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*Research article***Group rings that are weakly quadratent****Ömer KÜSMÜŞ\* and Murat LUZUM**

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**Abstract:** A ring  $R$  is said to be weakly quadratent if every element  $x \in R$  satisfies  $x^4 = x$  or  $x^4 = -x$  [8]. In this paper, we determine, for an associative and unital ring  $R$  and an arbitrary group  $G$ , exactly when the group ring  $RG$  is weakly quadratent. Specifically, we prove that  $RG$  is weakly quadratent if and only if either  $G$  is trivial and  $R$  is weakly quadratent or  $G$  is a non-trivial elementary abelian 3-group and  $R$  is a quadratent ring, that is, a subring of a direct product of copies of finite fields  $\mathbb{F}_2$  and  $\mathbb{F}_4$ ; in the latter case,  $RG$  is in fact quadratent. Besides, we establish connections between our results and the von Neumann regular group rings and, by means of explicit examples, demonstrate how far semiprimeness is from guaranteeing the desired properties. Our work extends the existing line of research on weakly Boolean, weakly tripotent, and weakly quadratent rings, as well as their analogues in group rings. Finally, we present several open research problems that may lead to further developments in this area.

**Keywords:** group rings; quadratent rings; weakly quadratent rings; Jacobson radical; semiprimitive rings

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**1. Introduction and preliminaries**

Rings whose elements satisfy a fixed polynomial identity of the form  $x^n = x$  form one of the oldest areas of study in ring theory. According to the theorem of Jacobson [1], every such ring is commutative and a complete structure theory is available for small exponents. The case  $n = 2$  is that of the Boolean rings; for  $n = 3$ , one obtains the tripotent rings, which are precisely the subdirect products of copies of the finite fields  $\mathbb{F}_2$  and  $\mathbb{Z}_3$ , while for  $n = 4$ , one arrives at the quadratent rings, built analogously from  $\mathbb{F}_2$  and  $\mathbb{F}_4$  (see [2] for the general identity  $x^n = x$ ). In recent years, considerable attention has been devoted to the weak versions of these conditions, in which every element is merely required to satisfy one of the two equations  $x^n = x$  or  $x^n = -x$ ; such classes arise naturally in the study of clean-type additive decompositions of rings (see [3, 4] and the references therein). In this sense, a central question in the

theory of group rings is to determine when a property of the coefficient ring is inherited by the group ring  $RG$  and conversely when such a property of  $RG$  reflects back to the coefficient ring  $R$ . Classical examples include Maschke's semisimplicity theorem and the von Neumann regularity criterion due to Auslander, Villamayor, and Connell [5,6]. The present paper is situated at the intersection of these two lines of research: we determine completely when a group ring satisfies the weak identity  $x^4 = \pm x$ , and we analyze the lifting property and radical-theoretical aspects.

Throughout the paper all rings are assumed to be associative with an identity element 1 which differs from 0, and all subrings are unital. For a ring  $R$  and a (multiplicatively written) group  $G$ , the group ring  $RG$  consists of all formal sums  $\alpha = \sum_{g \in G} a_g g$  with  $a_g \in R$  and finite support  $\text{supp}(\alpha) = \{g \in G : a_g \neq 0\}$ , endowed with the obvious addition and with the multiplication extending that of  $G$  by  $R$ -bilinearity. We freely use the following two standard facts: the elements of  $G$  form a free  $R$ -basis of  $RG$ , so that two elements of  $RG$  are equal exactly when all their coefficients coincide; and  $R$  is embeddable into  $RG$  as a unital subring via  $r \mapsto r \cdot 1_G$ , while the augmentation map  $\varepsilon: RG \rightarrow R$ ,  $\varepsilon(\sum a_g g) = \sum a_g$ , is a surjective ring homomorphism [7]. More generally, every normal subgroup  $N \trianglelefteq G$  gives rise to a surjective ring homomorphism  $RG \rightarrow R(G/N)$  obtained via the natural projection  $G \rightarrow G/N$ ; its kernel is the relative augmentation ideal  $\Delta(G, N)$ , so that  $RG/\Delta(G, N) \cong R(G/N)$ . In particular,  $\Delta(G) := \Delta(G, G) = \ker \varepsilon$  is the augmentation ideal of  $RG$ . Usually,  $\mathbb{F}_n$  denotes the field with  $n$  elements and  $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$  the ring of integers modulo  $n$ ; in particular,  $\mathbb{Z}_p \cong \mathbb{F}_p$  for every prime  $p$ , and we may use the two notations interchangeably. Throughout the study,  $C_n$  denotes a cyclic group of order  $n$ .

We first recall the main definition from [8], which plays a central role throughout this paper as well.

**Definition 1.1.** *A ring  $R$  is said to be quadratent if  $x^4 = x$  for every  $x \in R$  and weakly quadratent provided that for every  $x \in R$ , either  $x^4 = x$  or  $x^4 = -x$ .*

Quadratent rings are classified in [2, 8] as in the following proposition [8, Proposition 1.1].

**Proposition 1.2.** *A ring  $R$  holds  $x^4 = x$  for all  $x \in R$  if and only if  $R$  is embeddable as a subring of a direct product of copies of the fields  $\mathbb{F}_2$  and  $\mathbb{F}_4$ .*

Danchev also established the following characterization of weakly quadratent direct products [8].

**Lemma 1.3.** *For arbitrary rings  $A$  and  $B$ , the direct product  $A \times B$  is weakly quadratent if and only if both  $A$  and  $B$  are weakly quadratent and at least one of them is quadratent.*

The classification in Proposition 1.2. is indeed a part of the studies which are devoted to rings satisfying  $x^n = x$  or  $x^n = -x$  for small  $n$ : the case  $n = 2$  (weakly Boolean rings) was settled in [3], the case  $n = 3$  (weakly tripotent rings) in [9] (see also [4]), and the case  $n = 4$  in [8]. More recently, the transfer of such investigation from a ring  $R$  to the group ring  $RG$  has been carried out for several related ring classes, such as commutative weakly tripotent group rings and commutative nil-neat group rings [10, 11].

The paper is organized as follows. Section 2 contains Theorem 2.1 and its proof. In Section 3 we complement the classification by a study of the corresponding lifting problem: given an ideal  $I$  of  $RG$  such that the factor ring  $RG/I$  is weakly quadratent, when is  $RG$  itself weakly quadratent? We prove that weakly quadratent rings are semiprimitive (Lemma 3.1) and deduce that the weakly quadratent property never lifts modulo a non-zero nil ideal (Corollary 3.2) — in sharp contrast with the classical lifting of idempotents modulo nil ideals. On the positive side, we establish complete lifting criteria

for ideals generated by central idempotents (Corollary 3.4) and for the relative augmentation ideals  $\Delta(G, N)$  (Theorem 3.5). In Section 4 we take a closer look at semiprimitivity: every weakly quadratent group ring is semiprimitive, and by combining a classical regularity criterion of Connell with a series of examples we measure how far semiprimitivity is from being sufficient; we also compute the Jacobson radical of the group rings responsible for the failure of lifting and combine it with the semiprimitivity of weakly quadratent rings into a four-fold equivalence (Corollary 4.6). Section 5 collects some open problems. In the next section, we introduce our main results related to characterize weakly quadratent group rings.

## 2. Main results

**Theorem 2.1.** *Let  $R$  be a ring and let  $G$  be a group.  $RG$  is weakly quadratent if and only if exactly one of the following conditions holds:*

- (i)  $G = \{1\}$  and  $R$  is weakly quadratent (so that  $R$  is as described in Theorem 2.3 in [8]);
- (ii)  $G \neq \{1\}$  is an elementary abelian 3-group and  $R$  is a quadratent ring. In this case  $RG$  is itself quadratent.

It should be noted that in contrast with [10, 11], no commutativity is assumed either on  $R$  or on  $G$ ; the commutativity of  $RG$  (and hence the abelianness of  $G$ ) is derived from the defining identity via a classical theorem of Jacobson [1] (see also [12, Chapter 3]). Furthermore, these properties are generally not preserved when passing to group rings over a nontrivial group  $G$ . The only nontrivial cases arise in characteristic 2 for  $R$ .

We begin with two general observations which do the bulk of the work and which are independent of the characteristic.

**Lemma 2.2.** *Every weakly quadratent ring is reduced, that is, it contains no non-zero nilpotent elements.*

*Proof.* Assume  $z \neq 0$  is nilpotent and choose  $n \geq 2$  minimal with  $z^n = 0$ . Put  $w = z^{\lceil n/2 \rceil}$ . Since  $\lceil n/2 \rceil < n$ , the minimality of  $n$  gives  $w \neq 0$ , whereas  $2\lceil n/2 \rceil \geq n$  yields  $w^2 = 0$  and hence  $w^4 = 0$ . The weakly quadratent condition applied to  $w$  now forces  $0 = w^4 = \pm w$ , so  $w = 0$ , a contradiction as established in [8].  $\square$

**Lemma 2.3.** *If the group ring  $RG$  is weakly quadratent, then every element of  $G$  has order 1 or 3.*

*Proof.* Let  $g \in G$ . Since  $g$  is a unit of  $RG$ , the relation  $g^4 = g$  (resp.  $g^4 = -g$ ) gives  $g^3 = 1$  (resp.  $g^3 = -1$ ); in either case  $g^6 = 1$ , so  $\text{ord}(g) \in \{1, 2, 3, 6\}$ .

Suppose first that some  $h \in G$  has order 2. Then,  $h^4 = 1$ , and the weakly quadratent condition applied to  $h$  yields  $h = 1$  or  $h = -1$ , i.e.  $1_G = \pm h$  inside  $RG$ . Comparing the coefficients of the basis element  $1_G$  on both sides (recall that  $G$  is a free  $R$ -basis of  $RG$  and that  $h \neq 1_G$ ) gives  $1 = 0$  in  $R$ , a contradiction. Hence,  $G$  contains no element of order 2; consequently, it contains no element of order 6 either, since the cube of such an element would have order 2. Therefore,  $\text{ord}(g) \in \{1, 3\}$  for every  $g \in G$ .  $\square$

We can now eliminate the characteristics 3 and 7. The underlying reasons are different. In characteristic 3, the obstruction is due to nilpotency, whereas in characteristic 7, it is due to the linear independence of the group elements.

**Lemma 2.4.** *If  $RG$  is weakly quadratent and  $\text{char}(R) = 3$ , then  $G = \{1\}$ ; consequently,  $RG = R \cong \mathbb{Z}_3$ .*

*Proof.* Assume  $g \in G$  with  $g \neq 1$ . By Lemma 2.3, we have  $g^3 = 1$ . Since 1 and  $g$  commute and  $\text{char}(RG) = \text{char}(R) = 3$ , the binomial formula gives

$$(g - 1)^3 = g^3 - 3g^2 + 3g - 1 = g^3 - 1 = 0,$$

so  $g - 1$  is a nilpotent element of  $RG$ . But  $RG$  is reduced by Lemma 2.2; whence,  $g = 1$ , a contradiction. Thus,  $G = \{1\}$  and  $RG = R$  is a weakly quadratent ring of characteristic 3; Theorem 2.3. in [8] yields  $R \cong \mathbb{Z}_3$ .  $\square$

**Lemma 2.5.** *If  $RG$  is weakly quadratent and  $\text{char}(R) = 7$ , then  $G = \{1\}$ ; consequently,  $RG = R \cong \mathbb{Z}_7$ .*

*Proof.* Assume  $g \in G$ ,  $g \neq 1$ . By Lemma 2.3,  $\text{ord}(g) = 3$ , so  $1, g, g^2$  are pairwise distinct elements of  $G$  and hence  $R$ -linearly independent in  $RG$ . Using  $g^3 = 1$ ,  $g^4 = g$  and  $\text{char}(R) = 7$  we compute

$$(1 + g)^4 = 1 + 4g + 6g^2 + 4g^3 + g^4 = (1 + 4) + (4 + 1)g + 6g^2 = 5 + 5g + 6g^2.$$

On the other hand, the weakly quadratent condition requires  $(1 + g)^4 = 1 + g$  or  $(1 + g)^4 = -(1 + g) = 6 + 6g$ . Comparing the coefficients of  $g^2$  (linear independence!) gives  $6 = 0$  in  $R$  in both cases, which is false in characteristic 7. This contradiction shows  $G = \{1\}$ , so  $RG = R$  is a weakly quadratent ring of characteristic 7 and Theorem 2.3. in [8] gives  $R \cong \mathbb{Z}_7$ .  $\square$

**Corollary 2.6.** *If  $RG$  is weakly quadratent and  $G \neq \{1\}$ , then  $\text{char}(R) = 2$ .*

*Proof.* Since  $R$  is a unital subring of  $RG$ , the ring  $R$  is weakly quadratent as well, and  $\text{char}(R) = \text{char}(RG) \in \{2, 3, 6, 7, 14\}$  by Theorem 2.3. in [8] (applied to  $RG$ ). The values 3 and 7 are excluded by Lemmas 2.4 and 2.5. Suppose  $\text{char}(R) = 6$ . By [8, Lemma 2.1] we may write  $R \cong R_1 \times R_2$  with  $\text{char}(R_1) = 2$  and  $\text{char}(R_2) = 3$ , whence  $RG \cong R_1G \times R_2G$ . The canonical projection  $RG \twoheadrightarrow R_2G$  is a surjective ring homomorphism, and the weakly quadratent condition, being an element-wise identity, passes to homomorphic images; hence,  $R_2G$  is a weakly quadratent group ring with characteristic 3 and Lemma 2.4 forces  $G = \{1\}$ , a contradiction. The case  $\text{char}(R) = 14$  is excluded in the same manner, using Lemma 2.5 instead. Therefore,  $\text{char}(R) = 2$ .  $\square$

The next lemma settles the structure of the pair  $(R, G)$  in the other characteristic. Recall Jacobson's classical theorem [1] (see also [12, Theorem 3.2.2]): if  $n \geq 2$  is a fixed integer and a ring  $S$  satisfies the identity  $x^n = x$ , then  $S$  is commutative.

**Lemma 2.7.** *Assume that  $RG$  is weakly quadratent,  $\text{char}(R) = 2$  and  $G \neq \{1\}$ . Then,  $R$  is quadratent (hence, by Proposition 1.2, a product of copies of  $\mathbb{F}_2$  and  $\mathbb{F}_4$ ) and  $G$  is an elementary abelian 3-group. Moreover,  $RG$  satisfies the identity  $x^4 = x$ .*

*Proof.* In characteristic 2 we have  $-x = x$  for all  $x \in RG$ , so the weakly quadratent condition amounts to the identity  $x^4 = x$ ; that is,  $RG$  is quadratent. By Jacobson's theorem,  $RG$  is commutative. Two consequences follow at once. First,  $R \subseteq RG$  is commutative and, being closed under the identity, is quadratent. Second, for all  $g, h \in G$  the equality  $gh = hg$  holds in  $RG$ ; since  $gh$  and  $hg$  are basis elements, the equality already holds in  $G$ , i.e.  $G$  is abelian. By Lemma 2.3 every non-trivial element of  $G$  has order 3; an abelian group of exponent dividing 3 is a vector space over  $\mathbb{F}_3$ , that is, an elementary abelian 3-group.  $\square$

We now turn to the sufficiency. The key computation is the following decomposition in the finite case, which rests on the semisimplicity of  $FC_3$  in characteristic 2 and on the splitting behaviour of  $x^3 - 1$  over  $\mathbb{F}_2$  and  $\mathbb{F}_4$ .

**Lemma 2.8.** *Let  $H$  be a finite elementary abelian 3-group and let  $F \in \{\mathbb{F}_2, \mathbb{F}_4\}$ . Then,  $FH$  is isomorphic to a finite direct product of copies of the fields  $\mathbb{F}_2$  and  $\mathbb{F}_4$ ; in particular,  $FH$  is quadratent.*

*Proof.* Write  $H \cong C_3^m$  and induct on  $m$ , the case  $m = 0$  being trivial.

For  $m = 1$  we have  $FC_3 \cong F[x]/(x^3 - 1)$ . In characteristic 2 the polynomial  $x^3 - 1 = (x - 1)(x^2 + x + 1)$  is square-free, its two factors being relatively prime, so the Chinese remainder theorem applies. Over  $\mathbb{F}_2$ , the factor  $x^2 + x + 1$  is irreducible; whence,

$$\mathbb{F}_2 C_3 \cong \mathbb{F}_2[x]/(x - 1) \times \mathbb{F}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_2 \times \mathbb{F}_4,$$

while over  $\mathbb{F}_4$  we have the complete splitting  $x^2 + x + 1 = (x - \theta)(x - \theta^2)$ , where  $\theta$  is a primitive third root of unity, so that

$$\mathbb{F}_4 C_3 \cong \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_4.$$

For the inductive step, write  $C_3^m = C_3^{m-1} \times C_3$  and use the standard natural isomorphisms  $S(H' \times C_3) \cong (SH')C_3$  and  $(S_1 \times S_2)H' \cong S_1 H' \times S_2 H'$  for finite groups. By the induction hypothesis  $FC_3^{m-1} \cong \prod_j F_j$  with each  $F_j \in \{\mathbb{F}_2, \mathbb{F}_4\}$ ; whence,

$$FC_3^m \cong \left( \prod_j F_j \right) C_3 \cong \prod_j F_j C_3,$$

and each factor is, by the case  $m = 1$ , again a direct product of copies of  $\mathbb{F}_2$  and  $\mathbb{F}_4$ .

Finally, since every element of finite fields  $\mathbb{F}_2$  and of  $\mathbb{F}_4$  satisfies  $x^4 = x$ , and this element-wise identity is inherited by direct products; thus,  $FH$  is quadratent.  $\square$

**Proposition 2.9.** *Let  $R$  be a quadratent ring and let  $G$  be an arbitrary (possibly infinite) elementary abelian 3-group. Then,  $RG$  is (weakly) quadratent.*

*Proof.* Let  $\alpha \in RG$  and put  $H = \langle \text{supp}(\alpha) \rangle \leq G$ . Being a finitely generated abelian group of exponent dividing 3,  $H$  is a finite elementary abelian 3-group, and clearly  $\alpha \in RH \subseteq RG$ . Since the identity  $x^4 = x$  can be verified inside  $RH$ , it suffices to show that  $RH$  is quadratent for every finite elementary abelian 3-group  $H$ .

By Proposition 1.2 there is an embedding  $R \hookrightarrow \prod_{i \in I} F_i$  with  $F_i \in \{\mathbb{F}_2, \mathbb{F}_4\}$ , which induces an embedding  $RH \hookrightarrow (\prod_{i \in I} F_i)H$ . Furthermore, the natural map

$$\Phi: \left( \prod_{i \in I} F_i \right) H \longrightarrow \prod_{i \in I} (F_i H), \quad \Phi \left( \sum_{h \in H} (a_h^{(i)})_i h \right) = \left( \sum_{h \in H} a_h^{(i)} h \right)_i,$$

is a ring homomorphism and is injective: if  $\Phi(\alpha) = 0$ , then for every  $i \in I$  and every  $h \in H$  the coefficient  $a_h^{(i)}$  vanishes, because  $H$  is a free basis of each  $F_i H$ ; hence,  $\alpha = 0$ . (For finite  $H$  the map  $\Phi$  is even an isomorphism, but injectivity is all we need.) Composing, we obtain an embedding

$$RH \hookrightarrow \prod_{i \in I} F_i H.$$

By Lemma 2.8, each  $F_i H$  is quadratent, and the element-wise identity  $x^4 = x$  passes to direct products and to subrings. Therefore,  $RH$  and hence  $RG$  are quadratent.  $\square$

Assembling the pieces, we arrive at our main result.

**Theorem 2.10.** *Let  $R$  be a ring and  $G$  a group. The group ring  $RG$  is weakly quadratent if and only if exactly one of the following conditions holds:*

- (i)  $G = \{1\}$  and  $R$  is weakly quadratent, that is,  $R$  is isomorphic to one of either  $K$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_7$ ,  $K \times \mathbb{Z}_3$  or  $K \times \mathbb{Z}_7$ , where  $K$  is also a quadratent ring;
- (ii)  $G \neq \{1\}$  is an elementary abelian 3-group and  $R$  is a quadratent ring, i.e. is isomorphic to a subring of a direct product of copies of  $\mathbb{F}_2$  and  $\mathbb{F}_4$ . In this case  $RG$  is itself quadratent.

*Proof.* ( $\Leftarrow$ ) If (i) holds, then  $RG = R$  and there is nothing to prove; if (ii) holds, apply Proposition 2.9.

( $\Rightarrow$ ) Suppose  $RG$  is weakly quadratent. If  $G = \{1\}$ , then  $RG = R$  is weakly quadratent and Theorem 2.3. in [8] yields (i). If  $G \neq \{1\}$ , then  $\text{char}(R) = 2$  by Corollary 2.6, and Lemma 2.7 shows that  $R$  is quadratent, that  $G$  is an elementary abelian 3-group, and that  $RG$  is quadratent, which is exactly (ii).  $\square$

**Remark 2.11.** (a) *The smallest weakly quadratent group rings over a non-trivial group are*

$$\mathbb{Z}_2 C_3 \cong \mathbb{Z}_2 \times \mathbb{F}_4 \quad \text{and} \quad \mathbb{F}_4 C_3 \cong \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_4,$$

*as computed in Lemma 2.8.*

(b) *The group ring  $\mathbb{Z}_3 C_3$  is not weakly quadratent due to the fact that since  $g$  is a generator of  $C_3$ ,  $(g-1)^3 = 0$  while  $g-1 \neq 0$ , so  $\mathbb{Z}_3 C_3$  is not reduced, in violation of Lemma 2.2. This illustrates concretely why the components  $\mathbb{Z}_3$  (and similarly  $\mathbb{Z}_7$ ) of Theorem 2.3. in [8] disappear over a non-trivial group: for instance,  $(K \times \mathbb{Z}_3)C_3 \cong KC_3 \times \mathbb{Z}_3 C_3$  contains the offending factor  $\mathbb{Z}_3 C_3$  as a homomorphic image.*

(c) *We emphasize again that Theorem 2.10 assumes no commutativity of  $R$  or  $G$ ; the commutativity of  $RG$  in case (ii) is forced by the identity  $x^4 = x$  via Jacobson's theorem. In this respect our result strengthens the results in [10, 11], where the coefficient rings and groups are assumed commutative from the outset.*

### 3. Lifting the weakly quadratent property

Since the defining condition  $x^4 = \pm x$  is an element-wise identity, it always descends from  $RG$  to every factor ring  $RG/I$ . In this section we investigate the converse (i.e., lifting problem): if  $I$  is a (two-sided) ideal of  $RG$  such that  $RG/I$  is weakly quadratent, when is  $RG$  itself weakly quadratent? Before proceeding further, we point out an important consequence concerning the lifting problem modulo the Jacobson radical  $J(RG)$ . In fact, every weakly quadratent ring is semiprimitive: Indeed, let  $x$  be an element in the Jacobson radical. If  $x^4 = x$ , then  $x(1 - x^3) = 0$ ; since  $x^3$  is also in the Jacobson radical, the element  $1 - x^3$  is a unit, and multiplying on the right by its inverse gives  $x = 0$ . If  $x^4 = -x$ , the same argument applies with  $1 + x^3$  in place of  $1 - x^3$  as established in [8]. Thus, a nontrivial lifting problem from  $RG/J(RG)$  to  $RG$  can arise only when  $J(RG) \neq 0$ . However, in this case  $RG$  itself cannot be weakly quadratent. On the other hand, if  $J(RG) = 0$ , then  $RG/J(RG) \cong RG$ , and the lifting is trivial. Hence, weak quadratentness cannot be lifted nontrivially modulo the Jacobson radical. We therefore examine the conditions under which the nontrivial case  $J(RG) \neq 0$  may occur. The following lemma which complements Lemma 2.2 records the relevant consequences.

**Lemma 3.1.** *Let  $G$  be a non-trivial group. If  $RG$  is not  $J$ -semisimple, then either*

- (1)  $R$  is not isomorphic to a subring of a direct product of copies of  $\mathbb{F}_2$  and  $\mathbb{F}_4$ , or  
 (2)  $G$  differs from an elementary abelian 3-group.

*Proof.* Assuming the negations of 1. and 2. in the lemma, we can prove the result by considering the contrapositive of the statement in the lemma. Suppose that

$$R \cong S \leq \prod_{i \in I} F_i, \quad F_i \in \{\mathbb{F}_2, \mathbb{F}_4\},$$

and that  $G$  is an elementary abelian 3-group.

Since  $R$  is a direct product of fields, we have that  $R$  is  $J$ -semisimple. Moreover, for a prime integer  $p$ , using the set

$$J_p(R) = \{r \in R \mid pr \in J(R)\}$$

which is defined in [13] for  $p = 3$ , we obtain that

$$J_3(R) = \{r \in R \mid 3r \in J(R) = \{0\}\} = \{0\}.$$

By Karpilovsky's description [13],

$$J(RG) = J(R)G + \langle r(g-1) \mid g \in G_3, r \in J_3(R) \rangle,$$

it follows that  $RG$  is  $J$ -semisimple (see also [14]). □

Let us note that reducedness (Lemma 2.2) and semiprimitivity (Lemma 3.1) are logically independent necessary conditions, as the reduced non-semiprimitive ring which is localization of  $\mathbb{Z}$  at a prime ideal  $\langle p \rangle$  and the semiprimitive non-reduced ring  $M_2(\mathbb{F}_2)$  which is the ring of  $2 \times 2$  matrices whose entries are from the field  $\mathbb{F}_2$  show for instances.

The following result provides a conceptual explanation for the failure of lifting the weakly quadratent property over nonzero nil ideals.

**Corollary 3.2.** *Let  $RG$  be a commutative group ring and let  $I$  be a nonzero nil ideal of  $RG$ . Then,  $RG$  is not weakly quadratent. That is, the weakly quadratent property cannot be lifted modulo nonzero nil ideals in commutative group rings.*

*Proof.* Immediate from Lemma 3.1, since  $I \subseteq J(RG)$  and  $I \neq \{0\}$  force  $J(RG) \neq \{0\}$ . □

**Example 3.3.** *Let  $S = \mathbb{Z}_3 C_3$  with  $C_3 = \langle g \rangle$ , and let  $I = \Delta(C_3)$  be the augmentation ideal. The ring  $S$  is commutative and  $I$  is generated by  $g - 1$ , where  $(g - 1)^3 = 0$  by the computation in the proof of Lemma 2.4; hence,  $I$  is a non-zero nilpotent ideal (indeed  $I^3 = \{0\}$ ), while  $S/I \cong \mathbb{Z}_3$  is weakly quadratent. By Corollary 3.2 (or directly by Lemma 2.4),  $S$  is not weakly quadratent.*

In combination with the explicit descriptions of the Jacobson radical of a group ring available in the literature (see, e.g., [13]), Corollary 3.2 produces a wealth of concrete situations in which lifting is impossible; Example 3.3 is merely the simplest one.

**Corollary 3.4.** *Let  $RG$  be a commutative group ring, let  $e \in RG$  be a non-trivial idempotent, and say  $I = eRG$ . If  $RG/I$  is weakly quadratent, then  $RG$  is weakly quadratent if and only if the ring  $eRG$  is weakly quadratent and at least one of the rings  $eRG$  and  $RG/I$  is quadratent.*

*Proof.* Using the fact that  $RG/I \cong (1 - e)RG$  and combining the canonical Peirce decomposition of  $RG$  with Lemma 1.3 clarify the result.  $\square$

We now turn to the ideals which are specific to group rings. Observe first that an ideal of the form  $I_0G$  with  $I_0 \trianglelefteq R$  leads to  $RG/I_0G \cong (R/I_0)G$  and thereby reduces the group ring lifting problem to lifting the quadratentness identity from  $R/I_0$  to  $R$ . The latter may fail even over reduced, semiprimitive commutative base rings: for instance,  $\mathbb{F}_2[t]/(t^2 + t) \cong \mathbb{F}_2 \times \mathbb{F}_2$  is quadratent, whereas the polynomial ring  $\mathbb{F}_2[t]$  obviously is not (as  $t^4 \neq \pm t$ ). For the relative augmentation ideals  $\Delta(G, N)$ , however, our classification yields a complete and explicit answer.

**Theorem 3.5.** *Given a ring  $R$ , a group  $G$  and  $N \trianglelefteq G$  which is normal in  $G$ , and suppose that the factor ring  $RG/\Delta(G, N) \cong R(G/N)$  is weakly quadratent. Then,*

- (1) *If  $N \neq G$ , then  $R$  is necessarily quadratent, and  $RG$  is weakly quadratent if and only if  $G$  is an elementary abelian 3-group.*
- (2) *If  $N = G$ , then  $RG$  is weakly quadratent if and only if either  $G = \{1\}$  or  $G$  is not trivial elementary abelian 3-group and  $R$  is quadratent.*

*Proof.* (1) Since  $N \neq G$ , the group  $G/N$  is non-trivial. Applying Theorem 2.10 to the weakly quadratent group ring  $R(G/N)$ , we infer that  $R$  is quadratent and  $G/N$  is an elementary abelian 3-group. In particular  $G \neq \{1\}$ , so a second application of Theorem 2.10, this time to  $RG$ , shows that  $RG$  is weakly quadratent precisely when  $G$  is a (non-trivial) elementary abelian 3-group, since  $R$  is already known to be quadratent.

(2) Here  $R(G/N) \cong R$ , and the assertion is a literal restatement of Theorem 2.10.  $\square$

**Remark 3.6.** *We illustrate the sharpness of Theorem 3.5 by three examples; in each of them the quotient in question is weakly quadratent while the group ring itself is not, so that lifting genuinely fails.*

(a) *Let  $R = \mathbb{F}_2$ ,  $G = C_9$  and let  $N \cong C_3$  be the unique subgroup of order 3. Then,  $R(G/N) \cong \mathbb{F}_2C_3$  is weakly quadratent by Theorem 2.10, but  $\mathbb{F}_2C_9$  is not, since  $G$  contains an element of order 9 (Lemma 2.3). Hence, the condition in Theorem 3.5(1) must be imposed on all of  $G$  and not merely on  $G/N$ .*

(b) *Let  $R = \mathbb{F}_2$  and let  $G$  be the extraspecial group of order 27 and exponent 3 (the Heisenberg group over  $\mathbb{F}_3$ ), with  $N = Z(G) \cong C_3$ , so that  $G/N \cong C_3 \times C_3$ . Every element of  $G$  has order 1 or 3, and both  $N$  and  $G/N$  are elementary abelian 3-groups; moreover  $R(G/N) \cong \mathbb{F}_2(C_3 \times C_3)$  is weakly quadratent by Theorem 2.10. Nevertheless,  $\mathbb{F}_2G$  is a non-commutative ring and therefore cannot satisfy the identity  $x^4 = x$ ; as  $\text{char}(\mathbb{F}_2G) = 2$ , it is thus not weakly quadratent either. In particular, the elementary abelian condition in Theorem 3.5(1) cannot be verified on  $N$  and  $G/N$  separately (see also [14]).*

(c) *Let  $R = \mathbb{Z}_3$  and  $G = C_3$ , with  $N = G$ . Then,  $R$  is weakly quadratent, whereas  $RG = \mathbb{Z}_3C_3$  is not (Example 3.3); here the obstruction is indeed  $R$ , in accordance with Theorem 3.5(2). The same situation occurs for the coefficient rings  $K \times \mathbb{Z}_3$  and  $K \times \mathbb{Z}_7$  as can be seen in Theorem 2.3. of [8] (see also [14]).*

#### 4. Semiprimitivity of weakly quadratent group rings

Lemma 3.1 applies, in particular, to group rings and thus immediately yields the following.

**Corollary 4.1.** *Every weakly quadratent group ring  $RG$  is semiprimitive and reduced.*

*Proof.* Combining Lemmas 2.2 and 3.1 finishes the proof (see also [14]).  $\square$

**Remark 4.2.** *When  $G \neq \{1\}$ ,  $RG$  is quadratent by Theorem 2.10, hence commutative (by Jacobson's theorem) and reduced (by Lemma 2.2), and therefore a subdirect product of the integral domains  $RG/P$ , where  $P$  runs over the minimal prime ideals of  $RG$ ; each such quotient possesses the identity  $x^4 = x$  and, being a domain, is isomorphic to  $\mathbb{F}_2$  or to  $\mathbb{F}_4$  by the proof of Proposition 1.2. Since surjective ring homomorphisms map the Jacobson radical into the Jacobson radical, and  $J(\mathbb{Z}_2) = J(\mathbb{F}_4) = \{0\}$ , we conclude again that  $RG$  is  $J$ -semisimple.*

Recall that a ring  $S$  is von Neumann regular if for every  $x \in S$  there exists  $y \in S$  with  $x = xyx$ ; von Neumann regular rings are semiprimitive (see, e.g., [15]). By a classical theorem essentially due to Auslander, Villamayor and Connell (see [5]; cf. also [6]), the group ring  $RG$  is von Neumann regular if, and only if,  $R$  is von Neumann regular,  $G$  is locally finite and the order of every element of  $G$  is invertible in  $R$ .

**Proposition 4.3.** *If  $RG$  is not a semiprimitive group ring and  $G$  is locally finite group all of whose elements have odd order, then  $R$  is not quadratent.*

*Proof.* Assume that  $RG$  is not semiprimitive. We argue by contraposition. Suppose, on the contrary, that  $R$  is quadratent. By Proposition 1.2,  $R$  is von Neumann regular and  $\text{char}(R) = 2$ . Consequently, every odd integer is invertible in  $R$ . Since  $G$  is locally finite and all of its elements have odd order, Connell's criterion implies that  $RG$  is von Neumann regular. Hence,  $RG$  is Jacobson semisimple, contradicting the assumption. Therefore,  $R$  cannot be quadratent.  $\square$

Every elementary abelian 3-group is locally finite, and all of its elements have odd order. Therefore, Proposition 4.3. not only includes the group rings appearing in Theorem 2.1(ii), but also applies to a much broader class of group rings. This makes the following observation important. Among the large class of semiprimitive, and even regular, group rings described by Proposition 4.3, only a small part is weakly quadratent. To belong to this class, a group ring must satisfy three separate conditions: the exponent condition, the sign condition given in Lemma 1.3, and commutativity. The next example shows that these three conditions are independent of one another. Hence, semiprimitivity by itself is far from being sufficient to guarantee that a group ring is weakly quadratent.

**Example 4.4.** *The following group rings are all semisimple, and therefore semiprimitive. However, none of them is weakly quadratent.*

(a) Over  $\mathbb{F}_2$  the polynomial  $x^5 - 1$  factors as  $(x - 1)(x^4 + x^3 + x^2 + x + 1)$  with irreducible quartic factor; whence,

$$\mathbb{F}_2 C_5 \cong \mathbb{F}_2 \times \mathbb{F}_{16}.$$

*This ring is semisimple, reduced and of characteristic 2, but it is not weakly quadratent: the group  $C_5$  contains an element of order 5, contradicting Lemma 2.3 (equivalently,  $x^4 = x$  fails in  $\mathbb{F}_{16}$ ).*

(b) Since  $7 \equiv 1 \pmod{3}$ , the polynomial  $x^3 - 1$  splits over  $\mathbb{F}_7$  with the distinct roots 1, 2, 4, so that

$$\mathbb{F}_7 C_3 \cong \mathbb{F}_7 \times \mathbb{F}_7 \times \mathbb{F}_7,$$

*a semisimple artinian ring each of whose simple factors  $\mathbb{F}_7$  is weakly quadratent by Theorem 2.3. in [8]. Nevertheless no factor is quadratent, so Lemma 1.3 forbids the product from being weakly*

quadratent; concretely, for  $x = (1, 3, 1)$  one has  $x^4 = (1, 4, 1) \neq \pm x$ , because  $3^4 = 4$  in  $\mathbb{F}_7$ . This is the semiprimitive version of Lemma 2.5.

(c) Let  $G$  be the extraspecial group of order 27, as in Remark 3.6(b). Since  $\text{char}(\mathbb{F}_2) = 2$  does not divide  $|G| = 27$ , Maschke's theorem shows that  $\mathbb{F}_2 G$  is semisimple; moreover, every element of  $G$  has order 1 or 3. However,  $\mathbb{F}_2 G$  is non-commutative and hence not weakly quadratent.

We finally return to the lifting problem of Section 3 and identify the Jacobson radical of the group rings which prevent lifting there and hence strengthening Example 3.3.

**Proposition 4.5.** *Let  $G$  be an elementary abelian 3-group. Then,*

$$J(\mathbb{Z}_3 G) = \Delta(G).$$

*Proof.* For the inclusion  $\Delta(G) \subseteq J(\mathbb{Z}_3 G)$  it suffices, by the fact that nil one-sided ideals lie in the Jacobson radical (see [15]), to show that  $\Delta(G)$  is nil. Let  $\alpha \in \Delta(G)$  and put  $H = \langle \text{supp}(\alpha) \rangle$ , a finite elementary abelian 3-group; since  $\varepsilon(\alpha) = 0$  we have  $\alpha \in \Delta(H)$ . Now  $\Delta(H)$  is generated, as an ideal of the commutative ring  $\mathbb{Z}_3 H$ , by the finitely many elements  $h - 1$  with  $h \in H$ , each of which satisfies  $(h - 1)^3 = 0$  by the computation in the proof of Lemma 2.4; consequently  $\Delta(H)$  is a sum of finitely many nilpotent ideals  $(h - 1)\mathbb{Z}_3 H$  and is therefore itself nilpotent. In particular  $\alpha$  is nilpotent, and  $\Delta(G)$  is nil.

Conversely,  $\mathbb{Z}_3 G / \Delta(G) \cong \mathbb{Z}_3$  is a field, so  $\Delta(G)$  is a maximal ideal and hence contains  $J(\mathbb{Z}_3 G)$ . The two inclusions give the asserted equality [13].  $\square$

The importance of Proposition 4.5 becomes clear when it is considered together with the semiprimitivity condition in Lemma 3.1. Together, these two results show exactly how the group rings  $(\mathbb{Z}_3 G)$  differ from weakly quadratent group rings. In particular, they explain this difference using only Jacobson radical.

**Corollary 4.6.** *For an arbitrary elementary abelian 3-group  $G$ , the following conditions are equivalent:*

- (i)  $\mathbb{Z}_3 G$  is weakly quadratent;
- (ii)  $\mathbb{Z}_3 G$  is semiprimitive;
- (iii)  $\Delta(G) = \{0\}$ ;
- (iv)  $G = \{1\}$ .

*Proof.* (i) $\Rightarrow$ (ii) is precisely Lemma 3.1. (ii) $\Rightarrow$ (iii) follows from Proposition 4.5, which identifies  $\Delta(G)$  with the vanishing Jacobson radical  $J(\mathbb{Z}_3 G)$ . (iii) $\Rightarrow$ (iv): if there existed  $g \in G$  with  $g \neq 1$ , then  $0 \neq g - 1 \in \Delta(G)$ . Finally, (iv) $\Rightarrow$ (i): for trivial  $G$  we have  $\mathbb{Z}_3 G = \mathbb{Z}_3$ , which is weakly quadratent by Theorem 2.3. in [8] (see also [14]).  $\square$

Corollary 4.6 gives another proof of the characteristic (3) part of our classification for elementary abelian groups. Unlike the proof based on Lemma 2.3, this argument relies only on properties of the Jacobson radical. In fact, the only reason why  $\mathbb{Z}_3 G$  is not weakly quadratent is that its Jacobson radical is nonzero, that is,  $J(\mathbb{Z}_3 G) = \Delta(G) \neq 0$ .

## 5. Conclusions

In this paper, the notion of weakly quadratent elements has been studied in the setting of group rings  $RG$ . Based on this notion, several classification results have been obtained, together with conditions on the ring  $R$  and the group  $G$ . More generally, as related researchers know, characterizing rings in terms of the properties of their elements is a broad area of research in classical ring theory. Now, we present several open research problems that we believe may lead to interesting new results. Our aim is to draw the reader's attention to the importance of this topic and to encourage further research in this direction. In view of [8, Problem 2.5] and of Theorems 2.10 and 3.5, the following questions arise naturally and thus they may be future studies.

**Problem 5.1.** *For a fixed natural number  $n \geq 2$ , describe those pairs  $(R, G)$  for which every element  $x$  of the group ring  $RG$  satisfies  $x^n = x$  or  $x^n = -x$ .*

The cases  $n = 2$  and  $n = 3$  can be extracted from [3] and [10], respectively, while the case  $n = 4$  is settled by Theorem 2.10; the general case appears to be open, as is already the corresponding ring theoretical question.

**Problem 5.2.** *Characterize the weakly quadratent twisted group rings  $R^\gamma G$ , where  $\gamma$  is a 2-cocycle.*

**Problem 5.3.** *It would be interesting to determine for which group rings  $RG$  the identity  $x^n = \pm x$  can be lifted from a quotient ring to the group ring itself. More precisely, if a quotient ring  $RG/I$  satisfies the identity  $x^n = \pm x$ , under what conditions does the group ring  $RG$  also satisfy the same identity?*

Answering these questions could lead to a better understanding of characterization of rings by element-wise identities and would naturally extend the results obtained in this paper.

## Author contributions

Both authors contributed equally to the preparation of this manuscript. Both authors have read and approved the final version of the manuscript for publication.

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The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that no conflicts of interest in this paper.

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