edges incident with v ;
 - 5: **while** $MG_{\zeta}(\mathcal{Q}) \neq \emptyset$ **do**
 - 6: Put the vertex v_0 with maximum degree into $r_{\zeta}(\mathcal{Q})$;
 - 7: Remove all edges incident with vertex v_0 ;
 - 8: **end while**
 - 9: For any vertex v in $r_{\zeta}(\mathcal{Q})$, if $r_{\zeta}(\mathcal{Q}) - \{v\}$ can cover all edges incident with v , then remove vertex v from $r_{\zeta}(\mathcal{Q})$;
 - 10: Return $r_{\zeta}(\mathcal{Q})$.
-

Theorem 3.6. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, then the result $r_{\zeta}(\mathcal{Q})$ output by Algorithm 2 is a GAR of $\mathcal{Q}$.

Proof. It can be seen from Theorem 3.5 that a minimal transversal of induced hypergraph $G_{\zeta}(\mathcal{Q})$ corresponds to a GAR of $\mathcal{Q}$. In order to prove this result, it is only necessary to explain that the result $r_{\zeta}(\mathcal{Q})$ output by Algorithm 2 is a minimal transversal of induced hypergraph $G_{\zeta}(\mathcal{Q})$. According

to steps 3–8 of Algorithm 2, it is clear that $r_\zeta(\mathcal{Q})$ is a minimal transversal of induced graph $G_\zeta(\mathcal{Q})$, because $r_\zeta(\mathcal{Q})$ covers all edges of $G_\zeta(\mathcal{Q})$. Let's use the counter-proof method to assume that $r_\zeta(\mathcal{Q})$ is not a minimal transversal. That is, there is $v \in r_\zeta(\mathcal{Q})$ so that the vertex set $r_\zeta(\mathcal{Q}) - \{v\}$ can still cover all edges of $G_\zeta(\mathcal{Q})$. However, it can be seen from step 9 of Algorithm 2 that there is no such $v \in r_\zeta(\mathcal{Q})$, so that the vertex set $r_\zeta(\mathcal{Q}) - \{v\}$ can still cover all edges of $G_\zeta(\mathcal{Q})$. Therefore, the theorem is proved. $\square$

It is worth noting that the Boolean-reasoning-based method (Algorithm 1) can obtain all GARs of an FFDC, whereas the hypergraph-based minimal transversal method (Algorithm 2) yields only one GAR.

4. Distributional reduction of IFFDCs

In this section, two granular distribution reduction theories are introduced, one based on rough sets and the other on hypergraph theory.

4.1. Boolean reasoning representation of granular distribution reduction

Definition 4.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, for any $x_i, x_j \in O$, we denote

$$D_\tau^C(x_i, x_j) = \begin{cases} \{a \in C \mid \tilde{M}(x_i, a) > \tilde{M}(x_j, a)\}, & \tau_C^*(x_j) \not\leq \tau_C^*(x_i), \\ \emptyset, & \text{others,} \end{cases} \quad (4.1)$$

$D_\tau^C(x_i, x_j)$ is referred to as the granular distribution difference attribute set. And $\mathcal{D}_\tau = \{D_\tau^C(x_i, x_j) \mid x_i, x_j \in O\}$ is referred to as the granular distribution difference matrix, and written $\mathcal{D}_\tau^0 = \{D_\tau^A(x_i, x_j) \mid D_\tau^A(x_i, x_j) \neq \emptyset\}$. $f_\tau(\mathcal{Q}) = \bigwedge \{\forall a \mid a \in D_\tau^C(x_i, x_j), x_i, x_j \in O\}$ is referred to as the granular distribution difference formula.

Lemma 4.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, $A \subseteq C$. A is a GDCS of $\mathcal{Q}$ if and only if for any $x_i, x_j \in O$, if $\tau_C^*(x_i) \not\leq \tau_C^*(x_j)$, there is $[x_i]_{R_A} \not\subseteq [x_j]_{R_A}$.

Proof. ($\Rightarrow$) If $[x_i]_{R_A} \subseteq [x_j]_{R_A}$, then we have $\tau_A^*(x_i) \leq \tau_A^*(x_j)$, and since A is a GDCS of $\mathcal{Q}$, we have $\tau_C^*(x_i) = \tau_A^*(x_i)$ and $\tau_C^*(x_j) = \tau_A^*(x_j)$, so there is $\tau_C^*(x_i) \leq \tau_C^*(x_j)$, which is contradictory to $\tau_C^*(x_i) \not\leq \tau_C^*(x_j)$, therefore, there is $[x_i]_{R_A} \not\subseteq [x_j]_{R_A}$.

($\Leftarrow$) To prove that A is a GDCS of $\mathcal{Q}$, it is necessary to show that for any $x \in O$, there is $\tau_C^*(x) = \tau_A^*(x)$. By Property 2.1, $\tau_C^*(x) \leq \tau_A^*(x)$ holds. Therefore, it is sufficient to prove that $\tau_A^*(x) \leq \tau_C^*(x)$, i.e. $[x]_{R_A} \subseteq [x]_{R_C}$. For any $x_i \in [x]_{R_A}$, there is $[x_i]_{R_A} \subseteq [x]_{R_A}$. Therefore, there is $\tau_C^*(x_i) \leq \tau_C^*(x)$, and so there is $[x_i]_{R_C} \subseteq [x]_{R_C}$, and then there is $x_i \in [x]_{R_C}$. That is, it is proved that $[x]_{R_A} \subseteq [x]_{R_C}$. Thus, there is $\tau_C^*(x) = \tau_A^*(x)$, i.e., it is proved that A is a GDCS of $\mathcal{Q}$. $\square$

Theorem 4.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, $A \subseteq C$. A is a GDCS of $\mathcal{Q}$ if and only if for any $D_\tau^C(x_i, x_j) \in \mathcal{D}_\tau^0$, there is $A \cap D_\tau^C(x_i, x_j) \neq \emptyset$.

Proof. ($\Rightarrow$) Since $D_\tau^C(x_i, x_j) \neq \emptyset$, we have $\tau_C^*(x_i) \not\leq \tau_C^*(x_j)$. Since A is a GDCS of $\mathcal{Q}$, there are $\tau_A^*(x_i) = \tau_C^*(x_i)$ and $\tau_A^*(x_j) = \tau_C^*(x_j)$, and $\tau_A^*(x_i) \not\leq \tau_A^*(x_j)$, and hence, there exists $a \in A$, such that $\tilde{M}(x_i, a) > \tilde{M}(x_j, a)$, i.e., $a \in D_\tau^C(x_i, x_j)$, and then there is $a \in A \cap D_\tau^C(x_i, x_j)$, therefore, $A \cap D_\tau^C(x_i, x_j) \neq \emptyset$.

($\Leftarrow$) Let $D_\tau^C(x_i, x_j) \neq \emptyset$, by Definition 4.1 we know that $\tau_C^*(y) \not\leq \tau_C^*(x)$, and since $A \cap D_\tau^C(x_i, x_j) \neq \emptyset$, we get that there is at least one $a \in A \cap D_\tau^C(x_i, x_j)$, such that $\tilde{M}(x_i, a) > \tilde{M}(x_j, a)$ ($a \in A$), and hence there is $\tilde{M}(x_i, a) \not\leq \tilde{M}(x_j, a)$, that is $[x_j]_{R_A} \not\subseteq [x_i]_{R_A}$, and by Lemma 4.1, A is the GDCS of $\mathcal{Q}$. $\square$

Theorem 4.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC,

$$f_{\tau}(\mathcal{Q}) = \bigvee_m^p \left(\bigwedge_{n=1}^q a_n \right) \quad (4.2)$$

is the minimal disjunctive normal form for the granular distribution difference formula $f_{\tau}(\mathcal{Q})$ of $\mathcal{Q}$, where $a_n \in \mathcal{D}_{\tau}^0$, noting $C_m = \{a_n | n = 1, 2, \dots, q\}$, and $R_{\tau}(\mathcal{Q})$ is the set of all GDRs of $\mathcal{Q}$, then $\{C_m(m = 1, 2, \dots, p)\} = R_{\tau}(\mathcal{Q})$.

Proof. The proof is analogous to that of Theorem 3.2. $\square$

Definition 4.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, M_{τ} is the $|O| \times |O|$ Boolean matrix. M_{τ} is termed a granular distribution matrix of $\mathcal{Q}$, if $M_{\tau}(i, :)$ is a granular distribution function of the object x_i , i.e.,: $M_{\tau}(i, :) = \tau_C^*(x_i)$, where $M_{\tau}(i, :)$ denotes the i -th row vector of the matrix M_{τ} .

Example 4.1. (Continue from Example 2.3) From Example 2.3, it follows that $\mathcal{Q}$ is an IFFDC. By Definitions 2.4 and 4.1, the following result is obtained.

$$\tau_C^*(x_1) = \left(\frac{2}{5}, \frac{1}{5}, 0, 0, \frac{2}{5} \right), \quad (4.3)$$

$$\tau_C^*(x_2) = \left(\frac{1}{5}, \frac{1}{5}, 0, 0, \frac{1}{5} \right), \quad (4.4)$$

$$\tau_C^*(x_3) = \left(0, 0, \frac{1}{5}, 0, \frac{1}{5} \right), \quad (4.5)$$

$$\tau_C^*(x_4) = \left(0, 0, \frac{1}{5}, \frac{1}{5}, \frac{2}{5} \right), \quad (4.6)$$

$$\tau_C^*(x_5) = \left(\frac{1}{5}, \frac{1}{5}, 0, 0, \frac{2}{5} \right). \quad (4.7)$$

The granular distribution matrix M_{τ} is given as follows:

$$M_{\tau} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 & 0 & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{2}{5} \end{bmatrix}_{5 \times 5}. \quad (4.8)$$

And the granular distribution difference matrix is shown in Table 6.

Table 6. The granular distribution difference matrix for $\mathcal{Q}$.

$D_{\tau}^C(x_i, x_j)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	a_4	a_3a_4	a_4
x_2	$a_1a_3a_5$	$\emptyset$	$a_3a_4a_5$	$a_3a_4a_5$	a_4a_5
x_3	$a_1a_2a_5$	a_1a_2	$\emptyset$	a_3a_5	a_1a_2
x_4	$a_1a_2a_5$	a_1a_2	$\emptyset$	$\emptyset$	a_1a_2
x_5	$a_1a_2a_5$	$\emptyset$	a_3	a_3a_5	$\emptyset$

The granular distribution difference function is

$$f_{\tau}(\mathcal{Q}) = a_4 \wedge (a_3 \vee a_4) \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_2) \quad (4.9)$$

$$\wedge (a_1 \vee a_2 \vee a_5) \wedge (a_3 \vee a_5) \wedge a_3 \quad (4.10)$$

$$= (a_1 \wedge a_3 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \quad (4.11)$$

Let $C_1 = \{a_1, a_3, a_4\}$ and $C_2 = \{a_2, a_3, a_4\}$. It is easy to verify that for any $x_i \in O$, there are $\tau_{C_j}^*(x_i) = \tau_C^*(x_i)$, $j = 1, 2$. Therefore, C_1 and C_2 are GDCSs of $\mathcal{Q}$. And any true subset of C_1 and C_2 is not a GDCS of $\mathcal{Q}$. Therefore, C_1 and C_2 are GDRs of $\mathcal{Q}$.

An algorithm for calculating the granular distribution matrix is designed according to Definition 4.2, i.e., **Algorithm 3**.

Algorithm 3. Computing granular distribution matrix.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$, where $|O| = m$;

Output: M_{τ} //the granular distribution matrix of $\mathcal{Q}$.

- 1: Calculate the object granular matrix M_C of $\mathcal{Q}$, decision granular matrix M_D ;
 - 2: Replicate $M_C(i, :)$ m times to get M_{CC} ;
 - 3: Form a one-dimensional array I by combining $M_{CC}(i, :)$ with the number of M_D that have 1 elements in the same position in each row;
 - 4: Let $M_{\tau}(x_i) = I'$;// I' is the transpose of I ;
 - 5: Return M_{τ} .
-

According to Definition 4.1, construct an algorithm for computing the granular distribution difference matrix, i.e., **Algorithm 4**.

Algorithm 4. Computing granular distribution difference matrix.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$, where $|O| = m$, $|C| = n$;

Output: $\mathcal{D}_{\tau}$ //the granular distribution difference matrix of $\mathcal{Q}$.

- 1: Calculate the granular distribution matrix M_{τ} of $\mathcal{Q}$ according to **Algorithm 3**;
 - 2: Replicate M_{τ} m times, and form a new matrix M_{τ}^B ;
 - 3: Let $L = M_{\tau}^B - M_{\tau}$;
 - 4: If $L(j, :)$ is all 0, then $\mathcal{D}_{\tau}(i, j) = \emptyset$;
 - 5: If $L(j, :)$ is not all 0, put $\{a \in C | \tilde{M}(x_i, a) > \tilde{M}(x_j, a)\}$ into $\mathcal{D}_{\tau}(i, j)$;
 - 6: Return $\mathcal{D}_{\tau}$.
-

The above theoretical results provide the basis for developing the Boolean-reasoning-based attribute reduction method. Specifically, the reduction conditions established in Theorems 4.3 can be represented by the corresponding minimal disjunctive normal form, whose solutions determine the attribute subsets satisfying the reduction criterion. Based on these theoretical results, we therefore develop **Algorithm 5** to compute all GDRs of an IFFDC.

Algorithm 5. Computing granular distribution reduction of IFFDC.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$;

Output: $r_\tau(\mathcal{Q})$ // the granular distribution reduction of $\mathcal{Q}$.

1: Calculate the granular distribution difference matrix $\mathcal{D}_\tau$ of $\mathcal{Q}$ according to **Algorithm 4**;

2: Calculation $f_\tau(\mathcal{Q}) = \bigwedge_{(x,y) \in O \times O} \bigvee D_\tau(x, y)$;

3: Calculate $f_\tau(\mathcal{Q}) = \bigvee_{i=1}^t (\bigwedge_{j=1}^{s_i} a_j)$;

4: Let $N_k = \{a_j | j \leq s_i\}$, $r_\tau(\mathcal{Q}) = \{N_k | k \leq t\}$;

5: Return $r_\tau(\mathcal{Q})$.

4.2. Hypergraph representation of granular distribution reduction

Definition 4.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, $\mathcal{D}_\tau$ is the granular distribution difference matrix of $\mathcal{Q}$. Noting that $V = C$, $E = \{e \in \mathcal{D}_\tau^0 : e \neq \emptyset\}$, the graph $G_\tau(\mathcal{Q}) = (V, E)$ is referred to as the granular distribution induced graph derived from $\mathcal{Q}$.

All minimal transversals of $G_\tau(\mathcal{Q})$ are denoted by $\mathcal{T}G_\tau(\mathcal{Q})$, and all GDRs of $\mathcal{Q}$ are denoted by $\mathcal{R}_\tau(\mathcal{Q})$.

Theorem 4.4. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC. If $G_\tau(\mathcal{Q}) = (V, E)$ is the granular distribution induced graph of $\mathcal{Q}$, then $\mathcal{T}G_\tau(\mathcal{Q}) = \mathcal{R}_\tau(\mathcal{Q})$.

Proof. The proof is analogous to that of Theorem 3.5. □

Example 4.2. (Continue from Example 4.1) Table 6 shows the granular distribution difference matrix of $\mathcal{Q}$ in Example 2.3. From Definition 4.3, we obtain its corresponding granular distribution induced graph $G_\tau(\mathcal{Q}) = (V, E)$, where $V = \{a_1, a_2, a_3, a_4, a_5\}$, $E = \{e_1 = (a_3), e_2 = (a_4), e_3 = (a_3, a_4), e_4 = (a_3, a_5), e_5 = (a_4, a_5), e_6 = (a_1, a_2), e_7 = (a_1, a_2, a_5), e_8 = (a_1, a_3, a_5), e_9 = (a_3, a_4, a_5)\}$, see Figure 5.

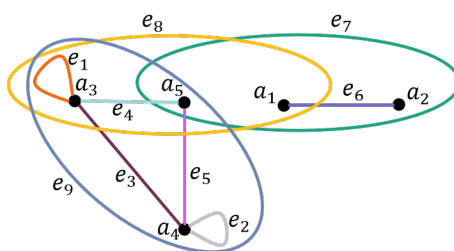


Figure 5. The granular distribution induced graph $G_\tau(\mathcal{Q})$ of $\mathcal{Q}$.

By Lemma 3.4 it follows that all minimal transversals of the graph $G_\tau(\mathcal{Q})$ can be obtained by the following equation

$$f_{G_\tau}(\mathcal{Q}) = a_4 \wedge (a_3 \vee a_4) \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_2) \quad (4.12)$$

$$\wedge (a_1 \vee a_2 \vee a_5) \wedge (a_3 \vee a_5) \wedge a_3 \quad (4.13)$$

$$= (a_1 \wedge a_3 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \quad (4.14)$$

Clearly, the minimal transversals of the graph $G_\tau(\mathcal{Q})$ conformity with the GDRs of $\mathcal{Q}$ computed in Example 4.1. This is another example of Theorem 4.4.

The established theoretical results show that a GDR can be characterized by a minimal transversal of the granular distribution induction graph, thereby providing a hypergraph-based characterization of the reduction problem. This characterization allows a GDR to be obtained by searching for a minimal transversal of the corresponding hypergraph. Accordingly, **Algorithm 6** is constructed to compute one GDR of an IFFDC.

Algorithm 6. Computing the minimal transversal algorithm of $G_\tau(\mathcal{Q})$.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$;

Output: $r_\tau(\mathcal{Q})$ // the minimal transversal of $G_\tau(\mathcal{Q})$.

- 1: Generate the granular distribution induced graph $G_\tau(\mathcal{Q})$ of $\mathcal{Q}$;
 - 2: Generate the incidence matrix $\mathcal{D}_C$ for graph $G_\tau(\mathcal{Q})$;
 - 3: Put all the vertices with rings into $r_\tau(\mathcal{Q})$;
 - 4: For any vertex v in $r_\tau(\mathcal{Q})$, remove all edges incident with v ;
 - 5: **while** $\mathcal{D}_C \neq \emptyset$ **do**
 - 6: Put the vertex v_0 with maximum degree into $r_\tau(\mathcal{Q})$;
 - 7: Delete all edges incident with vertex v_0 .
 - 8: **end while**
 - 9: For any vertex v in $r_\tau(\mathcal{Q})$, if $r_\tau(\mathcal{Q}) - \{v\}$ covers all the edges incident with v , then remove vertex v from $r_\tau(\mathcal{Q})$;
 - 10: Return $r_\tau(\mathcal{Q})$.
-

Theorem 4.5. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, then the result $r_\tau(\mathcal{Q})$ output by Algorithm 6 is a GDR of $\mathcal{Q}$.

Proof. The proof is analogous to that of Theorem 3.6. □

It is worth noting that the Boolean-reasoning-based method and its corresponding algorithm (Algorithm 5) can obtain all GDRs of an FFDC, whereas the minimal transversal method for hypergraphs and its corresponding algorithm (Algorithm 6) yield only one GDR of the FFDC.

5. Maximum distribution reduction of IFFDCs

In this section, two granular maximum distribution reduction theories are introduced, one based on rough sets and the other on hypergraph theory.

5.1. Boolean reasoning representation of granular maximum distribution reduction

Definition 5.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, for any $x_i, x_j \in O$, we denote

$$D_\lambda^C(x_i, x_j) = \begin{cases} \{a \in C | \tilde{M}(x_i, a) > \tilde{M}(x_j, a)\}, & \lambda_C^*(x_j) \neq \lambda_C^*(x_i), \\ \emptyset, & \text{others,} \end{cases} \quad (5.1)$$

$D_\lambda^C(x_i, x_j)$ is referred to as the granular maximum distribution difference attribute set. And $\mathcal{D}_\lambda = \{D_\lambda^C(x_i, x_j) | x_i, x_j \in O\}$ is referred to as the granular maximum distribution difference matrices, and written $\mathcal{D}_\lambda^0 = \{D_\lambda^C(x_i, x_j) | D_\lambda^C(x_i, x_j) \neq \emptyset\}$. $f_\lambda(\mathcal{Q}) = \bigwedge \{\bigvee a | a \in D_\lambda^C(x_i, x_j), x_i, x_j \in O\}$ is referred to as the granular maximum distribution difference formula.

Theorem 5.1. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, $A \subseteq C$. A is the GMDCS of $\mathcal{Q}$ if and only if for $D_\lambda^C(x_i, x_j) \in \mathcal{D}_\lambda^0$, there is $A \cap D_\lambda^C(x_i, x_j) \neq \emptyset$.

Proof. The proof is analogous to that of Theorem 3.1. $\square$

Theorem 5.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC,

$$f_{\lambda}(\mathcal{Q}) = \bigvee_m^p \left(\bigwedge_{n=1}^q a_n \right) \quad (5.2)$$

is a minimal analysis paradigm for the granular maximum distribution difference formula, where $a_n \in \mathcal{D}_{\lambda}^0$, noting that $D_m = \{a_n | n = 1, 2, \dots, q\}$, and that $\mathcal{R}_{\lambda}(\mathcal{Q})$ is the set of all GMDRs of $\mathcal{Q}$, then

$$\{D_m(m = 1, 2, \dots, p)\} = \mathcal{R}_{\lambda}(\mathcal{Q}).$$

Proof. The proof is analogous to that of Theorem 3.2. $\square$

Definition 5.2. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, M_{λ} is a $1 \times |O|$ vector, then M_{λ} is referred to as the granular maximum distribution vector of $\mathcal{Q}$, if $M_{\lambda}(i)$ is the granular maximum distribution function of x_i , i.e., $M_{\lambda}(i) = \lambda_C^*(x_i)$, where $M_{\lambda}(i)$ denotes the i -th column element of the vector M_{λ} .

Example 5.1. (Continue from Example 2.3) From Example 2.3, it follows that $\mathcal{Q}$ is an IFFDC. By Definition 2.4, the corresponding result can be obtained as follows.

$$\lambda_A^*(x_1) = \lambda_A^*(x_4) = \lambda_A^*(x_5) = \frac{2}{5}, \quad (5.3)$$

$$\lambda_A^*(x_2) = \lambda_A^*(x_3) = \frac{1}{5}. \quad (5.4)$$

The granular maximum distribution difference matrix is shown in Table 7.

Table 7. The granular maximum distribution difference matrix for $\mathcal{Q}$.

$D_{\lambda}^C(x, y)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	a_4	$\emptyset$	$\emptyset$
x_2	$a_1 a_3 a_5$	$\emptyset$	$\emptyset$	$a_3 a_4 a_5$	$a_4 a_5$
x_3	$a_1 a_2 a_5$	$\emptyset$	$\emptyset$	$a_3 a_5$	$a_1 a_2$
x_4	$\emptyset$	$a_1 a_2$	$\emptyset$	$\emptyset$	$\emptyset$
x_5	$\emptyset$	$\emptyset$	a_3	$\emptyset$	$\emptyset$

The granular maximum distribution vector is

$$M_{\lambda} = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5} \right). \quad (5.5)$$

The granular maximum distribution difference function is

$$\begin{aligned} f_{\lambda}(\mathcal{Q}) &= a_4 \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_5) \\ &\quad \wedge (a_3 \vee a_5) \wedge (a_1 \vee a_2) \wedge a_3 \\ &= (a_1 \wedge a_3 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \end{aligned} \quad (5.6)$$

Let $D_1 = \{a_1, a_3, a_4\}$ and $D_2 = \{a_2, a_3, a_4\}$. It is easy to verify that for any $x_i \in O$, there are $\tau_{D_j}^*(x_i) = \tau_C^*(x_i)$, $j = 1, 2$. Therefore D_1 and D_2 are GMDCSs of $\mathcal{Q}$. And any proper subset of D_1 and D_2 is not a GMDCS of $\mathcal{Q}$. Therefore, D_1 and D_2 are GMDRs of $\mathcal{Q}$.

According to Definition 5.1 and Theorem 5.2, we design the algorithm to compute the granular maximum distribution difference matrix, i.e., **Algorithm 7**, which is constructed based on Boolean reasoning.

Algorithm 7. Computing the granular maximum distribution difference matrix.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$, where $|O| = m$, $|C| = n$;

Output: $\mathcal{D}_\lambda$ // the granular maximum distribution difference matrix of $\mathcal{Q}$.

```

1: Calculate the granular distribution matrix  $M_\tau$  of  $\mathcal{Q}$  using Algorithm 3;
2: Take the maximum value of  $M_\tau(i, :)$  in  $i$ -th row is  $M_\lambda(i)$ ;
3: for  $i = 1$  to  $m$  do
4:   for  $j = 1$  to  $m$  do
5:      $K = M_\lambda(i) - M_\lambda(j)$ ;
6:     if  $K = 0$  then
7:        $\mathcal{D}_\lambda(i, j) = \emptyset$ ;
8:     else
9:       Put  $a$  satisfying the condition:  $\{a \in C | \tilde{M}(x_i, a) > \tilde{M}(x_j, a)\}$  into  $\mathcal{D}_\lambda(i, j)$ ;
10:    end if
11:  end for
12: end for
13: Return  $\mathcal{D}_\lambda$ .
```

According to Theorem 5.2, the attribute reduction criterion can be formulated as a Boolean reasoning problem and represented by the corresponding minimal disjunctive normal form. Each solution of this form determines an attribute subset satisfying the reduction criterion, thereby establishing a direct connection between the theoretical characterization and the computation of reducts. Accordingly, **Algorithm 8** is developed to obtain all GMDRs of an IFFDC.

Algorithm 8. Computing the granular maximum distribution reduction of IFFDCs.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$;

Output: $r_\lambda(\mathcal{Q})$ // the granular maximum distribution reduction of $\mathcal{Q}$.

```

1: Calculate granular maximum distribution difference matrix  $\mathcal{D}_\lambda(\mathcal{Q})$  of  $\mathcal{Q}$  by using Algorithm 7;
2: Calculate  $f_\lambda(\mathcal{Q}) = \bigwedge_{(x,y) \in O \times O} \vee D_\lambda(x, y)$ ;
3: Calculate  $f_\lambda(\mathcal{Q}) = \bigvee_{i=1}^t (\bigwedge_{j=1}^{s_i} a_j)$ ;
4: Let  $N_k = \{a_j | j \leq s_i\}$ ,  $r_\lambda(\mathcal{Q}) = \{N_k | k \leq t\}$ ;
5: Return  $r_\lambda(\mathcal{Q})$ .
```

5.2. Hypergraph representation of granular maximum distribution reduction

Definition 5.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, $\mathcal{D}_\lambda(\mathcal{Q})$ is the granular maximum distribution difference matrix of $\mathcal{Q}$. Noting that $V = C$, $E = \{e \in D_\mathcal{Q}^C : e \neq \emptyset\}$, the graph $G_\lambda(\mathcal{Q}) = (V, E)$ is referred to as the granular maximum distribution induced graph derived from $\mathcal{Q}$.

All minimal transversals of $G_\lambda(\mathcal{Q})$ are denoted by $\mathcal{T}G_\lambda(\mathcal{Q})$, and all GMDRs of $\mathcal{Q}$ are denoted by $\mathcal{R}_\lambda(\mathcal{Q})$.

Theorem 5.3. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC. If $G_\lambda(\mathcal{Q}) = (V, E)$ is the granular maximum distribution induced graph, then $\mathcal{T}G_\lambda(\mathcal{Q}) = \mathcal{R}_\lambda(\mathcal{Q})$.

Proof. The proof is analogous to that of Theorem 3.5. $\square$

Example 5.2. (Continue from Example 5.1) Table 7 shows the granular maximum distribution difference matrix of $\mathcal{Q}$ in Example 2.3. From Definition 5.3, we obtain its corresponding granular maximum distribution induced graph $G_\lambda(\mathcal{Q}) = (V, E)$, where

$$\begin{aligned} V &= \{a_1, a_2, a_3, a_4, a_5\}, \\ E &= \{e_1 = (a_3), e_2 = (a_4), e_3 = (a_3, a_4, a_5), e_4 = (a_3, a_5), \\ &\quad e_5 = (a_4, a_5), e_6 = (a_1, a_2), e_7 = (a_1, a_2, a_5), e_8 = (a_1, a_3, a_5)\}, \end{aligned}$$

see Figure 6 for details.

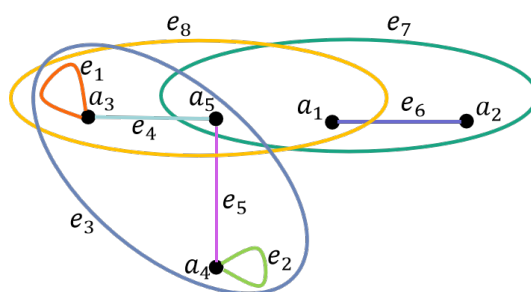


Figure 6. The granular maximum distribution induced graph $G_\lambda(\mathcal{Q})$ of $\mathcal{Q}$.

By Lemma 3.4 it follows that all minimal transversals of the graph $G_\lambda(\mathcal{Q})$ can be obtained by the following equation

$$\begin{aligned} f_{G_\lambda}(\mathcal{Q}) &= a_4 \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_5) \\ &\quad \wedge (a_3 \vee a_5) \wedge (a_1 \vee a_2) \wedge a_3 \\ &= (a_1 \wedge a_3 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \end{aligned} \quad (5.7)$$

Clearly, the minimal transversal of the graph $G_\lambda(\mathcal{Q})$ coincides with the GMDR of $\mathcal{Q}$.

According to the preceding theoretical results, the computation of a GMDR is equivalent to finding a minimal transversal of the granular maximum distribution induction graph. This equivalence provides a direct computational route from the theoretical reduction criterion to a hypergraph-based solution. Accordingly, **Algorithm 9** is developed to determine one GMDR of an IFFDC.

Algorithm 9. Algorithms for computing the minimal transversal algorithm of $G_\lambda(\mathcal{Q})$.

Input: An IFFDC $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$;

Output: $r_\lambda(\mathcal{Q})$ // the minimal transversal of $G_\lambda(\mathcal{Q})$.

1. Calculate granular maximum distribution difference matrix $\mathcal{D}_\lambda(\mathcal{Q})$ of $\mathcal{Q}$ by **Algorithm 7**;
 2. Generate the granular maximum distribution induced graph $G_\lambda(\mathcal{Q})$ of $\mathcal{Q}$ according to Definition 5.3;
 3. Generate the incidence matrix $\mathcal{D}_G$ for graph $G_\lambda(\mathcal{Q})$;
 4. Put all the vertices with rings into $r_\lambda(\mathcal{Q})$;
 5. For any vertex v in $r_\lambda(\mathcal{Q})$, remove all edges incident with v ;
 6. **while** $\mathcal{D}_G \neq \emptyset$ **do**
 - 7: Put the vertex v_0 with maximum degree into $r_\lambda(\mathcal{Q})$;
 - 8: Delete all edges incident with vertex v_0 .
 9. **end while**
 10. For any vertex v in $r_\lambda(\mathcal{Q})$, if $r_\lambda(\mathcal{Q}) - \{v\}$ covers all the edges incident with v , then remove vertex v from $r_\lambda(\mathcal{Q})$;
 11. Return $r_\lambda(\mathcal{Q})$.
-

Theorem 5.4. Let $\mathcal{Q} = (O, C, \tilde{M}, D, \tilde{N})$ be an IFFDC, then the result $r_\lambda(\mathcal{Q})$ output by Algorithm 9 is a GMDR of $\mathcal{Q}$.

Proof. The proof is analogous to that of Theorem 3.6. □

It is worth noting that the Boolean-reasoning-based method and its corresponding algorithm (Algorithm 8) can obtain all GMDRs of an FFDC, whereas the minimal transversal method for hypergraphs and its corresponding algorithm (Algorithm 9) yield only one GMDR of the FFDC.

6. Conclusions

To our knowledge, while numerous attribute reduction approaches exist for CFFDCs, only a limited number have been developed for IFFDCs. This paper presents the first application of hypergraph transversal theory to attribute reduction in the IFFDCs. We define the distribution, allocation, maximum distribution, and upper approximation sum functions, and accordingly establish their corresponding reductions. And, we discuss the relationships between the various reductions and establish that the allocation reduction is equivalent to the upper approximation sum reduction, and that the distribution reduction is necessarily both the maximum distribution reduction and the allocation reduction. Furthermore, we construct a reduction algorithm based on rough set inclusion and a reduction algorithm based on hypergraph transversal theory, and prove that these two algorithms are equivalent.

Previous studies have applied vertex cover theory mainly to CFFDCs. In contrast, this study extends its application to attribute reduction in FFC, providing a graph-theoretic perspective on attribute optimization in FFCA. Nevertheless, the proposed framework is currently limited to a single granularity level, and the two developed algorithms have not yet been systematically compared through extensive data experiments.

Future work will focus on two directions. First, we will extend the proposed framework to multi-granulation FFCs and develop corresponding reduction algorithms, together with comprehensive experiments to evaluate their effectiveness and efficiency against existing methods.

Second, we will investigate attribute reduction for dynamic FFCs, with particular attention to efficient reduction mechanisms under evolving data.

From an application perspective, attribute reduction in FFCs provides an effective means of eliminating redundant attributes while preserving essential information, and thus has potential for uncertain information mining. For instance, in intelligent fault diagnosis, it can identify informative fault-related features from fuzzy and incomplete sensor data, thereby reducing model dimensionality and computational complexity while improving the efficiency and interpretability of fault identification.

Author contributions

Jing Zhang: contributed to formal analysis, investigation, and original draft preparation; Zengtai Gong: contributed to conceptualization, supervision, project administration, and funding acquisition; Huai Yu: contributed to literature review and methodology. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

AI tool “ChatGPT” was used for language polishing in the “Introduction” and “Conclusions” sections.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. R. Wille, Restructuring lattice theory: an approach based on hierarchies of concepts, In: I. Rival, *Ordered sets*, Dordrecht-Boston, 1982, 445–470. https://doi.org/10.1007/978-94-009-7798-3_15
2. H. Xia, Y. Chen, H. C. Gao, Z. Z. Li, Y. P. Chen, Concept lattice-based semantic web service matchmaking, *Communication Software and Networks, International Conference*, 2010, 439–443. <https://doi.org/10.1109/ICCSN.2010.69>
3. K. H. Yuan, D. Q. Miao, Y. Y. Yao, H. Y. Zhang, X. R. Zhao, Feature selection using zentropy-based uncertainty measure, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 2246–2260. <https://doi.org/10.1109/TFUZZ.2023.3347757>

4. C. L. Zhang, G. M. Xue, W. H. Xu, H. L. Zhi, Y. F. Zhou, E. C. C. Tsang, Association-based concept-cognitive learning for classification: fusing knowledge with distance metric learning, *Inf. Fusion*, **124** (2025), 103386. <https://doi.org/10.1016/j.inffus.2025.103386>
5. Y. N. Li, J. Long, Z. Yang, Asymmetric cross-modal hashing based on formal concept analysis, *Proc. AAAI Conf. Artif. Intell.*, **39** (2025), 1392–1401. <https://doi.org/10.1609/aaai.v39i2.32129>
6. C. F. Zou, D. Q. Zhang, J. F. Wan, M. M. Hassan, J. Lloret, Using concept lattice for personalized recommendation system design, *IEEE Syst. J.*, **11** (2017), 305–314. <https://doi.org/10.1109/JSYST.2015.2457244>
7. W. X. Zhang, L. Wei, J. J. Qi, Attribute reduction theory and approach of concept lattices, *Sci. China Ser. E Inf. Sci.*, **35** (2005), 628–639.
8. W. Z. Wu, Y. Leung, J. S. Mi, Granular computing and knowledge reduction in formal contexts, *IEEE Trans. Knowl. Data Eng.*, **21** (2009), 1461–1474. <https://doi.org/10.1109/TKDE.2008.223>
9. J. K. Chen, J. S. Mi, Y. J. Lin, A graph approach for knowledge reduction in formal contexts, *Knowl. Based Syst.*, **148** (2018), 177–188. <https://doi.org/10.1016/j.knosys.2018.02.039>
10. L. A. Zadeh, *Fuzzy sets and information granularity*, North-Holland, 1979.
11. A. Burusco, R. Fuentes-Gonzalez, The study of the L-fuzzy concept lattices, *Mathw. Soft Comput.*, **3** (1994), 209–218.
12. A. Burusco, R. Fuentes-González, Concept lattices defined from implication operators, *Fuzzy Sets Syst.*, **114** (2000), 431–436. [https://doi.org/10.1016/S0165-0114\(98\)00182-1](https://doi.org/10.1016/S0165-0114(98)00182-1)
13. S. B. Yahia, K. Arour, A. Slimani, A. Jaoua, Discovery of compact rules in relational databases, *Inf. Sci. J.*, **3** (2000), 497–511.
14. S. Krajčí, Cluster based efficient generation of fuzzy concepts, *Neural Netw. World*, **5** (2003), 521–530.
15. G. Addison, A. Izadpanahi, S. Saha, M. Winter, L-fuzzy concept analysis using fuzzy categories, *Fuzzy Sets Syst.*, **460** (2023), 72–102. <https://doi.org/10.1016/j.fss.2022.11.008>
16. Z. Zhang, Constructing L-fuzzy concept lattices without fuzzy Galois closure operation, *Fuzzy Sets Syst.*, **333** (2018), 71–86. <https://doi.org/10.1016/j.fss.2017.05.002>
17. O. Kridlo, M. Ojeda-Aciego, Extending formal concept analysis using intuitionistic L-fuzzy sets, *IEEE International Conference on Fuzzy Systems*, IEEE, 2017. <https://doi.org/10.1109/FUZZ-IEEE.2017.8015570>
18. C. L. Zhang, E. C. C. Tsang, W. H. Xu, Y. D. Lin, Incremental concept-cognitive learning approach for concept classification oriented to weighted fuzzy concepts, *Knowl. Based Syst.*, **260** (2023), 110093. <https://doi.org/10.1016/j.knosys.2022.110093>
19. M. W. Shao, Y. Leung, X. Z. Wang, W. Z. Wu, Granular reducts of formal fuzzy contexts, *Knowl. Based Syst.*, **114** (2016), 156–166. <https://doi.org/10.1016/j.knosys.2016.10.010>
20. Z. T. Gong, J. Zhang, Granular reduction in formal fuzzy contexts: graph representation, graph approach and its algorithm, *Artif. Intell. Rev.*, **56** (2023), 15101–15127. <https://doi.org/10.1007/s10462-023-10523-2>

21. H. L. Fan, F. Hao, L. K. Zhang, J. Li, L. J. Guo, W. H. Xu, et al., Binary relations-preserving incremental pseudo-equiconcept reduction for symmetric formal context, *Expert Syst. Appl.*, **276** (2025), 127086. <https://doi.org/10.1016/j.eswa.2025.127086>
22. Z. T. Gong, L. L. He, An attribute reduction method of fuzzy formal contexts based on fuzzy hypergraphs in rough sets, *Int. J. Mach. Learn. Cybern.*, **17** (2026), 166. <https://doi.org/10.1007/s13042-025-02969-3>
23. D. Y. Niu, J. S. Mi, Reductions of concept lattices based on Boolean formal contexts, *Int. J. Approx. Reason.*, **179** (2025), 109372. <https://doi.org/10.1016/j.ijar.2025.109372>
24. Q. Hu, K. Y. Qin, H. Yang, B. B. Xue, A novel approach to attribute reduction and rule acquisition of formal decision context, *Appl. Intell.*, **53** (2023), 13834–13851. <https://doi.org/10.1007/s10489-022-04139-2>
25. Q. Zhang, J. J. Qi, L. Wei, S. Y. Zhao, Attribute combination reduction in formal concept analysis: a theoretical characterization, *Int. J. Approx. Reason.*, **186** (2025), 109498. <https://doi.org/10.1016/j.ijar.2025.109498>
26. Y. D. Lin, J. J. Li, A. H. Tan, J. Zhang, Granular matrix-based knowledge reductions of formal fuzzy contexts, *Int. J. Mach. Learn. Cybern.*, **11** (2020), 643–656. <https://doi.org/10.1007/s13042-019-01022-4>
27. Z. L. Li, J. S. Mi, B. Xie, Attribute reduction in inconsistent decision formal contexts, *Comput. Sci.*, **46** (2019), 257–260.
28. C. C. Huang, J. H. Li, S. M. Dias, Attribute significance, consistency measure and attribute reduction in formal concept analysis, *Neural Netw. World*, **26** (2016), 607–623. <https://doi.org/10.14311/NNW.2016.26.035>
29. C. Berge, *Graphs and Hypergraphs*, North-Holland, 1973.
30. S. Listrovoy, S. Minukhin, The solution algorithms for problems on the minimal vertex cover in networks and the minimal cover in Boolean matrixes, *Int. J. Comput. Sci. Issues*, **9** (2012), 8–15.



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