



*Research article***Risk propagation in heterogeneous supply chain networks with saturated contagion pressure and environmental uncertainty****Hui Li¹, Shufen Wei², Xinyu Ye³, Xinze Lian^{4,*} and Feng Rao^{2,*}**¹ School of Economics and Management, Nanjing Tech University, Nanjing 211816, China² School of Mathematical and Physical Sciences, Nanjing Tech University, Nanjing 211816, China³ Human Resource Institute, Curtin University, Singapore 117684, Singapore⁴ School of Data Science and Artificial Intelligence, Wenzhou University of Technology, Wenzhou 325000, China*** Correspondence:** Email: xinzelian@163.com, raofeng2002@163.com.

Abstract: This paper investigates risk contagion within a heterogeneous supply chain network by formulating a degree-indexed susceptible-infected-recovered-susceptible (SIRS) stochastic dynamical system. The proposed system incorporates nonlinear saturated incidence rates and environmental perturbations to capture the complex interplay between network topology and external uncertainty. We first perform a deterministic analysis, derive the basic reproduction number $\mathcal{R}_0$ using the next-generation matrix approach, and establish stability criteria for both risk-free and endemic risk equilibria. Then, we extend the proposed framework to a stochastic environment by incorporating proportional (multiplicative) noise. Mathematically, we verify the existence and uniqueness of a global positive solution to ensure the system's well-posedness. By constructing Lyapunov functions, we derive stochastic thresholds $\mathcal{R}_0^s$ and $\mathcal{R}_0^p$ that dictates the system's asymptotic behavior. Our results show that if $\mathcal{R}_0^s < 1$, the risk may disappear, while when $\mathcal{R}_0^p > 1$, the system converges to a unique ergodic stationary distribution, which means that the risk will continue to exist under a stable probability state. Numerical simulations utilizing the Euler-Maruyama scheme validate these theoretical thresholds and further elucidate the degree-temporal evolution of contagion across the network. The findings suggest that risk managers should transition from traditional deterministic point forecasting toward distribution-based buffering strategies, prioritizing hub-centric resilience interventions to effectively mitigate cascading failures within highly coupled systems.

Keywords: supply chain risk contagion; complex networks; stochastic differential equations; saturated incidence; stationary distribution

Mathematics Subject Classification: 05C82, 37A25, 60H10

1. Introduction

In the modern context of industrial globalization, supply chain networks have evolved from linear sequential structures into ecosystems characterized by high interconnectivity [1, 2], interdependence, and significant structural heterogeneity [3, 4]. While this complex topology greatly enhances operational efficiency [5, 6], it also acts as a catalyst for the rapid propagation of systemic shocks [7, 8], allowing localized distress to escalate into large-scale, cascading collapses that affect the entire network [9, 10]. A critical challenge in contemporary risk management is that risk contagion is neither a static nor a purely deterministic process [11]; rather, it is driven by the intricate interplay between network architecture, specifically the pivotal role of highly connected hub firms, and the environmental stochasticity inherent in market fluctuations and policy shifts [12–14]. Furthermore, as contagion pressure intensifies, the adaptive behavioral responses of firms often trigger non-linear saturation effects, such as precautionary hedging and operational contraction, thereby fundamentally altering the critical thresholds for risk dissipation [15, 16].

However, effective risk governance is increasingly hindered by the structural heterogeneity [17] and stochasticity of these industrial networks. Empirical evidence suggests that supply chains typically exhibit a scale-free architecture [18], where a small cohort of hub enterprises facilitates extensive connectivity while the vast majority of nodes remains peripheral [19–21]. This uneven distribution of links implies that hubs act as disproportionate amplifiers of contagion pressure, effectively lowering the barrier for systemic outbreaks. For example, by formulating the degree-dependent risk propagation model, Chen et al. [22] demonstrated that structural heterogeneity significantly amplifies systemic vulnerability, where the uneven distribution of connectivity accelerates contagion and severely impairs logistics-transportation efficiency. Despite this, conventional risk management frameworks predominantly rely on static or deterministic assessments, which are inherently inadequate for capturing the nonlinear behavioral responses of firms and the jagged manifestation of risk under persistent environmental noise [23–25]. This theoretical limitation underscores the urgent need for a dynamic stochastic methodology that can not only map heterogeneously distributed transmission pathways but also identify the precise stochastic thresholds governing the transition between risk eradication and endemicity. For example, Goh et al. [26] presented a stochastic model of the multi-stage global supply chain network problem, incorporating a set of related risks, namely, supply, demand, exchange, and disruption. In [27], the authors proposed a production-planning method for multi-echelon supply chains facing both demand uncertainty and unreliable production facilities, explicitly considering the global and cumulative impacts on total system performance. Additionally, drawing on stochastic process theory, Ma and Hou [28] addressed the dynamic characteristics of operational state changes in supply chain firms during risk propagation.

To bridge these theoretical gaps, researchers sought to synthesize complex network science with compartmental epidemiological modeling [18, 20, 29]. One main stream uses topological metrics to characterize how supply chain architectures, most notably scale-free properties and hub-centricity, influence the velocity and depth of risk diffusion [30–32]. For example, Hearnshaw et al. [29] characterized efficient supply chains as scale-free networks, integrating real-world features into an improved model and linking complex-network concepts like resilience, cascading failure, and adaptability to observed supply-chain phenomena. In response to the increasing complexity of global supply chains, Perera et al. [31] proposed a fitness-based generative framework for more realistic

topology reproduction. A parallel stream employs epidemiological frameworks, such as susceptible-infected-recovered (SIR) and susceptible-infected-recovered-susceptible (SIRS) models, to derive the basic reproduction number ($\mathcal{R}_0$), providing a critical threshold for the transition between systemic recovery and persistent distress [13, 33, 34]. For example, Li et al. [35] applied an SIR model to characterize risk propagation and recovery in supply-chain networks. To understand risk propagation within supply chain networks, [36] compared the traditional single-layer SIR model with the widely adopted double-layer SIR-unaware-aware-unaware (SIR-UAU) model, presenting simulation results that depict node state changes over time. However, a significant proportion of these models operate under deterministic dynamics, overlooking a fundamental industrial reality, environmental stochasticity [37]. In real-world supply chains, demand volatility and exogenous shocks act not merely as transient disturbances but as continuous perturbations that can fundamentally shift the system's stability and threshold behavior [38].

Despite recent progress in stochastic SIRS modelling on heterogeneous networks, two issues remain unresolved. First, the interplay between structural heterogeneity and state-dependent noise has not been characterized through a threshold quantity; most existing analyses either treat the deterministic heterogeneous mean-field (HMF) kernel $\Theta(t)$ alone or add noise with restrictive forms, leaving the effect of state-dependent multiplicative perturbations on the stochastic threshold unexamined. Second, the transition from point stability to measure stability has received little attention in supply chain risk literature, even though the latter is the more relevant object for risk managers who design resilience strategies under long-term fluctuations rather than point predictions.

Motivated by these limitations, this paper develops a degree-indexed SIRS stochastic dynamical framework that integrates structural heterogeneity with nonlinear saturated contagion pressure under environmental uncertainty. The works of this paper are threefold

- (1) We formulate a stochastic SIRS system on a heterogeneous network that captures the intricate interplay between topological connectivity and behavioral saturation effects. Unlike conventional linear models, our framework considers the precautionary inhibition of firms as risk intensifies, providing a more realistic depiction of contagion dynamics within hub-and-spoke industrial ecosystems.
- (2) We integrate in a single framework (i) the state-dependent multiplicative noise, (ii) the saturated incidence rate, and (iii) the composite dependency index. Within the degree-based mean-field framework, this coupling yields the stochastic threshold and establishes the criteria for both risk extinction and convergence to a unique ergodic stationary distribution. This analysis identifies the transition from deterministic point-stability to probabilistic measure-stability, defining the mathematical boundary for chronic risk prevalence.
- (3) We implement an Euler-Maruyama scheme to simulate stochastic risk paths and visualize the stationary risk landscape across the degree spectrum. Our findings provide insights into how environmental volatility modulates contagion, advocating for a strategic shift toward distribution-based risk buffering and targeted hub-centric resilience interventions to truncate systemic cascades.

The remainder of this paper is organized as follows. Section 2 constructs the stochastic SIRS system within the supply chain context. Section 3 establishes the mathematical foundation by proving the existence, uniqueness, and stability of a positive solution of the deterministic system. For the stochastic system, Section 4 conducts a stochastic dynamics analysis to derive the threshold conditions

for almost sure risk extinction and to prove the existence of a unique stationary distribution. Section 5 provides numerical simulations. Finally, Section 6 concludes the study.

2. Model formulation

In this section, we formulate a stochastic dynamical system to characterize risk contagion within a heterogeneous supply chain network. The topology is represented as a complex graph in which nodes denote individual enterprises and edges encapsulate transactional or credit linkages. To account for structural heterogeneity, in particular, the small number of hub enterprises with disproportionately high connectivity that is characteristic of modern supply chains, we adopt degree-based mean-field theory [29, 39–41]. To capture the multi-stage evolution of systemic disruptions under environmental stochasticity (e.g., demand volatility and market shocks), we further partition enterprises into three mutually exclusive operational states following the SIRS compartmental framework.

Enterprises are categorized into n degree classes according to their connectivity $k \in \{1, 2, \dots, n\}$. At any time t , each enterprise of degree k resides in one of the following three states:

- Susceptible (S_k): healthy enterprises operating at nominal capacity. They maintain standard operations but remain vulnerable to contagion through transactional links with distressed counterparts.
- Distressed (I_k): enterprises experiencing active operational disruptions, manifesting as cascading delays, liquidity shortages, or order cancellations across business ties.
- Recovered (R_k): enterprises that have successfully mitigated disruptions through restorative interventions (e.g., contingency sourcing, emergency financing). These firms acquire temporary structural immunity, which erodes gradually under evolving market conditions.

The transitions between these states are governed by industrial dynamics. New firms enter the market at rate b , firms exit the network due to natural turnover or restructuring at rate d , and recovered firms lose their temporary protection and become susceptible again at rate γ .

2.1. Composite degree and contagion kernel

The degree k in our heterogeneous network is defined as a composite dependency index rather than mere adjacency

$$k_i = w_1 \cdot \text{trade}_i + w_2 \cdot \text{finance}_i + w_3 \cdot \text{info}_i, \quad (2.1)$$

where trade_i , finance_i , and info_i denote normalized trade volume, financial exposure, and information coupling of firm i , respectively, and the weights satisfy

$$w_1 + w_2 + w_3 = 1.$$

Under this specification, the contagion pressure $\Theta(t)$ perceived by a node, the probability that a randomly chosen link leads to a distressed neighbour in the heterogeneous topology, is

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k(t)}{N_k(t)}, \quad (2.2)$$

where P_k is the degree distribution and

$$\langle k \rangle = \sum_{k=1}^n k P_k$$

is the average connectivity. The kernel $\Theta(t)$ aggregates the systemic dependency burden impinging on each node.

2.2. Saturated incidence rate

To reflect the behavioural response of firms, specifically that healthy firms tighten credit terms and diversify suppliers as systemic risk rises, we adopt the saturated incidence rate

$$\frac{m k S_k(t) \Theta(t)}{1 + \alpha \Theta(t)},$$

where m is the transmission coefficient and α is the saturation factor. The denominator $1 + \alpha \Theta(t)$ captures diminishing marginal transmission as precautionary measures intensify, ensuring that contagion is asymptotically self-limiting even under arbitrarily large network pressure.

2.3. Stochastic extension

Real supply chains are perpetually exposed to exogenous environmental noise such as demand volatility and macro-policy shifts. To capture these uncertainties, we perturb the state transitions by white noise that is proportional to the current state [42–44], yielding the stochastic differential system

$$\begin{cases} dS_k(t) = \left[b - dS_k(t) - \frac{mkS_k(t)\Theta(t)}{1 + \alpha\Theta(t)} + \gamma R_k(t) \right] dt + \sigma_{1k} S_k(t) dB_{1k}(t), \\ dI_k(t) = \left[\frac{mkS_k(t)\Theta(t)}{1 + \alpha\Theta(t)} - (d + \mu)I_k(t) \right] dt + \sigma_{2k} I_k(t) dB_{2k}(t), \\ dR_k(t) = [\mu I_k(t) - (d + \gamma)R_k(t)] dt + \sigma_{3k} R_k(t) dB_{3k}(t), \end{cases} \quad (2.3)$$

where $B_{ik}(t)$ ($i = 1, 2, 3$) are independent standard Brownian motions and σ_{ik} denotes the noise intensity. The term $\sigma_{ik}X_k$ reflects a phenomenon consistent with empirical observation: operational volatility varies with firm size, meaning that large firms experience greater absolute shocks than small firms. This phenomenon is known as the proportional volatility hypothesis in financial and supply chain data analysis and has been confirmed [45]. The total population

$$N_k(t) = S_k(t) + I_k(t) + R_k(t)$$

evolves jointly with the three compartments, subject to the bounded initial conditions

$$\begin{aligned} 0 \leq S_k(0), I_k(0), R_k(0) \leq \frac{b}{d}, \\ S_k(0) + I_k(0) + R_k(0) = N_k(0), \end{aligned} \quad (2.4)$$

for all $k \in \{1, 2, \dots, n\}$.

The full parameter set of the system (2.3), together with managerial interpretations, empirical proxies, and baseline numerical values used in subsequent simulations, is consolidated in Table 1.

Table 1. Parameters of the stochastic supply chain risk contagion system (2.3): managerial interpretation, empirical proxy, and numerical values used in simulations.

Parameter	Interpretation	Empirical proxy	Baseline range	Source
b	Market entry rate; new enterprises entering the supply chain.	Firm birth rate, market expansion data.	0.6 [0.1, 0.8]	[41, 46]
d	Natural exit rate; baseline turnover unrelated to contagion.	Bankruptcy rate, industry churn data.	0.3 [0.1, 0.5]	[29]
m	Transmission coefficient; probability of distress spreading.	Trade credit default probability.	0.15 [0.01, 0.2]	[37, 47]
α	Saturation factor; diminishing marginal transmission as firms enhance self-protection.	Inventory turnover, hedging ratio.	0.5 [0, 1.0]	[48]
μ	Recovery rate; efficiency of restructuring or financial bailout.	Mean recovery time, insurance claim frequency.	0.1 [0.05, 0.4]	[37, 47]
γ	Resilience erosion rate; loss of temporary credit buffers.	Buffer MTM, hedge expiry, drill gap.	0.1 [0.01, 0.2]	[49]
σ_{ik}	Diffusion intensity; magnitude of external financial shocks.	Interest-rate volatility, policy-shift index.	0.05/0.4 [0.01, 0.6]	[50]
k_i	Composite dependency degree (Eq (2.1)).	Weighted trade + finance + information index.	—	—
P_k	Degree distribution; structural topology of the supply chain.	Empirical network adjacency matrix.	$\sim k^{-\eta}$	[18, 41]
$\langle k \rangle$	Average connectivity; mean number of transactional links per firm.	Mean degree in transaction network.	10	[41]
κ	Heterogeneity index $\langle k^2 \rangle / \langle k \rangle$.	Top-1% share of total dependency mass.	~ 30	[41]
$\Theta(t)$	Contagion pressure kernel (Eq (2.2)).	Weighted fraction of distressed neighbours.	—	—
S_k, I_k, R_k	Operational states of a degree- k firm.	Capacity utilisation, stockout rate, backup-supplier count.	—	—
N_k	Total population of degree- k firms, $S_k + I_k + R_k$.	Industry census by size class.	b/d	—
η	Power-law exponent of degree distribution.	Empirical degree distribution tail exponent.	2.5 [2.0, 3.0]	[18, 41]
n	Maximal degree in the network.	Structural upper bound on firm connectivity.	200 [50, 500]	[41]

Note: Composite dependency index: $k_i = w_1 \text{trade}_i + w_2 \text{finance}_i + w_3 \text{info}_i$ with $w_1 + w_2 + w_3 = 1$. Saturated incidence rate: $\frac{mkS_k\Theta}{1+\alpha\Theta}$. State-dependent noise: $\sigma_{ik}(X_k - X^*)$ captures endogenous volatility amplification [45].

Remark 2.1. For large supply chain networks, a degree-indexed representation is a natural choice that aligns with first principles. In practical applications, we can usually only obtain the degree distribution P_k from commercial credit records and supplier disclosure information, but not the complete adjacency matrix. Therefore, ignoring high-order details while preserving the heterogeneity of node connections is not only consistent with the actual situation of data availability but also facilitates mathematical analysis. Under the assumption of annealed networks, nodes of the same degree share the same dynamic state, and the spread timescale exceeds the network mixing time. This approximation becomes accurate under the thermodynamic limit and remains quantitatively accurate

for finite networks [39, 41]. For this reason, the degree-based framework is the tool for analyzing large-scale supply chain risk propagation when only first-order connectivity information is available [29].

3. Preliminary analysis

To establish a baseline for stochastic analysis, we first examine the deterministic version of system (2.3). Setting $\sigma_{ik} = 0$ reduces the deterministic risk propagation dynamics to the following ordinary differential equations

$$\begin{cases} \frac{dS_k(t)}{dt} = b - dS_k(t) - \frac{mkS_k(t)\Theta(t)}{1 + \alpha\Theta(t)} + \gamma R_k(t), \\ \frac{dI_k(t)}{dt} = \frac{mkS_k(t)\Theta(t)}{1 + \alpha\Theta(t)} - (d + \mu)I_k(t), \\ \frac{dR_k(t)}{dt} = \mu I_k(t) - (d + \gamma)R_k(t). \end{cases} \quad (3.1)$$

To ensure that the proposed supply chain risk model is economically and systematically well-posed, it is essential to demonstrate that the state variables remain non-negative and bounded within a feasible region. By summing the equations in system (3.1), the evolution of the total number of nodes with degree k is governed by

$$\frac{dN_k(t)}{dt} = b - dN_k(t), \quad k = 1, 2, \dots, n. \quad (3.2)$$

Solving this differential equation yields

$$N_k(t) = N_k(0)e^{-dt} + \frac{b}{d}(1 - e^{-dt}).$$

It follows immediately that if $N_k(0) = \frac{b}{d}$, then $N_k(t) \equiv \frac{b}{d}$ for all $t \geq 0$. Furthermore, for any initial condition $N_k(0) \geq 0$, we have

$$\limsup_{t \rightarrow \infty} N_k(t) = \frac{b}{d}.$$

Based on the above analysis, we define the following biologically and economically feasible region for system (3.1).

Lemma 3.1. *The closed set*

$$\Gamma = \left\{ (S_k, I_k, R_k) \in \mathbb{R}_+^{3n} : S_k + I_k + R_k = \frac{b}{d}, \quad k = 1, \dots, n \right\}$$

is a positively invariant set for the deterministic system (3.1) with respect to the initial condition (2.4).

Remark 3.2. *Lemma 3.1 establishes the structural consistency and well-posedness of the proposed system. From a managerial standpoint, the non-negativity of state variables (S_k, I_k, R_k) ensures that the distribution of firms across different risk states remains economically meaningful throughout the contagion process. Furthermore, the invariance of region Γ encapsulates the dynamic equilibrium between market entry (b) and organic attrition (d). This boundedness implies that risk propagation*

occurs within a defined market carrying capacity, $N_k = \frac{b}{d}$, preventing divergent or unrealistic growth. Consequently, this provides a stable framework for the longitudinal assessment of systemic vulnerability and the long-term effectiveness of risk-mitigation strategies.

The equilibria of system (3.1) are obtained by setting the time derivatives to zero, which leads to the following algebraic system

$$\begin{cases} b - dS_k - \frac{mkS_k\Theta}{1 + \alpha\Theta} + \gamma R_k = 0, \\ \frac{mkS_k\Theta}{1 + \alpha\Theta} - (d + \mu)I_k = 0, \\ \mu I_k - (d + \gamma)R_k = 0, \end{cases} \quad (3.3)$$

for $k = 1, 2, \dots, n$. In the absence of systemic risk, system (3.1) always admits a risk-free equilibrium (RFE), denoted by $\mathbf{E}^0$. At this state, the infectious and recovered compartments vanish, i.e., $I_k = R_k = 0$ for all $k \in \{1, \dots, n\}$. Under this condition, the contagion pressure Θ naturally reduces to zero. Substituting these into the first equation of (3.3) yields $S_k^0 = \frac{b}{d}$. Therefore, the RFE is defined as

$$\mathbf{E}^0 = \left(\underbrace{\frac{b}{d}, 0, 0}_{k=1}, \underbrace{\frac{b}{d}, 0, 0}_{k=2}, \dots, \underbrace{\frac{b}{d}, 0, 0}_{k=n} \right) \in \mathbb{R}_+^{3n}.$$

The stability of the risk-free equilibrium $\mathbf{E}^0$ and the potential for risk persistence are governed by the basic reproduction number $\mathcal{R}_0$. This threshold parameter represents the expected number of secondary infections produced by a single distressed node in a fully susceptible enterprise. To derive $\mathcal{R}_0$, we employ the next-generation matrix method.

Let $\mathbf{I} = (I_1, I_2, \dots, I_n)^T$ be the vector of distressed compartments. The linearized dynamics of risk propagation near the RFE can be described by the operator $\mathcal{F} - \mathcal{V}$. For each degree class k , the rate of new contagion incidents ($\mathcal{F}_k$) and the rate of risk resolution ($\mathcal{V}_k$) are given by

$$\mathcal{F}_k = \frac{mkS_k(t)\Theta(t)}{1 + \alpha\Theta(t)}, \quad \mathcal{V}_k = (d + \mu)I_k(t).$$

At the risk-free equilibrium $\mathbf{E}^0$, we have $S_k^0 = \frac{b}{d}$ and $N_k^0 = \frac{b}{d}$. The linear approximation of the contagion pressure Θ around $\mathbf{E}^0$ is

$$\Theta = \frac{1}{\langle k \rangle} \sum_{j=1}^n jP_j \frac{I_j}{b/d} = \frac{d}{b\langle k \rangle} \sum_{j=1}^n jP_j I_j.$$

Substituting S_k^0 and Θ into $\mathcal{F}_k$, and noting that the saturation term $(1 + \alpha\Theta)^{-1} \rightarrow 1$ when $\Theta \rightarrow 0$, we derive the Jacobian matrices F and V of size $n \times n$

$$F = \left[\frac{\partial \mathcal{F}_k}{\partial I_j} \right]_{\mathbf{E}^0} = \frac{mkd}{b\langle k \rangle} \cdot \frac{b}{d} \cdot jP_j = \frac{mkjP_j}{\langle k \rangle},$$

$$V = \left[\frac{\partial \mathcal{V}_k}{\partial I_j} \right]_{\mathbf{E}^0} = (d + \mu)\delta_{kj},$$

where δ_{kj} is the Kronecker delta. The next-generation matrix $K = FV^{-1}$ is then defined as:

$$K_{kj} = \frac{mk_j P_j}{\langle k \rangle (d + \mu)}.$$

The basic reproduction number $\mathcal{R}_0$ is the spectral radius $\rho(K)$. Since K is a rank-one matrix, its spectral radius is equal to its trace. Thus, we calculate

$$\mathcal{R}_0 = \sum_{k=1}^n K_{kk} = \sum_{k=1}^n \frac{mk^2 P_k}{\langle k \rangle (d + \mu)} = \frac{m}{\langle k \rangle (d + \mu)} \sum_{k=1}^n k^2 P_k.$$

By utilizing the statistical moments of the network's degree distribution, where $\langle k \rangle = \sum k P_k$ and $\langle k^2 \rangle = \sum k^2 P_k$, the final expression for $\mathcal{R}_0$ is obtained

$$\mathcal{R}_0 = \frac{m \langle k^2 \rangle}{(d + \mu) \langle k \rangle}. \quad (3.4)$$

Remark 3.3. The expression for $\mathcal{R}_0$ identifies the critical threshold for risk persistence within the supply chain network. It illustrates that the systemic risk potential is jointly determined by the intrinsic contagion parameters and the network's structural topology. Specifically, $\mathcal{R}_0$ is positively correlated with the transmission rate m and the heterogeneity ratio $\frac{\langle k^2 \rangle}{\langle k \rangle}$. This ratio indicates that in supply chains with a high degree of variance, where a few hub enterprises possess a disproportionately large number of transactional links, the risk of a systemic outbreak is significantly amplified. Conversely, the risk is more likely to be contained when the resolution rate μ or the natural attrition rate d increases, as both factors effectively shorten the duration during which a distressed firm can propagate risk to its partners.

The threshold parameter $\mathcal{R}_0$ not only quantifies the initial propagation potential but also dictates the asymptotic stability of the system near the risk-free state. The following theorem establishes the local stability conditions for $\mathbf{E}^0$.

Theorem 3.4. For the deterministic supply chain system (3.1), the risk-free equilibrium $\mathbf{E}^0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$.

While Theorem 3.4 guarantees the network's recovery from minor risk perturbations, a more robust assessment of supply chain resilience requires ensuring that the system can eliminate distress regardless of the initial scale of the contagion. To this end, we establish the global asymptotic stability of the risk-free equilibrium $\mathbf{E}^0$ using the Lyapunov direct method.

Theorem 3.5. If $\mathcal{R}_0 \leq 1$, the risk-free equilibrium $\mathbf{E}^0$ of system (3.1) is globally asymptotically stable in the feasible region Γ .

The detailed proofs of Theorems 3.4 and 3.5 are relegated to the appendix.

Remark 3.6. Theorem 3.5 provides a condition for the containment of systemic risk within the supply chain. While local stability only guarantees recovery from minor disruptions, the global asymptotic stability of $\mathbf{E}^0$ implies that as long as $\mathcal{R}_0 \leq 1$, the network is theoretically capable of eliminating distress regardless of the initial scale or severity of the outbreak. From a managerial standpoint, this

result underscores the inherent long-term resilience of the system, suggesting that the risk-free state is a global attractor. However, it should be noted that this conclusion is derived based on the deterministic dynamics; the transient duration and the specific path of risk dissipation remain dependent on the interplay between the network topology and the recovery mechanisms.

While the above content established the conditions under which distress can be eradicated, it is equally crucial to understand the long-term behavior of the network when the threshold of containment is exceeded. When $\mathcal{R}_0 > 1$, the risk-free equilibrium $\mathbf{E}^0$ loses its stability, and a state of permanent risk persistence may emerge. In this section, we investigate the existence and stability of the endemic equilibrium $\mathbf{E}^*$, representing a scenario where risk remains a feature of the supply chain ecosystem.

The risk-persistent equilibrium $\mathbf{E}^*$ corresponds to the positive steady-state of the system where financial distress persists across the network. From Eq (3.3), we can express the density of recovered firms in terms of the distressed ones

$$R_k^* = \frac{\mu}{d + \gamma} I_k^*. \quad (3.5)$$

Substituting (3.5) into the equilibrium total node constraint $S_k^* + I_k^* + R_k^* = \frac{b}{d}$, the density of susceptible firms is given by

$$S_k^* = \frac{b}{d} - \left(1 + \frac{\mu}{d + \gamma}\right) I_k^* = \frac{b}{d} - \frac{d + \gamma + \mu}{d + \gamma} I_k^*. \quad (3.6)$$

Next, substituting (3.6) into the second equation of Eq (3.3), we obtain

$$\frac{mk\Theta^*}{1 + \alpha\Theta^*} \left(\frac{b}{d} - \frac{d + \gamma + \mu}{d + \gamma} I_k^* \right) = (d + \mu) I_k^*.$$

Solving for I_k^* , we derive the equilibrium density of distressed firms for each degree class k :

$$I_k^* = \frac{bmk\Theta^* (d + \gamma)}{d[(d + \mu)(1 + \alpha\Theta^*)(d + \gamma) + mk\Theta^*(d + \gamma + \mu)]}. \quad (3.7)$$

The global contagion pressure Θ^* must satisfy the self-consistency condition defined by the network topology. Substituting I_k^* and $N_k^* = \frac{b}{d}$ into the definition of Θ^*

$$\Theta^* = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{dI_k^*}{b} = \frac{d}{b\langle k \rangle} \sum_{k=1}^n k P_k I_k^*.$$

Substituting (3.7) into this equation yields:

$$\Theta^* = \frac{1}{\langle k \rangle} \sum_{k=1}^n \frac{mk^2 P_k \Theta^* (d + \gamma)}{(d + \mu)(1 + \alpha\Theta^*)(d + \gamma) + mk\Theta^*(d + \gamma + \mu)}. \quad (3.8)$$

To find the non-zero equilibrium ($\Theta^* > 0$), we divide both sides of (3.8) by Θ^* . Let $G(\Theta^*)$ be a function defined as

$$G(\Theta^*) = \frac{1}{\langle k \rangle} \sum_{k=1}^n \frac{mk^2 P_k (d + \gamma)}{(d + \mu)(1 + \alpha\Theta^*)(d + \gamma) + mk\Theta^*(d + \gamma + \mu)}.$$

It is evident that $G(\Theta^*)$ is a strictly monotonically decreasing function of Θ^* . The equation $G(\Theta^*) = 1$ has a unique positive root $\Theta^* \in (0, 1)$, which requires

(i) At the limit $\Theta^* \rightarrow 0$, we have

$$G(0) = \frac{1}{\langle k \rangle} \sum_{k=1}^n \frac{mk^2 P_k}{d + \mu} = \frac{m\langle k^2 \rangle}{(d + \mu)\langle k \rangle} = \mathcal{R}_0.$$

Thus, $G(0) > 1$ is equivalent to $\mathcal{R}_0 > 1$.

(ii) Since $\lim_{\Theta^* \rightarrow \infty} G(\Theta^*) = 0$, the condition $\lim_{\Theta^* \rightarrow \infty} G(\Theta^*) < 1$ is always satisfied.

Based on the mean value theorem, there exists a unique positive solution Θ^* if and only if $\mathcal{R}_0 > 1$. Consequently, the risk-persistent equilibrium $\mathbf{E}^*$ exists and is unique under the condition that the basic reproduction number exceeds unity.

Theorem 3.7. *If $\mathcal{R}_0 > 1$, the system (3.1) admits a unique positive risk-persistent equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, \dots, S_n^*, I_n^*, R_n^*)$ in Γ . The components of $\mathbf{E}^*$ are given by*

$$\begin{cases} S_k^* = \frac{b}{d} - \frac{d + \gamma + \mu}{d + \gamma} I_k^*, \\ I_k^* = \frac{bmk\Theta^*(d + \gamma)}{d[(d + \mu)(1 + \alpha\Theta^*)(d + \gamma) + mk\Theta^*(d + \gamma + \mu)]}, \\ R_k^* = \frac{\mu}{d + \gamma} I_k^*, \end{cases} \quad (3.9)$$

where the contagion pressure Θ^* is the unique positive solution to the self-consistency equation

$$1 = \frac{1}{\langle k \rangle} \sum_{k=1}^n \frac{mk^2 P_k (d + \gamma)}{(d + \mu)(1 + \alpha\Theta^*)(d + \gamma) + mk\Theta^*(d + \gamma + \mu)}. \quad (3.10)$$

If $\mathcal{R}_0 \leq 1$, the only equilibrium within Γ is the risk-free equilibrium $\mathbf{E}^0$.

Remark 3.8. *Theorem 3.7 shows the qualitative shift in the supply chain's long-term behavior as $\mathcal{R}_0$ crosses the threshold. When $\mathcal{R}_0 > 1$, the system transitions from a risk-free state to one of permanent risk prevalence. From a modeling perspective, the uniqueness of Θ^* suggests that for a given set of structural and operational parameters, the contagion pressure will theoretically stabilize at a specific, non-zero level.*

It is important to note that while the saturation factor α does not prevent the emergence of the persistent equilibrium, it plays a crucial role in moderating the steady-state intensity. Specifically, a higher α , reflecting enhanced defensive responses or information transparency, reduces the magnitude of I_k^* at equilibrium compared to linear incidence models. Furthermore, the self-consistency Eq (3.10) highlights that the persistence of risk is disproportionately sustained by highly connected nodes (k^2), indicating that hub enterprises serve as the structural backbone for chronic financial distress within the network.

When $\mathcal{R}_0 > 1$, the system (3.1) moves away from the risk-free state and settles into a steady state of risk. The following theorem establishes the local stability of this equilibrium.

Theorem 3.9. *If $\mathcal{R}_0 > 1$, the unique risk-persistent equilibrium $\mathbf{E}^* = (S_k^*, I_k^*, R_k^*)$ of system (3.1) is locally asymptotically stable in Γ .*

The proof of Theorem 3.9 is given in the Appendix.

Remark 3.10. Theorem 3.9 reveals the structural stability of the risk-persistent state when the threshold is breached. From a managerial perspective, the local asymptotic stability of $\mathbf{E}^*$ implies that once the distress becomes endemic ($\mathcal{R}_0 > 1$), the supply chain tends to settle into a state of risk prevalence. In this regime, minor external fluctuations or transient restorative actions are unlikely to spontaneously steer the network back to a risk-free state, as the system possesses an intrinsic tendency to return to the equilibrium $\mathbf{E}^*$. However, this stability is local in nature, meaning it pertains to the system's recovery from small, marginal perturbations. Furthermore, while the saturation factor α ensures the stability of $\mathbf{E}^*$ by moderating the marginal contagion pressure, the specific time required for the system to converge to this steady state remains sensitive to the underlying network degree distribution P_k .

4. Stochastic dynamics analysis

While the preceding deterministic analysis characterizes the fundamental thresholds and steady states of risk propagation, real-world supply chain networks are perpetually exposed to stochastic shocks, such as market volatility, liquidity crises, and policy shifts. To capture these environmental uncertainties, we extend our framework to the stochastic system (2.3), where risk evolution is perturbed by Brownian motion. Before investigating the long-term probabilistic properties of the system, it is mathematically and managerially imperative to establish that the model is well-posed. This requires proving that for any positive initial condition, the system admits a unique, non-negative global solution that remains within a bounded domain, thereby ensuring that the enterprise densities are both logically consistent and economically meaningful over an infinite time horizon.

Theorem 4.1. For any given initial condition $X(0) = (S_k(0), I_k(0), R_k(0))_{k=1}^n \in \mathbb{R}_+^{3n}$, the system (2.3) admits a unique global solution $X(t) = (S_k(t), I_k(t), R_k(t))_{k=1}^n$ on $t \in [0, \infty)$ almost surely. Moreover, the solution remains inside the positive cone $\mathbb{R}_+^{3n}$ with probability one.

Proof. Consider the drift and diffusion coefficients of system (2.3). Since $N_k(t) = S_k(t) + I_k(t) + R_k(t) > 0$ on $\mathbb{R}_+^{3n}$, the contagion pressure

$$\Theta(X) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k}{N_k}$$

is continuously differentiable on $\mathbb{R}_+^{3n}$. The nonlinear incidence rate $\frac{mkS_k\Theta}{1+\alpha\Theta}$ and all linear terms are locally Lipschitz continuous on $\mathbb{R}_+^{3n}$. The multiplicative diffusion coefficient $\sigma_{ik}X_k$ are smooth and linear, hence also locally Lipschitz.

For any initial value $X(0) \in \mathbb{R}_+^{3n}$, there exists a unique maximal local solution $X(t)$ on the random interval $t \in [0, \tau_e)$, where τ_e denotes the explosion time [50]. To prove that this solution is global (i.e., $\tau_e = \infty$ almost surely) and that the positive cone $\mathbb{R}_+^{3n}$ is invariant, let $\varpi_0 > 1$ be sufficiently large such that all initial components lie within $[\frac{1}{\varpi_0}, \varpi_0]$. For each integer $\varpi \geq \varpi_0$, we define

$$\tau_\varpi = \inf \left\{ t \in [0, \tau_e) : \min_{1 \leq k \leq n} \{S_k(t), I_k(t), R_k(t)\} \leq \frac{1}{\varpi} \text{ or } \max_{1 \leq k \leq n} \{S_k(t), I_k(t), R_k(t)\} \geq \varpi \right\}, \quad (4.1)$$

with $\inf \emptyset = \infty$. The sequence $\{\tau_\varpi\}_{\varpi \geq \varpi_0}$ is strictly non-decreasing. Set $\tau_\infty := \lim_{\varpi \rightarrow \infty} \tau_\varpi$. Clearly, $\tau_\infty \leq \tau_e$ a.s. Thus, it needs to prove that $\tau_\infty = \infty$ a.s.

Define a C^2 -Lyapunov function $V: \mathbb{R}_+^{3n} \rightarrow \mathbb{R}_+$ by

$$V(X) = \sum_{k=1}^n \left[(S_k - 1 - \ln S_k) + (I_k - 1 - \ln I_k) + (R_k - 1 - \ln R_k) \right]. \quad (4.2)$$

The auxiliary function $u - 1 - \ln u$ is strictly positive for all $u \in (0, \infty) \setminus \{1\}$ and vanishes if and only if $u = 1$. Then, $V(X) \geq 0$ on $\mathbb{R}_+^{3n}$. Moreover, $u - 1 - \ln u \rightarrow +\infty$ as $u \rightarrow 0^+$ or $u \rightarrow +\infty$.

Applying Itô's formula to $V(X)$, we have for each component $u \in \{S_k, I_k, R_k\}$ with $du = f_u dt + \sigma u dB$. Summing over all $k \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} \mathcal{L}V = & \sum_{k=1}^n \left[\left(1 - \frac{1}{S_k}\right) \left(b - dS_k - \frac{mkS_k\Theta}{1 + \alpha\Theta} + \gamma R_k\right) + \frac{1}{2}\sigma_{1k}^2 + \left(1 - \frac{1}{I_k}\right) \left(\frac{mkS_k\Theta}{1 + \alpha\Theta} - (d + \mu)I_k\right) \right. \\ & \left. + \frac{1}{2}\sigma_{2k}^2 + \left(1 - \frac{1}{R_k}\right) (\mu I_k - (d + \gamma)R_k) + \frac{1}{2}\sigma_{3k}^2 \right], \end{aligned} \quad (4.3)$$

and regroup $\mathcal{L}V$ as

$$\begin{aligned} \mathcal{L}V = & \sum_{k=1}^n \left\{ b - d(S_k + I_k + R_k) - \frac{b}{S_k} + d + \frac{mk\Theta}{1 + \alpha\Theta} - \frac{\gamma R_k}{S_k} \right. \\ & \left. - \frac{mkS_k\Theta}{(1 + \alpha\Theta)I_k} + (d + \mu) - \frac{\mu I_k}{R_k} + (d + \gamma) + \frac{1}{2}(\sigma_{1k}^2 + \sigma_{2k}^2 + \sigma_{3k}^2) \right\}. \end{aligned} \quad (4.4)$$

Notice that $b - d(S_k + I_k + R_k) = b - dN_k < b$. Since $I_k < N_k = S_k + I_k + R_k$, we have $0 < \frac{I_k}{N_k} < 1$, which provides an upper bound

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k(t)}{N_k(t)} < \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k = 1.$$

This implies $\frac{mk\Theta}{1 + \alpha\Theta} < mk\Theta < mk$. Then, we obtain

$$\begin{aligned} \mathcal{L}V & \leq \sum_{k=1}^n \left[b + (d + mk) + (d + \mu) + (d + \gamma) + \frac{1}{2}(\sigma_{1k}^2 + \sigma_{2k}^2 + \sigma_{3k}^2) \right] \\ & = \sum_{k=1}^n \left[b + 3d + mk + \mu + \gamma + \frac{1}{2}(\sigma_{1k}^2 + \sigma_{2k}^2 + \sigma_{3k}^2) \right] \\ & =: C < \infty, \end{aligned} \quad (4.5)$$

where C is a positive constant independent of the state X and time t .

For any fixed time $T > 0$, integrating Itô's formula from 0 to $\tau_\varpi \wedge T$ yields

$$\mathbb{E}[V(X(\tau_\varpi \wedge T))] = V(X(0)) + \mathbb{E}\left[\int_0^{\tau_\varpi \wedge T} \mathcal{L}V(X(s)) ds\right] \leq V(X(0)) + CT. \quad (4.6)$$

For any sample path in the event $\{\tau_\varpi \leq T\}$, by the definition of the stopping time τ_ϖ in (4.1), at least one component of $X(\tau_\varpi)$ reaches either $\frac{1}{\varpi}$ or ϖ . Consequently,

$$V(X(\tau_\varpi)) \geq \zeta_\varpi := \left(\frac{1}{\varpi} - 1 - \ln \frac{1}{\varpi} \right) \wedge (\varpi - 1 - \ln \varpi).$$

It follows from (4.6) that

$$\begin{aligned} V(X(0)) + CT &\geq \mathbb{E} [V(X(\tau_{\varpi} \wedge T)) \mathbf{1}_{\{\tau_{\varpi} \leq T\}}] \\ &= \mathbb{E} [V(X(\tau_{\varpi})) \mathbf{1}_{\{\tau_{\varpi} \leq T\}}] \\ &\geq \zeta_{\varpi} \mathbb{P}(\tau_{\varpi} \leq T). \end{aligned} \quad (4.7)$$

Suppose that $\mathbb{P}(\tau_{\infty} < \infty) > 0$. Then there exist constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}(\tau_{\infty} \leq T) > \varepsilon$. Since $\tau_{\varpi} \leq \tau_{\infty}$, we have $\{\tau_{\infty} \leq T\} \subseteq \{\tau_{\varpi} \leq T\}$ for all $\varpi \geq \varpi_0$, which implies

$$\mathbb{P}(\tau_{\varpi} \leq T) \geq \mathbb{P}(\tau_{\infty} \leq T) > \varepsilon.$$

Substituting this into (4.7) gives

$$V(X(0)) + CT \geq \varepsilon \zeta_{\varpi}.$$

Noting that $\lim_{\varpi \rightarrow \infty} \zeta_{\varpi} = +\infty$, taking $\varpi \rightarrow \infty$ on both sides leads to

$$\infty > V(X(0)) + CT \geq \infty,$$

which is a contradiction.

Therefore, we have $\tau_{\infty} = \infty$ almost surely. Since $\tau_e \geq \tau_{\infty}$, it follows that $\tau_e = \infty$ a.s. Hence, the solution $X(t)$ exists globally on $[0, \infty)$ and almost surely never exits the open positive cone $\mathbb{R}_+^{3n}$. This completes the proof. $\square$

Remark 4.2. *Theorem 4.1 establishes the fundamental well-posedness of the stochastic supply chain system. From a managerial perspective, the guaranteed non-negativity and global existence of the solution ensure that the enterprise densities remain within an economically meaningful domain, preventing logically inconsistent scenarios such as negative firm populations or infinite risk escalation in finite time. Mathematically, this result provides the necessary theoretical foundation for subsequent investigations into the system's long-term asymptotic behavior. It should be noted, however, that while this theorem ensures the existence of the trajectory, the specific path and stability of the network remain subject to the intensity of the environmental noise σ_{ik} .*

In the following, we derive the threshold conditions under which the risk will vanish from the supply chain network almost surely. We define a stochastic threshold parameter, denoted as $\mathcal{R}_0^s$, which incorporates both network topology and environmental noise.

Theorem 4.3. *Let $X(t) = (S_k(t), I_k(t), R_k(t))_{k=1}^n$ be the unique global positive solution of system (2.3) originating from any initial condition satisfying (2.4) in $\mathbb{R}_+^{3n}$. Define the network stochastic threshold parameter by*

$$\mathcal{R}_0^s := \frac{m\langle k^2 \rangle}{\langle k \rangle (d + \mu + \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2)}. \quad (4.8)$$

If $\mathcal{R}_0^s < 1$, then the network contagion pressure $\Theta(t)$ and the density of disrupted enterprises $I_k(t)$ in every degree tier $k \in \{1, 2, \dots, n\}$ vanish exponentially almost surely. Specifically,

$$\limsup_{t \rightarrow \infty} \frac{\ln \Theta(t)}{t} \leq (d + \mu + \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2) (\mathcal{R}_0^s - 1) < 0 \quad \text{a.s.}, \quad (4.9)$$

and for each $k \in \{1, \dots, n\}$,

$$\limsup_{t \rightarrow \infty} \frac{\ln I_k(t)}{t} < 0 \quad a.s. \quad (4.10)$$

Thus, systemic risk is eradicated from the supply chain network almost surely.

Proof. For each degree class $k \in \{1, \dots, n\}$, define the proportion of disrupted enterprises as $u_k(t) := \frac{I_k(t)}{N_k(t)}$. Clearly, $u_k(t) \in (0, 1)$ since $S_k(t), I_k(t), R_k(t) > 0$ almost surely by Theorem 4.1.

Let $s_k(t) := \frac{S_k(t)}{N_k(t)}$ and $r_k(t) := \frac{R_k(t)}{N_k(t)}$, satisfying $s_k(t) + u_k(t) + r_k(t) = 1$. The aggregate compartment $N_k(t)$ satisfies

$$dN_k(t) = [b - dN_k(t)] dt + \sigma_{1k} S_k(t) dB_{1k}(t) + \sigma_{2k} I_k(t) dB_{2k}(t) + \sigma_{3k} R_k(t) dB_{3k}(t).$$

Applying the Itô formula to $u_k = f(I_k, N_k) = \frac{I_k}{N_k}$, we obtain

$$\begin{aligned} du_k(t) &= \frac{1}{N_k} dI_k - \frac{u_k}{N_k} dN_k - \frac{1}{N_k^2} d\langle I_k, N_k \rangle_t + \frac{u_k}{N_k^2} d\langle N_k, N_k \rangle_t \\ &= \left[\frac{mks_k(t)\Theta(t)}{1 + \alpha\Theta(t)} - (d + \mu)u_k(t) - u_k(t) \left(\frac{b}{N_k(t)} - d \right) + \Psi_k(t) \right] dt + d\tilde{M}_k(t) \\ &= \left[\frac{mks_k(t)\Theta(t)}{1 + \alpha\Theta(t)} - \mu u_k(t) - \frac{b}{N_k(t)} u_k(t) + \Psi_k(t) \right] dt + d\tilde{M}_k(t), \end{aligned} \quad (4.11)$$

where the martingale increment is

$$d\tilde{M}_k(t) = \sigma_{2k} u_k(t)(1 - u_k(t)) dB_{2k}(t) - \sigma_{1k} u_k(t) s_k(t) dB_{1k}(t) - \sigma_{3k} u_k(t) r_k(t) dB_{3k}(t),$$

and the quadratic cross-variation term $\Psi_k(t)$ is given by

$$\begin{aligned} \Psi_k(t) &= -\sigma_{2k}^2 u_k^2(t) + u_k(t) \left[\sigma_{1k}^2 s_k^2(t) + \sigma_{2k}^2 u_k^2(t) + \sigma_{3k}^2 r_k^2(t) \right] \\ &= -\sigma_{2k}^2 u_k^2(t)(1 - u_k(t)) + u_k(t) \left[\sigma_{1k}^2 s_k^2(t) + \sigma_{3k}^2 r_k^2(t) \right]. \end{aligned}$$

From

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k(t)}{N_k(t)},$$

then $\Theta(t) = \sum_{k=1}^n w_k u_k(t)$ with degree-weights $w_k := \frac{kP_k}{\langle k \rangle} > 0$ such that $\sum_{k=1}^n w_k = 1$. Taking the stochastic differential of $\Theta(t)$:

$$d\Theta(t) = \left[\sum_{k=1}^n w_k \frac{mks_k(t)\Theta(t)}{1 + \alpha\Theta(t)} - \sum_{k=1}^n w_k \left(\mu + \frac{b}{N_k(t)} \right) u_k(t) + \sum_{k=1}^n w_k \Psi_k(t) \right] dt + dM_0(t), \quad (4.12)$$

where $M_0(t) = \sum_{k=1}^n w_k \tilde{M}_k(t)$ is a continuous local martingale with $M_0(0) = 0$.

Next, apply Itô's formula to $\ln \Theta(t)$

$$d \ln \Theta(t) = \frac{1}{\Theta(t)} d\Theta(t) - \frac{1}{2\Theta^2(t)} d\langle \Theta, \Theta \rangle_t. \quad (4.13)$$

and estimate the constituent parts of (4.13), we obtain

(1) Since $s_k(t) \leq 1$ and $1 + \alpha\Theta(t) \geq 1$, then

$$\frac{1}{\Theta(t)} \sum_{k=1}^n w_k \frac{m k s_k(t) \Theta(t)}{1 + \alpha\Theta(t)} \leq \sum_{k=1}^n \frac{k P_k}{\langle k \rangle} m k = \frac{m \langle k^2 \rangle}{\langle k \rangle}. \quad (4.14)$$

(2) Along the trajectory toward risk extinction (i.e., $u_k(t) \rightarrow 0$), the dynamics of $N_k(t)$ are asymptotically decoupled from the infection rate $u_k(t)$. In this case, $N_k(t)$ is governed by an ergodic linear stochastic system with stationary mean $\mathbb{E}[N_k] = \frac{b}{d}$ and the time average satisfies $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{b}{N_k(s)} ds = d$. Then we derive

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left[-\frac{1}{\Theta(s)} \sum_{k=1}^n w_k \left(\mu + \frac{b}{N_k(s)} \right) u_k(s) \right] ds \leq -(d + \mu). \quad (4.15)$$

(3) For

$$d\langle \Theta, \Theta \rangle_t = \sum_{k=1}^n w_k^2 \sigma_{2k}^2 u_k^2(t) (1 - u_k(t))^2 dt + \sum_{k=1}^n w_k^2 u_k^2(t) [\sigma_{1k}^2 s_k^2(t) + \sigma_{3k}^2 r_k^2(t)] dt,$$

in the vicinity of extinction ($u_k \rightarrow 0$, where $(1 - u_k)^2 \rightarrow 1$), then

$$\frac{d\langle \Theta, \Theta \rangle_t}{2\Theta^2(t) dt} \geq \frac{\sum_{k=1}^n w_k^2 \sigma_{2k}^2 u_k^2(t)}{2(\sum_{k=1}^n w_k u_k(t))^2} (1 - o(1)).$$

Let $\xi_k := w_k u_k(t) > 0$,

$$\left(\sum_{k=1}^n \xi_k \right)^2 \leq n \sum_{k=1}^n \xi_k^2 \implies \frac{\sum_{k=1}^n \xi_k^2}{(\sum_{k=1}^n \xi_k)^2} \geq \frac{1}{n}, \quad (4.16)$$

we obtain the uniform lower bound

$$\frac{1}{2\Theta^2(t)} \frac{d\langle \Theta, \Theta \rangle_t}{dt} \geq \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 - \varepsilon(t), \quad (4.17)$$

where $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$ as $u_k(t) \rightarrow 0$.

Integrating (4.13) from 0 to t and dividing by t :

$$\begin{aligned} \frac{\ln \Theta(t) - \ln \Theta(0)}{t} &\leq \frac{m \langle k^2 \rangle}{\langle k \rangle} - \frac{1}{t} \int_0^t \sum_{k=1}^n w_k \left(\mu + \frac{b}{N_k(s)} \right) \frac{u_k(s)}{\Theta(s)} ds \\ &\quad - \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 + \frac{1}{t} \int_0^t \frac{\sum_k w_k \Psi_k(s)}{\Theta(s)} ds + \frac{M(t)}{t}, \end{aligned} \quad (4.18)$$

where

$$M(t) = \int_0^t \frac{1}{\Theta(s)} dM_0(s)$$

is a continuous local martingale. Since $\frac{u_k(s)}{\Theta(s)} \leq \frac{1}{w_k}$, the integrand of the martingale quadratic variation satisfies

$$\frac{d\langle M, M \rangle_t}{dt} = \frac{1}{\Theta^2(t)} \frac{d\langle M_0, M_0 \rangle_t}{dt} \leq \max_{1 \leq k \leq n} \{ \sigma_{1k}^2 + \sigma_{2k}^2 + \sigma_{3k}^2 \} < \infty.$$

By the strong law of large numbers for local martingales [50],

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = 0 \quad \text{a.s.} \quad (4.19)$$

Taking the superior limit as $t \rightarrow \infty$ on both sides of (4.18) and applying (4.14), (4.15), and (4.17), thus

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln \Theta(t)}{t} &\leq \frac{m\langle k^2 \rangle}{\langle k \rangle} - (d + \mu) - \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 \\ &= \left(d + \mu + \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 \right) \left[\frac{m\langle k^2 \rangle}{\langle k \rangle \left(d + \mu + \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 \right)} - 1 \right] \\ &= \left(d + \mu + \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 \right) (\mathcal{R}_0^s - 1). \end{aligned} \quad (4.20)$$

Because $\mathcal{R}_0^s < 1$, the right-hand side is strictly negative, establishing (4.9).

From the definition of $\Theta(t)$, for any degree class $k \in \{1, \dots, n\}$,

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k(t)}{N_k(t)} \geq \left(\frac{k P_k}{\langle k \rangle} \right) \frac{I_k(t)}{N_k(t)} > 0.$$

Rearranging yields

$$I_k(t) \leq \left(\frac{\langle k \rangle}{k P_k} \right) N_k(t) \Theta(t).$$

Taking logarithms and dividing by t ,

$$\frac{\ln I_k(t)}{t} \leq \frac{\ln \langle k \rangle - \ln(k P_k)}{t} + \frac{\ln N_k(t)}{t} + \frac{\ln \Theta(t)}{t}. \quad (4.21)$$

Taking the superior limit as $t \rightarrow \infty$ on both sides of (4.21),

$$\limsup_{t \rightarrow \infty} \frac{\ln I_k(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{\ln \Theta(t)}{t} \leq \left(d + \mu + \frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2 \right) (\mathcal{R}_0^s - 1) < 0 \quad \text{a.s.}$$

This establishes (4.10) and completes the proof. $\square$

Remark 4.4. The obtained stochastic threshold parameter $\mathcal{R}_0^s$ is a sufficient condition for risk extinction in Theorem 4.3. The noise-induced extinction mechanism stems from the correction $-\frac{1}{2n} \min_{1 \leq k \leq n} \sigma_{2k}^2$ acting on the contagion pressure $\Theta(t)$, which is naturally diluted by the network tier diversity ($1/n$). This dissipative damping relies on the state-dependent multiplicative noise structure $\sigma_{2k} I_k dB_{2k}(t)$, which vanishes as $I_k \rightarrow 0$ and preserves the invariance of the risk-free boundary. In contrast, additive noise or jump shocks do not produce this negative logarithmic drift and would preclude almost-sure extinction.

The following theorem establishes the existence of a stationary distribution, which implies that the distress becomes a chronic feature of the network.

Theorem 4.5. Let $X(t) = (S_k(t), I_k(t), R_k(t))_{k=1}^n$ be the unique global positive solution of system (2.3) with initial state $X(0) \in \mathbb{R}_+^{3n}$. Define the stochastic persistence threshold parameter by

$$\mathcal{R}_0^p := \frac{m\langle k^2 \rangle}{\langle k \rangle (d + \mu + \frac{1}{2} \max_{1 \leq k \leq n} \sigma_{2k}^2)}. \quad (4.22)$$

If $\mathcal{R}_0^p > 1$, then the system (2.3) admits a unique stationary distribution $\pi(\cdot)$ and it has the ergodicity property. Therefore, disruption risk persists in the supply chain network as a stable stochastic stationary state almost surely.

Proof. To establish the existence of a unique ergodic stationary distribution, we verify the two conditions of Khasminskii's criterion [51]: (i) The diffusion matrix associated with system (2.3) is uniformly elliptic on every compact subset of $\mathbb{R}_+^{3n}$; (ii) There exists a bounded open domain $D \subset \mathbb{R}_+^{3n}$ with smooth boundary such that $\bar{D} \subset \mathbb{R}_+^{3n}$, and a non-negative C^2 -Lyapunov function $\mathcal{V}: \mathbb{R}_+^{3n} \rightarrow \mathbb{R}_+$ satisfying $\mathcal{L}\mathcal{V}(X) \leq -1$ for all $X \in \mathbb{R}_+^{3n} \setminus D$.

The diffusion matrix of (2.3) is $\mathbf{A}(X) = \text{diag}(\mathbf{A}_1(X), \dots, \mathbf{A}_n(X)) \in \mathbb{R}^{3n \times 3n}$, where

$$\mathbf{A}_k(X) = \text{diag}(\sigma_{1k}^2 S_k^2, \sigma_{2k}^2 I_k^2, \sigma_{3k}^2 R_k^2), \quad k = 1, \dots, n. \quad (4.23)$$

Let K be any compact subset of $\mathbb{R}_+^{3n}$; then there exist constants $\delta_K, \Delta_K > 0$ such that $\delta_K \leq S_k, I_k, R_k \leq \Delta_K$ for all $1 \leq k \leq n$. For any $\zeta = (\zeta_1, \dots, \zeta_{3n})^\top \in \mathbb{R}^{3n}$,

$$\zeta^\top \mathbf{A}(X) \zeta = \sum_{k=1}^n (\sigma_{1k}^2 S_k^2 \zeta_{3k-2}^2 + \sigma_{2k}^2 I_k^2 \zeta_{3k-1}^2 + \sigma_{3k}^2 R_k^2 \zeta_{3k}^2) \geq \lambda_{\min} |\zeta|^2,$$

where $\lambda_{\min} = \delta_K^2 \min_{1 \leq k \leq n} \{\sigma_{1k}^2, \sigma_{2k}^2, \sigma_{3k}^2\} > 0$. Hence, $\mathbf{A}(X)$ is uniformly elliptic on K , verifying condition (i).

We construct a C^2 -Lyapunov function $\mathcal{V}: \mathbb{R}_+^{3n} \rightarrow \mathbb{R}_+$

$$\mathcal{V}(X) = \mathcal{W} \mathcal{V}_1(X) + \sum_{k=1}^n \left(-\ln S_k - \ln I_k - \ln R_k \right) + K \sum_{k=1}^n \frac{1}{\theta + 1} N_k^{\theta+1} + C_0, \quad (4.24)$$

where $N_k = S_k + I_k + R_k$. The constants $\mathcal{W}, K, C_0 > 0$ will be determined later. The parameter $\theta \in (0, 1)$ is chosen sufficiently small such that

$$d - \frac{1}{2} \theta \bar{\sigma}^2 > 0, \quad \text{with} \quad \bar{\sigma}^2 := \max_{1 \leq k \leq n} \{\sigma_{1k}^2, \sigma_{2k}^2, \sigma_{3k}^2\}. \quad (4.25)$$

Define

$$\mathcal{V}_1(X) = -\ln \Theta, \quad \text{with} \quad \Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k(t)}{N_k(t)}. \quad (4.26)$$

Let $u_k = \frac{I_k}{N_k}$, $s_k = \frac{S_k}{N_k}$, and $r_k = \frac{R_k}{N_k}$. Applying Itô's formula to u_k yields

$$du_k(t) = \left[\frac{mk s_k(t) \Theta(t)}{1 + \alpha \Theta(t)} - (d + \mu) u_k(t) - u_k(t) \left(\frac{b}{N_k(t)} - d \right) + \Psi_k(t) \right] dt + d\tilde{M}_k(t),$$

where $\Psi_k(t) = \mathcal{O}(u_k(t))$ denotes higher-order infinitesimal terms as $u_k \rightarrow 0$, and $\widetilde{M}_k(t)$ is a continuous local martingale and satisfies

$$d\langle \widetilde{M}_k, \widetilde{M}_k \rangle_t \leq \sigma_{2k}^2 u_k^2 (1 - u_k)^2 dt + \mathcal{O}(u_k^2) dt.$$

Noticing that $\Theta(t) = \sum_{k=1}^n w_k u_k(t)$ with $w_k = \frac{kP_k}{\langle k \rangle}$, applying Itô's formula to $\mathcal{V}_1 = -\ln \Theta$ gives

$$\mathcal{L}\mathcal{V}_1 = -\frac{1}{\Theta} \sum_{k=1}^n w_k \left[\frac{mks_k\Theta}{1 + \alpha\Theta} - (d + \mu)u_k \right] + \frac{1}{2\Theta^2} \frac{d\langle \Theta, \Theta \rangle_t}{dt} + \mathcal{O}(\Theta). \quad (4.27)$$

In the regime $\Theta \rightarrow 0$, we have $u_k \rightarrow 0$, $I_k \rightarrow 0$, $s_k = 1 - u_k - r_k \rightarrow 1$, and $\frac{1}{1 + \alpha\Theta} \rightarrow 1$. Thus,

$$-\frac{1}{\Theta} \sum_{k=1}^n w_k \frac{mks_k\Theta}{1 + \alpha\Theta} \leq -\frac{m\langle k^2 \rangle}{\langle k \rangle} + \mathcal{O}(\Theta)$$

and $\frac{1}{\Theta} \sum_k w_k (d + \mu)u_k = d + \mu$. Since $\sum_{k=1}^n w_k^2 u_k^2 \leq (\sum_{k=1}^n w_k u_k)^2 = \Theta^2$,

$$\frac{1}{2\Theta^2} \frac{d\langle \Theta, \Theta \rangle_t}{dt} \leq \frac{1}{2\Theta^2} \sum_{k=1}^n w_k^2 \sigma_{2k}^2 u_k^2 \leq \frac{\bar{\sigma}_2^2}{2}, \quad \text{with } \bar{\sigma}_2^2 := \max_{1 \leq k \leq n} \sigma_{2k}^2.$$

Substituting these estimates into (4.27) yields

$$\begin{aligned} \mathcal{L}\mathcal{V}_1 &\leq -\frac{m\langle k^2 \rangle}{\langle k \rangle} + (d + \mu) + \frac{1}{2}\bar{\sigma}_2^2 + C_\Theta \Theta \\ &= -\left(d + \mu + \frac{1}{2}\bar{\sigma}_2^2\right)(\mathcal{R}_0^p - 1) + C_\Theta \Theta, \end{aligned} \quad (4.28)$$

where $C_\Theta > 0$ is a uniform positive constant. When $\mathcal{R}_0^p > 1$, we define

$$\lambda_0 := \left(d + \mu + \frac{1}{2}\bar{\sigma}_2^2\right)(\mathcal{R}_0^p - 1) > 0. \quad (4.29)$$

Applying Itô's formula to $-\ln S_k, -\ln I_k, -\ln R_k$:

$$\mathcal{L}(-\ln S_k) = -\frac{b}{S_k} + d + \frac{mk\Theta}{1 + \alpha\Theta} - \frac{\gamma R_k}{S_k} + \frac{1}{2}\sigma_{1k}^2 \leq -\frac{b}{S_k} + d + mk + \frac{1}{2}\sigma_{1k}^2, \quad (4.30)$$

$$\mathcal{L}(-\ln I_k) = -\frac{mkS_k\Theta}{(1 + \alpha\Theta)I_k} + (d + \mu) + \frac{1}{2}\sigma_{2k}^2, \quad (4.31)$$

$$\mathcal{L}(-\ln R_k) = -\frac{\mu I_k}{R_k} + (d + \gamma) + \frac{1}{2}\sigma_{3k}^2. \quad (4.32)$$

For each $k \in \{1, \dots, n\}$, summing the equations of system (2.3) gives

$$dN_k = (b - dN_k)dt + \sum_{i=1}^3 \sigma_{ik} X_{ik} dB_{ik}.$$

Hence,

$$\begin{aligned}\mathcal{L}\left(\frac{1}{\theta+1}N_k^{\theta+1}\right) &= N_k^\theta(b - dN_k) + \frac{1}{2}\theta N_k^{\theta-1}(\sigma_{1k}^2 S_k^2 + \sigma_{2k}^2 I_k^2 + \sigma_{3k}^2 R_k^2) \\ &\leq bN_k^\theta - \left(d - \frac{1}{2}\theta\bar{\sigma}^2\right)N_k^{\theta+1}.\end{aligned}\quad (4.33)$$

By condition (4.25), $d - \frac{1}{2}\theta\bar{\sigma}^2 > 0$. Because $\theta + 1 > \theta$, as $N_k \rightarrow \infty$, this term decays to $-\infty$ at the power rate $N_k^{\theta+1}$.

Let $\epsilon \in (0, 1)$ be a sufficiently small constant. Define the bounded open domain

$$D_\epsilon = \left\{X \in \mathbb{R}_+^{3n} : \epsilon < S_k < \frac{1}{\epsilon}, \epsilon^3 < I_k < \frac{1}{\epsilon}, \epsilon^4 < R_k < \frac{1}{\epsilon}, \forall k = 1, \dots, n, \text{ and } \Theta > \epsilon\right\}. \quad (4.34)$$

The complement $\mathbb{R}_+^{3n} \setminus D_\epsilon$ is completely partitioned into

$$\begin{aligned}\mathcal{D}_1 &= \{X \in \mathbb{R}_+^{3n} : \Theta \leq \epsilon\}, \\ \mathcal{D}_{2,k} &= \{X \in \mathbb{R}_+^{3n} : S_k \leq \epsilon\}, \quad (1 \leq k \leq n), \\ \mathcal{D}_{3,k} &= \{X \in \mathbb{R}_+^{3n} : I_k \leq \epsilon^3, \Theta > \epsilon\}, \quad (1 \leq k \leq n), \\ \mathcal{D}_{4,k} &= \{X \in \mathbb{R}_+^{3n} : R_k \leq \epsilon^4, I_k > \epsilon^3\}, \quad (1 \leq k \leq n), \\ \mathcal{D}_5 &= \left\{X \in \mathbb{R}_+^{3n} : \max_{1 \leq k \leq n} N_k \geq \frac{1}{\epsilon}\right\}.\end{aligned}$$

We verify that $\mathcal{L}^\mathcal{V}(X) \leq -1$ on each subregion.

- (1) On $\mathcal{D}_5$: There exists at least one k such that $N_k \geq \frac{1}{\epsilon}$. From (4.33), the term $-K(d - \frac{1}{2}\theta\bar{\sigma}^2)N_k^{\theta+1}$ dominates all bounded positive terms in $\mathcal{L}^\mathcal{V}$ and tends to $-\infty$ as $\epsilon \rightarrow 0$. By selecting a sufficiently large constant K , we obtain $\mathcal{L}^\mathcal{V}(X) \leq -1$ on $\mathcal{D}_5$.
- (2) On $\mathcal{D}_1 \setminus \mathcal{D}_5$: All components are bounded above by $\frac{1}{\epsilon}$, so all terms in $\sum_k \mathcal{L}(-\ln S_k - \ln I_k - \ln R_k) + K \sum_k \mathcal{L}(\frac{1}{\theta+1}N_k^{\theta+1})$ are bounded above by a uniform constant $C_1 > 0$. Applying (4.28),

$$\mathcal{L}^\mathcal{V}(X) \leq -\mathcal{W}\lambda_0 + \mathcal{W}C_\Theta\epsilon + C_1.$$

First choose $\mathcal{W} > \frac{C_1+2}{\lambda_0}$. Then choose $\epsilon \in (0, 1)$ small enough such that $\mathcal{W}C_\Theta\epsilon \leq 1$. We obtain

$$\mathcal{L}^\mathcal{V}(X) \leq -(C_1 + 2) + 1 + C_1 = -1 \quad \text{on } \mathcal{D}_1 \setminus \mathcal{D}_5.$$

- (3) On $\mathcal{D}_{2,k} \setminus \mathcal{D}_5$: By (4.30), $\mathcal{L}(-\ln S_k) \leq -\frac{b}{\epsilon} + C_S$. As $\epsilon \rightarrow 0$, $-\frac{b}{\epsilon} \rightarrow -\infty$. Because all other terms remain bounded above on $\mathbb{R}_+^{3n} \setminus \mathcal{D}_5$, choosing ϵ sufficiently small ensures $\mathcal{L}^\mathcal{V}(X) \leq -1$.
- (4) On $\mathcal{D}_{3,k} \setminus (\mathcal{D}_5 \cup \mathcal{D}_{2,k})$: In this domain, $S_k > \epsilon$, $\Theta > \epsilon$, and $I_k \leq \epsilon^3$. By (4.31),

$$-\frac{mkS_k\Theta}{(1+\alpha\Theta)I_k} \leq -\frac{mk\epsilon \cdot \epsilon}{(1+\alpha)\epsilon^3} = -\frac{mk}{(1+\alpha)\epsilon} \rightarrow -\infty, \quad \text{as } \epsilon \rightarrow 0.$$

This $-O(\frac{1}{\epsilon})$ singularity strictly dominates all bounded terms, ensuring $\mathcal{L}^\mathcal{V}(X) \leq -1$.

(5) On $\mathcal{D}_{4,k} \setminus (\mathcal{D}_5 \cup \mathcal{D}_{3,k})$: In this domain, $I_k > \epsilon^3$ and $R_k \leq \epsilon^4$. By (4.32),

$$-\frac{\mu I_k}{R_k} \leq -\frac{\mu \epsilon^3}{\epsilon^4} = -\frac{\mu}{\epsilon} \rightarrow -\infty, \quad \text{as } \epsilon \rightarrow 0.$$

This negative singular term dominates all bounded terms, which guarantees $\mathcal{L}\mathcal{V}(X) \leq -1$.

Combining cases (1)–(5), there exists $\epsilon > 0$ such that $\mathcal{L}\mathcal{V}(X) \leq -1$ for all $X \in \mathbb{R}_+^{3n} \setminus D_\epsilon$. Together with the uniform ellipticity of $\mathbf{A}(X)$ established on $\bar{D}_\epsilon$ in Part 1, Khasminskii's theorem [51] implies that the Markov diffusion process $X(t)$ is ergodic and admits a unique stationary distribution $\pi(\cdot)$ supported on $\mathbb{R}_+^{3n}$. The proof is complete. $\square$

Remark 4.6. Theorem 4.5 establishes the persistence threshold $\mathcal{R}_0^p$, derived from an upper-bound estimate of $\max_{1 \leq k \leq n} \sigma_{2k}^2$. When $\mathcal{R}_0^p > 1$, environmental fluctuations cannot eliminate disruption cascades, so the system admits a unique ergodic stationary distribution $\pi(\cdot)$ in which risk persists. The factor $\frac{\langle k^2 \rangle}{\langle k \rangle}$ in $\mathcal{R}_0^p$ confirms that degree heterogeneity promotes risk persistence by sustaining macroscopic contagion pressure $\Theta(t)$ across trading tiers. By contrast, the extinction threshold $\mathcal{R}_0^s$ follows from a lower-bound estimate of the quadratic variation involving $\min_{1 \leq k \leq n} \sigma_{2k}^2$. If $\mathcal{R}_0^s < 1$, disruption risk in the supply chain network becomes almost surely extinct. Note the intermediate region $\mathcal{R}_0^p < 1 < \mathcal{R}_0^s$, where current theory leaves the long-term dynamics of disruption risk undetermined.

5. Numerical simulations

In this section, we present some numerical simulations that support the main findings of this paper.

We consider a supply chain network with $n = 200$ degree classes, following a scale-free distribution $P_k = ck^{-\eta}$, where $\eta = 2.5$ represents a typical power-law infrastructure with dominant hub nodes [18]. The average connectivity $\langle k \rangle$ and the second moment $\langle k^2 \rangle$ are calculated accordingly. The stochastic system (2.3) is discretized with a time step $\Delta t = 0.01$ as follows:

$$\begin{cases} S_k(j+1) = S_k(j) + [b - dS_k(j) - \frac{mkS_k(j)\Theta(j)}{1 + \alpha\Theta(j)} + \gamma R_k(j)]\Delta t + \sigma_{1k}S_k(j)\sqrt{\Delta t}\zeta_{1j}, \\ I_k(j+1) = I_k(j) + [\frac{mkS_k(j)\Theta(j)}{1 + \alpha\Theta(j)} - (d + \mu)I_k(j)]\Delta t + \sigma_{2k}I_k(j)\sqrt{\Delta t}\zeta_{2j}, \\ R_k(j+1) = R_k(j) + [\mu I_k(j) - (d + \gamma)R_k(j)]\Delta t + \sigma_{3k}R_k(j)\sqrt{\Delta t}\zeta_{3j}, \end{cases} \quad (5.1)$$

where $\zeta_{ij} \sim N(0, 1)$ are independent standard normal random variables. All parameters follow the baseline values consolidated in Table 1; the ranges listed there are used in the subsequent sensitivity analysis.

To compare the sensitivity of deterministic thresholds $\mathcal{R}_0$ with stochastic thresholds $\mathcal{R}_0^s$ and $\mathcal{R}_0^p$, we conduct partial rank correlation coefficient (PRCC) analysis based on Latin hypercube sampling (sample size 2000).

Figure 1 compares the parameter sensitivities across the three thresholds. The transmission rate m exhibits a strong positive correlation (PRCC > 0.5), whereas the power-law exponent η , recovery rate μ , and exit rate d are significantly negatively correlated (PRCC < -0.5). This demonstrates that higher degree heterogeneity (smaller η , indicating prominent hub suppliers) and lower recovery

capacity critically exacerbate systemic risk. Notably, environmental noise σ_I displays a pronounced hierarchy: it has no effect on $\mathcal{R}_0$ (PRCC = 0), a mild damping effect on $\mathcal{R}_0^s$ due to network-tier diversification ($1/n$), but a strong negative impact on $\mathcal{R}_0^p$ (PRCC < -0.5), confirming that market volatility substantially suppresses risk persistence. In contrast, parameters b , γ , and α have negligible PRCC values ($|\text{PRCC}| \approx 0$), verifying that entry flux, resilience restructuring, and saturation defense do not govern the initial invasion/persistence boundaries.

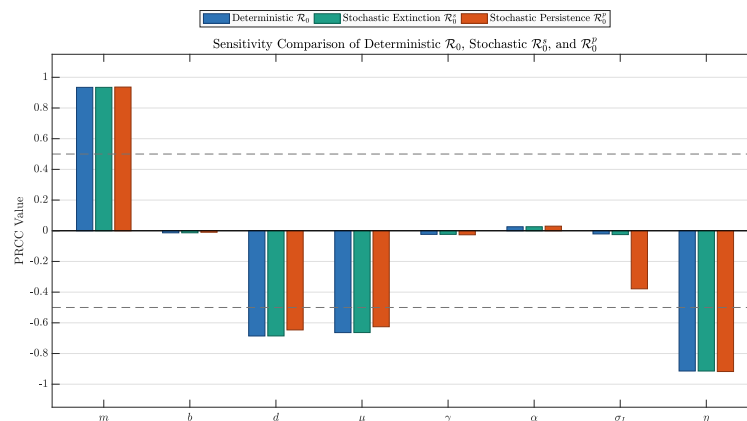


Figure 1. PRCCs depicting the sensitivity of the deterministic threshold $\mathcal{R}_0$, the stochastic extinction threshold $\mathcal{R}_0^s$, and the stochastic persistence threshold $\mathcal{R}_0^p$ with respect to system parameters ($N = 2000$). Blue, green, and orange bars represent the PRCC values for $\mathcal{R}_0$, $\mathcal{R}_0^s$, and $\mathcal{R}_0^p$, respectively. Dashed horizontal lines at ± 0.5 indicate the benchmark for strong statistical correlation.

To explore the asymptotic behavior of the supply chain network in the persistence regime ($\mathcal{R}_0^p > 1$), stochastic simulations are performed across a spectrum of noise intensities: $\sigma_I \in \{0.05, 0.15, 0.40\}$, corresponding to threshold values $\mathcal{R}_0^p$ of 2.05, 2.00, and 1.71, respectively. The values of the other parameters are given in Table 1. Figure 2a illustrates the sample trajectories of the aggregate distress density $\bar{I}(t)$ relative to the deterministic steady state I^* (indicated by the dashed black line). The results demonstrate that while the deterministic framework predicts a monotonic convergence to the endemic equilibrium $\mathbf{E}^*$ (where $\mathcal{R}_0 = 2.0516 > 1$), the stochastic counterpart exhibits enduring oscillations around this deterministic attractor, signifying a state of ergodic persistence. Specifically, as the value of σ_I increases, the amplitude of the fluctuations expands, and the frequency of instantaneous deviations becomes more pronounced. This phenomenon suggests that environmental volatility acts as a continuous perturbative force that prevents the system from settling at a fixed point, yet the trajectories remain anchored to the neighborhood of I^* , signifying the ergodic nature of the contagion process.

Figure 2b presents the corresponding stationary probability density functions (PDFs) estimated from the long-term simulation data. For a low noise level ($\sigma_I = 0.05$), the distribution is highly concentrated with a sharp peak near the deterministic equilibrium, indicating that the risk level remains relatively predictable. Conversely, as σ_I rises to 0.40, the probability measure becomes increasingly dispersed, exhibiting a lower peak and heavier tails. This transition from a concentrated to a broad distribution

numerically validates the existence of a unique stationary distribution $\pi(\cdot)$ as stipulated in Theorem 4.5.



Figure 2. Ergodic persistence and stationary distribution of the stochastic supply chain system (2.3). (a) Sample trajectories of the aggregate distress density $\bar{I}(t)$ under varying environmental noise intensities $\sigma_I \in \{0.05, 0.15, 0.40\}$. The dashed black line represents the deterministic risk level I^* . (b) Probability density functions of the stationary distribution for different noise levels. The other parameters' values are listed in Table 1.

From a risk management perspective, these results illustrate that while the average level of distress might be governed by the structural parameters of the network, the instantaneous exposure is highly sensitive to external uncertainty. The broadening of the PDF under high volatility implies an increased likelihood of extreme risk events that deviate significantly from the mean. Therefore, these findings suggest that in volatile market environments, effective supply chain resilience strategies should focus not only on the expected contagion level but also on managing the variance and extreme fluctuations inherent in the stable probabilistic steady state.

To further investigate the interplay between network topology and stochastic dynamics, we compare the steady-state distributions of susceptible (S_k), distressed (I_k), and recovered (R_k) enterprises across degree classes. Based on the parameters in Table 1, which ensure risk persistence, a uniform noise intensity of $\sigma_S = \sigma_I = \sigma_R = 0.15$ is applied to evaluate the system's resilience to environmental perturbations.

Figure 3 illustrates the densities of S_k , I_k , and R_k enterprises as a function of the node degree k . It is observed that the deterministic equilibrium $\mathbf{E}^*$ (solid lines) characterizes the steady state of the supply chain's risk profile. Notably, the time-averaged densities derived from the stochastic simulations (dashed lines) exhibit a high degree of alignment with the deterministic steady states across all degree

classes. This numerical result demonstrates that the ergodic mean of the stochastic contagion process converges to the underlying deterministic steady state despite continuous environmental noise.

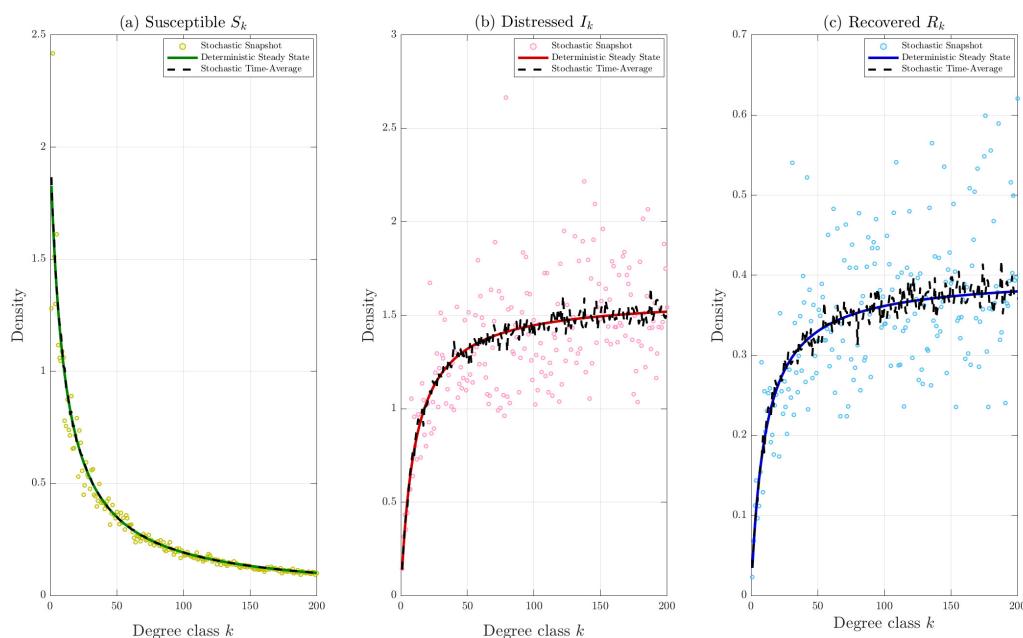


Figure 3. Steady-state distribution of enterprise densities across degree classes $k \in [1, 200]$. Panels (a)–(c) show susceptible (S_k), distressed (I_k), and recovered (R_k) compartments, respectively. Solid colored lines are the analytical equilibrium $\mathbf{E}^*$ of the deterministic system (3.1); black dashed lines are time-averaged densities $\bar{X}_k$ from stochastic simulations. Light scatter points denote an instantaneous snapshot of the SDE system (2.3) at $t = 600$.

Furthermore, the scatter points reveal that the magnitude of stochastic deviations is not uniformly distributed across the network. High-degree classes (hub enterprises) exhibit a wider dispersion of risk states relative to the mean, whereas low-degree nodes remain more tightly clustered around the equilibrium. This suggests that while connectivity k dictates the average exposure to distress, it also amplifies the sensitivity of core entities to external market fluctuations. These findings underscore the importance of incorporating both network heterogeneity and environmental uncertainty into supply chain resilience assessments.

6. Conclusions

This study addresses risk contagion in heterogeneous supply chain networks by constructing and analyzing a stochastic dynamical system that incorporates a nonlinear saturated incidence rate and state-dependent environmental perturbations. The well-posedness of the system, global existence, uniqueness, positivity, and boundedness of solutions, has been established. The threshold quantities govern the long-term risk dynamics: the deterministic basic reproduction number $\mathcal{R}_0$, obtained by the next-generation matrix method, and the stochastic thresholds $\mathcal{R}_0^s$ and $\mathcal{R}_0^p$, derived from the Itô correction term that arises in the analysis of the multiplicative noise structure of the system (2.3).

From a dynamical perspective, the analysis reveals a positive correlation between $\mathcal{R}_0$ and the second moment of the network degree distribution, $\langle k^2 \rangle$. This dependence enters through the HMF kernel

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_k \frac{k P_k I_k}{N_k}$$

and quantifies the dual role of hub nodes in risk propagation: they serve as the core of system liquidity while simultaneously acting as super-spreaders of systemic risk. Higher heterogeneity

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

therefore lowers the outbreak threshold and increases the system's sensitivity to initial perturbations.

These stochastic thresholds satisfy $\mathcal{R}_0^p \leq \mathcal{R}_0^s \leq \mathcal{R}_0$, with equality holding only in the deterministic limit $\sigma_{ik} \rightarrow 0$. This relationship indicates that the state-dependent noise structure introduces stochastic points of interruption and persistence into the propagation chain, thereby statistically suppressing or maintaining the long-term contagion process. According to Theorem 4.3, if the condition $\mathcal{R}_0^s < 1$ holds, systemic risk is eliminated from the supply chain network almost surely; conversely, when $\mathcal{R}_0^p > 1$, the system possesses a unique ergodic stationary distribution $\pi(\cdot)$ in which the risk persists.

The mathematical results provide the following decision-making insights for supply chain risk management.

- (1) The $\mathcal{R}_0$ dependence on κ implies that the marginal benefit of resilience investment is concentrated at high-degree nodes. A direct corollary of $\mathcal{R}_0 \propto \langle k^2 \rangle$ is that injecting liquidity or hedging capacity into the top-ranked firms by degree, rather than distributing it uniformly across the network, yields a disproportionately reduction in $\mathcal{R}_0$ and therefore in the long-run prevalence of systemic distress.
- (2) The condition $\mathcal{R}_0^s \leq \mathcal{R}_0$ indicates that when the determinism $\mathcal{R}_0$ is only slightly higher than 1, the Itô correction term will reduce $\mathcal{R}_0^s$ to below 1, thus leading to extinction. Risk managers operating in such regimes can therefore identify a quantitative window, defined by the gap $\mathcal{R}_0 - \mathcal{R}_0^s$, in which stochastic volatility exerts a clearing effect. This insight is a threshold-level conclusion, applicable to deterministic systems in a marginally critical state.
- (3) The persistence threshold $\mathcal{R}_0^p$ makes the role of network heterogeneity in supply chain risk. Because $\langle k^2 \rangle$ enters the numerator, hub-targeted interventions and recovery-rate investment are the natural levers for managers, while disruption buffers should be calibrated to the effective branching factor $\frac{\langle k^2 \rangle}{\langle k \rangle}$ rather than to the average degree. The presence of $\max_{1 \leq k \leq n} \sigma_{2k}^2$ in the denominator indicates that environmental volatility has only a small stabilizing effect, which should not be relied upon as a primary intervention.

The above conclusions are based on two modeling choices that limit the framework to a well-defined scope of application. (i) The degree-based formulation ignores community structure, degree of correlation, and clustering effects among adjacent firms in the supply chain (especially in supply chains with significant hierarchical structures or strong homogeneous mixing characteristics among core firms); these characteristics may accelerate or inhibit risk contagion on a magnitude scale, and their impact exceeds the scope that the kernel function $\Theta(t)$ can capture. For the threshold quantities such as $\mathcal{R}_0^s$ and $\mathcal{R}_0^p$ that depend primarily on the degree distribution, however, the reduction

remains quantitatively accurate. (ii) The composite dependency index k_i and the transition rates $(\mu, \gamma, \sigma_{ik})$ are not directly observable and must be inferred from firm-level financial disclosures and supply chain surveys. Calibrating the model against industry-level disruption data and incorporating adaptive network evolution through time-dependent topology extensions are natural next steps.

Three data sources will be used for the calibration exercise: trade credit and inter-firm transaction registries such as Compustat Segment and Organisation for Economic Co-operation and Development (OECD) Trade in Value Added (TiVA), which provide firm-level network edges and exposure values to estimate the composite dependency index k_i ; operational-disruption event logs from Resilinc or business-continuity databases, which provide time-stamped firm-level disruption onset and recovery times to estimate the transition rates μ and γ ; and high-frequency market-volatility proxies such as the Chicago board options exchange volatility index, credit default swap spreads, and commodity futures, which provide time series for the state-dependent noise intensity σ_{ik} . Combining these three sources within a Bayesian calibration framework would yield posterior estimates of the model parameters and enable a direct empirical test of the predicted stochastic threshold $\mathcal{R}_0^s$ against observed system behavior.

Author contributions

Hui Li: writing—original draft, conceptualization; Shufen Wei: visualization; Xinyu Ye: visualization; Xinze Lian: writing—review and editing, visualization, supervision; Feng Rao: writing—original draft, review and editing, validation, supervision, methodology. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix

A.1. Proof of Theorem 3.4

Proof. To evaluate the local stability of $\mathbf{E}^0$, we examine the Jacobian matrix of system (3.1) evaluated at the risk-free state. At $\mathbf{E}^0 = (\frac{b}{d}, 0, 0, \dots, \frac{b}{d}, 0, 0)$, the linearized dynamics can be decomposed into n blocks of 3×3 matrices. However, since the variables S_k are decoupled from the infectious dynamics when $I_k, R_k \rightarrow 0$, we can focus on the linearized subsystem of the infectious and recovered compartments (I_k, R_k) .

The Jacobian matrix $J(\mathbf{E}^0)$ takes the following block structure

$$J(\mathbf{E}^0) = \begin{pmatrix} A & \mathbf{0} \\ B & C \end{pmatrix},$$

where the eigenvalues of the blocks associated with S_k are all $-d < 0$. The remaining eigenvalues are determined by the $2n \times 2n$ sub-matrix corresponding to the (I_k, R_k) components

$$\begin{aligned} \frac{dI_k}{dt} &= \frac{mkb}{d} \left(\frac{d}{b\langle k \rangle} \sum_{j=1}^n jP_j I_j \right) - (d + \mu)I_k, \\ \frac{dR_k}{dt} &= \mu I_k - (d + \gamma)R_k. \end{aligned}$$

This linear subsystem can be written as $\dot{\mathbf{X}} = M\mathbf{X}$, where $\mathbf{X} = (I_1, \dots, I_n, R_1, \dots, R_n)^T$. The stability is governed by the matrix M :

$$M = \begin{pmatrix} F - V & 0 \\ \mu E & -(d + \gamma)E \end{pmatrix},$$

where F and V are the matrices derived in the calculation of $\mathcal{R}_0$, and E is the identity matrix. The eigenvalues of M are the union of the eigenvalues of $-(d + \gamma)E$ and the eigenvalues of $F - V$.

According to the properties of the next-generation matrix, the eigenvalues of $F - V$ all have negative real parts if and only if $\rho(FV^{-1}) < 1$. Since $\mathcal{R}_0 = \rho(FV^{-1})$, we conclude:

- (1) If $\mathcal{R}_0 < 1$, all eigenvalues of $J(\mathbf{E}^0)$ have negative real parts, rendering $\mathbf{E}^0$ locally asymptotically stable.
- (2) If $\mathcal{R}_0 > 1$, there exists at least one eigenvalue with a positive real part, making $\mathbf{E}^0$ unstable.

The proof is complete. □

A.2. Proof of Theorem 3.5

Proof. Consider the following Lyapunov function candidate $L(t)$, which represents the weighted aggregate of the distressed nodes across the entire network

$$L(t) = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k I_k(t).$$

Taking the time derivative of $L(t)$ along the trajectories of system (3.1), we obtain

$$\frac{dL}{dt} = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \left[\frac{mk S_k \Theta}{1 + \alpha \Theta} - (d + \mu) I_k \right].$$

Since $S_k \leq \frac{b}{d}$ and $\frac{1}{1 + \alpha \Theta} \leq 1$ for all $\Theta \geq 0$, it follows that

$$\begin{aligned} \frac{dL}{dt} &\leq \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \left[mk \frac{b}{d} \Theta - (d + \mu) I_k \right] \\ &= \left(\frac{mb \langle k^2 \rangle}{d \langle k \rangle} \right) \Theta - (d + \mu) \Theta \frac{b}{d} \\ &= \frac{b(d + \mu)}{d} \Theta \left(\frac{m \langle k^2 \rangle}{(d + \mu) \langle k \rangle} - 1 \right) \\ &= \frac{b(d + \mu)}{d} \Theta (\mathcal{R}_0 - 1). \end{aligned}$$

If $\mathcal{R}_0 < 1$, then $\frac{dL}{dt} \leq 0$. Furthermore, $\frac{dL}{dt} = 0$ if and only if $\Theta = 0$, which corresponds to the risk-free equilibrium $\mathbf{E}^0$. According to the LaSalle's invariance principle, every trajectory in Γ converges to the largest invariant set where $\frac{dL}{dt} = 0$. Consequently, $\mathbf{E}^0$ is globally asymptotically stable when $\mathcal{R}_0 \leq 1$. $\square$

A.3. Proof of Theorem 3.9

Proof. To analyze the local stability of $\mathbf{E}^*$, we examine the linearized dynamics of the system under small perturbations. Let $\Delta S_k(t) = S_k(t) - S_k^*$, $\Delta I_k(t) = I_k(t) - I_k^*$, and $\Delta R_k(t) = R_k(t) - R_k^*$. Utilizing the invariant manifold property $S_k + I_k + R_k = N = \frac{b}{d}$, we have $\Delta S_k = -(\Delta I_k + \Delta R_k)$, which allows us to reduce the system's dimensionality to $2n$.

For convenience, let

$$f(\Theta) = \frac{\Theta}{1 + \alpha \Theta}$$

denote the non-linear incidence function, and

$$f'(\Theta) = \frac{1}{(1 + \alpha \Theta)^2}$$

its derivative. The linearized equations for the distressed and recovered compartments of degree k are given by

$$\begin{cases} \frac{d(\Delta I_k)}{dt} = mk [-f(\Theta^*)(\Delta I_k + \Delta R_k) + S_k^* f'(\Theta^*) \Delta \Theta] - (d + \mu) \Delta I_k, \\ \frac{d(\Delta R_k)}{dt} = \mu \Delta I_k - (d + \gamma) \Delta R_k, \end{cases} \quad (\text{A.1})$$

where

$$\Delta\Theta = \frac{1}{N\langle k \rangle} \sum_{j=1}^n jP_j \Delta I_j$$

represents the perturbation in the aggregate contagion pressure.

Assume the perturbations take the exponential form $(\Delta I_k, \Delta R_k) = (\xi_{1,k}, \xi_{2,k})e^{\lambda t}$, where λ is an eigenvalue of the Jacobian matrix. Substituting this into the second equation of (A.1) yields

$$\xi_{2,k} = \frac{\mu}{\lambda + d + \gamma} \xi_{1,k}. \quad (\text{A.2})$$

Substituting (A.2) into the first equation of (A.1) and rearranging the terms, we obtain

$$\xi_{1,k} \left[\lambda + (d + \mu) + mkf(\Theta^*) \left(1 + \frac{\mu}{\lambda + d + \gamma} \right) \right] = mkS_k^* f'(\Theta^*) \Delta\Theta.$$

Define the characteristic coefficient for degree class k as

$$A_k(\lambda) = \lambda + (d + \mu) + mkf(\Theta^*) \frac{\lambda + d + \gamma + \mu}{\lambda + d + \gamma}.$$

Then the perturbation amplitude for the distressed nodes is expressed as

$$\xi_{1,k} = \frac{mkS_k^* f'(\Theta^*) \Delta\Theta}{A_k(\lambda)}. \quad (\text{A.3})$$

By multiplying both sides of (A.3) by $\frac{kP_k}{N\langle k \rangle}$ and summing over all $k \in \{1, \dots, n\}$, we derive the characteristic equation for the system

$$Q(\lambda) \triangleq \frac{f'(\Theta^*)}{N\langle k \rangle} \sum_{k=1}^n \frac{mk^2 P_k S_k^*}{A_k(\lambda)} = 1. \quad (\text{A.4})$$

The stability of $\mathbf{E}^*$ requires all roots λ of $Q(\lambda) = 1$ to have negative real parts. Suppose there exists a root λ such that $\text{Re}(\lambda) \geq 0$. Considering the modulus of $A_k(\lambda)$, for $\text{Re}(\lambda) \geq 0$, we observe that $|\lambda + d + \mu| \geq d + \mu$ and

$$\left| \frac{\lambda + d + \gamma + \mu}{\lambda + d + \gamma} \right| = \left| 1 + \frac{\mu}{\lambda + d + \gamma} \right| \geq 1 + \frac{\mu}{0 + d + \gamma} = \frac{d + \gamma + \mu}{d + \gamma}.$$

Thus, it follows that $|A_k(\lambda)| \geq A_k(0)$ for any $\text{Re}(\lambda) \geq 0$. Consequently, we have $|Q(\lambda)| \leq Q(0)$. From the equilibrium conditions (3.9), the term $A_k(0)$ can be simplified as

$$A_k(0) = (d + \mu) + mkf(\Theta^*) \frac{d + \gamma + \mu}{d + \gamma} = \frac{mkS_k^* f(\Theta^*)}{I_k^*},$$

where the last equality is obtained by the second equation of (3.3), which obtained I_k^* . Substituting $A_k(0)$ back into $Q(0)$, we find

$$Q(0) = \frac{f'(\Theta^*)}{N\langle k \rangle} \sum_{k=1}^n \frac{mk^2 P_k S_k^*}{mkS_k^* f(\Theta^*)/I_k^*} = \frac{f'(\Theta^*)}{f(\Theta^*)} \left(\frac{1}{N\langle k \rangle} \sum_{k=1}^n kP_k I_k^* \right).$$

By the definition of the steady-state contagion pressure, $\Theta^* = \frac{1}{N\langle k \rangle} \sum k P_k I_k^*$, the expression reduces to

$$Q(0) = \frac{f'(\Theta^*)\Theta^*}{f(\Theta^*)}.$$

Substituting the specific forms of $f(\Theta^*) = \frac{\Theta^*}{1+\alpha\Theta^*}$ and $f'(\Theta^*) = \frac{1}{(1+\alpha\Theta^*)^2}$, we obtain

$$Q(0) = \frac{\Theta^*}{(1+\alpha\Theta^*)^2} \frac{1+\alpha\Theta^*}{\Theta^*} = \frac{1}{1+\alpha\Theta^*} < 1.$$

Since $|Q(\lambda)| \leq Q(0) < 1$ for all $\text{Re}(\lambda) \geq 0$, the characteristic equation $Q(\lambda) = 1$ has no roots with non-negative real parts. Therefore, all eigenvalues λ must have negative real parts, proving that $\mathbf{E}^*$ is locally asymptotically stable. $\square$



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