



*Research article***Local M -fractional deformation of the (2+1)-dimensional AMTZ system: Coordinate equivalence, Lax integrability, and exact solutions****Zh.R. Myrzakulova¹ and A. Taishiyeva^{2,*}**¹ Faculty of Mechanics and Mathematics, L.N. Gumilyov Eurasian National University, 13 Kazhymukan S., Astana, Kazakhstan² Faculty of Physics, Mathematics and Information Technology, Kh. Dosmukhamedov Atyrau University, 1 Student Avenue, Atyrau, Kazakhstan*** Correspondence:** Email: aigulustaz@gmail.com.

Abstract: We investigate a local M -deformation of the $(2 + 1)$ -dimensional Akbota–Myrzakulov–Tolkynay–Zhaidary (AMTZ) system and clarify its integrability and traveling-wave structure. For sufficiently smooth functions on the positive domain, we establish an exact two-parameter coordinate equivalence between the local M -AMTZ and classical AMTZ systems for $0 < \alpha \leq 1$ and $\eta > 0$. The same transformation shows that the corresponding Lax representation, zero-curvature condition, and conservation structure are inherited from the classical AMTZ system under coordinate pullback.

A general traveling-wave reduction is derived while retaining the integration constant, leading to a generalized Duffing-type equation with an independent constant term. Its first integral is a quartic polynomial and provides a phase-plane description of the reduced dynamics. Generic root configurations lead to elliptic traveling-wave solutions, for which an explicit Weierstrass representation is obtained, whereas repeated-root degenerations yield elementary traveling-wave profiles.

The analysis further shows that the parameters α and η do not modify the intrinsic reduced dynamics. Instead, they determine the coordinate representation and the traveling-wave phase, with α producing a nonlinear power-law reparametrization and η contributing a constant scaling. Thus, the local M -parameters do not by themselves generate distinct intrinsic traveling-wave families. The results provide a unified coordinate-equivalence framework that distinguishes inherited integrability structures from their local M -representation.

Keywords: local M -fractional derivative; AMTZ system; coordinate equivalence; Lax pair; zero-curvature condition; traveling-wave reduction; Riccati method; exact solutions

Mathematics Subject Classification: 35Q51, 35C08, 34A08, 37K10

1. Introduction

Integrable nonlinear evolution equations play an important role in mathematical physics owing to the existence of Lax representations, conservation laws, Hamiltonian structures, and broad classes of exact solutions [1–3]. Inverse-scattering, Darboux, and Hirota-type methods provide systematic tools for the analysis of soliton, periodic, and rational wave structures [4–6]. In higher dimensions, classical examples such as the Kadomtsev–Petviashvili and Davey–Stewartson systems demonstrate the rich interaction between integrability and multidimensional nonlinear wave dynamics [7–9].

A more recent $(2+1)$ -dimensional integrable model is the Akbota–Myrzakulov–Tolkynay–Zhaidary (AMTZ) system [10]. A Lax representation, gauge equivalence with the Kuralay–Ma–Myrzakulov system, integrable reductions, and several classes of exact solutions were established in [11], while subsequent work studied soliton, bifurcation, and chaotic regimes of related reductions [12]. These results make the AMTZ system a useful model for examining how integrable structures behave under local fractional-type deformations.

Fractional calculus provides several frameworks for modifying classical differential models. Standard derivatives such as the Caputo and Riemann–Liouville operators are nonlocal, whereas a number of fractional-type derivatives are local [13–17]. The M -derivative introduced by Sousa and Oliveira [18] belongs to the latter class. For classically differentiable functions, it reduces to a weighted first-order differential operator. Various M -fractional and truncated M -fractional nonlinear models have consequently been studied using traveling-wave and related analytical methods [19–24].

A different direction is represented by fractional integrable systems in which fractional or nonlocal operators modify the spectral or dispersive structure itself. Fractional integrable extensions of fundamental nonlinear wave equations have been developed within inverse-scattering frameworks [25, 26], together with fractional integrable hierarchies, multisoliton constructions, and models with fractional dispersion [27–29]. Related exact-solution studies include fractional coupled Hirota systems, modified Korteweg–de Vries–sine-Gordon models, and fractional Korteweg–de Vries-type and nonlinear Schrödinger-type equations [30–37].

The local M -framework considered here has a different structural character. For sufficiently smooth functions on the positive domain, the M -derivative can be transformed into ordinary differentiation by an invertible reparametrization of the independent variable. Consequently, it is important to distinguish structures inherited from the classical AMTZ system from properties associated with its representation in the original local M -variables. In particular, a Lax representation obtained through the same coordinate transformation should be regarded as an induced structure rather than an independent source of integrability.

Motivated by this distinction, we investigate the local M -deformation of the AMTZ system in the general two-parameter setting. The main contribution is a structural characterization of this deformation based on an exact natural-coordinate equivalence with the classical AMTZ system. The equivalence is established for the general parameters $0 < \alpha \leq 1$ and $\eta > 0$ on the positive domain and is used to determine precisely which structures are inherited under coordinate pullback.

More specifically, the results fall into three categories. First, the two-parameter coordinate equivalence and the resulting interpretation of the local M -AMTZ model provide the structural basis of the present analysis. Second, the Lax representation, zero-curvature condition, and conservation structure are shown to be inherited from the classical AMTZ system under this transformation and

are not claimed to be new independent integrability structures. Third, the traveling-wave reduction, its quartic first integral, and its elementary degenerations are analyzed within the same coordinate-equivalence framework in order to clarify their role in the local M -representation.

In the traveling-wave analysis, the integration constant is retained so that the zero-constant reduction appears explicitly as a special subclass. The general reduced equation is described through its first integral, effective potential, equilibrium structure, and quartic quadrature. Generic root configurations lead to elliptic solutions, whereas repeated-root configurations yield elementary degenerations. For the zero-constant subclass, representative hyperbolic, trigonometric, and rational profiles are obtained by a Riccati expansion and verified by direct substitution. These profiles are not claimed to be new classical AMTZ traveling-wave families; rather, they provide simple representatives of distinct degenerate quartic configurations and are compared with the existing AMTZ literature.

The analysis also clarifies the roles of the two local M -parameters. Neither α nor η enters the autonomous reduced equation or its phase-plane dynamics. They act through the transformation of the independent variables, with α producing a nonlinear power-law reparametrization and η producing a constant scaling. Thus, varying these parameters changes the coordinate representation of a traveling wave but does not by itself generate a new intrinsic solution family.

The remainder of the paper is organized as follows. Section 2 introduces the local M -derivative and the associated natural coordinates. Section 3 establishes the two-parameter coordinate equivalence between the local M -AMTZ and classical AMTZ systems. Section 4 presents the induced Lax and zero-curvature structures, and Section 5 derives a conservation law and its coordinate pullback. Section 6 develops the general traveling-wave reduction, first integral, phase-plane structure, and elliptic-function representation. Section 7 treats the special Riccati subclass and compares the retained elementary representatives with classical AMTZ solutions. Section 8 clarifies the roles of the local M -parameters and illustrates the coordinate collapse of the traveling-wave profiles. Section 9 summarizes the main conclusions.

2. Preliminaries: Local M -derivative and natural coordinates

We recall the local M -derivative introduced by Sousa and Oliveira [18]. For $0 < \alpha \leq 1$ and $\eta > 0$, it is defined by

$$D_M^{\alpha,\eta} f(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(xE_\eta(\varepsilon x^{-\alpha})) - f(x)}{\varepsilon}, \quad (2.1)$$

where

$$E_\eta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\eta k + 1)}.$$

For $f \in C^1((0, \infty))$, this operator reduces to

$$D_M^{\alpha,\eta} f(x) = \frac{x^{1-\alpha}}{\Gamma(\eta + 1)} f'(x). \quad (2.2)$$

For functions of several variables, we use

$$D_{M,x}^{\alpha,\eta} = \frac{x^{1-\alpha}}{\Gamma(\eta + 1)} \partial_x, \quad D_{M,y}^{\alpha,\eta} = \frac{y^{1-\alpha}}{\Gamma(\eta + 1)} \partial_y, \quad D_{M,t}^{\alpha,\eta} = \frac{t^{1-\alpha}}{\Gamma(\eta + 1)} \partial_t.$$

The smooth representation (2.2) admits an exact coordinate interpretation.

Proposition 2.1. *Let $0 < \alpha \leq 1$, $\eta > 0$, and $x > 0$. Define*

$$X = \frac{\Gamma(\eta + 1)}{\alpha} x^\alpha.$$

If $F(x) = \widetilde{F}(X(x))$ is sufficiently smooth, then

$$D_{M,x}^{\alpha,\eta} F = \widetilde{F}_X,$$

and hence

$$D_{M,x}^{\alpha,\eta} = \partial_X.$$

Proof. Since

$$\frac{dX}{dx} = \Gamma(\eta + 1)x^{\alpha-1},$$

the chain rule and (2.2) give

$$D_{M,x}^{\alpha,\eta} = \frac{x^{1-\alpha}}{\Gamma(\eta + 1)} \frac{dX}{dx} \partial_X = \partial_X.$$

□

For the three independent variables of the AMTZ system, define

$$X = \frac{\Gamma(\eta + 1)}{\alpha} x^\alpha, \quad Y = \frac{\Gamma(\eta + 1)}{\alpha} y^\alpha, \quad T = \frac{\Gamma(\eta + 1)}{\alpha} t^\alpha. \quad (2.3)$$

Then

$$D_{M,x}^{\alpha,\eta} = \partial_X, \quad D_{M,y}^{\alpha,\eta} = \partial_Y, \quad D_{M,t}^{\alpha,\eta} = \partial_T. \quad (2.4)$$

For $x, y, t > 0$, the transformation (2.3) is invertible, with

$$x = \left(\frac{\alpha X}{\Gamma(\eta + 1)} \right)^{1/\alpha}, \quad y = \left(\frac{\alpha Y}{\Gamma(\eta + 1)} \right)^{1/\alpha}, \quad t = \left(\frac{\alpha T}{\Gamma(\eta + 1)} \right)^{1/\alpha}.$$

Repeated and mixed local M -derivatives transform accordingly; in particular,

$$\left(D_{M,x}^{\alpha,\eta} \right)^2 D_{M,y}^{\alpha,\eta} F = \widetilde{F}_{XXY}.$$

Thus, in the smooth setting considered here, α determines the nonlinear power-law reparametrization of the independent variables, whereas η contributes only the constant scaling factor $\Gamma(\eta + 1)$. After establishing the general (α, η) coordinate identity, we use the normalized case $\eta = 1$ in subsequent calculations for notational economy. Then

$$X = \frac{x^\alpha}{\alpha}, \quad Y = \frac{y^\alpha}{\alpha}, \quad T = \frac{t^\alpha}{\alpha}, \quad (2.5)$$

and

$$D_{M,x}^\alpha = \partial_X, \quad D_{M,y}^\alpha = \partial_Y, \quad D_{M,t}^\alpha = \partial_T. \quad (2.6)$$

Remark 2.1. For $\alpha = 1$ and arbitrary $\eta > 0$,

$$D_{M,x}^{1,\eta} = \frac{1}{\Gamma(\eta + 1)} \partial_x, \quad X = \Gamma(\eta + 1)x.$$

Hence the operator coincides directly with ordinary differentiation in the original coordinate at $(\alpha, \eta) = (1, 1)$, whereas for general $\eta > 0$ it is ordinary differentiation with respect to the scaled natural coordinate.

Remark 2.2. For differentiable functions, the local M -derivative considered here is a local weighted first-order differential operator. This distinguishes it from nonlocal fractional operators such as the Riemann–Liouville and Caputo derivatives, whose definitions involve integral kernels [14]. No intrinsic memory effect is therefore associated with the local M -operator in the smooth setting considered here.

The transformation (2.3) is a smooth diffeomorphism of $(0, \infty)^3$ onto itself. For $0 < \alpha < 1$, derivatives of the coordinate map, such as

$$\frac{dX}{dx} = \Gamma(\eta + 1)x^{\alpha-1},$$

become unbounded as $x \rightarrow 0^+$, with analogous behavior in y and t . Therefore, the coordinate-equivalence results below are formulated on the open positive domain; boundary behavior requires separate consideration.

3. Local M -deformation and coordinate equivalence of the AMTZ system

We consider the classical $(2 + 1)$ -dimensional Akbota–Myrzakulov–Tolkynay–Zhaidary (AMTZ) system [11]:

$$\begin{aligned} q_t - \frac{1}{b}uq_x + \frac{\beta}{b}qq_y - \beta r_y &= 0, \\ r_t - \frac{1}{b}ur_x + \frac{\beta}{b}rq_y - \frac{\beta}{4ab}q_{xy} &= 0, \\ u_x + \frac{\beta}{2}q_y &= 0, \end{aligned} \tag{3.1}$$

where $q = q(x, y, t)$, $r = r(x, y, t)$, $u = u(x, y, t)$, and $a, b, \beta \in \mathbb{R} \setminus \{0\}$.

Replacing the ordinary derivatives in (3.1) by the local M -operators introduced in Section 2 gives the local M -AMTZ system

$$\begin{aligned} D_{M,t}^{\alpha,\eta}q - \frac{1}{b}uD_{M,x}^{\alpha,\eta}q + \frac{\beta}{b}qD_{M,y}^{\alpha,\eta}q - \beta D_{M,y}^{\alpha,\eta}r &= 0, \\ D_{M,t}^{\alpha,\eta}r - \frac{1}{b}uD_{M,x}^{\alpha,\eta}r + \frac{\beta}{b}rD_{M,y}^{\alpha,\eta}q - \frac{\beta}{4ab}(D_{M,x}^{\alpha,\eta})^2 D_{M,y}^{\alpha,\eta}q &= 0, \\ D_{M,x}^{\alpha,\eta}u + \frac{\beta}{2}D_{M,y}^{\alpha,\eta}q &= 0. \end{aligned} \tag{3.2}$$

3.1. Coordinate equivalence

Theorem 3.1 (Coordinate equivalence). *Let*

$$\Omega_+ = (0, \infty)^3, \quad 0 < \alpha \leq 1, \quad \eta > 0,$$

and suppose that $q, r, u \in C^3(\Omega_+)$. Define

$$X = \frac{\Gamma(\eta + 1)}{\alpha} x^\alpha, \quad Y = \frac{\Gamma(\eta + 1)}{\alpha} y^\alpha, \quad T = \frac{\Gamma(\eta + 1)}{\alpha} t^\alpha. \quad (3.3)$$

Writing

$$q(x, y, t) = Q(X, Y, T), \quad r(x, y, t) = R(X, Y, T), \quad u(x, y, t) = U(X, Y, T),$$

the local M-AMTZ system (3.2) is equivalent on Ω_+ to

$$\begin{aligned} Q_T - \frac{1}{b} U Q_X + \frac{\beta}{b} Q Q_Y - \beta R_Y &= 0, \\ R_T - \frac{1}{b} U R_X + \frac{\beta}{b} R Q_Y - \frac{\beta}{4ab} Q_{XXY} &= 0, \\ U_X + \frac{\beta}{2} Q_Y &= 0, \end{aligned} \quad (3.4)$$

which is precisely the classical AMTZ system (3.1) expressed in the natural coordinates (X, Y, T) .

Proof. By (2.4),

$$D_{M,x}^{\alpha,\eta} = \partial_X, \quad D_{M,y}^{\alpha,\eta} = \partial_Y, \quad D_{M,t}^{\alpha,\eta} = \partial_T.$$

Hence

$$D_{M,t}^{\alpha,\eta} q = Q_T, \quad D_{M,x}^{\alpha,\eta} q = Q_X, \quad D_{M,y}^{\alpha,\eta} q = Q_Y,$$

with analogous identities for r and u . Moreover,

$$(D_{M,x}^{\alpha,\eta})^2 D_{M,y}^{\alpha,\eta} q = Q_{XXY}.$$

Substitution into (3.2) gives (3.4) term by term.

The transformation (3.3) is a diffeomorphism of Ω_+ onto itself, with inverse

$$x = \left(\frac{\alpha X}{\Gamma(\eta + 1)} \right)^{1/\alpha}, \quad y = \left(\frac{\alpha Y}{\Gamma(\eta + 1)} \right)^{1/\alpha}, \quad t = \left(\frac{\alpha T}{\Gamma(\eta + 1)} \right)^{1/\alpha}.$$

Hence the correspondence is invertible. This completes the proof. $\square$

Corollary 3.1 (Correspondence of solutions). *Every C^3 -solution (Q, R, U) of the classical AMTZ system on the positive domain generates a C^3 -solution of the local M-AMTZ system through*

$$q(x, y, t) = Q(X, Y, T), \quad r(x, y, t) = R(X, Y, T), \quad u(x, y, t) = U(X, Y, T),$$

where (X, Y, T) are given by (3.3). Conversely, every C^3 -solution of (3.2) on Ω_+ corresponds to a classical AMTZ solution under the inverse transformation.

For the remainder of the calculations, we use the normalized case $\eta = 1$ for notational economy, so that

$$X = \frac{x^\alpha}{\alpha}, \quad Y = \frac{y^\alpha}{\alpha}, \quad T = \frac{t^\alpha}{\alpha},$$

while the general case $\eta > 0$ follows from Theorem 3.1 by constant rescaling of the natural coordinates.

Remark 3.1 (Structural interpretation). *Theorem 3.1 shows that, in the smooth positive-domain setting, the local M -deformation is an invertible coordinate representation of the classical AMTZ system. The parameter α determines the nonlinear power-law reparametrization of the independent variables, whereas η contributes a constant scaling of the natural coordinates.*

Accordingly, structures obtained solely through this coordinate transformation are inherited from the classical AMTZ system rather than independent new structures. In particular, the spectral and zero-curvature representations considered below are coordinate pullbacks of their classical AMTZ counterparts.

Remark 3.2 (Domain and boundary). *The equivalence is established on the open positive domain $\Omega_+ = (0, \infty)^3$. For $0 < \alpha < 1$, derivatives of the coordinate map, such as*

$$\frac{dX}{dx} = \Gamma(\eta + 1)x^{\alpha-1},$$

become unbounded as $x \rightarrow 0^+$, with analogous behavior in y and t . Therefore, extensions to the coordinate planes or to negative values of the independent variables require additional assumptions and are not considered here.

Theorem 3.1 provides the structural basis for the induced Lax representation considered next.

4. Induced Lax representation and zero-curvature structure

The coordinate equivalence established in Theorem 3.1 also determines the spectral interpretation of the local M -AMTZ system. The Lax representation considered below is obtained by pulling back the classical AMTZ spectral problem [11] under the natural-coordinate transformation and is therefore an inherited rather than an independent integrability structure.

4.1. Induced local M -spectral problem

Let

$$\Phi = \Phi(x, y, t; \lambda)$$

denote a two-component eigenfunction, where λ is the spectral parameter. In the normalized case $\eta = 1$, the induced local M -spectral problem is

$$D_{M,x}^\alpha \Phi = \mathcal{U}\Phi, \tag{4.1}$$

$$D_{M,t}^\alpha \Phi = \beta \lambda D_{M,y}^\alpha \Phi + \mathcal{B}\Phi, \tag{4.2}$$

where $0 < \alpha \leq 1$.

The matrix $\mathcal{U}$ is given by

$$\mathcal{U} = \begin{pmatrix} 0 & a \\ b\lambda^2 + q\lambda + r & 0 \end{pmatrix} = (b\lambda^2 + q\lambda)\Sigma + \mathcal{Q}, \quad (4.3)$$

where

$$\Sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} 0 & a \\ r & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix $\mathcal{B}$ is polynomial in the spectral parameter:

$$\mathcal{B} = \mathcal{B}_2\lambda^2 + \mathcal{B}_1\lambda + \mathcal{B}_0, \quad (4.4)$$

with

$$\mathcal{B}_2 = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = u\Sigma, \quad \mathcal{B}_1 = \begin{pmatrix} 0 & 0 \\ b^{-1}uq & 0 \end{pmatrix} = b^{-1}uq\Sigma,$$

and

$$\mathcal{B}_0 = \begin{pmatrix} \frac{\beta}{4b}D_{M,y}^\alpha q & \frac{a}{b}u \\ \frac{1}{b}ur + \frac{\beta}{4ab}D_{M,x}^\alpha D_{M,y}^\alpha q & -\frac{\beta}{4b}D_{M,y}^\alpha q \end{pmatrix}. \quad (4.5)$$

Equivalently,

$$\mathcal{B}_0 = \frac{\beta}{4b}D_{M,y}^\alpha q \sigma_3 + \frac{u}{b}\mathcal{Q} + \frac{\beta}{4ab}D_{M,x}^\alpha D_{M,y}^\alpha q \Sigma.$$

Under the normalized natural-coordinate transformation

$$X = \frac{x^\alpha}{\alpha}, \quad Y = \frac{y^\alpha}{\alpha}, \quad T = \frac{t^\alpha}{\alpha},$$

we write

$$q(x, y, t) = Q(X, Y, T), \quad r(x, y, t) = R(X, Y, T), \quad u(x, y, t) = U(X, Y, T).$$

Since

$$D_{M,x}^\alpha = \partial_X, \quad D_{M,y}^\alpha = \partial_Y, \quad D_{M,t}^\alpha = \partial_T,$$

Eqs (4.1)–(4.2) become

$$\Phi_X = \mathcal{U}^{(c)}\Phi, \quad (4.6)$$

$$\Phi_T = \beta\lambda\Phi_Y + \mathcal{B}^{(c)}\Phi, \quad (4.7)$$

where

$$\mathcal{U}^{(c)} = \begin{pmatrix} 0 & a \\ b\lambda^2 + Q\lambda + R & 0 \end{pmatrix},$$

and

$$\mathcal{B}^{(c)} = \mathcal{B}_2^{(c)}\lambda^2 + \mathcal{B}_1^{(c)}\lambda + \mathcal{B}_0^{(c)},$$

with

$$\mathcal{B}_2^{(c)} = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}, \quad \mathcal{B}_1^{(c)} = \begin{pmatrix} 0 & 0 \\ b^{-1}UQ & 0 \end{pmatrix},$$

and

$$\mathcal{B}_0^{(c)} = \begin{pmatrix} \frac{\beta}{4b}Q_Y & \frac{a}{b}U \\ \frac{1}{b}UR + \frac{\beta}{4ab}Q_{XY} & -\frac{\beta}{4b}Q_Y \end{pmatrix}.$$

Thus, in the natural coordinates, the induced local M -spectral problem coincides with the classical AMTZ Lax representation [11].

4.2. Induced zero-curvature representation

The compatibility condition of (4.1)–(4.2) is

$$D_{M,t}^\alpha \mathcal{U} - D_{M,x}^\alpha \mathcal{B} - \beta \lambda D_{M,y}^\alpha \mathcal{U} + [\mathcal{U}, \mathcal{B}] = 0. \quad (4.8)$$

Indeed, applying $D_{M,t}^\alpha$ to (4.1) and $D_{M,x}^\alpha$ to (4.2), and using the commutativity of the local M -operators with respect to distinct independent variables, yields (4.8).

Under the natural-coordinate transformation, (4.8) becomes

$$\mathcal{U}_T^{(c)} - \mathcal{B}_X^{(c)} - \beta \lambda \mathcal{U}_Y^{(c)} + [\mathcal{U}^{(c)}, \mathcal{B}^{(c)}] = 0. \quad (4.9)$$

This is precisely the zero-curvature condition of the classical AMTZ Lax representation. Equating coefficients of the powers of λ in (4.9) reproduces the classical AMTZ system (3.4). Therefore, by Theorem 3.1, (4.8) is equivalent to the local M -AMTZ system (3.2).

The same conclusion holds for arbitrary $\eta > 0$ under the general natural-coordinate transformation (2.3). Hence the Lax representation and zero-curvature condition are inherited from the classical AMTZ system by coordinate pullback.

5. Conservation law and its coordinate pullback

The coordinate equivalence established in Theorem 3.1 transports conservation laws of the classical AMTZ system to the local M -AMTZ representation. We illustrate this correspondence by means of one explicit conservation law and its associated conserved integral.

5.1. An explicit AMTZ conservation law

In the natural coordinates (X, Y, T) , the first and third equations of (3.4) give

$$Q_T - \frac{1}{b}UQ_X + \frac{\beta}{b}QQ_Y - \beta R_Y = 0, \quad (5.1)$$

and

$$U_X + \frac{\beta}{2}Q_Y = 0.$$

Hence

$$U_X = -\frac{\beta}{2}Q_Y,$$

so that

$$-\frac{1}{b}(UQ)_X + \frac{\beta}{4b}(Q^2)_Y = -\frac{1}{b}UQ_X + \frac{\beta}{b}QQ_Y.$$

Therefore, (5.1) takes the divergence form

$$Q_T + \partial_X \left(-\frac{1}{b} U Q \right) + \partial_Y \left(\frac{\beta}{4b} Q^2 - \beta R \right) = 0. \quad (5.2)$$

Thus, we may take

$$\rho = Q, \quad J = -\frac{1}{b} U Q, \quad K = \frac{\beta}{4b} Q^2 - \beta R$$

as the conserved density and the corresponding fluxes. If the boundary flux vanishes and the integral converges, then

$$\mathcal{I}(T) = \iint_{\Omega_{XY}} Q(X, Y, T) dX dY, \quad \frac{d\mathcal{I}}{dT} = 0. \quad (5.3)$$

5.2. Pullback to the local M -AMTZ variables

For the normalized case $\eta = 1$,

$$\partial_T = D_{M,t}^\alpha, \quad \partial_X = D_{M,x}^\alpha, \quad \partial_Y = D_{M,y}^\alpha.$$

Hence (5.2) pulls back to

$$D_{M,t}^\alpha q + D_{M,x}^\alpha \left(-\frac{1}{b} u q \right) + D_{M,y}^\alpha \left(\frac{\beta}{4b} q^2 - \beta r \right) = 0. \quad (5.4)$$

The same identity follows directly from the first and third equations of (3.2), using

$$D_{M,x}^\alpha u = -\frac{\beta}{2} D_{M,y}^\alpha q$$

and the Leibniz rule.

5.3. Weighted conserved integral in the original variables

For arbitrary $\eta > 0$, the spatial natural coordinates satisfy

$$X = \frac{\Gamma(\eta + 1)}{\alpha} x^\alpha, \quad Y = \frac{\Gamma(\eta + 1)}{\alpha} y^\alpha,$$

and hence

$$dX = \Gamma(\eta + 1) x^{\alpha-1} dx, \quad dY = \Gamma(\eta + 1) y^{\alpha-1} dy.$$

Therefore,

$$dX dY = \Gamma^2(\eta + 1) x^{\alpha-1} y^{\alpha-1} dx dy.$$

Consequently, the conserved integral (5.3) is represented in the original variables by

$$\mathcal{I}_M(t) = \Gamma^2(\eta + 1) \iint_{\Omega_{xy}} q(x, y, t) x^{\alpha-1} y^{\alpha-1} dx dy. \quad (5.5)$$

The two representations are related by

$$\mathcal{I}_M(t) = \mathcal{I} \left(\frac{\Gamma(\eta + 1)}{\alpha} t^\alpha \right).$$

Thus, under the same convergence and vanishing-flux assumptions,

$$\frac{dI}{dT} = 0 \quad \implies \quad D_{M,t}^{\alpha,\eta} I_M(t) = 0.$$

For $\eta = 1$, $\Gamma^2(\eta + 1) = \Gamma^2(2) = 1$.

The factor

$$x^{\alpha-1}y^{\alpha-1}$$

is the Jacobian weight induced by the natural-coordinate transformation, not a new conserved density. Hence both the differential conservation law and the associated conserved integral are inherited from the classical AMTZ system under coordinate pullback.

6. Traveling-wave reduction and the generalized Duffing-type equation

We now consider traveling-wave reductions in the natural coordinates. The general integration constant is retained throughout the reduction, so that the zero-constant case considered later appears only as a special subclass.

6.1. Traveling-wave reduction

For the normalized case $\eta = 1$, let

$$\omega = kX + \mu Y - \tau T = \frac{1}{\alpha} (kx^\alpha + \mu y^\alpha - \tau t^\alpha), \quad (6.1)$$

where $k, \mu, \tau \in \mathbb{R}$. We assume $k \neq 0$ and $\mu \neq 0$ unless explicitly stated otherwise. We seek solutions of the form

$$q(x, y, t) = Q(\omega), \quad r(x, y, t) = R(\omega), \quad u(x, y, t) = U(\omega).$$

Since

$$D_{M,x}^\alpha \omega = k, \quad D_{M,y}^\alpha \omega = \mu, \quad D_{M,t}^\alpha \omega = -\tau,$$

the chain rule gives

$$D_{M,x}^\alpha Q = kQ', \quad D_{M,y}^\alpha Q = \mu Q', \quad D_{M,t}^\alpha Q = -\tau Q',$$

and

$$(D_{M,x}^\alpha)^2 D_{M,y}^\alpha Q = k^2 \mu Q'''.$$

Substitution into (3.2) yields

$$\begin{cases} -\tau Q' - \frac{k}{b} U Q' + \frac{\beta \mu}{b} Q Q' - \beta \mu R' = 0, \\ -\tau R' - \frac{k}{b} U R' + \frac{\beta \mu}{b} R Q' - \frac{\beta k^2 \mu}{4ab} Q''' = 0, \\ kU' + \frac{\beta \mu}{2} Q' = 0. \end{cases} \quad (6.2)$$

The third equation integrates to

$$U(\omega) = C_1 - \frac{\beta\mu}{2k}Q(\omega). \quad (6.3)$$

Substitution into the first equation of (6.2) and a further integration give

$$R(\omega) = \frac{3}{4b}Q^2(\omega) - \frac{b\tau + kC_1}{b\beta\mu}Q(\omega) + C_2. \quad (6.4)$$

6.2. Generalized Duffing-type equation

Substituting (6.3) and (6.4) into the second equation of (6.2) and integrating once with respect to ω yields

$$Q'' + \mathcal{A}Q + \mathcal{B}Q^2 + CQ^3 + \mathcal{D} = 0, \quad (6.5)$$

where

$$\mathcal{A} = -\frac{4a[(C_1k + b\tau)^2 + C_2b\beta^2\mu^2]}{b\beta^2k^2\mu^2},$$

$$\mathcal{B} = \frac{6a(C_1k + b\tau)}{b\beta k^2\mu}, \quad C = -\frac{2a}{bk^2},$$

and

$$\mathcal{D} = \frac{4ab}{\beta k^2\mu}D_0.$$

Here D_0 is the integration constant generated by the final integration of the traveling-wave equation.

Equation (6.5) is an autonomous generalized Duffing-type equation containing linear, quadratic, cubic, and constant terms. In particular,

$$\mathcal{D} = 0 \quad \Longleftrightarrow \quad D_0 = 0$$

is an additional restriction and does not follow from the M -AMTZ system or from the traveling-wave ansatz. Hence $\mathcal{D} = 0$ will be used below only as the special subclass associated with the Riccati-expansion solutions.

Neither α nor η enters the coefficients $\mathcal{A}, \mathcal{B}, C$ and $\mathcal{D}$, as a direct consequence of the coordinate equivalence established in Theorem 3.1. For arbitrary $\eta > 0$, the same reduced equation is obtained with

$$\omega = \frac{\Gamma(\eta + 1)}{\alpha}(kx^\alpha + \mu y^\alpha - \tau t^\alpha). \quad (6.6)$$

Thus, α and η affect the representation of the traveling wave in the original variables but not the intrinsic autonomous dynamics of (6.5).

6.3. First integral and phase-plane structure

To analyze the general case, including $\mathcal{D} \neq 0$, we multiply (6.5) by Q' and integrate once with respect to ω . This gives

$$\frac{1}{2}(Q')^2 + V(Q) = E, \quad (6.7)$$

where $E \in \mathbb{R}$ is an integration constant and

$$V(Q) = \frac{\mathcal{A}}{2}Q^2 + \frac{\mathcal{B}}{3}Q^3 + \frac{C}{4}Q^4 + \mathcal{D}Q$$

is the effective potential. Equivalently,

$$(Q')^2 = P_4(Q), \quad P_4(Q) = 2E - \mathcal{A}Q^2 - \frac{2\mathcal{B}}{3}Q^3 - \frac{C}{2}Q^4 - 2\mathcal{D}Q. \quad (6.8)$$

Thus, the general traveling-wave dynamics is governed by a quartic polynomial.

Introducing $p = Q'$, Eq. (6.5) is equivalent to the planar autonomous system

$$Q' = p, \quad p' = -(\mathcal{A}Q + \mathcal{B}Q^2 + CQ^3 + \mathcal{D}).$$

Its equilibrium points satisfy

$$(Q, p) = (Q_*, 0), \quad F(Q_*) = 0,$$

where

$$F(Q) = \mathcal{A}Q + \mathcal{B}Q^2 + CQ^3 + \mathcal{D}. \quad (6.9)$$

Linearization at $(Q_*, 0)$ gives

$$v^2 = -F'(Q_*), \quad F'(Q_*) = \mathcal{A} + 2\mathcal{B}Q_* + 3CQ_*^2.$$

Hence a simple equilibrium is a center when $F'(Q_*) > 0$ and a saddle when $F'(Q_*) < 0$. The degenerate case $F'(Q_*) = 0$ requires higher-order analysis.

The number of real equilibria is determined by

$$CQ^3 + \mathcal{B}Q^2 + \mathcal{A}Q + \mathcal{D} = 0.$$

For $C \neq 0$, its discriminant is

$$\Delta_{\text{eq}} = 18\mathcal{A}\mathcal{B}C\mathcal{D} - 4\mathcal{B}^3\mathcal{D} + \mathcal{A}^2\mathcal{B}^2 - 4C\mathcal{A}^3 - 27C^2\mathcal{D}^2.$$

Accordingly, $\Delta_{\text{eq}} > 0$ gives three distinct real equilibria, $\Delta_{\text{eq}} < 0$ gives one real equilibrium, and $\Delta_{\text{eq}} = 0$ corresponds to a multiple equilibrium. Together with the energy levels of (6.7), this determines the qualitative phase-plane structure.

As a representative example with $\mathcal{D} \neq 0$, consider

$$\mathcal{A} = -1, \quad \mathcal{B} = 0, \quad C = 1, \quad \mathcal{D} = 0.2.$$

The equilibrium equation becomes

$$Q^3 - Q + 0.2 = 0,$$

with three real roots

$$Q_1 \approx -1.0880, \quad Q_2 \approx 0.2091, \quad Q_3 \approx 0.8789.$$

Since $F'(Q_1) > 0$, $F'(Q_3) > 0$, and $F'(Q_2) < 0$, the two outer equilibria are centers, whereas the middle equilibrium is a saddle. The corresponding phase portrait is shown in Figure 1.

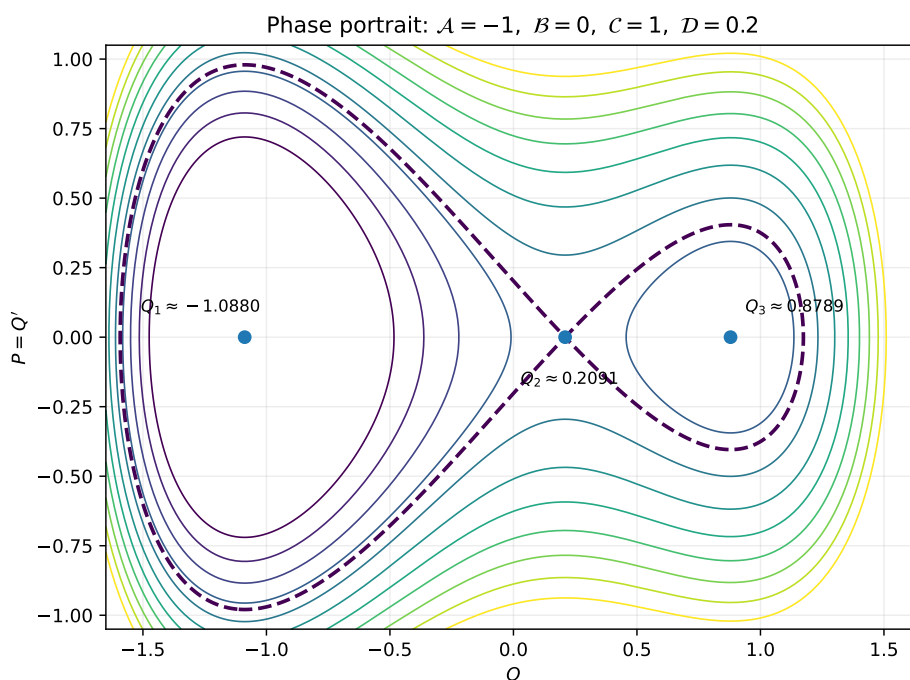


Figure 1. Representative phase portrait of the generalized Duffing-type reduction for $\mathcal{A} = -1$, $\mathcal{B} = 0$, $\mathcal{C} = 1$, and $\mathcal{D} = 0.2$. The two outer equilibrium points are centers, while the middle equilibrium is a saddle. The trajectories are level curves of the first integral, and the dashed level through the saddle represents the separatrix.

6.4. Quartic root structure and elliptic traveling waves

The first integral (6.8) gives the quadrature

$$\omega - \omega_0 = \pm \int_{Q_{\text{ref}}}^Q \frac{ds}{\sqrt{P_4(s)}}, \quad (6.10)$$

on every admissible interval where $P_4(Q) \geq 0$. Hence the traveling-wave orbits are determined by the root structure of the quartic polynomial P_4 .

For a nonsingular quartic with distinct roots, (6.10) is an elliptic integral. An explicit elliptic-function representation can be obtained by choosing a simple root Q_r of P_4 , that is, a point such that $P_4(Q_r) = 0$ and $P'_4(Q_r) \neq 0$. Then the solution of

$$(Q')^2 = P_4(Q)$$

can be written in the Weierstrass form

$$Q(\omega) = Q_r + \frac{P'_4(Q_r)}{4\wp(\omega - \omega_1; g_2, g_3) - \frac{1}{6}P''_4(Q_r)}, \quad (6.11)$$

where ω_1 is an integration constant and

$$g_2 = \frac{(P'_4(Q_r))^2}{48} - \frac{P_4(Q_r)P''_4(Q_r)}{24}, \quad (6.12)$$

$$g_3 = -\frac{(P_4''(Q_r))^3}{1728} + \frac{P_4'(Q_r)P_4''(Q_r)P_4'''(Q_r)}{576} - \frac{(P_4'(Q_r))^2 P_4''''(Q_r)}{384}. \quad (6.13)$$

Indeed, the substitution

$$Q = Q_r + \frac{1}{z}$$

reduces the quartic equation to a cubic equation for z , which can be transformed to the standard Weierstrass equation

$$(\wp')^2 4\wp^3 = g_2\wp - g_3.$$

Thus, when

$$\Delta_\wp = g_2^3 - 27g_3^2 \neq 0,$$

the corresponding traveling wave is genuinely elliptic.

The real form of the solution depends on the number and ordering of the real roots of P_4 . In particular, when four distinct real roots are present, bounded periodic motion may occur between adjacent turning points on intervals where $P_4(Q) > 0$. Other nonsingular root configurations lead to the corresponding elliptic branches.

When $\Delta_\wp = 0$, the elliptic curve degenerates and the quartic possesses repeated roots. Depending on the multiplicity and ordering of these roots, the elliptic representation reduces to elementary hyperbolic, trigonometric, or rational forms. Accordingly, the Riccati representatives considered in the next section should be interpreted as special degenerate cases of the general quartic first-integral structure rather than as the general traveling-wave solution.

The constant term $\mathcal{D}$ enters both the equilibrium polynomial (6.9) and the quartic (6.8). Therefore, imposing $\mathcal{D} = 0$ changes the equilibrium and root structures and defines only a special subclass of the general traveling-wave reduction.

Finally, neither the first integral nor the elliptic invariants depend on α or η . These parameters enter only through the traveling phase (6.6); therefore, they modify the coordinate representation of a given orbit but not its equilibrium, energy, phase-plane, or elliptic structure.

7. Riccati-expansion solutions for the special case $\mathcal{D} = 0$

We now restrict the general traveling-wave equation (6.5) to the special case $\mathcal{D} = 0$. This restriction is not imposed by the M -AMTZ system itself; it selects a special subclass of the general quartic first-integral problem within which particular parameter and energy configurations admit elementary Riccati-based representations. Thus,

$$Q'' + \mathcal{A}Q + \mathcal{B}Q^2 + \mathcal{C}Q^3 = 0. \quad (7.1)$$

7.1. Riccati expansion and compatibility conditions

We consider

$$Q(\omega) = a_0 + a_1\phi(\omega) + \frac{b_1}{\phi(\omega)}, \quad (7.2)$$

where

$$\phi' = \sigma + \phi^2, \quad \sigma \in \{-1, 0, 1\}. \quad (7.3)$$

Substituting (7.2) into (7.1) and equating the coefficients of ϕ^j , $j = -3, \dots, 3$, to zero yields

$$\begin{cases} a_1(Ca_1^2 + 2) = 0, \\ a_1^2(\mathcal{B} + 3Ca_0) = 0, \\ a_1(\mathcal{A} + 2\mathcal{B}a_0 + 3Ca_0^2 + 3Ca_1b_1 + 2\sigma) = 0, \\ \mathcal{A}a_0 + \mathcal{B}a_0^2 + 2\mathcal{B}a_1b_1 + Ca_0^3 + 6Ca_0a_1b_1 = 0, \\ b_1(\mathcal{A} + 2\mathcal{B}a_0 + 3Ca_0^2 + 3Ca_1b_1 + 2\sigma) = 0, \\ b_1^2(\mathcal{B} + 3Ca_0) = 0, \\ b_1(Cb_1^2 + 2\sigma^2) = 0. \end{cases} \quad (7.4)$$

The two-sided case $a_1b_1 \neq 0$ produces additional representations, which we do not pursue here. Since the purpose is not to construct an exhaustive Riccati catalog but to exhibit representative elementary degenerations of the general quartic reduction, we retain the simpler one-sided subclass $b_1 = 0$.

For $a_1 \neq 0$ and $b_1 = 0$, Eq (7.4) gives

$$a_0 = -\frac{\mathcal{B}}{3C}, \quad a_1^2 = -\frac{2}{C}, \quad \mathcal{A} = \frac{\mathcal{B}^2}{3C} - 2\sigma, \quad (7.5)$$

together with

$$\mathcal{B}(9\mathcal{A}C - 2\mathcal{B}^2) = 0.$$

Combining this condition with (7.5) gives

$$\mathcal{B}(\mathcal{B}^2 - 18C\sigma) = 0. \quad (7.6)$$

Thus,

$$\mathcal{B} = 0 \quad \text{or} \quad \mathcal{B}^2 = 18C\sigma.$$

Real-valued nonconstant representatives require

$$C < 0.$$

These reduced parameter conditions are compatible with the original traveling-wave parameters. For the one-sided representatives with $\mathcal{B} = 0$, the expressions for the coefficients of the reduced equation give

$$C_1k + b\tau = 0, \quad \mathcal{A} = -\frac{4aC_2}{k^2}, \quad C = -\frac{2a}{bk^2}. \quad (7.7)$$

Hence $C < 0$ requires $a/b > 0$, while the choices

$$C_2 = -\frac{k^2}{2a}, \quad C_2 = \frac{k^2}{2a}, \quad C_2 = 0$$

yield, respectively,

$$\mathcal{A} = 2, \quad \mathcal{A} = -2, \quad \mathcal{A} = 0.$$

These cases correspond, respectively, to the hyperbolic ($\mathcal{A} = 2$), trigonometric ($\mathcal{A} = -2$), and rational ($\mathcal{A} = 0$) representatives considered below.

7.2. Elementary Riccati representatives

Set

$$z = \omega - \omega_0.$$

Hyperbolic branch. For $\sigma = -1$, take

$$\phi(\omega) = -\tanh z.$$

Equation (7.6) admits the two real alternatives

$$\mathcal{B} = 0 \quad \text{or} \quad \mathcal{B}^2 = -18C.$$

To give a compact representative of the hyperbolic degeneration, we retain the simplest case $\mathcal{B} = 0$, for which

$$\mathcal{A} = 2, \quad a_0 = 0, \quad a_1 = \varepsilon \sqrt{-\frac{2}{C}}, \quad \varepsilon = \pm 1.$$

Hence

$$Q_h(z) = -\varepsilon \sqrt{-\frac{2}{C}} \tanh z. \quad (7.8)$$

This solution is real and nonsingular for all $z \in \mathbb{R}$. Its limiting values

$$Q_h(-\infty) = \varepsilon \sqrt{-\frac{2}{C}}, \quad Q_h(+\infty) = -\varepsilon \sqrt{-\frac{2}{C}}$$

are equilibria of (7.1); hence the profile represents a heteroclinic orbit of the reduced phase-plane system.

Trigonometric branch. For $\sigma = 1$, take

$$\phi(\omega) = \tan z.$$

The second alternative in (7.6) would require $\mathcal{B}^2 = 18C$, which cannot hold for real $\mathcal{B}$ since $C < 0$. Hence the real one-sided branch necessarily has

$$\mathcal{B} = 0, \quad \mathcal{A} = -2.$$

Then

$$a_0 = 0, \quad a_1 = \varepsilon \sqrt{-\frac{2}{C}}, \quad \varepsilon = \pm 1,$$

and therefore

$$Q_{tr}(z) = \varepsilon \sqrt{-\frac{2}{C}} \tan z. \quad (7.9)$$

It is periodic on its intervals of regularity and has poles at

$$z = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}.$$

Rational branch. For $\sigma = 0$, take

$$\phi(\omega) = -\frac{1}{z}.$$

The compatibility equations reduce to

$$\mathcal{A} = \mathcal{B} = 0, \quad a_0 = 0, \quad a_1^2 = -\frac{2}{C},$$

and therefore

$$Q_r(z) = -\varepsilon \sqrt{-\frac{2}{C}} \frac{1}{z}. \quad (7.10)$$

The profile is real for $C < 0$ and has a simple pole at $z = 0$.

The relation of these three representatives to the quartic first integral (6.8) can be made explicit.

For the hyperbolic branch,

$$\mathcal{A} = 2, \quad \mathcal{B} = \mathcal{D} = 0, \quad C < 0,$$

and the solution (7.8) lies on the energy level

$$E = -\frac{1}{C}.$$

Hence

$$P_4(Q) = -\frac{2}{C} - 2Q^2 - \frac{C}{2}Q^4 = -\frac{C}{2}\left(Q^2 + \frac{2}{C}\right)^2. \quad (7.11)$$

Since $C < 0$, the two real roots

$$Q = \pm \sqrt{-\frac{2}{C}}$$

are both double. Thus, the hyperbolic profile is a double-root degeneration of the quartic first integral and gives the heteroclinic orbit connecting these two equilibria.

For the trigonometric branch,

$$\mathcal{A} = -2, \quad \mathcal{B} = \mathcal{D} = 0, \quad C < 0,$$

and (7.9) lies on the energy level

$$E = -\frac{1}{C}.$$

Therefore,

$$P_4(Q) = -\frac{2}{C} + 2Q^2 - \frac{C}{2}Q^4 = -\frac{C}{2}\left(Q^2 - \frac{2}{C}\right)^2. \quad (7.12)$$

Because $C < 0$, the roots

$$Q = \pm i \sqrt{-\frac{2}{C}}$$

form a complex-conjugate pair, each of multiplicity two. This explains the trigonometric degeneration and the occurrence of poles in the corresponding real profile.

For the rational branch,

$$\mathcal{A} = \mathcal{B} = \mathcal{D} = 0, \quad E = 0,$$

so that

$$P_4(Q) = -\frac{C}{2}Q^4. \quad (7.13)$$

Hence $Q = 0$ is a quadruple root, which gives the rational degeneration.

Thus, the three retained representatives correspond, respectively, to two real double roots, a complex-conjugate pair of double roots, and a quadruple root of the quartic polynomial. They therefore describe distinct elementary degenerations of the general quartic first-integral structure and do not exhaust the general traveling-wave solution space described in Subsection 6.4.

7.3. Direct residual verification

As an independent symbolic verification, define the residual of (7.1) by

$$\mathcal{R}[Q](z) = Q'' + \mathcal{A}Q + \mathcal{B}Q^2 + CQ^3.$$

Direct substitution of the three representatives gives

$$\mathcal{R}[Q_h] = Q_h'' + 2Q_h + CQ_h^3 \equiv 0,$$

$$\mathcal{R}[Q_{tr}] = Q_{tr}'' - 2Q_{tr} + CQ_{tr}^3 \equiv 0,$$

and

$$\mathcal{R}[Q_r] = Q_r'' + CQ_r^3 \equiv 0.$$

Thus, all three representatives satisfy the reduced equation identically on their respective domains.

7.4. Interpretation, nonredundancy, and relation to classical AMTZ solutions

The factorizations established in Subsection 7.2 show that the retained representatives correspond to distinct degenerations of the quartic first integral: Two real double roots for the hyperbolic branch, a complex-conjugate pair of double roots for the trigonometric branch, and a quadruple root at $Q = 0$ for the rational branch. Hence they are nonredundant at the level of the quartic root structure.

These elementary profiles are not claimed here as new traveling-wave families of the classical AMTZ system. Hyperbolic, trigonometric, and rational traveling-wave solution types have already been reported in studies of the classical AMTZ equation [10–12]. Their role here is to provide simple representatives of distinct quartic degenerations within the coordinate-equivalence framework of the local M -AMTZ system.

Table 1 summarizes their simplified forms, parameter restrictions, relation to the classical AMTZ literature, and status within the present work.

Table 1. Comparison and non-redundancy of the retained Riccati representatives.

Representative	Simplified form	Parameter conditions	Closest classical result	AMTZ	Status
Hyperbolic	$-\varepsilon \sqrt{-\frac{2}{C}} \tanh z$	$\mathcal{A} = 2, \mathcal{B} = 0, C < 0$	Hyperbolic/soliton families reported in [11, 12]		Not distinct as a local- M family; real double-root degeneration
Trigonometric	$\varepsilon \sqrt{-\frac{2}{C}} \tan z$	$\mathcal{A} = -2, \mathcal{B} = 0, C < 0$	Trigonometric wave families reported in [10, 12]	traveling-	Not distinct as a local- M family; complex double-root degeneration
Rational	$-\varepsilon \sqrt{-\frac{2}{C}} \frac{1}{z}$	$\mathcal{A} = \mathcal{B} = 0, C < 0$	Rational families reported in [10–12]	traveling-wave	Not distinct as a local- M family; quadruple-root degeneration

Finally, since the local M -AMTZ system is related to the classical AMTZ system by the natural-coordinate transformation, the retained profiles are interpreted as coordinate pullbacks of classical traveling-wave profiles rather than as new intrinsic families generated by the local M -parameters. The parameters α and η enter only through the traveling phase (6.6) and therefore do not alter the intrinsic reduced solution families.

8. Role of the parameters α and η : Coordinate reparametrization

The coordinate equivalence established in Theorem 3.1, together with the natural coordinates (3.3), determines the role of the local M -parameters. For arbitrary $\eta > 0$,

$$X = \frac{\Gamma(\eta + 1)}{\alpha} x^\alpha, \quad Y = \frac{\Gamma(\eta + 1)}{\alpha} y^\alpha, \quad T = \frac{\Gamma(\eta + 1)}{\alpha} t^\alpha,$$

and hence

$$D_{M,x}^{\alpha,\eta} = \partial_X, \quad D_{M,y}^{\alpha,\eta} = \partial_Y, \quad D_{M,t}^{\alpha,\eta} = \partial_T.$$

Thus, within the smooth positive-domain setting considered here, α and η act through a reparametrization of the independent variables rather than through a modification of the intrinsic AMTZ dynamics.

8.1. Effect on the traveling-wave dynamics

In the natural coordinates, the traveling-wave phase is

$$\omega = kX + \mu Y - \tau T,$$

whereas in the original variables it is given by (6.6).

Neither α nor η appears explicitly in the coefficients $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ of the reduced equation

$$Q'' + \mathcal{A}Q + \mathcal{B}Q^2 + \mathcal{C}Q^3 + \mathcal{D} = 0.$$

Consequently, for fixed reduced parameters, varying α or η does not alter the equilibrium points, their local stability classification, the phase portrait, the energy levels, the quartic root structure, or the classification of the traveling-wave orbits.

The two parameters affect only the representation of a fixed profile $Q(\omega)$ in the original variables: α produces a nonlinear power-law reparametrization, whereas η contributes the constant scaling factor $\Gamma(\eta + 1)$.

8.2. Coordinate collapse

The coordinate character of the local M -parameters can be illustrated using the hyperbolic representative (7.8),

$$Q_h(z) = -\varepsilon \sqrt{-\frac{2}{C}} \tanh z, \quad z = \omega - \omega_0.$$

In the original variables, this profile is represented as

$$q_{\alpha,\eta}(x, y, t) = Q_h \left[\frac{\Gamma(\eta + 1)}{\alpha} (kx^\alpha + \mu y^\alpha - \tau t^\alpha) - \omega_0 \right]. \quad (8.1)$$

Thus, different values of α and η may produce visibly different curves when the same reduced solution is plotted against the original coordinates, even though the underlying profile Q_h is unchanged.

Figure 2 illustrates this effect. Panel (a) shows the hyperbolic profile in the original coordinate x for several admissible pairs (α, η) , with the remaining variables and reduced parameters fixed. The curves are stretched or compressed because the map from x to the natural coordinate depends on α and η .

In the natural coordinates, however, the same solution has the parameter-independent form

$$q(X, Y, T) = Q_h(kX + \mu Y - \tau T - \omega_0).$$

Consequently, when the corresponding profiles are expressed in terms of the traveling-wave phase ω , they collapse onto the same curve, as shown in Figure 2(b).

This collapse directly illustrates the coordinate nature of the local M -parameters: They enter the reconstruction of the solution through the phase (6.6), but not through the autonomous profile equation itself. The same interpretation applies to the other traveling-wave branches, so additional stretching plots would be redundant.

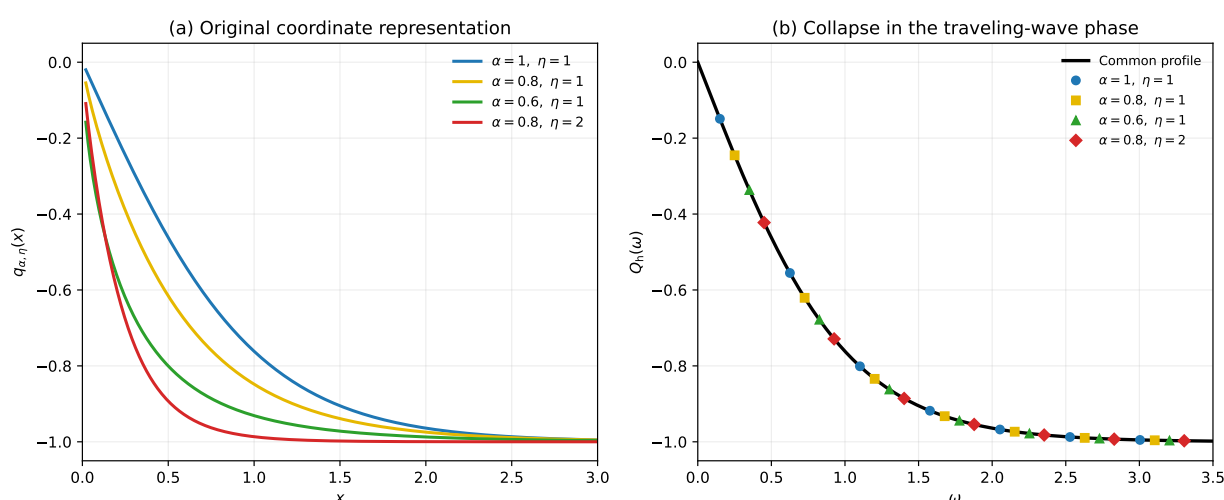


Figure 2. Coordinate-collapse illustration for the hyperbolic traveling-wave representative. (a) In the original coordinate x , different admissible values of α and η produce different coordinate representations of the same reduced profile. (b) When expressed in terms of the traveling-wave phase ω , all parameter choices collapse onto the same curve. The colored markers identify the corresponding parameter pairs.

8.3. Classical limit and interpretation

For fixed $\eta > 0$, the limit $\alpha \rightarrow 1$ gives

$$D_{M,x}^{1,\eta} = \frac{1}{\Gamma(\eta+1)} \partial_x, \quad X = \Gamma(\eta+1)x,$$

with analogous relations for y, Y and t, T . Thus, $\alpha = 1$ with arbitrary $\eta > 0$ corresponds to a constant rescaling of the independent variables rather than to ordinary differentiation with respect to the original coordinates.

The classical AMTZ formulation in the original variables is recovered directly at

$$(\alpha, \eta) = (1, 1),$$

because $\Gamma(2) = 1$. Therefore,

$$X = x, \quad Y = y, \quad T = t, \quad \omega = kx + \mu y - \tau t.$$

Accordingly, different values of α or η applied to a fixed profile $Q(\omega)$ are not counted as distinct intrinsic traveling-wave families. They represent the same reduced solution in different coordinates. In the smooth local setting considered here, the M -parameters do not introduce an additional memory mechanism; their effect is coordinate-based.

Therefore, the novelty of an explicit traveling-wave profile must be assessed at the level of the corresponding classical AMTZ reduction rather than through variation of α or η alone.

9. Conclusions

In this work, we investigated a local M -deformation of the $(2 + 1)$ -dimensional AMTZ system in the full two-parameter setting

$$0 < \alpha \leq 1, \quad \eta > 0.$$

For sufficiently smooth functions on the positive domain, we established an exact coordinate equivalence between the local M -AMTZ system and the classical AMTZ system through the natural variables

$$X = \frac{\Gamma(\eta + 1)}{\alpha} x^\alpha, \quad Y = \frac{\Gamma(\eta + 1)}{\alpha} y^\alpha, \quad T = \frac{\Gamma(\eta + 1)}{\alpha} t^\alpha.$$

This two-parameter equivalence provides the structural basis for the interpretation of the local M -deformation.

In particular, the local M -spectral problem and its zero-curvature condition are coordinate pullbacks of the corresponding classical AMTZ Lax representation. They therefore express inherited Lax integrability rather than an independent fractional integrable structure. Likewise, an explicit conservation law was transported to the original local M -variables, where the associated conserved integral acquires the Jacobian factor generated by the natural-coordinate transformation.

The traveling-wave reduction was analyzed without suppressing the final integration constant $\mathcal{D}$. Consequently, $\mathcal{D} = 0$ appears only as a special subclass of the general reduced equation

$$Q'' + \mathcal{A}Q + \mathcal{B}Q^2 + CQ^3 + \mathcal{D} = 0.$$

Its first integral is governed by a quartic polynomial whose root structure determines the equilibrium configuration, phase-plane dynamics, and traveling-wave behavior. For nonsingular quartics, the corresponding quadrature is elliptic and admits an explicit representation in terms of the Weierstrass elliptic function, whereas repeated-root configurations lead to elementary degenerations.

Within the special case $\mathcal{D} = 0$, representative hyperbolic, trigonometric, and rational solutions were obtained by a Riccati expansion and verified by direct symbolic substitution. Their quartic factorizations show that they correspond, respectively, to two real double roots, a complex-conjugate pair of double roots, and a quadruple root. These profiles are therefore non-redundant as distinct elementary degenerations of the quartic first-integral structure, but they are not claimed as new traveling-wave families of the classical AMTZ system.

The roles of the two local M -parameters are also clarified. Neither α nor η enters the autonomous reduced equation, its phase-plane dynamics, or the associated elliptic structure. The parameter α produces a nonlinear power-law reparametrization of the independent variables, whereas η contributes a constant scaling through $\Gamma(\eta + 1)$. Consequently, varying these parameters changes the coordinate representation and traveling-wave phase but does not by itself generate distinct intrinsic solution families. This coordinate interpretation is illustrated by the collapse of the traveling-wave profiles in Figure 2.

The classical AMTZ system in the original variables is recovered directly at

$$(\alpha, \eta) = (1, 1).$$

For $\alpha = 1$ and arbitrary $\eta > 0$, the local M -operator differs from the ordinary derivative only by the corresponding constant coordinate scaling.

The principal contribution of the present work is therefore the structural characterization of the local M -AMTZ deformation within a unified coordinate-equivalence framework. This framework separates the genuinely established two-parameter coordinate equivalence from the Lax and conservation structures inherited from the classical AMTZ system, while placing the general traveling-wave reduction, its elliptic representation, and its elementary degenerations within the same setting.

Thus, within the smooth positive-domain regime considered here, the local M -deformation preserves the intrinsic AMTZ dynamics while changing its representation through an invertible coordinate transformation. In this sense, its mathematical role is distinct from that of genuinely nonlocal fractional operators, for which the nonlocal kernel may alter the underlying dynamical structure.

Author contributions

Zhaidary Myrzakulova: Conceptualization, methodology, formal analysis, investigation, writing – original draft, writing – review & editing, supervision; Aigul Taishiyeva: Methodology, formal analysis, validation, visualization, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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