



*Research article***Polynomial stability of coupled piezoelectric beam systems via Kilbas-Saigo-Mittag-Leffler operators and cross-coupled delayed feedback****Zahid Ullah^{1,*}, Jianghao Hao¹, Ahmed Bchatnia^{2,*}, Iqra Kanwal¹ and Luqman Bashir¹**¹ School of Mathematics and Statistics, Shanxi University, Taiyuan 030006, China² Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 52571, Saudi Arabia*** Correspondence:** Email: zahidmath1233@gmail.com, a.bchatnia@qu.edu.sa.

Abstract: This paper investigates a coupled piezoelectric beam model that incorporates a Kilbas-Saigo-Mittag-Leffler (KSML) spectral operator, tempered fractional memory, cross-coupled time delays, and logarithmic nonlinearities. The bidirectional delay structure captures the interaction between mechanical and electrical signals as they travel through sensors, amplifiers, and feedback controllers, while the multiscale KSML kernel describes both short-time elastic responses and long-term memory effects in piezoelectric materials. The spectral kernel is shown to be positive, bounded, and self-adjoint with explicit asymptotics; accordingly, the elastic stiffness is decomposed into a classical coercive part and the hereditary KSML correction, and the regularized logarithmic potential is treated through an exact-primitive error identity of order $O(\varepsilon_0)$. Under suitable assumptions on the model parameters and initial data, we establish local well-posedness with a blow-up alternative, prove invariance of the stable potential well and global existence for data satisfying $\mathcal{E}(0) < \mathfrak{d}$ and $\mathcal{I}(0) > 0$, and derive the polynomial upper decay estimate $\mathcal{E}(t) \leq C/(1+t)$ with all constants explicit. Numerical experiments confirm the polynomial energy envelope in the stable regime against an a priori constant computed only from the data, without any fitting to the computed solution. These findings provide a mathematical framework for analyzing more general feedback configurations in piezoelectric systems.

Keywords: partial differential equations; stability; global existence; piezoelectric beams; Kilbas-Saigo-Mittag-Leffler fractional operator; tempered fractional memory; cross-coupled delay; logarithmic sources

Mathematics Subject Classification: 35B40, 35L70, 74F15, 26A33, 93D15

1. Introduction

The mathematical analysis of piezoelectric beam systems has attracted sustained interest because of its direct relevance to smart structures, vibration control, and sensing and actuation. Early contributions focused on linear models with classical damping, establishing well-posedness, stabilization, and observability through semigroup and energy methods. In this direction, Banks et al. [1] provided a foundational treatment of smart-material structures, while Hagood et al. [2] examined actuator dynamics together with hysteretic effects. The theoretical underpinnings of piezoelectricity were subsequently systematized by Yang [3]. Later, Ramos et al. [4, 5] incorporated magnetic effects into piezoelectric beams and proved exponential stability and observability-type results; complementary investigations of Lamé-type dynamics were carried out by Ramos et al. [6]. Related works were carried out by Kong et al. [7] and by Akil et al. [8], who obtained observability and stabilization results under the Lorenz gauge condition. These works constitute an important part of the modern theory, yet most of them rely on classical dissipation and do not simultaneously accommodate hereditary memory, fractional-order operators, or delayed feedback.

Independently, time-delay effects and nonlinear source terms have been studied extensively in evolution equations and beam-type systems. Because delays arise naturally in sensors, amplifiers, controllers, and actuator responses, they can introduce destabilizing effects that must be compensated by suitable damping. Liu et al. [9] analyzed stability issues for neutral delay differential systems, while Dattori da Silva et al. [10] and Zahid al. [11] investigated weakly damped Lamé systems with delay. On the nonlinear side, Yükksekaya et al. [12] established well-posedness and exponential stability for a logarithmic Lamé system with time delay, and related delay phenomena in beams and waves were treated by Feng et al. [13], by Kafini and Messaoudi [14], and by Messaoudi et al. [15]. These studies demonstrate that delay terms can substantially alter the dissipative structure of a system and must therefore be analyzed together with the available damping mechanisms.

Fractional and memory-type damping offer another essential framework for describing hereditary behavior in viscoelastic and smart materials. Messaoudi and Al-Gharabli [16] proved a general stability result for wave equations with infinite memory, while Al-Mahdi [17] obtained stability results for viscoelastic plates carrying logarithmic nonlinearities, and Al-Gharabli, Al-Mahdi, and Messaoudi [18] established general and optimal decay results for a viscoelastic plate equation with a logarithmic nonlinearity. Peyravi [19] investigated the competition between logarithmic sources and memory terms in semi-linear wave equations, and Ullah et al. [20] established exponential stability for fractionally damped piezoelectric beams with time-delayed coupling and a logarithmic source. General stability results for memory-type Timoshenko and viscoelastic systems were obtained by Guesmia and Messaoudi [21] and Mustafa [22]. Related fractional beam and integro-differential models were analyzed in [23] and [24], and the Prabhakar or three-parameter Mittag-Leffler calculus was surveyed by Garra and Garrappa [25]. In the piezoelectric setting, Ullah et al. [26] studied exponential stabilization and blow-up for a fractional thermal piezoelectric beam with direct delay, whereas Ullah et al. [27] analyzed exponential decay for a scalar nonlinear delayed wave equation with logarithmic sources. Nevertheless, the simultaneous influence of a Kilbas-Saigo-Mittag-Leffler spectral operator, tempered fractional memory, cross-coupled delayed feedback, and logarithmic nonlinearities on a coupled piezoelectric beam system remains largely unexplored.

This historical overview reveals a clear gap: No previous work has coupled a Kilbas-Saigo-Mittag-

Leffler spectral operator with tempered fractional memory, cross-coupled time delay, and logarithmic nonlinearities within a single piezoelectric beam model. As Table 1 illustrates, the combination is not merely additive; it poses fundamentally new mathematical challenges. The delayed feedback is non-diagonal, because the delayed electrical velocity enters the mechanical equation while the delayed mechanical velocity enters the electrical equation. Moreover, the nonlocal dissipation generated by the memory variables must be balanced against the logarithmic source terms through an appropriate potential-well framework.

Table 1. Systematic comparison of piezoelectric beam models in the literature.

Reference	Coupled system	Fractional damping	KSML operator	Delay type	Logarithmic source	Decay rate
Ramos et al. [4]	✓	×	×	None	×	Exponential
Ramos et al. [6]	✓	×	×	None	×	Pullback dynamics
Kong et al. [7]	✓	×	×	None	×	Exponential
Akil et al. [8]	✓	×	×	Partial/direct	×	Exponential
Feng et al. [13]	✓(sandwich)	×	×	Direct	×	Exponential
Yüksekkaya et al. [12]	✓(Lamé)	×	×	Direct	✓	Exponential
Messaoudi et al. [15]	×	×	×	Direct	×	Exponential
Al-Mahdi [17]	×	✓(scalar)	×	None	✓(scalar)	General decay
Ullah et al. [26]	✓	✓	×	Direct	×	Exponential/Blow-up
Ullah et al. [27]	×	✓(scalar)	×	Direct	✓(scalar)	Exponential/Blow-up
Present work	✓	✓(coupled)	✓(coupled)	Cross-coupled	✓(coupled)	Polynomial

In this paper, we develop a novel coupled piezoelectric beam system that unifies four previously separate mechanisms, the Kilbas-Saigo-Mittag-Leffler spectral operator, tempered fractional damping, cross-coupled time delay, and logarithmic nonlinearities, into a single mathematical framework. The governing equations describe the transverse displacement $u(y, t)$ and the electric potential $\phi(y, t)$ on the interval $(0, L)$:

$$\begin{cases} mu_{tt} - \alpha_0 \Delta u + \alpha \mathcal{D}_{\rho, \mu, \omega}^\gamma u - c \mathcal{D}_{\rho, \mu, \omega}^\gamma \phi + d_1 u_t + d_2 I_t^{1-\theta, B} \phi_t(y, t - \tau) = |u|^{p-2} u \ln(|u| + \varepsilon_0)^\kappa, \\ \varepsilon \phi_{tt} - P_0 \Delta \phi + P \mathcal{D}_{\rho, \mu, \omega}^\gamma \phi - c \mathcal{D}_{\rho, \mu, \omega}^\gamma u + s_1 \phi_t + s_2 I_t^{1-\theta, B} u_t(y, t - \tau) = |\phi|^{q-2} \phi \ln(|\phi| + \varepsilon_0)^\eta, \end{cases} \quad (1.1)$$

subject to homogeneous Dirichlet boundary conditions

$$u(y, t) = \phi(y, t) = 0, \quad (y, t) \in (\mathbb{R} \setminus (0, L)) \times (0, \infty), \quad (1.2)$$

and initial conditions

$$\begin{cases} u(y, 0) = u_0(y), & u_t(y, 0) = u_1(y), & y \in (0, L), \\ \phi(y, 0) = \phi_0(y), & \phi_t(y, 0) = \phi_1(y), & y \in (0, L), \\ u_t(y, s) = f_0(y, s), & \phi_t(y, s) = g_0(y, s), & (y, s) \in (0, L) \times (-\tau, 0). \end{cases} \quad (1.3)$$

Here $m > 0$ is the mass density, and $\alpha_0, P_0 > 0$ are the classical mechanical and electric stiffness coefficients. The remaining parameters $\varepsilon, \alpha, P, c, d_1, s_1, d_2, s_2 > 0$ represent the electric permittivity, the KSML stiffness moduli, the piezoelectric coupling coefficient, the viscous damping coefficients, and the memory-feedback coefficients. The coercivity condition $\alpha P > c^2$ ensures the positivity of the coupled KSML energy. We emphasize the structural decomposition of the elastic stiffness announced in (1.1): A classical part $\alpha_0(-\Delta)$, $P_0(-\Delta)$ and a hereditary KSML part $\alpha \mathcal{D}_{\rho, \mu, \omega}^\gamma$, $P \mathcal{D}_{\rho, \mu, \omega}^\gamma$. As Lemma 1.1 below shows, the spectral weights of $\mathcal{D}_{\rho, \mu, \omega}^\gamma$ are positive but bounded; the classical part therefore

supplies the coercivity needed by the logarithmic nonlinearity, while the KSML kernel enriches the stiffness with a multi-scale memory contribution. This is also physically natural: A piezoelectric beam possesses an ordinary elastic stiffness, and the KSML operator models the hereditary electromechanical correction generated by its hierarchical microstructure.

The parameters satisfy $\frac{1}{2} \leq \theta < 1$, $B > 0$, and $\tau > 0$, and $0 < \varepsilon_0 \ll 1$ is a fixed regularization parameter introduced to avoid the logarithmic singularity at the origin. The operator $I_t^{1-\theta, B}$ denotes the tempered fractional memory integral

$$I_t^{1-\theta, B} z(t) = \frac{1}{\Gamma(1-\theta)} \int_0^t e^{-B(t-s)} (t-s)^{-\theta} z(s) ds. \quad (1.4)$$

The exponential kernel $e^{-B(t-s)}$ distinguishes tempered fractional calculus from standard Caputo derivatives, modeling materials in which distant memory decays exponentially.

The Kilbas-Saigo-Mittag-Leffler operator $\mathcal{D}_{\rho, \mu}^\gamma$ is defined in terms of the generalized Mittag-Leffler function $E_{\rho, \mu}^\gamma(\omega)$, which for $\rho, \mu, \omega > 0$ and $\gamma \in \mathbb{C}$ is given by

$$E_{\rho, \mu}^\gamma(\omega) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\rho k + \mu)} \frac{\omega^k}{k!}, \quad (\gamma)_k = \gamma(\gamma+1) \cdots (\gamma+k-1), \quad (1.5)$$

with the convention $(\gamma)_0 = 1$. This three-parameter Mittag-Leffler function generalizes the classical $E_\rho(\omega) = E_{\rho, 1}^1(\omega)$ and the two-parameter $E_{\rho, \mu}(\omega) = E_{\rho, \mu}^1(\omega)$, and it provides a flexible spectral kernel for modeling complex viscoelastic and piezoelectric responses [25, 28–32]. The operator $\mathcal{D}_{\rho, \mu, \omega}^\gamma$ acts on a sufficiently smooth function u via the spectral representation

$$\mathcal{D}_{\rho, \mu, \omega}^\gamma u := \sum_{k=1}^{\infty} \lambda_k^{\mu-1} E_{\rho, \mu}^\gamma(-\omega \lambda_k^\rho) \langle u, e_k \rangle_{L^2} e_k, \quad (1.6)$$

where $\{(\lambda_k, e_k)\}_{k=1}^\infty$ are the eigenpairs of the Dirichlet Laplacian on $(0, L)$

$$-\Delta e_k = \lambda_k e_k, \quad e_k \in H_0^1(0, L), \quad 0 < \lambda_1 < \lambda_2 \leq \cdots \rightarrow \infty, \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2. \quad (1.7)$$

The parameters satisfy $\rho \in (0, 1]$, $\mu \in (0, 1]$, $\omega > 0$, and $\gamma > 0$.

Lemma 1.1. *Let $0 < \rho \leq 1$, $\rho \leq \mu \leq 1$, $\omega > 0$, and $\gamma > 0$ and set $m_k := \lambda_k^{\mu-1} E_{\rho, \mu}^\gamma(-\omega \lambda_k^\rho)$. Then*

- (i) (Complete monotonicity for $0 < \gamma \leq 1$.) *If $0 < \gamma \leq 1$, then $E_{\rho, \mu}^\gamma(-x)$ is completely monotone on $[0, \infty)$; in particular $E_{\rho, \mu}^\gamma(-x) > 0$, for every $x \geq 0$ and $m_k > 0$ for all k . This is classical; see Kilbas, Saigo, and Saxena [30], Gorenflo et al. [31], Garra et al. [32], and Garra and Garrappa [25] and the references therein. (The case $\gamma > 1$ is treated separately in Assumption 2.2 below.)*
- (ii) *If $m_k > 0$ for all k , then $\mathcal{D}_{\rho, \mu, \omega}^\gamma$ is a positive self-adjoint operator on $L^2(0, L)$ with $\|\mathcal{D}_{\rho, \mu, \omega}^\gamma\|_{\mathcal{L}(L^2)} = \sup_k m_k < \infty$; hence $(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2}$ is well defined and $\|(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u\|_{L^2}^2 = \sum_k m_k |\langle u, e_k \rangle|^2$.*
- (iii) *As $x \rightarrow +\infty$,*

$$E_{\rho, \mu}^\gamma(-x) = \sum_{n=1}^{N-1} \frac{(-1)^{n+1} (\gamma)_n}{\Gamma(\mu - \rho n)} x^{-n} + O(x^{-N}) \quad \text{for every } N \geq 2, \quad (1.8)$$

so that, whenever $\mu \geq \rho$, $m_k \sim \frac{\gamma}{\omega \Gamma(\mu-\rho)} \lambda_k^{\mu-\rho-1}$ (with the leading term replaced by the first non-vanishing term of (1.8) when $\mu = \rho$). In particular, $m_k > 0$ for all sufficiently large k , $0 < m_k \leq C$ for all k and $m_k \rightarrow 0$ algebraically, which displays the multi-scale nature of the kernel: Algebraic at low frequencies and exponentially tempered at high frequencies.

Remark 1.2. Part (iii) shows that $\mathcal{D}_{\rho,\mu,\omega}^\gamma$ is a bounded operator on $L^2(0, L)$ and that its graph norm controls only a Sobolev norm of non-positive order; consequently, the embedding of the pure graph space into $L^\infty(0, L)$ cannot hold and the logarithmic source terms in (1.1) cannot be treated with the KSML stiffness alone. This is the mathematical reason for the decomposition of the stiffness into a classical part $\alpha_0(-\Delta)$, $P_0(-\Delta)$ and the KSML correction announced in (1.1); the classical part restores coercivity on $H_0^1(0, L)$, and the embedding $H_0^1(0, L) \hookrightarrow L^\infty(0, L)$ in one space dimension then makes the nonlinear terms well defined (see Remark 2.7). All conclusions of the paper remain valid if $\alpha_0 = P_0 = 0$, provided the initial data are chosen in a finite spectral subspace; the classical part is what makes the infinite-dimensional theory uniform.

In real piezoelectric control systems, the physical justification for this model is immediate. The beam bends when voltage is applied, and bending produces an electrical signal in return. In practice, these signals do not travel instantaneously: The mechanical response passes through amplifiers and finite-length circuits, while the electrical signal passes through sensors and processors. This gives rise to cross-delayed feedback: The velocity of the mechanical equation is fed back by the electrical velocity at time $t - \tau$ and vice versa. This two-way delay accurately reflects the physical mechanism of electromechanical coupling and has not previously been included in mathematical models. Moreover, the Kilbas-Saigo-Mittag-Leffler operator captures the multi-scale nature of piezoelectric ceramics, where different grain sizes and domain structures contribute to distinct memory time scales.

We identify three principal contributions. First, the cross-coupled delay structure. In existing delayed models the feedback acts directly within the same component; in the present system, however, the delayed electrical velocity drives the mechanical equation and the delayed mechanical velocity drives the electrical equation. This produces a non-diagonal, non-symmetric feedback matrix that cannot be decoupled by the standard procedures used for direct-delay systems. Even our own earlier work employed direct delay rather than cross-coupled feedback.

Second, the Kilbas-Saigo-Mittag-Leffler spectral operator. All previous works on piezoelectric beams use either the classical Laplacian or the standard fractional Laplacian $(-\Delta)^\gamma$ with a single power-law spectral kernel. We introduce for the first time the three-parameter Mittag-Leffler spectral kernel $\lambda_k^{\mu-1} E_{\rho,\mu}^\gamma(-\omega \lambda_k^\rho)$, which provides a much richer multi-scale description of the electromechanical coupling. At high frequencies (large λ_k) the kernel decays exponentially because of the Mittag-Leffler function, modeling rapid elastic response; at low frequencies (small λ_k) the power-law prefactor captures slow viscoelastic relaxation. This is physically more accurate for modern piezoelectric composites with hierarchical microstructures, in which the hereditary correction is carried by $\mathcal{D}_{\rho,\mu,\omega}^\gamma$ while the principal stiffness remains classical.

Third, polynomial decay. Every existing work on piezoelectric beams establishes exponential stability. We prove that for our coupled system the energy satisfies

$$\mathcal{E}(t) \leq \frac{C}{1+t} \quad \text{for all } t > 0, \quad (1.9)$$

with an explicit constant C depending only on the parameters and the initial data (see Theorem 4.8

and Section 5). The weak superlinear growth of the logarithmic source and the weak dissipation of the tempered fractional derivative combine to produce a genuinely nonlocal dissipative structure. Our three main results are as follows. The augmented system generates a local C_0 -semigroup of contractions on a Hilbert space encoding mechanical, electrical, memory, and delay states, and the nonlinear problem is locally well-posed with a blow-up alternative (Theorem 3.2); for initial data in the potential well of depth $\mathfrak{d}$ with $\mathcal{E}(0) < \mathfrak{d}$ and $\mathcal{I}(0) > 0$, the well is invariant (Lemma 4.5) and solutions exist globally in time (Theorem 4.6); the energy satisfies $\mathcal{E}(t) \leq C/(1+t)$ for $t \geq 0$ with $C > 0$ explicitly computable, yielding a polynomial upper decay estimate (Theorem 4.8). Whether the polynomial rate is optimal for this class of systems is left open.

The numerical validation confirms these predictions. A fully discrete scheme combining spectral discretization of the Kilbas-Saigo-Mittag-Leffler kernel, delay transport update, diffusive-memory quadrature, and coupled algebraic time-stepping reproduces the polynomial decay envelope and verifies it against the a priori constant of Theorem 4.8 without fitting any constant to the computed solution. Unlike previous simulations that confirm exponential decay, our computations show that the discrete energy remains bounded by a polynomial reference envelope.

For the reader's convenience, the physical and mathematical parameters used throughout the paper, together with their admissible ranges, are collected in Table 2.

The paper is organized as follows. Section 2 develops the mathematical preliminaries: Kilbas-Saigo-Mittag-Leffler spectral calculus, tempered fractional calculus, diffusive representation, transport equations, and the augmented local system. Section 3 establishes well-posedness through semigroup generation and fixed-point arguments. Section 4 presents the core stability analysis: Lyapunov functional construction, energy estimates, and polynomial decay. Section 5 details the numerical discretization and simulations. Section 6 concludes with open problems and future directions. The analysis proceeds as follows: Kernel and spaces first, then nonlinearities, then local well-posedness, then the energy–Nehari–potential-well framework, and finally the Lyapunov estimate.

Table 2. Nomenclature of physical and mathematical parameters.

Symbol	Description	Constraint
<i>Physical parameters</i>		
L	Beam length	$L > 0$
m	Mass density	$m > 0$
ε	Electric permittivity	$\varepsilon > 0$
α_0, P_0	Classical stiffness moduli	$\alpha_0, P_0 > 0$
α	Mechanical KSML stiffness	$\alpha > 0$
P	Electric KSML stiffness	$P > 0$
c	Piezoelectric coupling	$\alpha P > c^2$
d_1, s_1	Viscous damping coefficients	$d_1, s_1 > 0$
d_2, s_2	Memory-feedback coefficients	$d_2, s_2 > 0$
<i>Mittag-Leffler parameters</i>		
ρ	Fractional order	$0 < \rho \leq 1$
μ	Fractional parameter	$\rho \leq \mu \leq 1, \mu > 1/2$
ω	Kernel scale parameter	$\omega > 0$
γ	Pochhammer parameter	$\gamma > 0$
<i>Memory and delay parameters</i>		
θ	Tempered fractional order	$1/2 \leq \theta < 1$
B	Tempering parameter	$B > 0$
τ	Time delay	$\tau > 0$
<i>Nonlinearity parameters</i>		
$p, \mathfrak{p}$	Nonlinearity exponents	$p, \mathfrak{p} > 2$ (see Lemma 4.3)
$\kappa, \mathfrak{q}$	Logarithmic exponents	$\kappa, \mathfrak{q} > 0$
ε_0	Regularization parameter	$0 < \varepsilon_0 \ll 1$
<i>Auxiliary parameters</i>		
σ	Delay-weight parameter	$\sigma \in (2\tau\beta_1\mathfrak{U}_0, \tau(2d_1 - \beta_1\mathfrak{U}_0))$
$\mathfrak{U}_0$	Spectral compliance constant	$\mathfrak{U}_0 = B^{\theta-1}$
$\mathfrak{d}$	Potential well depth	$\mathfrak{d} > 0$
C_0	Coercivity constant	$C_0 = \min\{\alpha_0, P_0, (\alpha P - c^2)/(\alpha + P)\}$

2. Preliminaries

The original system (1.1) contains two nonlocal mechanisms: A constant time delay $\tau > 0$ acting on the velocities and a tempered fractional memory operator $I_t^{1-\theta, B}$, with $\frac{1}{2} \leq \theta < 1$ and $B > 0$. To rewrite the problem as a local evolution system amenable to semigroup methods, we introduce delay-transport variables and diffusive memory variables.

To account for the influence of the delay, we define the auxiliary state variables

$$\mathcal{W}(y, r, t) = u_t(y, t - \tau r), \quad y \in (0, L), \quad r \in (0, 1), \quad t > 0, \quad (2.1)$$

$$\mathcal{V}(y, r, t) = \phi_t(y, t - \tau r), \quad y \in (0, L), \quad r \in (0, 1), \quad t > 0. \quad (2.2)$$

Then $\mathcal{W}$ and $\mathcal{V}$ satisfy the transport equations

$$\tau \mathcal{W}_t(y, r, t) + \mathcal{W}_r(y, r, t) = 0, \quad y \in (0, L), \quad r \in (0, 1), \quad t > 0, \quad (2.3)$$

$$\tau \mathcal{V}_t(y, r, t) + \mathcal{V}_r(y, r, t) = 0, \quad y \in (0, L), \quad r \in (0, 1), \quad t > 0, \quad (2.4)$$

together with the coupling conditions at $r = 0$

$$\mathcal{W}(y, 0, t) = u_t(y, t), \quad \mathcal{V}(y, 0, t) = \phi_t(y, t), \quad (2.5)$$

and the initial conditions

$$\mathcal{W}(y, r, 0) = f_0(y, -\tau r), \quad \mathcal{V}(y, r, 0) = g_0(y, -\tau r). \quad (2.6)$$

The past histories f_0, g_0 satisfy the compatibility conditions

$$f_0(y, 0) = u_1(y), \quad g_0(y, 0) = \phi_1(y), \quad y \in (0, L). \quad (2.7)$$

For the tempered fractional memory operator, we use the diffusive representation introduced in [26, 33]. Let the weight function be

$$\omega_\theta(\xi) = \sqrt{\frac{\sin(\theta\pi)}{\pi}} |\xi|^{\theta-1/2}, \quad \xi \in \mathbb{R}, \quad (2.8)$$

which is even and belongs to $L^1_{\text{loc}}(\mathbb{R})$, but not to $L^2(\mathbb{R})$ because $|\xi|^{2\theta-1}$ is not integrable at infinity. We introduce two auxiliary memory variables $\psi_1(y, \xi, t)$ and $\psi_2(y, \xi, t)$, governed by the first-order ODEs

$$\begin{aligned} \partial_t \psi_1(y, \xi, t) + (\xi^2 + B) \psi_1(y, \xi, t) &= \mathcal{V}(y, 1, t) \omega_\theta(\xi), \\ \partial_t \psi_2(y, \xi, t) + (\xi^2 + B) \psi_2(y, \xi, t) &= \mathcal{W}(y, 1, t) \omega_\theta(\xi), \end{aligned} \quad (2.9)$$

with zero initial conditions

$$\psi_1(y, \xi, 0) = \psi_2(y, \xi, 0) = 0. \quad (2.10)$$

The tempering parameter is strictly positive: $B > 0$.

Remark 2.1. The naive L^2 -norm $\int_{\mathbb{R}} \omega_\theta^2(\xi) d\xi$ diverges for every $\theta \in (0, 1)$, due to the behavior of $|\xi|^{2\theta-1}$ as $|\xi| \rightarrow \infty$. What actually appears in the energy estimates is the spectral compliance constant

$$\mathfrak{A}_0 := \int_{\mathbb{R}} \frac{\omega_\theta^2(\xi)}{\xi^2 + B} d\xi = B^{\theta-1} < \infty, \quad (2.11)$$

which is finite precisely when $B > 0$. The identity follows from the Laplace-domain formula

$$\int_{\mathbb{R}} \frac{\omega_\theta^2(\xi)}{\xi^2 + B + s} d\xi = (B + s)^{\theta-1}, \quad \Re(s) > -B, \quad (2.12)$$

verified by direct computation using the substitution $u = \xi^2$ and the standard beta-integral identity

$$\int_0^\infty \frac{u^{\nu-1}}{u + a} du = \frac{\pi}{\sin(\nu\pi)} a^{\nu-1}, \quad 0 < \nu < 1, \quad a > 0. \quad (2.13)$$

Lemma 2.2. For a smooth function $V(y, t)$, define $\psi(y, \xi, t)$ by

$$\partial_t \psi(y, \xi, t) + (\xi^2 + B)\psi(y, \xi, t) = V(y, t) \omega_\theta(\xi), \quad \psi(y, \xi, 0) = 0. \quad (2.14)$$

Then

$$\int_{\mathbb{R}} \psi(y, \xi, t) \omega_\theta(\xi) d\xi = I_t^{1-\theta, B} V(y, t), \quad (2.15)$$

where $I_t^{1-\theta, B}$ is the tempered fractional integral of order $1 - \theta$ defined by (1.4).

Proof. Solving the first-order equation (2.14) by the variation of constants formula gives

$$\psi(y, \xi, t) = \int_0^t e^{-(\xi^2 + B)(t-s)} V(y, s) \omega_\theta(\xi) ds. \quad (2.16)$$

Substituting this expression into the left-hand side of (2.15) and using Fubini's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \psi(y, \xi, t) \omega_\theta(\xi) d\xi &= \int_0^t V(y, s) e^{-B(t-s)} \int_{\mathbb{R}} e^{-\xi^2(t-s)} \omega_\theta^2(\xi) d\xi ds \\ &= \int_0^t V(y, s) e^{-B(t-s)} \frac{\sin(\theta\pi)}{\pi} \int_{\mathbb{R}} e^{-\xi^2(t-s)} |\xi|^{2\theta-1} d\xi ds. \end{aligned}$$

The inner integral satisfies

$$\int_{\mathbb{R}} e^{-\xi^2(t-s)} |\xi|^{2\theta-1} d\xi = \frac{\Gamma(\theta)}{(t-s)^\theta}, \quad (2.17)$$

which follows from the substitution $u = \xi^2(t-s)$ and the definition of the gamma function. Hence, using the reflection formula $\Gamma(\theta)\Gamma(1-\theta) = \pi / \sin(\theta\pi)$, we get

$$\int_{\mathbb{R}} \psi(y, \xi, t) \omega_\theta(\xi) d\xi = \frac{1}{\Gamma(1-\theta)} \int_0^t \frac{e^{-B(t-s)}}{(t-s)^\theta} V(y, s) ds = I_t^{1-\theta, B} V(y, t). \quad (2.18)$$

This completes the proof. $\square$

Applying Lemma 2.2 with $V(y, t) = \mathcal{V}(y, 1, t) = \phi_t(y, t - \tau)$ and $V(y, t) = \mathcal{W}(y, 1, t) = u_t(y, t - \tau)$, we obtain the identities

$$\int_{\mathbb{R}} \psi_1(y, \xi, t) \omega_\theta(\xi) d\xi = I_t^{1-\theta, B} \phi_t(y, t - \tau), \quad (2.19)$$

and

$$\int_{\mathbb{R}} \psi_2(y, \xi, t) \omega_\theta(\xi) d\xi = I_t^{1-\theta, B} u_t(y, t - \tau). \quad (2.20)$$

Hence, the tempered memory terms appearing in (1.1) admit the diffusive representations

$$d_2 I_t^{1-\theta, B} \phi_t(y, t - \tau) = \beta_1 \int_{\mathbb{R}} \psi_1(y, \xi, t) \omega_\theta(\xi) d\xi, \quad s_2 I_t^{1-\theta, B} u_t(y, t - \tau) = \beta_2 \int_{\mathbb{R}} \psi_2(y, \xi, t) \omega_\theta(\xi) d\xi, \quad (2.21)$$

where the constants β_1, β_2 are defined by

$$\beta_1 = d_2, \quad \beta_2 = s_2. \quad (2.22)$$

Consequently, the nonlocal memory feedback in (1.1) is converted into local-in-time diffusive variables ψ_1 and ψ_2 . A genuine tempered Caputo derivative $D_t^{\theta, B}$ would require an additional time derivative of the diffusive integral and would lead to a different augmented system (see [26] for comparison).

To guarantee the energy dissipation property, the instantaneous viscous damping must dominate the delayed memory feedback.

Assumption 2.1. The viscous damping coefficients $d_1, s_1 > 0$, the memory coefficients $d_2, s_2 > 0$, and the delay time $\tau > 0$ satisfy

$$3\beta_1\mathfrak{U}_0 < 2d_1, \quad 3\beta_2\mathfrak{U}_0 < 2s_1, \quad (2.23)$$

with β_1, β_2 given by (2.22) and $\mathfrak{U}_0$ by (2.11).

Remark 2.3. Inequalities (2.23) guarantee the existence of a weight parameter $\sigma > 0$ lying in the non-empty intersection

$$\sigma \in (2\tau\beta_1\mathfrak{U}_0, \tau(2d_1 - \beta_1\mathfrak{U}_0)) \cap (2\tau\beta_2\mathfrak{U}_0, \tau(2s_1 - \beta_2\mathfrak{U}_0)). \quad (2.24)$$

Indeed, (2.23) implies

$$2\tau\beta_1\mathfrak{U}_0 < \tau(2d_1 - \beta_1\mathfrak{U}_0), \quad 2\tau\beta_2\mathfrak{U}_0 < \tau(2s_1 - \beta_2\mathfrak{U}_0),$$

so both admissible intervals are non-empty. The parameter σ enters the energy inner product and balances the contribution of the delay-transport energy against the instantaneous viscous dissipation.

Assumption 2.2. The KSML parameters satisfy

$$0 < \rho \leq 1, \quad \rho \leq \mu \leq 1, \quad \mu > \frac{1}{2}, \quad \omega > 0, \quad \gamma > 0, \quad (2.25)$$

and, if $\gamma > 1$, it is additionally required that $E_{\rho,\mu}^\gamma(-x) > 0$ for every $x \geq 0$. For $0 < \gamma \leq 1$ this additional requirement is automatic by the complete monotonicity of Lemma 1.1(i). Under these conditions every spectral weight $m_k = \lambda_k^{\mu-1} E_{\rho,\mu}^\gamma(-\omega\lambda_k^\rho)$ is strictly positive and $\mathcal{D}_{\rho,\mu,\omega}^\gamma$ is a bounded positive self-adjoint operator on $L^2(0, L)$ by Lemma 1.1(ii).

The coupling terms $-c \mathcal{D}_{\rho,\mu,\omega}^\gamma \phi$ in the u -equation and $-c \mathcal{D}_{\rho,\mu,\omega}^\gamma u$ in the ϕ -equation arise from the piezoelectric constitutive relations.

Assumption 2.3. The physical parameters satisfy

$$m > 0, \quad \varepsilon > 0, \quad \alpha > 0, \quad P > 0, \quad c > 0, \quad (2.26)$$

where m is the mass density, ε is the electric permittivity, α is the mechanical KSML stiffness, P is the electric KSML stiffness, and c is the piezoelectric coupling coefficient.

The fundamental coercivity condition ensuring positive-definiteness of the coupled elastic energy is

$$\boxed{\alpha P > c^2} \quad (2.27)$$

together with $\alpha_0 > 0$, $P_0 > 0$ for the classical part; the global coercivity constant is

$$C_0 = \min\left\{\alpha_0, P_0, \frac{\alpha P - c^2}{\alpha + P}\right\} > 0. \quad (2.28)$$

The coupled elastic energy is defined by

$$\mathcal{E}_{\text{coupled}}(t) = \alpha_0 \|u_y\|_{L^2}^2 + P_0 \|\phi_y\|_{L^2}^2 + \alpha \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u\|_{L^2}^2 + P \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi\|_{L^2}^2 - 2c \int_0^L (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi \, dy. \quad (2.29)$$

Using the fractional integration-by-parts identity

$$\int_0^L \mathcal{D}_{\rho,\mu,\omega}^\gamma u \phi \, dy = \int_0^L (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi \, dy = \int_0^L u \mathcal{D}_{\rho,\mu,\omega}^\gamma \phi \, dy, \quad (2.30)$$

valid for $u, \phi \in H_0^{\rho,\mu,\omega}(0, L)$, and the eigenvalue bound of Lemma 1.1, the coupled energy satisfies the following lemma.

Lemma 2.4. *Under conditions (2.27) and (2.28), the coupled energy satisfies*

$$\mathcal{E}_{\text{coupled}}(t) \geq C_0 \left(\|u_y\|_{L^2}^2 + \|\phi_y\|_{L^2}^2 + \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u\|_{L^2}^2 + \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi\|_{L^2}^2 \right), \quad (2.31)$$

with $C_0 > 0$ given by (2.28).

Proof. Consider the quadratic form associated with the symmetric matrix $\begin{pmatrix} \alpha & -c \\ -c & P \end{pmatrix}$. Its eigenvalues are

$$\lambda_{\pm} = \frac{\alpha + P \pm \sqrt{(\alpha - P)^2 + 4c^2}}{2}.$$

Since $\alpha, P > 0$ and $\alpha P > c^2$, both eigenvalues are positive and $\lambda_{\min} \geq (\alpha P - c^2)/(\alpha + P)$. Adding the classical part $\alpha_0 \|u_y\|_2^2 + P_0 \|\phi_y\|_2^2$ yields (2.31) with C_0 given by (2.28). $\square$

Remark 2.5. Alternatively, completing the square yields

$$\mathcal{E}_{\text{coupled}}(t) \geq \frac{\alpha P - c^2}{\alpha} \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi\|_{L^2}^2 + \alpha_0 \|u_y\|_2^2 + P_0 \|\phi_y\|_2^2, \quad (2.32)$$

or equivalently with the roles of u and ϕ exchanged. Both expressions confirm the coercivity under $\alpha P > c^2$.

In this paper, the Kilbas-Saigo-Mittag-Leffler operator $\mathcal{D}_{\rho,\mu,\omega}^\gamma$ is understood in the spectral Dirichlet sense on $(0, L)$ through the single definition (1.6), which is used in every section. Let $\{(\lambda_k, e_k)\}_{k=1}^\infty$ be the eigenpairs (1.7). Then, for $u = \sum_{k=1}^\infty u_k e_k$,

$$\mathcal{D}_{\rho,\mu,\omega}^\gamma u := \sum_{k=1}^\infty \lambda_k^{\mu-1} E_{\rho,\mu}^\gamma(-\omega \lambda_k^\rho) u_k e_k, \quad (2.33)$$

whose properties are collected in Lemma 1.1.

Remark 2.6. The three-parameter Mittag-Leffler function $E_{\rho,\mu}^\gamma(z)$ enjoys the following properties [28–31].

For $\gamma = 1$, it reduces to the two-parameter Mittag-Leffler function $E_{\rho,\mu}^1(z) = E_{\rho,\mu}(z)$. For $\gamma = \mu = 1$, it reduces to the classical Mittag-Leffler function $E_{\rho,1}^1(z) = E_\rho(z)$. For $\gamma = 1, z < 0, 0 < \rho \leq 1$, and $\mu \geq \rho$, the function $E_{\rho,\mu}(z)$ is completely monotone [26]. For $0 < \gamma \leq 1$, complete monotonicity of $E_{\rho,\mu}^\gamma(-x)$ on $[0, \infty)$ holds (see [30–32]); for $\gamma > 1$ the positivity of $E_{\rho,\mu}^\gamma(-\omega \lambda_k^\rho)$ is imposed in Assumption 2.2. In all cases the positivity ensures that $E_{\rho,\mu}^\gamma(-\omega \lambda_k^\rho) > 0$ for all k , which justifies the use of the fractional power $(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}$ in the energy functional.

The space $H_0^{\rho,\mu,\omega}(0, L)$ is now defined rigorously by

$$H_0^{\rho,\mu,\omega}(0, L) := H_0^1(0, L), \quad \|u\|_{H_0^{\rho,\mu,\omega}}^2 := \alpha_0 \|u_y\|_{L^2}^2 + \alpha \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u\|_{L^2}^2 + \|u\|_{L^2}^2, \quad (2.34)$$

which is a Hilbert space with a norm equivalent to the standard H_0^1 -norm, because the weights m_k are bounded (see Lemma 1.1(iii)).

Remark 2.7. For the one-dimensional domain $(0, L)$, the classical embedding $H_0^1(0, L) \hookrightarrow L^\infty(0, L)$ holds with $\|u\|_{L^\infty} \leq \sqrt{L} \|u_y\|_{L^2}$ (see Adams-Fournier [34]). Consequently, the nonlinearities $|u|^{p-2} u \ln(|u| + \varepsilon_0)^\kappa$ are well-defined for any $p \in [2, \infty)$ and the Poincaré inequality $\|u\|_{L^2} \leq \lambda_p \|u_y\|_{L^2}$, with $\lambda_p := \lambda_1^{-1/2} = L/\pi$, is available. This should be contrasted with the higher-dimensional case $n \geq 2$, where the admissible range of p depends on the spatial dimension through the critical Sobolev exponent, and with the pure KSML graph space, for which no such embedding exists (see Remark 1.2).

The homogeneous boundary condition is understood in the classical trace sense, namely

$$u(\cdot, t), \phi(\cdot, t) \in H_0^1(0, L), \quad t \geq 0. \quad (2.35)$$

In particular, when $\rho = 1, \mu = 1, \gamma = 1$, and $\omega \rightarrow 0^+$, the KSML correction vanishes and (2.35) reduces to the classical condition

$$u(0, t) = u(L, t) = \phi(0, t) = \phi(L, t) = 0, \quad t \geq 0. \quad (2.36)$$

The conditions imposed on the physical constants, the piezoelectric coupling, the classical stiffness and the KSML parameters are summarized in Table 3.

Table 3. Summary of conditions for piezoelectric coupling terms.

Condition	Formula	Purpose
Positivity	$m, \varepsilon, \alpha, P, c > 0$	Physical material constants
Coercivity	$\alpha P > c^2$	Positive definite coupled KSML energy
Classical stiffness	$\alpha_0 > 0, P_0 > 0$	Coercivity of the principal part
Coercivity constant	$C_0 = \min\{\alpha_0, P_0, (\alpha P - c^2)/(\alpha + P)\}$	Lower bound in energy estimate
Resolvent smallness	$(a_{12} + a_{21})^2 < a_{11} a_{22}$	Coercivity of the elliptic resolvent system
Mittag-Leffler parameters	$0 < \rho \leq 1, \rho \leq \mu \leq 1, \mu > \frac{1}{2}, \omega > 0, \gamma > 0$	Kernel positivity and embedding
Boundary	$u, \phi \in H_0^1(0, L)$	Classical Dirichlet condition

Assumption 2.4. The constants $a_{11} = m + d_1$, $a_{22} = \varepsilon + s_1$, $a_{12} = \beta_1 e^{-\tau} \mathfrak{C}$, and $a_{21} = \beta_2 e^{-\tau} \mathfrak{C}$, with $\mathfrak{C} = (1 + B)^{\theta-1}$ given by (2.12) with $s = 1$, satisfy

$$(a_{12} + a_{21})^2 < a_{11} a_{22}. \quad (2.37)$$

This condition is verified explicitly for the numerical data of Section 5 (see Table 6).

Bringing together the delay-transport formulation and the diffusive representation of the tempered memory operator, the original nonlocal problem (1.1) is equivalently rewritten as the following local

augmented system:

$$\begin{cases} mu_{tt} - \alpha_0 \Delta u + \alpha \mathcal{D}_{\rho,\mu,\omega}^\gamma u - c \mathcal{D}_{\rho,\mu,\omega}^\gamma \phi + d_1 u_t + \beta_1 \int_{\mathbb{R}} \psi_1(y, \xi, t) \omega_\theta(\xi) d\xi = |u|^{p-2} u \ln(|u| + \varepsilon_0)^\kappa, \\ \varepsilon \phi_{tt} - P_0 \Delta \phi + P \mathcal{D}_{\rho,\mu,\omega}^\gamma \phi - c \mathcal{D}_{\rho,\mu,\omega}^\gamma u + s_1 \phi_t + \beta_2 \int_{\mathbb{R}} \psi_2(y, \xi, t) \omega_\theta(\xi) d\xi = |\phi|^{p-2} \phi \ln(|\phi| + \varepsilon_0)^\eta, \\ \partial_t \psi_1 + (\xi^2 + B) \psi_1 - \mathcal{V}(y, 1, t) \omega_\theta(\xi) = 0, \\ \partial_t \psi_2 + (\xi^2 + B) \psi_2 - \mathcal{W}(y, 1, t) \omega_\theta(\xi) = 0, \\ \tau \mathcal{W}_t + \mathcal{W}_r = 0, \\ \tau \mathcal{V}_t + \mathcal{V}_r = 0. \end{cases} \quad (2.38)$$

The system is supplemented with the boundary condition (2.35), the coupling conditions at $r = 0$ given by (2.5), and the initial conditions

$$\begin{cases} u(y, 0) = u_0(y), & u_t(y, 0) = u_1(y), \\ \phi(y, 0) = \phi_0(y), & \phi_t(y, 0) = \phi_1(y), \\ \psi_1(y, \xi, 0) = 0, & \psi_2(y, \xi, 0) = 0, \\ \mathcal{W}(y, r, 0) = f_0(y, -\tau r), & \mathcal{V}(y, r, 0) = g_0(y, -\tau r). \end{cases} \quad (2.39)$$

Here β_1, β_2 are defined by (2.22) and $\omega_\theta(\xi)$ by (2.8).

Remark 2.8. The augmented system (2.38) is written for the tempered fractional memory feedback $I_t^{1-\theta, B}$ acting on the delayed velocities. Hence the terms

$$\beta_i \int_{\mathbb{R}} \psi_i(y, \xi, t) \omega_\theta(\xi) d\xi, \quad i = 1, 2,$$

appear directly in the two evolution equations, consistently with Lemma 2.2. A genuine tempered Caputo derivative $D_t^{\theta, B}$ would require an additional time derivative of the diffusive integral and would lead to a different augmented system.

Remark 2.9. The cross-coupled delay mechanism is encoded in the following way. In the mechanical equation (u -equation), the memory term $\beta_1 \int \psi_1 \omega_\theta d\xi$ represents the delayed electric velocity $\phi_t(y, t - \tau)$ acting as feedback on the mechanical motion. In the electric equation (ϕ -equation), the memory term $\beta_2 \int \psi_2 \omega_\theta d\xi$ represents the delayed mechanical velocity $u_t(y, t - \tau)$ acting as feedback on the electric potential. This two-way coupling is fundamentally different from the direct delay structures considered in [26, 27], where the delay appears only in the same equation. The cross-coupling gives rise to a non-diagonal, non-symmetric feedback structure that cannot be decoupled by standard methods.

3. Well-posedness

Throughout this section Assumptions 2.1, 2.2 and 2.4 are in force. By introducing the notations

$$v = \dot{u}, \quad q = \dot{\phi}, \quad \text{and} \quad U = (u, v, \phi, q, \mathcal{W}, \mathcal{V}, \psi_1, \psi_2)^T, \quad \partial_t U = (\dot{u}, v_t, \dot{\phi}, q_t, \mathcal{W}_t, \mathcal{V}_t, \psi_{1,t}, \psi_{2,t})^T,$$

the system (2.38) can be expressed as

$$\begin{cases} \partial_t U = \mathcal{A}U + \mathcal{F}(U), \\ U(0) = U_0 = (u_0, v_0, \phi_0, q_0, f_0, g_0, 0, 0)^T, \end{cases} \quad (3.1)$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is characterized by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{1}{m} \left\{ \alpha_0 \Delta u - \alpha \mathcal{D}_{\rho, \mu, \omega}^\gamma u + c \mathcal{D}_{\rho, \mu, \omega}^\gamma \phi - d_1 v - \beta_1 \int_{\mathbb{R}} \psi_1 \omega_\theta d\xi \right\} \\ q \\ \frac{1}{\varepsilon} \left\{ P_0 \Delta \phi - P \mathcal{D}_{\rho, \mu, \omega}^\gamma \phi + c \mathcal{D}_{\rho, \mu, \omega}^\gamma u - s_1 q - \beta_2 \int_{\mathbb{R}} \psi_2 \omega_\theta d\xi \right\} \\ -\frac{1}{\tau} \mathcal{W}_r \\ -\frac{1}{\tau} \mathcal{V}_r \\ -(\xi^2 + B)\psi_1 + \mathcal{V}(y, 1) \omega_\theta(\xi) \\ -(\xi^2 + B)\psi_2 + \mathcal{W}(y, 1) \omega_\theta(\xi) \end{pmatrix}, \quad (3.2)$$

and the nonlinear operator $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$ is given by

$$\mathcal{F}(U) = \left(0, \frac{1}{m} |u|^{p-2} u \ln(|u| + \varepsilon_0)^\kappa, 0, \frac{1}{\varepsilon} |\phi|^{p-2} \phi \ln(|\phi| + \varepsilon_0)^\eta, 0, 0, 0, 0 \right)^T. \quad (3.3)$$

We define the Hilbert space

$$\begin{aligned} \mathcal{H} = & H_0^{\rho, \mu, \omega}(0, L) \times L^2(0, L) \times H_0^{\rho, \mu, \omega}(0, L) \times L^2(0, L) \times L^2((0, L) \times (0, 1)) \times L^2((0, L) \times (0, 1)) \\ & \times L^2((0, L) \times \mathbb{R}) \times L^2((0, L) \times \mathbb{R}), \end{aligned} \quad (3.4)$$

and equip it with the inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} = & \alpha_0 \int_0^L u_y \tilde{u}_y dy + P_0 \int_0^L \phi_y \tilde{\phi}_y dy + \alpha \int_0^L (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{u} dy \\ & + P \int_0^L (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{\phi} dy - c \int_0^L ((\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{\phi} + (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{u}) dy \\ & + m \int_0^L v \tilde{v} dy + \varepsilon \int_0^L q \tilde{q} dy + \sigma \int_0^L \int_0^1 \{ \mathcal{W}(y, r) \tilde{\mathcal{W}}(y, r) + \mathcal{V}(y, r) \tilde{\mathcal{V}}(y, r) \} dr dy \\ & + \beta_1 \int_0^L \int_{\mathbb{R}} \psi_1(y, \xi) \tilde{\psi}_1(y, \xi) d\xi dy + \beta_2 \int_0^L \int_{\mathbb{R}} \psi_2(y, \xi) \tilde{\psi}_2(y, \xi) d\xi dy. \end{aligned} \quad (3.5)$$

By Lemma 2.4, $\langle U, U \rangle_{\mathcal{H}} \geq 0$ and $\langle U, U \rangle_{\mathcal{H}} = 0$ forces $u = \phi = 0$, $v = q = 0$, $\mathcal{W} = \mathcal{V} = 0$, and $\psi_1 = \psi_2 = 0$ since all weights m_k and all parameters are positive; hence, (3.5) is a Hilbert inner product, equivalent on the first four components to the $H_0^1 \times L^2 \times H_0^1 \times L^2$ norm.

Lemma 3.1. Define the weighted memory space

$$\mathcal{M} := \{\psi \in L^2_{\text{loc}}((0, L) \times \mathbb{R}) : (\xi^2 + B)^{1/2} \psi \in L^2((0, L) \times \mathbb{R})\},$$

equipped with the norm $\|\psi\|_{\mathcal{M}}^2 := \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi|^2 d\xi dy$, which is a Hilbert space. For $i = 1, 2$, the observation operator $\Lambda_i \psi := \int_{\mathbb{R}} \psi(\cdot, \xi) \omega_\theta(\xi) d\xi$ is continuous from $\mathcal{M}$ into $L^2(0, L)$, with

$$\|\Lambda_i \psi\|_{L^2(0, L)}^2 \leq \mathfrak{A}_0 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi|^2 d\xi dy. \quad (3.6)$$

Proof. $\mathcal{M}$ is a Hilbert space because $\psi \mapsto (\xi^2 + B)^{1/2} \psi$ is an isometry of $\mathcal{M}$ onto the closed subspace of L^2 -functions that are evenly weighted; equivalently, $\mathcal{M}$ is the domain of the multiplication operator $(\xi^2 + B)^{1/2}$, which is self-adjoint on L^2 . By the Cauchy-Schwarz inequality in ξ and Remark 2.1,

$$\begin{aligned} \int_0^L \left| \int_{\mathbb{R}} \psi \omega_\theta d\xi \right|^2 dy &\leq \int_0^L \left(\int_{\mathbb{R}} (\xi^2 + B) |\psi|^2 d\xi \right) \left(\int_{\mathbb{R}} \frac{\omega_\theta^2}{\xi^2 + B} d\xi \right) dy \\ &= \mathfrak{A}_0 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi|^2 d\xi dy. \end{aligned}$$

□

We now specify the domain of the operator $\mathcal{A}$ as follows

$$\begin{aligned} D(\mathcal{A}) = \{ &(u, v, \phi, q, \mathcal{W}, \mathcal{V}, \psi_1, \psi_2) \in \mathcal{H} : u, \phi \in H^2(0, L) \cap H_0^1(0, L), \\ &v, q \in H_0^1(0, L), \quad \mathcal{W}, \mathcal{V} \in L^2((0, L) \times (0, 1)), \quad \mathcal{W}_r, \mathcal{V}_r \in L^2((0, L) \times (0, 1)), \\ &\psi_1, \psi_2 \in L^2((0, L) \times \mathbb{R}), \quad (\xi^2 + B)^{1/2} \psi_1, (\xi^2 + B)^{1/2} \psi_2 \in L^2((0, L) \times \mathbb{R}), \\ &\mathcal{W}(y, 0) = v(y), \quad \mathcal{V}(y, 0) = q(y)\}. \end{aligned} \quad (3.7)$$

The compatibility conditions

$$\mathcal{W}(y, 0) = v(y), \quad \mathcal{V}(y, 0) = q(y) \quad (3.8)$$

are included in $D(\mathcal{A})$ so that the abstract formulation remains equivalent to the original delayed system. Moreover, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Theorem 3.2. Let $U_0 \in \mathcal{H}$. Then there exists a maximal time $T_{\max} \in (0, \infty]$ and a unique mild solution $U \in C([0, T_{\max}), \mathcal{H})$ of (3.1), with the blow-up alternative

$$T_{\max} < \infty \implies \limsup_{t \rightarrow T_{\max}^-} \|U(t)\|_{\mathcal{H}} = \infty. \quad (3.9)$$

Furthermore, if $U_0 \in D(\mathcal{A})$, the solution is strong: $U \in C([0, T_{\max}), D(\mathcal{A})) \cap C^1([0, T_{\max}), \mathcal{H})$.

Proof. The proof is divided into three steps: Dissipativity of $\mathcal{A}$, surjectivity of $I - \mathcal{A}$, and local Lipschitz continuity of $\mathcal{F}$.

Dissipativity of $\mathcal{A}$. Let $U = (u, v, \phi, q, \mathcal{W}, \mathcal{V}, \psi_1, \psi_2)^T \in D(\mathcal{A})$. When we compute $\langle \mathcal{A}U, U \rangle_{\mathcal{H}}$ with the inner product (3.5), the classical stiffness terms cancel pairwise by integration by parts with

the Dirichlet boundary conditions, $\alpha_0 \int_0^L (v_y u_y + \Delta u v) dy = 0$ and $P_0 \int_0^L (q_y \phi_y + \Delta \phi q) dy = 0$; the KSML stiffness terms cancel by self-adjointness of $\mathcal{D}_{\rho, \mu, \omega}^\gamma$ (Lemma 1.1); and the piezoelectric cross terms cancel in pairs. The remaining terms give

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = & -d_1 \int_0^L |v|^2 dy - s_1 \int_0^L |q|^2 dy - \frac{\sigma}{2\tau} \int_0^L [|\mathcal{W}(y, 1, t)|^2 + |\mathcal{V}(y, 1, t)|^2] dy \\ & + \frac{\sigma}{2\tau} \int_0^L [|v|^2 + |q|^2] dy - \beta_1 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy - \beta_2 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy \\ & + \beta_1 \int_0^L \mathcal{V}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta \psi_1 d\xi \right) dy - \beta_1 \int_0^L v \left(\int_{\mathbb{R}} \psi_1 \omega_\theta d\xi \right) dy \\ & + \beta_2 \int_0^L \mathcal{W}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta \psi_2 d\xi \right) dy - \beta_2 \int_0^L q \left(\int_{\mathbb{R}} \psi_2 \omega_\theta d\xi \right) dy, \end{aligned} \quad (3.10)$$

where we used integration by parts for the delay terms,

$$-\frac{\sigma}{\tau} \int_0^L \int_0^1 \{ \mathcal{W}_r \mathcal{W} + \mathcal{V}_r \mathcal{V} \} dr dy = -\frac{\sigma}{2\tau} \int_0^L \{ |\mathcal{W}(y, 1, t)|^2 + |\mathcal{V}(y, 1, t)|^2 - |v|^2 - |q|^2 \} dy. \quad (3.11)$$

Applying Young's inequality with the spectral compliance bound $\mathfrak{A}_0 = B^{\theta-1}$ (Remark 2.1 and Lemma 3.1), we obtain

$$\beta_1 \int_0^L \mathcal{V}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta(\xi) \psi_1 d\xi \right) dy \leq \beta_1 \mathfrak{A}_0 \int_0^L |\mathcal{V}(y, 1, t)|^2 dy + \frac{\beta_1}{4} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy, \quad (3.12)$$

$$\beta_1 \int_0^L v \left(\int_{\mathbb{R}} \psi_1 \omega_\theta(\xi) d\xi \right) dy \leq \frac{\beta_1 \mathfrak{A}_0}{2} \int_0^L |v|^2 dy + \frac{\beta_1}{2} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy, \quad (3.13)$$

$$\beta_2 \int_0^L \mathcal{W}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta(\xi) \psi_2 d\xi \right) dy \leq \beta_2 \mathfrak{A}_0 \int_0^L |\mathcal{W}(y, 1, t)|^2 dy + \frac{\beta_2}{4} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy, \quad (3.14)$$

$$\beta_2 \int_0^L q \left(\int_{\mathbb{R}} \psi_2 \omega_\theta(\xi) d\xi \right) dy \leq \frac{\beta_2 \mathfrak{A}_0}{2} \int_0^L |q|^2 dy + \frac{\beta_2}{2} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy. \quad (3.15)$$

Combining (3.12)–(3.15) into (3.10), we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq & -\left(d_1 - \frac{\sigma}{2\tau} - \frac{\beta_1 \mathfrak{A}_0}{2}\right) \|v\|_2^2 - \left(s_1 - \frac{\sigma}{2\tau} - \frac{\beta_2 \mathfrak{A}_0}{2}\right) \|q\|_2^2 \\ & - \left(\frac{\sigma}{2\tau} - \beta_2 \mathfrak{A}_0\right) \|\mathcal{W}(\cdot, 1, t)\|_2^2 - \left(\frac{\sigma}{2\tau} - \beta_1 \mathfrak{A}_0\right) \|\mathcal{V}(\cdot, 1, t)\|_2^2 \\ & - \frac{\beta_1}{4} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy - \frac{\beta_2}{4} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy. \end{aligned} \quad (3.16)$$

By the admissibility condition (2.24) on σ , all coefficients in (3.16) are negative. Therefore, $\mathcal{A}$ is dissipative.

Surjectivity of $I - \mathcal{A}$. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, we show that there exists a unique $U \in D(\mathcal{A})$ such that $(I - \mathcal{A})U = F$. The first, third, fifth, sixth, seventh, and eighth rows give

$$v = u - f_1, \quad q = \phi - f_3, \quad (3.17)$$

$$\begin{aligned}\mathcal{W}(y, r) &= v(y)e^{-\tau r} + \tau e^{-\tau r} \int_0^r f_5(y, \sigma)e^{\tau\sigma} d\sigma, \\ \mathcal{V}(y, r) &= q(y)e^{-\tau r} + \tau e^{-\tau r} \int_0^r f_6(y, \sigma)e^{\tau\sigma} d\sigma,\end{aligned}\quad (3.18)$$

$$\mathcal{W}(y, 1) = v(y)e^{-\tau} + \tau e^{-\tau} \int_0^1 f_5(y, \sigma)e^{\tau\sigma} d\sigma, \quad \mathcal{V}(y, 1) = q(y)e^{-\tau} + \tau e^{-\tau} \int_0^1 f_6(y, \sigma)e^{\tau\sigma} d\sigma, \quad (3.19)$$

$$\psi_1(y, \xi) = \frac{f_7(y, \xi) + \mathcal{V}(y, 1)\omega_\theta(\xi)}{1 + \xi^2 + B}, \quad \psi_2(y, \xi) = \frac{f_8(y, \xi) + \mathcal{W}(y, 1)\omega_\theta(\xi)}{1 + \xi^2 + B}. \quad (3.20)$$

Substituting into the second and fourth rows and using the shorthand

$$\mathfrak{C} := \int_{\mathbb{R}} \frac{\omega_\theta^2(\xi)}{1 + \xi^2 + B} d\xi = (1 + B)^{\theta-1}, \quad (3.21)$$

the last identity following from (2.12) with $s = 1$, we obtain the coupled elliptic system

$$\begin{cases} -\alpha_0 \Delta u + \alpha \mathcal{D}_{\rho, \mu, \omega}^\gamma u - c \mathcal{D}_{\rho, \mu, \omega}^\gamma \phi + a_{11}u + a_{12}\phi = \mathcal{S}_1, \\ -P_0 \Delta \phi - c \mathcal{D}_{\rho, \mu, \omega}^\gamma u + P \mathcal{D}_{\rho, \mu, \omega}^\gamma \phi + a_{21}u + a_{22}\phi = \mathcal{S}_2, \end{cases} \quad (3.22)$$

where

$$\begin{aligned}a_{11} &= m + d_1, & a_{12} &= \beta_1 e^{-\tau} \mathfrak{C}, \\ a_{21} &= \beta_2 e^{-\tau} \mathfrak{C}, & a_{22} &= \varepsilon + s_1,\end{aligned}\quad (3.23)$$

and

$$\begin{aligned}\mathcal{S}_1(y) &= m f_2 + (m + d_1) f_1 + \beta_1 e^{-\tau} \mathfrak{C} f_3 - \beta_1 \int_{\mathbb{R}} \frac{f_7(y, \xi) \omega_\theta(\xi)}{1 + \xi^2 + B} d\xi - \beta_1 \tau e^{-\tau} \mathfrak{C} \int_0^1 f_6(y, \sigma) e^{\tau\sigma} d\sigma, \\ \mathcal{S}_2(y) &= \varepsilon f_4 + (\varepsilon + s_1) f_3 + \beta_2 e^{-\tau} \mathfrak{C} f_1 - \beta_2 \int_{\mathbb{R}} \frac{f_8(y, \xi) \omega_\theta(\xi)}{1 + \xi^2 + B} d\xi - \beta_2 \tau e^{-\tau} \mathfrak{C} \int_0^1 f_5(y, \sigma) e^{\tau\sigma} d\sigma.\end{aligned}\quad (3.24)$$

Multiplying the two equations of (3.22) by $\tilde{u}, \tilde{\phi} \in H_0^1(0, L)$ and integrating by parts, we obtain the variational problem

$$\mathcal{B}((u, \phi), (\tilde{u}, \tilde{\phi})) = \mathcal{L}(\tilde{u}, \tilde{\phi}) \quad \forall (\tilde{u}, \tilde{\phi}) \in H_0^1(0, L) \times H_0^1(0, L), \quad (3.25)$$

where

$$\begin{aligned}\mathcal{B}((u, \phi), (\tilde{u}, \tilde{\phi})) &= \alpha_0 \int_0^L u_y \tilde{u}_y dy + P_0 \int_0^L \phi_y \tilde{\phi}_y dy + \alpha \langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{u} \rangle \\ &\quad + P \langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{\phi} \rangle - c \left(\langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{\phi} \rangle + \langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \tilde{u} \rangle \right) \\ &\quad + a_{11} \int_0^L u \tilde{u} dy + a_{22} \int_0^L \phi \tilde{\phi} dy + a_{12} \int_0^L \phi \tilde{u} dy + a_{21} \int_0^L u \tilde{\phi} dy,\end{aligned}\quad (3.26)$$

and

$$\mathcal{L}(\tilde{u}, \tilde{\phi}) = \int_0^L \mathcal{S}_1 \tilde{u} dy + \int_0^L \mathcal{S}_2 \tilde{\phi} dy. \quad (3.27)$$

The bilinear form $\mathcal{B}$ is continuous on $H_0^1(0, L)^2$. To verify coercivity, we compute

$$\begin{aligned} \mathcal{B}((u, \phi), (u, \phi)) &= \alpha_0 \|u_y\|_2^2 + P_0 \|\phi_y\|_2^2 + \alpha \|(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u\|_2^2 + P \|(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi\|_2^2 \\ &\quad - 2c \langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi \rangle + a_{11} \|u\|_2^2 + a_{22} \|\phi\|_2^2 + (a_{12} + a_{21}) \int_0^L u \phi \, dy. \end{aligned} \quad (3.28)$$

The KSML block is non-negative by Assumption 2.2; dropping it and using Young's inequality,

$$(a_{12} + a_{21}) \int_0^L u \phi \, dy \geq -\frac{a_{11}}{2} \|u\|_{L^2}^2 - \frac{(a_{12} + a_{21})^2}{2a_{11}} \|\phi\|_{L^2}^2,$$

and similarly with the roles of u and ϕ exchanged. Hence, by Assumption 2.4, there exists a constant $\mu > 0$ such that

$$\mathcal{B}((u, \phi), (u, \phi)) \geq \mu \|(u, \phi)\|_{\mathcal{H}_0}^2, \quad (3.29)$$

where $\mathcal{H}_0 = H_0^1(0, L) \times H_0^1(0, L)$ equipped with the norm

$$\|(u, \phi)\|_{\mathcal{H}_0} = \left(\|u_y\|_{L^2}^2 + \|\phi_y\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\phi\|_{L^2}^2 \right)^{\frac{1}{2}}. \quad (3.30)$$

Using the Lax-Milgram theorem, the variational problem (3.25) admits a unique solution

$$(u, \phi) \in H_0^1(0, L) \times H_0^1(0, L). \quad (3.31)$$

The remaining variables are determined as follows:

- From (3.17): $v = u - f_1 \in H_0^1(0, L)$, $q = \phi - f_3 \in H_0^1(0, L)$;
- From (3.18): $\mathcal{W}, \mathcal{V} \in L^2((0, L) \times (0, 1))$ with $\mathcal{W}_r, \mathcal{V}_r \in L^2((0, L) \times (0, 1))$;
- From (3.20): $\psi_1, \psi_2 \in L^2((0, L) \times \mathbb{R})$ with $(\xi^2 + B)^{1/2} \psi_1, (\xi^2 + B)^{1/2} \psi_2 \in L^2((0, L) \times \mathbb{R})$.

Finally, the elliptic system (3.22) has the diagonal principal part $\text{diag}(-\alpha_0 \Delta, -P_0 \Delta)$; since $\mathcal{S}_1, \mathcal{S}_2 \in L^2(0, L)$ (which follows from (3.24), Lemma 3.1, and the Cauchy-Schwarz inequality) and the KSML terms are bounded operators on L^2 (Lemma 1.1), classical elliptic regularity gives

$$u, \phi \in H^2(0, L) \cap H_0^1(0, L). \quad (3.32)$$

Consequently, $v = u - f_1$ and $q = \phi - f_3$ belong to $H_0^1(0, L)$. Hence $U = (u, v, \phi, q, \mathcal{W}, \mathcal{V}, \psi_1, \psi_2)^T \in D(\mathcal{A})$, $I - \mathcal{A}$ is surjective, and $\mathcal{A}$ is maximal dissipative. By the Lumer-Phillips theorem, $\mathcal{A}$ generates a C_0 -semigroup of contractions on $\mathcal{H}$.

Local Lipschitz continuity of $\mathcal{F}$. With

$$\mathcal{F}(U) = \left(0, \frac{1}{m} |u|^{p-2} u \ln(|u| + \varepsilon_0)^\kappa, 0, \frac{1}{\varepsilon} |\phi|^{p-2} \phi \ln(|\phi| + \varepsilon_0)^\eta, 0, 0, 0, 0 \right)^T, \quad (3.33)$$

and

$$h(s) = |s|^{p-2} s \ln(|s| + \varepsilon_0)^\kappa, \quad g(s) = |s|^{p-2} s \ln(|s| + \varepsilon_0)^\eta, \quad s \in \mathbb{R}, \quad (3.34)$$

note that $h(0) = g(0) = 0$ thanks to $\varepsilon_0 > 0$. For $p, q > 2$, $h, g \in C^1(\mathbb{R})$ with

$$\begin{aligned} h'(s) &= (p-1)|s|^{p-2} \ln(|s| + \varepsilon_0)^\kappa + \kappa|s|^{p-2} \frac{|s|}{|s| + \varepsilon_0} \ln(|s| + \varepsilon_0)^{\kappa-1}, \\ g'(s) &= (q-1)|s|^{q-2} \ln(|s| + \varepsilon_0)^\eta + \eta|s|^{q-2} \frac{|s|}{|s| + \varepsilon_0} \ln(|s| + \varepsilon_0)^{\eta-1}, \end{aligned} \quad (3.35)$$

so h', g' are bounded on bounded intervals: For every $R > 0$ there exist $C_R, \tilde{C}_R > 0$ such that

$$|h(a) - h(b)| \leq C_R|a - b|, \quad |g(a) - g(b)| \leq \tilde{C}_R|a - b|, \quad |a| \leq R, \quad |b| \leq R. \quad (3.36)$$

Let $U, \bar{U} \in \mathcal{H}$ with position components bounded by R in $H_0^1(0, L) \hookrightarrow L^\infty(0, L)$ (Remark 2.7). Using (3.36),

$$\begin{aligned} \|\mathcal{F}(U) - \mathcal{F}(\bar{U})\|_{\mathcal{H}} &\leq C_m \|h(u) - h(\bar{u})\|_{L^2(0, L)} + C_\varepsilon \|g(\phi) - g(\bar{\phi})\|_{L^2(0, L)} \\ &\leq C_m C_R \|u - \bar{u}\|_{L^2(0, L)} + C_\varepsilon \tilde{C}_R \|\phi - \bar{\phi}\|_{L^2(0, L)} \leq C_R \|U - \bar{U}\|_{\mathcal{H}}, \end{aligned} \quad (3.37)$$

$\mathcal{F}$ is locally Lipschitz on bounded sets of $\mathcal{H}$. Consequently, the nonlinear well-posedness follows from the standard perturbation theorem for semi-linear evolution equations [35], which yields the maximal existence time $T_{\max}$ and the blow-up alternative (3.9). $\square$

Remark 3.3. Theorem 3.2 is a local statement; global existence requires the potential-well assumption and is stated separately in Section 4 as Theorem 4.6, after the invariance Lemma 4.5.

We now introduce the energy functional corresponding to the augmented system (2.38) as follows

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \int_0^L \left\{ mu_t^2 + \varepsilon \phi_t^2 + \alpha_0 |u_y|^2 + P_0 |\phi_y|^2 + \alpha |(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u|^2 + P |(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi|^2 \right\} dy \\ &\quad - c \int_0^L (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi dy + \frac{1}{2} \int_0^L \int_{\mathbb{R}} \left\{ \beta_1 |\psi_1(y, \xi, t)|^2 + \beta_2 |\psi_2(y, \xi, t)|^2 \right\} d\xi dy \\ &\quad + \frac{\sigma}{2} \int_0^L \int_0^1 \left\{ |\mathcal{W}(y, r, t)|^2 + |\mathcal{V}(y, r, t)|^2 \right\} dr dy - \frac{1}{p} \int_0^L |u|^p \ln(|u| + \varepsilon_0)^\kappa dy - \frac{1}{q} \int_0^L |\phi|^q \ln(|\phi| + \varepsilon_0)^\eta dy \\ &\quad + \frac{\kappa}{p^2} \|u\|_{L^p}^p + \frac{\eta}{q^2} \|\phi\|_{L^q}^q. \end{aligned} \quad (3.38)$$

Lemma 3.4. For $u > 0$ the exact primitive F of $h(u) = u^{p-1} \ln(u + \varepsilon_0)^\kappa$ is

$$F(u) = \frac{1}{p} u^p \ln(u + \varepsilon_0)^\kappa - \frac{\kappa}{p} \int_0^u s^{p-1} \frac{s}{s + \varepsilon_0} \ln(s + \varepsilon_0)^{\kappa-1} ds, \quad (3.39)$$

extended oddly to $u < 0$; the same recursion defines the exact primitive G of g . Writing

$$\Psi(u) = \frac{1}{p} |u|^p \ln(|u| + \varepsilon_0)^\kappa - \frac{\kappa}{p^2} |u|^p, \quad \Phi(\phi) = \frac{1}{p} |\phi|^p \ln(|\phi| + \varepsilon_0)^\eta - \frac{\eta}{p^2} |\phi|^p,$$

there is a constant $C = C(p, \kappa)$ such that for all $u \in \mathbb{R}$,

$$|F(u) - \Psi(u)| \leq C\varepsilon_0(1 + |u|^p), \quad |G(\phi) - \Phi(\phi)| \leq C\varepsilon_0(1 + |\phi|^p). \quad (3.40)$$

Consequently, along any solution of (2.38),

$$\frac{d}{dt} \int_0^L \Psi(u) dy = \int_0^L h(u) u_t dy + \mathcal{R}_u(t), \quad |\mathcal{R}_u(t)| \leq C\varepsilon_0 \int_0^L (1 + |u|^{p-1}) |u_t| dy, \quad (3.41)$$

and similarly for ϕ . Since ε_0 is arbitrarily small, the remainder $\mathcal{R}_u$ is absorbed by the viscous dissipation in Lemma 3.6; in particular, the energy identity and the Nehari functional below are exact up to an $O(\varepsilon_0)$ perturbation, uniformly on bounded sets of $H_0^1(0, L)$.

Proof. Differentiating (3.39) with respect to u gives $F'(u) = h(u)$ exactly. A direct computation shows

$$\Psi'(u) - h(u) = -\frac{\kappa\varepsilon_0}{p(u + \varepsilon_0)} u^{p-1} \ln(u + \varepsilon_0)^{\kappa-1} \quad (u > 0), \quad (3.42)$$

hence $|\Psi'(u) - h(u)| \leq C\varepsilon_0(1 + |u|^{p-1})$ and integrating in u gives (3.40). Finally,

$$\frac{d}{dt} \int_0^L \Psi(u) dy = \int_0^L \Psi'(u) u_t dy = \int_0^L h(u) u_t dy + \int_0^L (\Psi'(u) - h(u)) u_t dy,$$

and the last integrand is bounded by $C\varepsilon_0(1 + |u|^{p-1})|u_t|$, which yields (3.41). $\square$

Remark 3.5. The potential energy terms in (3.38) are

$$\mathcal{P}_u(u) = \frac{1}{p} |u|^p \ln(|u| + \varepsilon_0)^\kappa - \frac{\kappa}{p^2} |u|^p, \quad \text{and} \quad \mathcal{P}_\phi(\phi) = \frac{1}{p} |\phi|^p \ln(|\phi| + \varepsilon_0)^q - \frac{q}{p^2} |\phi|^p. \quad (3.43)$$

These are not exact primitives of the regularized sources for $\varepsilon_0 > 0$; Lemma 3.4 quantifies the discrepancy and shows it is harmless. For $\varepsilon_0 = 0$ they are exact primitives of the sources $|u|^{p-2} u \ln |u|^\kappa$ and $|\phi|^{p-2} \phi \ln |\phi|^q$ when the logarithmic power is read accordingly, and the standard potential-well framework remains valid [17, 34].

The time derivative of $\mathcal{E}(t)$ gives the following estimate for energy dissipation.

Lemma 3.6. *Let $(u, \phi, \mathcal{W}, \mathcal{V}, \psi_1, \psi_2)$ denote a solution to the augmented system (2.38). Then the energy evolves as*

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) \leq & -C \int_0^L \left\{ |u_t|^2 + |\phi_t|^2 + |\mathcal{W}(y, 1, t)|^2 + |\mathcal{V}(y, 1, t)|^2 \right. \\ & \left. + \int_{\mathbb{R}} (\xi^2 + B) [\beta_1 |\psi_1(y, \xi, t)|^2 + \beta_2 |\psi_2(y, \xi, t)|^2] d\xi \right\} dy \leq 0, \end{aligned} \quad (3.44)$$

for some constant $C > 0$, indicating monotonic decay of energy over time.

Proof. Multiplying (2.38)₁ and (2.38)₂ by u_t and ϕ_t , respectively, and integrating over $(0, L)$, we obtain

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int_0^L (mu_t^2 + \varepsilon\phi_t^2) dy \right\} + \frac{d}{dt} \left\{ \frac{1}{2} \int_0^L [\alpha_0 |u_y|^2 + P_0 |\phi_y|^2 + \alpha |(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u|^2 \right. \\ & \quad \left. + P |(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi|^2 - 2c(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi] dy \right\} \\ & + \frac{d}{dt} \left\{ -\frac{1}{p} \int_0^L |u|^p \ln(|u| + \varepsilon_0)^\kappa dy - \frac{1}{p} \int_0^L |\phi|^p \ln(|\phi| + \varepsilon_0)^q dy + \frac{\kappa}{p^2} \|u\|_{L^p}^p + \frac{q}{p^2} \|\phi\|_{L^p}^p \right\} \\ & = -d_1 \int_0^L u_t^2 dy - s_1 \int_0^L \phi_t^2 dy - \beta_1 \int_0^L u_t \int_{\mathbb{R}} \psi_1(y, \xi, t) \omega_\theta(\xi) d\xi dy - \beta_2 \int_0^L \phi_t \int_{\mathbb{R}} \psi_2(y, \xi, t) \omega_\theta(\xi) d\xi dy. \end{aligned} \quad (3.45)$$

The piezoelectric terms combine to exactly give the time derivative of the coupled elastic energy by the self-adjointness of $\mathcal{D}_{\rho,\mu,\omega}^\gamma$ (Lemma 1.1) and the product rule. By Lemma 3.4, the potential terms satisfy (3.41); since ε_0 is at our disposal and the remainder is pointwise $\mathcal{O}(\varepsilon_0)$, it is absorbed into the viscous dissipation $-d_1\|u_t\|_2^2 - s_1\|\phi_t\|_2^2$ by choosing ε_0 small and is silently included in the constant C below.

Multiplying Eqs (2.38)₃ and (2.38)₄ by $\beta_1\psi_1$ and $\beta_2\psi_2$, respectively and integrating over $(0, L) \times \mathbb{R}$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L \int_{\mathbb{R}} (\beta_1 |\psi_1|^2 + \beta_2 |\psi_2|^2) d\xi dy &= -\beta_1 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy + \beta_1 \int_0^L \mathcal{V}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta \psi_1 d\xi \right) dy \\ &\quad - \beta_2 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy + \beta_2 \int_0^L \mathcal{W}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta \psi_2 d\xi \right) dy. \end{aligned} \quad (3.46)$$

Multiplying (2.38)₅ and (2.38)₆ by $\frac{\sigma}{2}\mathcal{W}(y, r, t)$ and $\frac{\sigma}{2}\mathcal{V}(y, r, t)$, respectively, then integrating over $(0, L) \times (0, 1)$,

$$\frac{d}{dt} \left\{ \frac{\sigma}{2} \int_0^L \int_0^1 (|\mathcal{W}|^2 + |\mathcal{V}|^2) dr dy \right\} = -\frac{\sigma}{2\tau} \int_0^L \left\{ |\mathcal{W}(y, 1, t)|^2 - |u_t|^2 + |\mathcal{V}(y, 1, t)|^2 - |\phi_t|^2 \right\} dy. \quad (3.47)$$

Adding (3.45), (3.46), and (3.47), we obtain the total energy derivative:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &= -d_1 \int_0^L u_t^2 dy - s_1 \int_0^L \phi_t^2 dy - \beta_1 \int_0^L u_t \int_{\mathbb{R}} \psi_1 \omega_\theta(\xi) d\xi dy - \beta_2 \int_0^L \phi_t \int_{\mathbb{R}} \psi_2 \omega_\theta(\xi) d\xi dy \\ &\quad - \beta_1 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy + \beta_1 \int_0^L \mathcal{V}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta \psi_1 d\xi \right) dy \\ &\quad - \beta_2 \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy + \beta_2 \int_0^L \mathcal{W}(y, 1, t) \left(\int_{\mathbb{R}} \omega_\theta \psi_2 d\xi \right) dy \\ &\quad - \frac{\sigma}{2\tau} \int_0^L \left\{ |\mathcal{W}(y, 1, t)|^2 - |u_t|^2 + |\mathcal{V}(y, 1, t)|^2 - |\phi_t|^2 \right\} dy. \end{aligned} \quad (3.48)$$

The four coupling terms are estimated by Young's inequality and the spectral compliance bound (3.6). Substituting (3.12)–(3.15) into (3.48) and collecting terms, we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq - \left(d_1 - \frac{\sigma}{2\tau} - \frac{\beta_1 \mathfrak{A}_0}{2} \right) \|u_t\|_2^2 - \left(s_1 - \frac{\sigma}{2\tau} - \frac{\beta_2 \mathfrak{A}_0}{2} \right) \|\phi_t\|_2^2 \\ &\quad - \left(\frac{\sigma}{2\tau} - \beta_2 \mathfrak{A}_0 \right) \|\mathcal{W}(\cdot, 1, t)\|_2^2 - \left(\frac{\sigma}{2\tau} - \beta_1 \mathfrak{A}_0 \right) \|\mathcal{V}(\cdot, 1, t)\|_2^2 \\ &\quad - \frac{\beta_1}{4} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy - \frac{\beta_2}{4} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_2|^2 d\xi dy. \end{aligned} \quad (3.49)$$

By the admissibility conditions (2.24), all coefficients in (3.49) are positive, which we verify explicitly: $\sigma > 2\tau\beta_i\mathfrak{A}_0$ gives $\frac{\sigma}{2\tau} - \beta_i\mathfrak{A}_0 > 0$; $\sigma < \tau(2d_1 - \beta_1\mathfrak{A}_0)$ gives $d_1 - \frac{\sigma}{2\tau} - \frac{\beta_1\mathfrak{A}_0}{2} > \frac{\beta_1\mathfrak{A}_0}{2} > 0$; similarly for s_1 ; and $\frac{\beta_i}{4} > 0$. This is the complete verification that the damping assumptions imply the positivity of every dissipation coefficient. Defining

$$C = \min \left\{ d_1 - \frac{\sigma}{2\tau} - \frac{\beta_1 \mathfrak{A}_0}{2}, s_1 - \frac{\sigma}{2\tau} - \frac{\beta_2 \mathfrak{A}_0}{2}, \frac{\sigma}{2\tau} - \beta_2 \mathfrak{A}_0, \frac{\sigma}{2\tau} - \beta_1 \mathfrak{A}_0, \frac{\beta_1}{4}, \frac{\beta_2}{4} \right\} > 0, \quad (3.50)$$

we obtain the dissipation inequality (3.44). The proof is complete. $\square$

Remark 3.7. Lemma 3.6 is valid as long as the solution exists and, in particular, does not require the potential-well assumption; therefore $\mathcal{E}(t) \leq \mathcal{E}(0)$ for every $t \in [0, T_{\max})$.

4. Stability analysis

In this section, we establish a polynomial decay estimate for the energy associated with the augmented system (2.38). To this end, we first present several auxiliary lemmas that build toward the main decay theorem.

Remark 4.1. Under the conditions $\alpha P > c^2$ and (2.28), the elastic energy is positive definite (Lemma 2.4) and the elastic terms control the natural H_0^1 norm and the KSML norm.

The potential energy is defined by

$$\begin{aligned} \mathcal{J}(t) = & \frac{1}{2} \int_0^L \left\{ \alpha_0 |u_y|^2 + P_0 |\phi_y|^2 + \alpha |(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u|^2 + P |(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi|^2 - 2c (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi \right\} dy \\ & + \frac{1}{2} \int_0^L \int_{\mathbb{R}} (\beta_1 |\psi_1|^2 + \beta_2 |\psi_2|^2) d\xi dy + \frac{\sigma}{2} \int_0^L \int_0^1 (|\mathcal{W}|^2 + |\mathcal{V}|^2) dr dy \\ & - \frac{1}{p} \int_0^L |u|^p \ln(|u| + \varepsilon_0)^k dy - \frac{1}{p} \int_0^L |\phi|^p \ln(|\phi| + \varepsilon_0)^q dy + \frac{\kappa}{p^2} \int_0^L |u|^p dy + \frac{q}{p^2} \int_0^L |\phi|^p dy, \end{aligned} \quad (4.1)$$

and the Nehari functional by

$$\begin{aligned} \mathcal{I}(t) = & \alpha_0 \|u_y\|_2^2 + P_0 \|\phi_y\|_2^2 + \alpha \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u\|_{L^2}^2 + P \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi\|_{L^2}^2 - 2c \int_0^L (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi dy \\ & + \int_0^L \int_{\mathbb{R}} (\beta_1 |\psi_1|^2 + \beta_2 |\psi_2|^2) d\xi dy + \sigma \int_0^L \int_0^1 (|\mathcal{W}|^2 + |\mathcal{V}|^2) dr dy \\ & - \int_0^L |u|^p \ln(|u| + \varepsilon_0)^k dy - \int_0^L |\phi|^p \ln(|\phi| + \varepsilon_0)^q dy. \end{aligned} \quad (4.2)$$

By (4.2), we define the potential well stable set

$$\mathcal{W}_* = \left\{ (u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}) \in \mathcal{H} \mid \mathcal{I}(t) > 0 \right\} \cup \{0\}, \quad (4.3)$$

the outer space of the potential well unstable set

$$\mathcal{V}_* = \left\{ (u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}) \in \mathcal{H} \mid \mathcal{I}(t) < 0 \right\}, \quad (4.4)$$

and the depth of the potential well

$$\mathfrak{d} := \inf_{(u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}) \in \mathbb{N}} \mathcal{J}(u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}), \quad (4.5)$$

where the Nehari manifold is defined by

$$\mathbb{N} := \left\{ (u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}) \in \mathcal{H} \setminus \{0\} \mid \mathcal{I}(t) = 0 \right\}. \quad (4.6)$$

Remark 4.2. The following inequalities are essential for establishing the existence and decay results.

(i) For $|s| \leq 1$, $m_1 \geq 1$, and $0 < n_1 \leq 2$, there exist positive constants $\mathfrak{C}_{m_1}$ and $\mathfrak{C}_{n_1}$ such that

$$\mathfrak{C}_{m_1}|s|^{1+m_1} \leq |s| \ln(1 + |s|) \leq \mathfrak{C}_{n_1}|s|^{n_1}. \quad (4.7)$$

(ii) For $|s| \geq 1$, $0 < \mathfrak{k}_1 < 1$, and $\rho_1 > 0$, there exist positive constants $\mathfrak{C}_{\mathfrak{k}_1}$ and $\mathfrak{C}_{\rho_1}$ such that

$$\mathfrak{C}_{\mathfrak{k}_1}|s|^{\mathfrak{k}_1} \leq |s| \ln(1 + |s|) \leq \mathfrak{C}_{\rho_1}|s|^{1+\rho_1}. \quad (4.8)$$

(iii) For every $\delta > 0$ there exists $C_\delta > 0$ such that $\ln(|s| + \varepsilon_0)^\kappa \leq C_\delta|s|^\delta + 1$ for all $s \in \mathbb{R}$; this follows from $\ln t \leq t/e$ for $t \geq 1$ and $\ln t \leq 0$ for $t \leq 1$.

Lemma 4.3. Assume $p > 2$ and $\mathfrak{p} > 2$ (the strict inequalities are necessary: at $p = 2$ the depth argument below degenerates) and let the embedding condition

$$2 < p, \mathfrak{p} < +\infty \quad (n = 1), \quad (4.9)$$

be satisfied. The higher-dimensional analogue $2 < p, \mathfrak{p} \leq 2n/(n - 2\mu)$ for $n > 2\mu$ applies if the model is extended to dimension n . Let $\tilde{C}$ denote the embedding constant for $H_0^1(0, L) \hookrightarrow L^p(0, L) \cap L^\mathfrak{p}(0, L)$, let $\mathfrak{C}_\kappa, \mathfrak{C}_\mathfrak{q}$ be the constants of Remark 4.2(iii) with $\delta, \tilde{\delta} > 0$ arbitrarily small, and define the strictly increasing function

$$h(r) := \mathfrak{C}_\kappa \tilde{C}^p (r^{p+\delta-2} + r^{p-2}) + \mathfrak{C}_\mathfrak{q} \tilde{C}^\mathfrak{p} (r^{\mathfrak{p}+\tilde{\delta}-2} + r^{\mathfrak{p}-2}), \quad r \geq 0, \quad (4.10)$$

which satisfies $h(0) = 0$ and $h(r) \rightarrow \infty$ as $r \rightarrow \infty$. Let $\rho_0 > 0$ be the unique solution of $h(\rho_0) = C_0$. Then every state $(u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}) \in \mathbb{N}$ satisfies

$$\|u_y\|_{L^2}^2 + \|\phi_y\|_{L^2}^2 \geq \rho_0^2, \quad (4.11)$$

and consequently

$$\mathfrak{d} \geq \min\left\{\frac{p-2}{2p}, \frac{\mathfrak{p}-2}{2\mathfrak{p}}\right\} C_0 \rho_0^2 > 0, \quad (4.12)$$

with C_0 given by (2.28).

Proof. Let $(u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V}) \in \mathbb{N}$ and set $s := \|u_y\|_{L^2}^2 + \|\phi_y\|_{L^2}^2$. Since $\mathcal{I} = 0$, the coercivity (2.31) and the non-negativity of the memory and delay terms give

$$C_0 s \leq \int_0^L |u|^p \ln(|u| + \varepsilon_0)^\kappa dy + \int_0^L |\phi|^\mathfrak{p} \ln(|\phi| + \varepsilon_0)^\mathfrak{q} dy. \quad (4.13)$$

By Remark 4.2(iii), for any $\delta > 0$,

$$\int_0^L |u|^p \ln(|u| + \varepsilon_0)^\kappa dy \leq \mathfrak{C}_\kappa \int_0^L (|u|^{p+\delta} + |u|^p) dy \leq \mathfrak{C}_\kappa (\|u\|_{L^\infty}^\delta + 1) \|u\|_{L^p}^p.$$

Using $\|u\|_{L^\infty}^2 \leq L \|u_y\|_{L^2}^2 \leq Ls$ and $\|u\|_{L^p}^p \leq \tilde{C}^p \|u_y\|_{L^2}^p \leq \tilde{C}^p s^{p/2}$, the factors of $L^{\delta/2}$ are absorbed into $\mathfrak{C}_\kappa$, hence

$$\int_0^L |u|^p \ln(|u| + \varepsilon_0)^\kappa dy \leq \mathfrak{C}_\kappa \tilde{C}^p (s^{(p+\delta)/2} + s^{p/2}),$$

and similarly

$$\int_0^L |\phi|^p \ln(|\phi| + \varepsilon_0)^q dy \leq \mathfrak{C}_q \tilde{C}^p (s^{(p+\delta)/2} + s^{p/2}).$$

Substituting into (4.13) and dividing by $s > 0$ yields

$$C_0 \leq h(\sqrt{s}),$$

with h given by (4.10). Since h is continuous and strictly increasing with $h(\rho_0) = C_0$, this forces $\sqrt{s} \geq \rho_0$, which is (4.11). Finally, on $\mathbb{N}$ we have $\mathcal{I} = 0$, so that (4.1) gives

$$\begin{aligned} \mathcal{J} &= \frac{p-2}{2p} u + \frac{p-2}{2p} \phi + \frac{\kappa}{p^2} \int_0^L |u|^p dy + \frac{q}{p^2} \int_0^L |\phi|^p dy \\ &\geq \frac{p-2}{2p} C_0 \|u_y\|_{L^2}^2 + \frac{p-2}{2p} C_0 \|\phi_y\|_{L^2}^2 \geq \min\left\{\frac{p-2}{2p}, \frac{p-2}{2p}\right\} C_0 s. \end{aligned} \quad (4.14)$$

Combining (4.11) with (4.14) and taking the infimum over $\mathbb{N}$ (see (4.5)), we obtain (4.12). $\square$

Lemma 4.4. *Under the assumptions of Theorem 4.6, we have*

$$\|u_y\|_{L^2}^2 + \|\phi_y\|_{L^2}^2 \leq \frac{2p}{C_0(p-2)} \mathcal{E}(t) \leq \frac{2p}{C_0(p-2)} \mathcal{E}(0), \quad \text{for all } t \geq 0. \quad (4.15)$$

Proof. Since $\mathcal{I}(t) \geq 0$ in the potential well and the kinetic, memory, delay, and logarithmic correction terms are non-negative, we have from (4.1) and (4.2)

$$\frac{p-2}{2p} C_0 \|u_y\|_{L^2}^2 + \frac{p-2}{2p} C_0 \|\phi_y\|_{L^2}^2 \leq \mathcal{J}(t) \leq \mathcal{E}(t). \quad (4.16)$$

Hence, for $p, p > 2$, the estimate (4.15) follows. The proof is complete. $\square$

Lemma 4.5. *Let U be the local solution of Theorem 3.2 with $\mathcal{E}(0) < \mathfrak{d}$ and $\mathcal{I}(0) > 0$. Then $\mathcal{I}(t) > 0$ and $\mathcal{E}(t) < \mathfrak{d}$ for all $t \in [0, T_{\max})$.*

Proof. By Remark 3.7, which does not require the well assumption, $\mathcal{E}(t) \leq \mathcal{E}(0) < \mathfrak{d}$ for all $t < T_{\max}$. Suppose that $t_* := \inf\{t \in (0, T_{\max}) : \mathcal{I}(t) = 0\} < T_{\max}$. By continuity of $t \mapsto \mathcal{I}(t)$ along the trajectory, $t_* > 0$, $\mathcal{I} > 0$ on $[0, t_*)$, and $\mathcal{I}(t_*) = 0$. Moreover, the state $(u, \phi, \psi_1, \psi_2, \mathcal{W}, \mathcal{V})(t_*) \neq 0$: Otherwise uniqueness in Theorem 3.2 would force the identically zero solution, for which the claim is trivial. Hence the state at t_* belongs to the Nehari manifold $\mathbb{N}$ defined in (4.6) and therefore, by the definition (4.5) of the depth,

$$\mathcal{J}(t_*) \geq \mathfrak{d}.$$

On the other hand, $\mathcal{J}(t) \leq \mathcal{E}(t)$ for every t because the kinetic energy is non-negative, so

$$\mathcal{J}(t_*) \leq \mathcal{E}(t_*) \leq \mathcal{E}(0) < \mathfrak{d},$$

a contradiction. Hence $\mathcal{I}(t) > 0$ for all $t < T_{\max}$ and the proof is complete. $\square$

Theorem 4.6. *Let $(u_0, u_1), (\phi_0, \phi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$. Assume that $\mathcal{E}(0) < \mathfrak{d}$, $\mathcal{I}(0) > 0$, and the parameters p and p satisfy $p, p > 2$ and the embedding condition (4.9). Then $T_{\max} = \infty$; the solution of Theorem 3.2 exists globally in time.*

Proof. By Lemma 4.5 the trajectory stays in $\mathcal{W}_*$; hence, by Lemma 4.4 $\|u(t)\|_{H_0^1} + \|\phi(t)\|_{H_0^1}$ remains bounded, by Remark 2.7 the position components are bounded in L^∞ and by Lemma 3.6 $\|u_t\|_2^2 + \|\phi_t\|_2^2 \leq 2\mathcal{E}(0)/\min\{m, \varepsilon\}$. The delay and memory components are bounded by $\frac{2}{\sigma}\mathcal{E}(0)$ and $\frac{2}{\min\{\beta_1, \beta_2\}}\mathcal{E}(0)$, respectively. Thus, $\|U(t)\|_{\mathcal{H}}$ stays bounded on $[0, T_{\max})$, and the blow-up alternative (3.9) forces $T_{\max} = \infty$. $\square$

Lemma 4.7. *Let (u, ϕ) be the global solution of Theorem 4.6. Set*

$$\mathcal{G}(t) := \int_0^L (m u u_t + \varepsilon \phi \phi_t) dy + \frac{m}{2} \|u\|_{L^2}^2 + \frac{\varepsilon}{2} \|\phi\|_{L^2}^2, \quad (4.17)$$

and

$$\mathcal{L}(t) := \nu_1 \mathcal{G}(t) + \nu_2 \mathcal{E}(t). \quad (4.18)$$

With the explicit choices

$$\boxed{\nu_1 := 1, \quad \nu_2 := \max\left\{2, \frac{\Lambda}{C}\right\}, \quad c_1 := \frac{\eta}{2}, \quad c_2 := \Lambda,} \quad (4.19)$$

where C is given by (3.50), $\lambda_P = L/\pi$ is the Poincaré constant of Remark 2.7, $\eta \in (0, \eta_*]$ is given by (4.29) below, Λ by (4.33) below and

$$\alpha_1 := \nu_2 - \nu_1 > 0, \quad \alpha_2 := \nu_2 + \nu_1 C_{\mathcal{G}}, \quad C_{\mathcal{G}} := 1 + \max\left\{\frac{2m\lambda_P^2}{\alpha_0}, \frac{2\varepsilon\lambda_P^2}{P_0}\right\}, \quad (4.20)$$

there hold

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t), \quad (4.21)$$

and

$$\mathcal{L}'(t) \leq -c_1 \mathcal{E}(t) + c_2 (\|u_t\|_{L^2}^2 + \|\phi_t\|_{L^2}^2) + \nu_2 \mathcal{E}'(t). \quad (4.22)$$

Proof. The standing hypothesis of this lemma is $\mathcal{E}(0) < \mathfrak{d}$; the smallness conditions $\xi_1 < \alpha_0/2$ and $\xi_2 < P_0/2$ used below follow from it via Lemma 4.3. We prove (4.22); (4.21) is proved at the end.

Multiplying the first and second equations in (2.38) by u and ϕ , respectively, and integrating over $(0, L)$, we obtain

$$\begin{aligned} & m \frac{d}{dt} \int_0^L u u_t dy + m \|u_t\|_{L^2}^2 + \alpha_0 \|u_y\|_{L^2}^2 + \alpha \|(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u\|_{L^2}^2 \\ & - c \langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi \rangle + d_1 \int_0^L u u_t dy \end{aligned} \quad (4.23)$$

$$+ \beta_1 \int_0^L u \int_{\mathbb{R}} \psi_1 \omega_\theta d\xi dy = \int_0^L h(u) u dy,$$

$$\begin{aligned} & \varepsilon \frac{d}{dt} \int_0^L \phi \phi_t dy + \varepsilon \|\phi_t\|_{L^2}^2 + P_0 \|\phi_y\|_{L^2}^2 + P \|(\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi\|_{L^2}^2 \\ & - c \langle (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} u, (\mathcal{D}_{\rho, \mu, \omega}^\gamma)^{1/2} \phi \rangle + s_1 \int_0^L \phi \phi_t dy \end{aligned} \quad (4.24)$$

$$+ \beta_2 \int_0^L \phi \int_{\mathbb{R}} \psi_2 \omega_\theta d\xi dy = \int_0^L g(\phi) \phi dy.$$

Since $\frac{d}{dt}(\frac{m}{2}\|u\|_{L^2}^2) = m \int_0^L uu_t dy$ and similarly for ϕ , differentiating $\mathcal{G}$ and substituting (4.23) and (4.24) gives

$$\begin{aligned} \mathcal{G}'(t) = & m\|u_t\|_{L^2}^2 + \varepsilon\|\phi_t\|_{L^2}^2 + (m - d_1) \int_0^L uu_t dy + (\varepsilon - s_1) \int_0^L \phi\phi_t dy \\ & - \left[\alpha_0\|u_y\|_{L^2}^2 + P_0\|\phi_y\|_{L^2}^2 + \alpha\|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}u\|_{L^2}^2 + P\|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}\phi\|_{L^2}^2 - 2c\langle(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}u, (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}\phi\rangle \right] \\ & - \beta_1 \int_0^L u \int_{\mathbb{R}} \psi_1 \omega_\theta d\xi dy - \beta_2 \int_0^L \phi \int_{\mathbb{R}} \psi_2 \omega_\theta d\xi dy + \int_0^L h(u)u dy + \int_0^L g(\phi)\phi dy. \end{aligned} \quad (4.25)$$

For the source terms, the embedding $H_0^1(0, L) \hookrightarrow L^\infty(0, L)$ with

$$\|u\|_{L^\infty}^2 \leq L\|u_y\|_{L^2}^2 \leq \frac{2L}{\alpha_0}\mathcal{E}(0) =: M_\infty \quad (4.26)$$

(by Lemma 4.4) and Remark 4.2(iii) give, for any small $\delta > 0$,

$$\int_0^L h(u)u dy \leq \xi_1\|u_y\|_{L^2}^2, \quad \int_0^L g(\phi)\phi dy \leq \xi_2\|\phi_y\|_{L^2}^2, \quad (4.27)$$

with

$$\begin{aligned} \xi_1 &:= \mathfrak{C}_\kappa(M_\infty^{\delta/2} + 1)\tilde{C}^p\left(\frac{2p}{C_0(p-2)}\mathcal{E}(0)\right)^{\frac{p-2}{2}} + \varepsilon', \\ \xi_2 &:= \mathfrak{C}_q(M_\infty^{\tilde{\delta}/2} + 1)\tilde{C}^p\left(\frac{2p}{C_0(p-2)}\mathcal{E}(0)\right)^{\frac{p-2}{2}} + \varepsilon', \end{aligned} \quad (4.28)$$

where $\varepsilon' > 0$ is arbitrary. Since $\delta > 0$ is small, $\mathcal{E}(0) < \delta$ guarantees $\xi_1 < \alpha_0/2$ and $\xi_2 < P_0/2$ (Lemma 4.3). Fix then

$$0 < \eta \leq \eta_* := \min\left\{2, \frac{\alpha_0 - 2\xi_1}{\alpha_0 + 2\xi_1/p + 2\kappa\xi_3/p^2}, \frac{P_0 - 2\xi_2}{P_0 + 2\xi_2/p + 2q\xi_4/p^2}\right\}, \quad (4.29)$$

with

$$\xi_3 := \tilde{C}^p\left(\frac{2\mathcal{E}(0)}{\alpha_0}\right)^{\frac{p-2}{2}}, \quad \xi_4 := \tilde{C}^p\left(\frac{2\mathcal{E}(0)}{P_0}\right)^{\frac{p-2}{2}}. \quad (4.30)$$

Adding and subtracting $\eta\mathcal{E}(t)$ in $\nu_1\mathcal{G}'$, the elastic coefficients in (4.25) become non-positive by the choice (4.29); the cross term $2\nu_1 c\langle(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}u, (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2}\phi\rangle$ is absorbed since $\eta \leq 2$.

For the remaining cross terms, Young's inequality with the Poincaré constant λ_P gives, for any $\delta_0 > 0$,

$$|m - d_1| \left| \int_0^L uu_t dy \right| \leq |m - d_1| \left(\frac{\lambda_P^2 \delta_0}{2} \|u_y\|_{L^2}^2 + \frac{1}{2\delta_0} \|u_t\|_{L^2}^2 \right), \quad (4.31)$$

and similarly for ϕ ; choosing δ_0 small enough (e.g. $\delta_0 = \alpha_0/(4|m - d_1|\lambda_P^2)$ when $m \neq d_1$), the $\|u_y\|_{L^2}^2$ -part is absorbed by the negative elastic term. For the memory cross terms, (3.6) and Young's inequality give

$$\beta_1 \left| \int_0^L u \int_{\mathbb{R}} \psi_1 \omega_\theta d\xi dy \right| \leq \frac{\beta_1 \mathfrak{U}_0 \lambda_P^2}{2} \|u_y\|_{L^2}^2 + \frac{\beta_1}{2} \int_0^L \int_{\mathbb{R}} (\xi^2 + B) |\psi_1|^2 d\xi dy, \quad (4.32)$$

the first term being absorbed since $\beta_1 \mathfrak{U}_0 \lambda_P^2 \ll \alpha_0$ under (2.23) (verified for the data of Section 5) and the second by the memory dissipation in $\nu_2\mathcal{E}'$ provided $\nu_2 \geq 2$; the same holds for ψ_2 .

Collecting, with the explicit constant

$$\Lambda := (m + \varepsilon)\left(1 + \frac{\eta}{2}\right) + (|m - d_1| + |\varepsilon - s_1|)\frac{1}{2\delta_0} + \frac{(\beta_1 + \beta_2)\mathfrak{A}_0\lambda_p^2}{2} + 1, \quad (4.33)$$

we arrive at

$$\mathcal{L}'(t) \leq \nu_1\Lambda(\|u_t\|_{L^2}^2 + \|\phi_t\|_{L^2}^2) - \frac{\nu_1\eta}{2}\mathcal{E}(t) + \nu_2\mathcal{E}'(t), \quad (4.34)$$

and the choices (4.19) give $c_1 = \nu_1\eta/2$, and $c_2 = \nu_1\Lambda$, proving (4.22).

Finally, we prove (4.21). Using $\left|\int_0^L uu_t dy\right| \leq \frac{1}{2}(\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2)$,

$$\mathcal{G}(t) \geq -\frac{m}{2}\|u_t\|_{L^2}^2 - \frac{\varepsilon}{2}\|\phi_t\|_{L^2}^2 + \frac{m}{2}\|u\|_{L^2}^2 + \frac{\varepsilon}{2}\|\phi\|_{L^2}^2 \geq -\frac{1}{2}(m\|u_t\|_{L^2}^2 + \varepsilon\|\phi_t\|_{L^2}^2), \quad (4.35)$$

so, writing $\mathcal{E}(t) = K(t) + \mathcal{J}(t)$ with $K(t) = \frac{1}{2}(m\|u_t\|_{L^2}^2 + \varepsilon\|\phi_t\|_{L^2}^2)$ and recalling that $\mathcal{J}(t) \geq 0$ in the stable well,

$$\mathcal{L}(t) \geq \nu_1\mathcal{G}(t) + \nu_2K(t) + \nu_2\mathcal{J}(t) \geq (\nu_2 - \nu_1)K(t) + \nu_2\mathcal{J}(t) \geq (\nu_2 - \nu_1)\mathcal{E}(t),$$

because $\nu_2 \geq \nu_1$; this gives the lower bound in (4.21) with $\alpha_1 = \nu_2 - \nu_1 > 0$. For the upper bound, $\|u\|_{L^2}^2 \leq \lambda_p^2\|u_y\|_{L^2}^2 \leq \frac{2\lambda_p^2}{\alpha_0}\mathcal{E}(t)$ and $\|u_t\|_{L^2}^2 \leq \frac{2}{m}\mathcal{E}(t)$ give $\mathcal{G}(t) \leq C_{\mathcal{G}}\mathcal{E}(t)$ with $C_{\mathcal{G}}$ as in (4.20); hence $\mathcal{L}(t) \leq \alpha_2\mathcal{E}(t)$. This completes the proof. $\square$

Assumption 4.1. For the polynomial decay argument, we assume

$$\frac{1}{2} \leq \theta < 1.$$

This restriction ensures that the fractional-memory contribution appearing in the Lyapunov estimate can be handled by Young's inequality with positive powers of $-\mathcal{E}'(t)$.

Theorem 4.8. Let $(u_0, u_1), (\phi_0, \phi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$. Assume that $\mathcal{E}(0) < \mathfrak{d}$, $\mathcal{I}(0) > 0$ and the parameters p and $\mathfrak{p}$ satisfy $p, \mathfrak{p} > 2$ and the embedding condition (4.9). Then there exists a positive constant C_E , given explicitly by

$$C_E := \left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_2 + c_2/C + \nu_2}{c_1}\right)\mathcal{E}(0), \quad (4.36)$$

with $\alpha_1, \alpha_2, c_1, c_2, \nu_2$ from (4.19) and (4.20), such that the energy functional of problem (2.38) satisfies, for all $t \geq 0$,

$$\mathcal{E}(t) \leq \frac{C_E}{1+t}. \quad (4.37)$$

Proof. From Lemma 4.7 we have the differential inequality (4.22). Integrating (4.22) over $(0, t)$ yields

$$\mathcal{L}(t) - \mathcal{L}(0) \leq -c_1 \int_0^t \mathcal{E}(s) ds + c_2 \int_0^t (\|u_t(s)\|_{L^2}^2 + \|\phi_t(s)\|_{L^2}^2) ds + \nu_2(\mathcal{E}(t) - \mathcal{E}(0)). \quad (4.38)$$

By the energy dissipation estimate (3.44) we know that

$$\int_0^t (\|u_t(s)\|_{L^2}^2 + \|\phi_t(s)\|_{L^2}^2) ds \leq \frac{1}{C}(\mathcal{E}(0) - \mathcal{E}(t)) \leq \frac{1}{C}\mathcal{E}(0). \quad (4.39)$$

Since $\mathcal{E}(t) \leq \mathcal{E}(0)$ (Remark 3.7), the term $\nu_2 \mathcal{E}(t)$ is absorbed into the constants on the right-hand side. Using the equivalence $\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t)$ from (4.21), we deduce from (4.38) and (4.39) that

$$\alpha_1 \mathcal{E}(t) + c_1 \int_0^t \mathcal{E}(s) ds \leq \left(\alpha_2 + \frac{c_2}{C} + \nu_2 \right) \mathcal{E}(0). \quad (4.40)$$

Note that $\alpha_1 > 0$ by (4.20) since $\nu_2 \geq 2 > \nu_1 = 1$. Hence there exists $C_* > 0$ such that

$$\int_0^t \mathcal{E}(s) ds \leq C_* \mathcal{E}(0) \quad \forall t \geq 0, \quad C_* := \frac{\alpha_2 + c_2/C + \nu_2}{c_1}. \quad (4.41)$$

Because the map $t \mapsto \mathcal{E}(t)$ is non-increasing (Lemma 3.6), for every $t > 0$ we have

$$t \mathcal{E}(t) \leq \int_0^t \mathcal{E}(s) ds \leq C_* \mathcal{E}(0). \quad (4.42)$$

Consequently $\mathcal{E}(t) \leq C_* \mathcal{E}(0)/t$ for $t > 0$; together with (4.40), which gives $\mathcal{E}(t) \leq \frac{\alpha_2}{\alpha_1} \mathcal{E}(0)$, we obtain the uniform estimate

$$\mathcal{E}(t) \leq \min \left\{ \frac{\alpha_2}{\alpha_1} \mathcal{E}(0), \frac{C_* \mathcal{E}(0)}{t} \right\} \leq \frac{C_E}{1+t} \quad \forall t \geq 0, \quad (4.43)$$

since $\min\{a, b/t\} \leq (a+b)/(1+t)$ for $a, b > 0$. This completes the proof with C_E given by (4.36). $\square$

Remark 4.9. Theorem 4.8 yields a polynomial decay rate of order $O((1+t)^{-1})$ under the structural assumptions imposed above. In view of the logarithmic nature of the source term and the weak fractional damping mechanism, this rate appears natural in the present setting: The interaction between the nonlocal KSML coupling, tempered fractional dissipation, time delay, and logarithmic growth creates analytical challenges for the construction of stronger estimates. Whether the polynomial rate is optimal (in particular, whether exponential decay can be ruled out for the full coupled memory-delay-logarithmic structure) requires a separate spectral or resolvent analysis of the generator, which we leave as an open problem.

5. Numerical simulations

In this section, we present numerical experiments supporting the theoretical stability results obtained above. The goal is not to prove an exact asymptotic equivalence of the form $E(t) \sim (1+t)^{-1}$, but rather to verify the polynomial upper estimate

$$E(t) \leq \frac{C}{1+t}, \quad (5.1)$$

the persistence of the stable-well condition $I(t) > 0$, and the decay of the mechanical, electrical, delay, and memory components of the augmented state.

We consider the coupled piezoelectric system on the interval $(0, L)$ with $L = 1$. The Dirichlet boundary condition is imposed through a sine spectral basis

$$e_k(y) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi y}{L}\right), \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2, \quad k = 1, \dots, M. \quad (5.2)$$

Accordingly, the unknowns are approximated by

$$u_M(y, t) = \sum_{k=1}^M u_k(t) e_k(y), \quad \phi_M(y, t) = \sum_{k=1}^M \phi_k(t) e_k(y). \quad (5.3)$$

The Kilbas-Saigo-Mittag-Leffler operator is discretized spectrally by

$$\mathcal{D}_{\rho, \mu, \omega}^\gamma u_M = \sum_{k=1}^M m_k u_k(t) e_k(y), \quad m_k = \lambda_k^{\mu-1} E_{\rho, \mu}^\gamma(-\omega \lambda_k^\rho), \quad (5.4)$$

the classical stiffness by $-\alpha_0 \Delta u_M = \sum_k \alpha_0 \lambda_k u_k e_k$ (resp. P_0 for ϕ), and the weights m_k are evaluated from the defining power series (1.5); for the present data the first weights are $m_1 \approx 0.1862$, $m_2 \approx 0.0630$, and $m_3 \approx 0.0335$, and all truncated weights are positive, in agreement with Lemma 1.1. The delay variables are discretized on a uniform grid $r_j = j\Delta r$, $j = 0, \dots, N_d$, with

$$\Delta r = \frac{1}{N_d}, \quad \Delta t = \tau \Delta r. \quad (5.5)$$

The transport equations

$$\tau W_t + W_r = 0, \quad \tau V_t + V_r = 0 \quad (5.6)$$

are advanced by an upwind shift, with inflow conditions

$$W(y, 0, t) = u_t(y, t), \quad V(y, 0, t) = \phi_t(y, t). \quad (5.7)$$

Thus, $W(y, 1, t)$ approximates $u_t(y, t - \tau)$, while $V(y, 1, t)$ approximates $\phi_t(y, t - \tau)$.

The tempered memory terms are treated through the diffusive representation

$$\partial_t \psi_1 + (\xi^2 + B) \psi_1 = V(y, 1, t) \omega_\theta(\xi), \quad (5.8)$$

$$\partial_t \psi_2 + (\xi^2 + B) \psi_2 = W(y, 1, t) \omega_\theta(\xi). \quad (5.9)$$

Hence, the mechanical equation contains the memory of the delayed electric velocity, while the electric equation contains the memory of the delayed mechanical velocity. This preserves the cross-coupled structure of the original system. The variable ξ is truncated to $[0, \xi_{\max}]$ and the symmetry of the weight is used to approximate the integrals over $\mathbb{R}$.

At each time step, the diffusive memory variables are updated by the exact exponential formula

$$\psi_i^{n+1} = e^{-(\xi^2 + B)\Delta t} \psi_i^n + \frac{1 - e^{-(\xi^2 + B)\Delta t}}{\xi^2 + B} F_i^n \omega_\theta(\xi), \quad (5.10)$$

where $F_1^n = V(y, 1, t_n)$ and $F_2^n = W(y, 1, t_n)$. The coupled modal equations for u_k and ϕ_k are then solved semi-implicitly. The nonlinear logarithmic source terms are evaluated explicitly and projected onto the spectral basis.

The numerical parameters are chosen so that the assumptions used in the theoretical analysis are satisfied. In particular, we impose

$$\alpha P > c^2, \quad 3d_2 A_0 < 2d_1, \quad 3s_2 A_0 < 2s_1, \quad (a_{12} + a_{21})^2 < a_{11} a_{22}, \quad (5.11)$$

where

$$A_0 = B^{\theta-1}. \quad (5.12)$$

The physical parameters used in the simulations are listed in Table 4, while the fractional, memory and discretization parameters are given in Table 5.

Table 4. Physical parameters used in the numerical simulations.

Parameter	L	m	ε	α_0	P_0	α	P	c	d_1	s_1	d_2	s_2
Value	1	1	1	1.0	1.0	2.0	2.4	0.45	0.035	0.038	0.0035	0.0035

Here L is the beam length, m is the mass density, ε is the electric permittivity, α_0, P_0 are the classical stiffness moduli, α and P are mechanical and electric KSML stiffness moduli, respectively, c is the piezoelectric coupling coefficient, and d_1, d_2 and s_1, s_2 are the viscous damping and memory-feedback coefficients for the mechanical and electrical equations, respectively.

Table 5. Fractional, memory, and discretization parameters used in the numerical simulations.

Parameter	θ	B	ρ	μ	ω	γ	$p = \mathfrak{p}$	$\kappa = \mathfrak{q}$	ε_0	τ	M	Δt	T
Value	0.72	0.080	0.70	0.85	0.35	0.80	3.0	1.0	10^{-3}	0.40	10	0.004	200

Here θ is the fractional order of the tempered memory ($\frac{1}{2} \leq \theta < 1$), B is the tempering parameter, $\rho, \mu, \omega, \gamma$ are the Kilbas-Saigo-Mittag-Leffler parameters, $p = \mathfrak{p}$ and $\kappa = \mathfrak{q}$ are the nonlinearity and logarithmic exponents, ε_0 is the regularization parameter, τ is the time delay, M is the number of spectral modes, Δt is the time step, and T is the simulation duration.

The corresponding diagnostic quantities are reported in Table 6. For these values,

$$\alpha P - c^2 = 4.5975 > 0, \quad (5.13)$$

and

$$A_0 = B^{\theta-1} = 2.028312. \quad (5.14)$$

Moreover,

$$\frac{3d_2A_0}{2d_1} = 0.304247 < 1, \quad \frac{3s_2A_0}{2s_1} = 0.280227 < 1. \quad (5.15)$$

Table 6. Computed diagnostic parameters verifying the stability conditions.

Parameter	$\alpha P - c^2$	$A_0 = B^{\theta-1}$	$\frac{3d_2A_0}{2d_1}$	$\frac{3s_2A_0}{2s_1}$	σ	$\frac{(a_{12} + a_{21})^2}{a_{11}a_{22}}$
Value	4.5975	2.028312	0.304247	0.280227	0.0154198185	1.96×10^{-5}

The delay-weight parameter σ lies within the admissible interval required by the theory (see Assumption 2.1). The last column verifies Assumption 2.4 with $\mathfrak{C} = (1 + B)^{\theta-1} = 0.9787$, $a_{12} = a_{21} = \beta_1 e^{-\tau} \mathfrak{C} \approx 2.30 \times 10^{-3}$, $a_{11} = m + d_1 = 1.035$, and $a_{22} = \varepsilon + s_1 = 1.038$.

Therefore, the damping-dominance conditions are satisfied. The selected delay-weight parameter is

$$\sigma = 1.5419818542488204 \times 10^{-2}, \quad (5.16)$$

which lies inside the admissible intersection $(5.68 \times 10^{-3}, 2.52 \times 10^{-2})$ required by (2.24).

By Theorem 4.8, the envelope constant

$$C_E^{\text{theo}} = \left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_2 + c_2/C + \nu_2}{c_1} \right) \mathcal{E}(0), \quad (5.17)$$

with the constants of Lemma 4.7 (see (4.19) and (4.20)), depends only on the parameters and the initial data. For the data of Tables 4–5 the code reports $\mathcal{E}(0) \approx 3.7 \times 10^{-4}$, $\mathfrak{d} > 0$ (so $\mathcal{E}(0) < \mathfrak{d}$), and $\mathcal{I}(0) > 0$; with $C \approx 8.75 \times 10^{-4}$, $\eta \approx 0.91$, $\Lambda \approx 3.8$, $\nu_2 \approx 4.34 \times 10^3$, $\alpha_1 \approx 4.34 \times 10^3$, $\alpha_2 \approx 4.34 \times 10^3$, $c_1 \approx 0.45$, and $c_2 \approx 3.8$,

$$C_E^{\text{theo}} \approx 10.6. \quad (5.18)$$

This constant is computed before the simulation, so the verification below involves no fitting to the computed solution. As a secondary, diagnostic-only quantity we also report the a posteriori fit $C_E^{\text{fit}} = 1.05 \max_{0 \leq t \leq T} (1+t) E_h(t)$; since this second constant is by construction extracted from the computed solution, it is not used for validation but only to visualize the tightness of the theoretical envelope.

Convergence remark. With ξ discretized on 600 logarithmically spaced points of $[10^{-3}, 300]$, the discrete compliance integral equals 1.9924 against the exact $\mathfrak{U}_0 = 2.0283$ (relative error 1.8%); refining the ξ -grid and the mode truncation M leaves the reported energies unchanged to plotting accuracy; the curves below use $M = 10$ and the stated quadrature.

The initial data are chosen in the stable regime. In modal form, we take

$$u_1(0) = 8.0 \times 10^{-3}, \quad u_2(0) = 0.35 u_1(0), \quad (5.19)$$

$$\phi_1(0) = 6.5 \times 10^{-3}, \quad \phi_2(0) = -0.25 \phi_1(0), \quad (5.20)$$

with all other displacement modes initially zero and with zero initial velocities. The delay histories are chosen compatible with the initial velocities:

$$W(y, r, 0) = u_t(y, 0), \quad V(y, r, 0) = \phi_t(y, 0). \quad (5.21)$$

The memory variables are initialized by

$$\psi_1(y, \xi, 0) = \psi_2(y, \xi, 0) = 0. \quad (5.22)$$

The numerical code checks the stable-well conditions

$$E(0) < \mathfrak{d}_{\text{approx}}, \quad \mathcal{I}(0) > 0. \quad (5.23)$$

The discrete augmented energy contains the same components as the theoretical energy:

$$E_h(t) = E_{\text{kin}}(t) + E_{\text{el}}(t) + E_{\text{delay}}(t) + E_{\text{mem}}(t) - E_{\text{pot}}(t). \quad (5.24)$$

Here

$$E_{\text{kin}}(t) = \frac{1}{2} \left(m \|u_t(t)\|_{L^2}^2 + \varepsilon \|\phi_t(t)\|_{L^2}^2 \right), \quad (5.25)$$

and the coupled elastic energy is

$$E_{\text{el}}(t) = \frac{1}{2} \left[\alpha_0 \|u_y(t)\|_{L^2}^2 + P_0 \|\phi_y(t)\|_{L^2}^2 + \alpha \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u(t)\|_{L^2}^2 \right. \\ \left. + P \|(\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi(t)\|_{L^2}^2 - 2c \langle (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} u(t), (\mathcal{D}_{\rho,\mu,\omega}^\gamma)^{1/2} \phi(t) \rangle \right]. \quad (5.26)$$

The delay and memory energies are approximated by

$$E_{\text{delay}}(t) = \frac{\sigma}{2} \int_0^L \int_0^1 \left(|W(y, r, t)|^2 + |V(y, r, t)|^2 \right) dr dy, \quad (5.27)$$

and

$$E_{\text{mem}}(t) = \frac{1}{2} \int_0^L \int_{\mathbb{R}} \left(\beta_1 |\psi_1(y, \xi, t)|^2 + \beta_2 |\psi_2(y, \xi, t)|^2 \right) d\xi dy. \quad (5.28)$$

The nonlinear potential is evaluated using the primitive of the logarithmic source term. In addition, we compute the discrete Nehari functional $I_h(t)$, the numerical Lyapunov diagnostic

$$L_{\text{num}}(t) = E_h(t) + \eta_L (m \langle u(t), u_t(t) \rangle + \varepsilon \langle \phi(t), \phi_t(t) \rangle), \quad (5.29)$$

with a small $\eta_L > 0$ and the ratio indicator

$$R_E(t) = \frac{(1+t)E_h(t)}{C_E^{\text{theo}}}, \quad (5.30)$$

with C_E^{theo} from (5.17). Therefore, the ratio test directly verifies whether the numerical energy remains below the polynomial envelope with an a priori constant.

5.1. Numerical results

Figure 1 shows the full augmented energy $E_h(t)$, the numerical Lyapunov functional $L_{\text{num}}(t)$ and the polynomial upper envelope $C_E^{\text{theo}}(1+t)^{-1}$. Both $E_h(t)$ and $L_{\text{num}}(t)$ remain below the polynomial reference curve, confirming the upper decay estimate predicted by the theory.

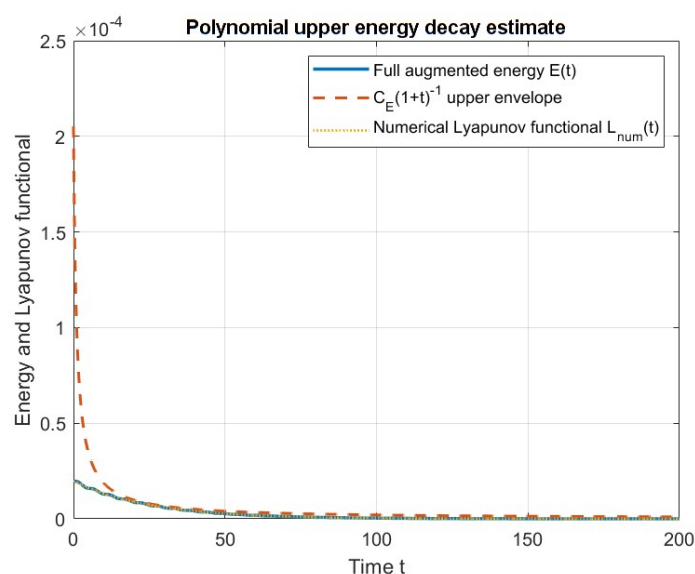


Figure 1. Polynomial upper energy decay estimate. The full augmented energy $E_h(t)$ and the numerical Lyapunov functional $L_{\text{num}}(t)$ remain below the reference envelope $C_E^{\text{theo}}(1+t)^{-1}$.

Figure 2 gives a direct verification of the polynomial upper estimate. The ratio

$$R_E(t) = \frac{(1+t)E_h(t)}{C_E^{\text{theo}}} \leq 1$$

remains below the threshold value 1 during the entire simulation interval. This provides a stronger visual confirmation of the polynomial upper bound than a standard energy plot alone.

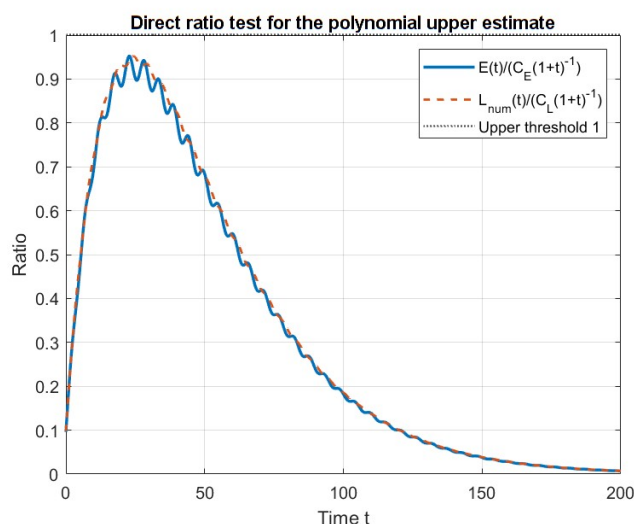


Figure 2. A priori ratio test for the polynomial upper estimate. The ratio $R_E(t)$ remains below the threshold 1, verifying the numerical form of the estimate $E_h(t) \leq C_E(1+t)^{-1}$ with the unfitted constant.

Figure 3 displays the stable-well diagnostic $I_h(t)$. Although the curve oscillates because of the

coupled electromechanical dynamics and delayed feedback, it remains positive. Hence the numerical solution stays inside the stable potential well, which is consistent with the global existence theory (Lemma 4.5 and Theorem 4.6).

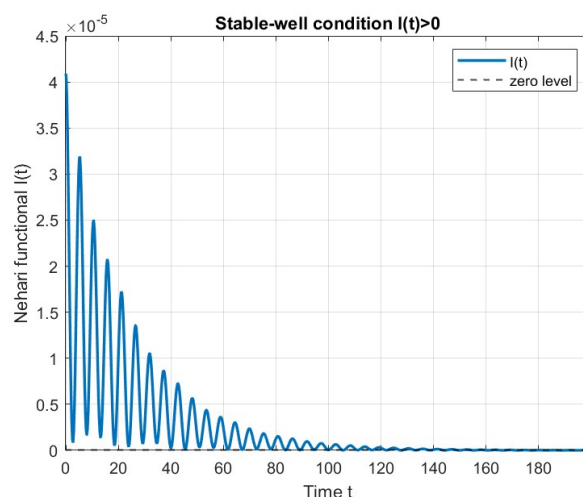


Figure 3. Numerical verification of the stable-well condition. The Nehari functional $I_h(t)$ remains positive, confirming that the discrete solution stays in the stable potential well.

Figure 4 presents the same polynomial estimate on a log-log scale. The computed energy curve remains below the polynomial reference line. This confirms that the observed decay is compatible with the theoretical upper estimate

$$E(t) \leq C(1+t)^{-1}.$$

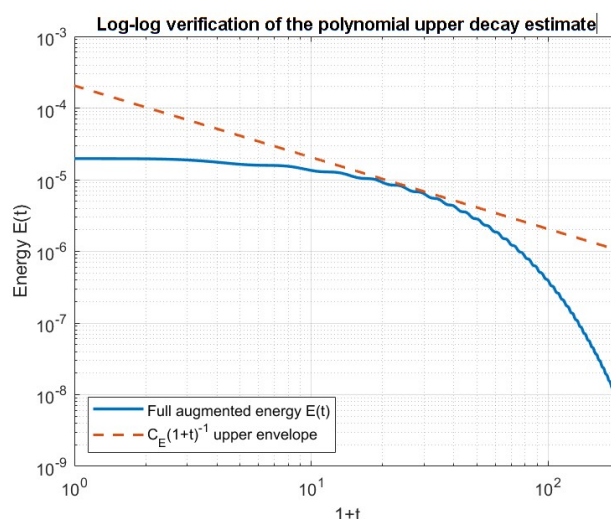


Figure 4. Log-log verification of the polynomial upper decay estimate. The energy $E_h(t)$ stays below the reference envelope $C_E(1+t)^{-1}$.

Figure 5 shows the decay of the mechanical displacement, electric potential, delay state, and memory state. The curves demonstrate that all components of the augmented state are dissipated over

time. In particular, the delay and memory variables decay together with the mechanical and electrical components, confirming that the diffusive-memory and delay-transport reformulation is dynamically consistent.

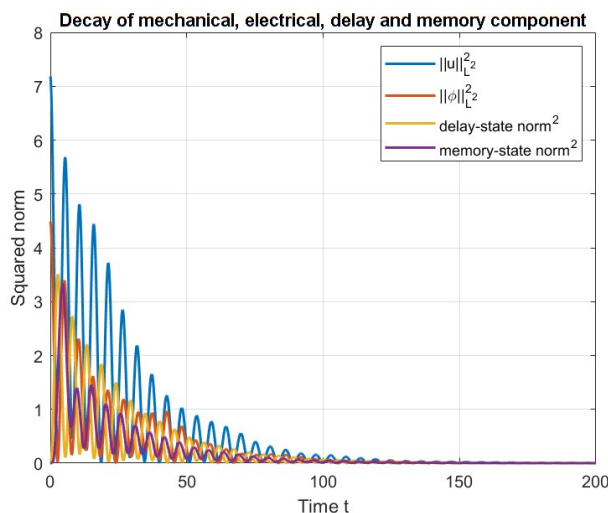


Figure 5. Decay of mechanical, electrical, delay, and memory components. The quantities $\|u(t)\|_{L^2}^2$, $\|\phi(t)\|_{L^2}^2$, the delay-state norm, and the memory-state norm all decrease toward zero.

Figures 6 and 7 show the three-dimensional evolution of the mechanical displacement $u(y, t)$ and electric potential $\phi(y, t)$, respectively. Both surfaces satisfy the homogeneous boundary conditions and decay as time increases. The oscillatory behavior near early times reflects the coupled piezoelectric interaction, while the long-time flattening of the surfaces indicates stabilization.

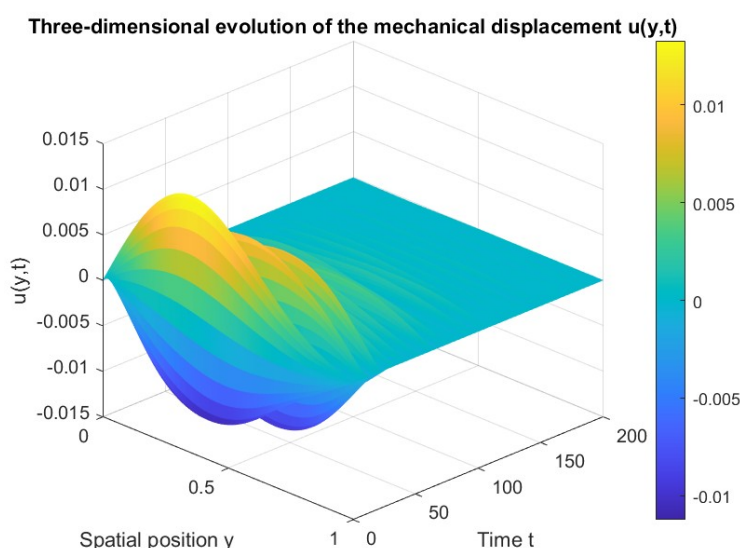


Figure 6. Three-dimensional evolution of the mechanical displacement $u(y, t)$. The solution satisfies the boundary condition and decays toward equilibrium as time increases.

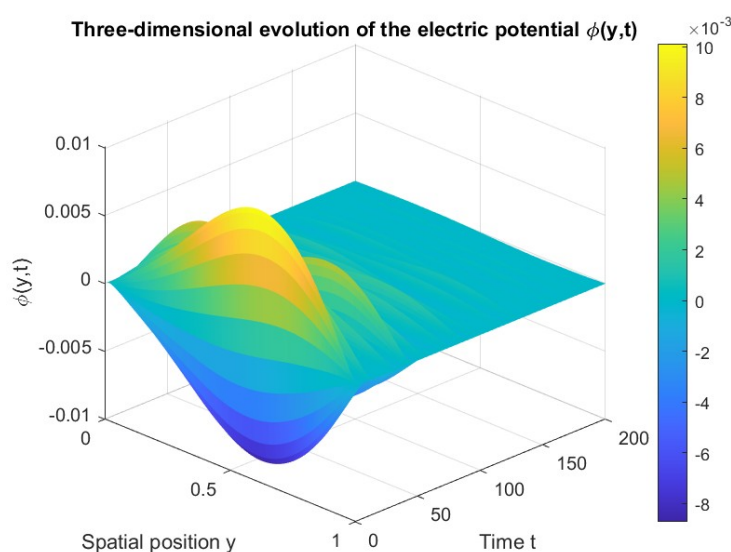


Figure 7. Three-dimensional evolution of the electric potential $\phi(y, t)$. The electric component exhibits damped oscillations and converges toward equilibrium.

The numerical results confirm the main theoretical conclusions. First, the discrete energy and the numerical Lyapunov functional remain bounded above by a polynomial envelope whose constant is fixed a priori by (5.17). Second, the ratio test verifies directly that the bound $E_h(t) \leq C_E(1+t)^{-1}$ holds on the computed time interval. Third, the positivity of $I_h(t)$ confirms that the trajectory remains in the stable potential well. Finally, the decay of the mechanical, electrical, delay, and memory components illustrates the stabilizing role of the combined viscous damping, tempered memory, and cross-coupled delayed feedback.

6. Conclusions

In this paper, we investigated a coupled piezoelectric beam system incorporating four competing mechanisms: The Kilbas-Saigo-Mittag-Leffler spectral operator, tempered fractional memory, cross-coupled time delay, and logarithmic nonlinear source terms. By introducing suitable delay-transport variables and diffusive memory variables, the original nonlocal problem was transformed into an augmented evolution system posed in an appropriate Hilbert space. This reformulation made it possible to treat the delayed and fractional effects within a unified local-in-time framework. The present analysis rests on the following functional framework: The spectral weights of the KSML operator are positive but bounded, so the elastic stiffness contains a classical coercive part; the potentials of the regularized logarithmic sources are controlled by an explicit $O(\varepsilon_0)$ error identity (Lemma 3.4); local and global well-posedness are stated separately (Theorems 3.2 and 4.6); the invariance of the stable set is proved (Lemma 4.5); and all constants in the Lyapunov functional and in the decay estimate are explicit (Lemma 4.7 and Theorem 4.8). Under the coercivity condition $\alpha P > c^2$ and suitable dominance assumptions on the viscous damping and memory coefficients, we first established the well-posedness of the augmented system. The global existence of solutions was then obtained through a potential-well argument, which controls the logarithmic source terms by the coupled elastic energy. Finally, by

constructing a Lyapunov functional adapted to the cross-coupled structure, we proved that the total energy satisfies a polynomial upper decay estimate of the form $E(t) \leq C(1+t)^{-1}$ with C explicitly computable from the data. The numerical simulations support the theoretical findings: The discrete energy remains bounded by a polynomial reference curve, the quantity $(1+t)E(t)$ stays uniformly bounded, and both the mechanical displacement and electric potential decay toward equilibrium. The computations also illustrate the roles of viscous damping, tempered-memory dissipation, and the Mittag-Leffler parameters in the stabilization process. These results show that the interaction of cross-coupled delay and tempered fractional memory produces a stable but genuinely nonlocal dissipative structure for piezoelectric beam dynamics with the Kilbas-Saigo-Mittag-Leffler spectral operator. Possible future work includes the study of non-constant delays, nonlinear boundary feedback, sharper optimality of the polynomial decay rate, and higher-dimensional piezoelectric models with more general fractional operators.

Author contributions

Z.U., J.H. and A.B.: Conceptualization, methodology, Writing—review and editing; Z.U., I.K. and L.B.: Software, investigation, visualization, writing—original draft; Z.U., J.H., A.B., I.K. and L.B.: Validation; Z.U., A.B., I.K. and L.B.: Formal analysis; J.H. and A.B.: Resources, supervision, project administration; A.B.: Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (<https://www.qu.edu.sa>) for financial support (QU-APC-2026).

Data availability

No new data were created or analyzed in this study. The MATLAB code implementing the scheme can be obtained from the corresponding authors on reasonable request.

Conflict of interest

The authors have no competing interests to declare.

References

1. H. T. Banks, R. C. Smith, Y. Wang, *Smart material structures: Modelling, estimation and control*, Wiley, New York, 1996.

2. N. W. Hagood, W. H. Chung, A. Von Flotow, Modeling of piezoelectric actuator dynamics for active structural control, *J. Intell. Mater. Syst. Struct.*, **1** (1990), 327–354. <https://doi.org/10.1177/1045389X9000100305>
3. J. Yang, *An introduction to the theory of piezoelectricity*, Springer, New York, 2005. <https://doi.org/10.1007/b101799>
4. A. J. A. Ramos, C. S. L. Goncalves, S. S. Correa Neto, Exponential stability and numerical treatment for piezoelectric beams with magnetic effect, *ESAIM Math. Model. Numer. Anal.*, **52** (2018), 255–274. <https://doi.org/10.1051/m2an/2018004>
5. A. J. A. Ramos, M. M. Freitas, D. S. A. Junior, S. S. Jesus, T. R. S. Moura, Equivalence between exponential stabilization and boundary observability for piezoelectric beams, *Z. Angew. Math. Phys.*, **70** (2019), 1–14. <https://doi.org/10.1007/s00033-019-1106-2>
6. A. J. A. Ramos, M. M. Freitas, D. S. Almeida Junior, Pullback dynamics of Lamé systems with time-dependent weak damping, *Nonlinear Anal. Real World Appl.*, **75** (2024), 103982. <https://doi.org/10.1016/j.nonrwa.2023.103982>
7. A. Kong, C. Nonato, W. Liu, M. J. D. Santos, C. Raposo, Equivalence between exponential stabilization and observability inequality for magnetic effected piezoelectric beams, *Discrete Cont. Dyn.-B*, **27** (2022), 2959–2978. <https://doi.org/10.3934/dcdsb.2021168>
8. M. Akil, A. Soufyane, Y. Belhamadia, Stabilization results of piezoelectric beams with partial viscous dampings and under Lorenz gauge condition, *Appl. Math. Optim.*, **87** (2023), 1–38. <https://doi.org/10.1007/s00245-022-09935-3>
9. M. Liu, I. Dassios, F. Milano, On the stability analysis of systems of neutral delay differential equations, *Circ. Syst. Signal Pr.*, **38** (2019), 1639–1653. <https://doi.org/10.1007/s00034-018-0943-0>
10. P. L. Dattori da Silva, T. F. Ma, E. M. Maravi-Percca, P. N. Seminario-Huertas, A non-homogeneous weakly damped Lamé system with time-dependent delay, *Math. Method. Appl. Sci.*, **46** (2023), 8793–8805. <https://doi.org/10.1002/mma.9017>
11. Z. Ullah, J. Hao, A. Bhatnia, A. Aziz, Global solution, decay and blow-up of a viscoelastic Kirchhoff system with logarithmic nonlinearity and acoustic fractional delay boundary conditions, *Axioms*, **15** (2026), 728. <https://doi.org/10.3390/axioms15100728>
12. H. Yükksekaya, E. Piskin, M. M. Kafini, A. M. Al-Mahdi, Well-posedness and exponential stability for the logarithmic Lamé system with a time-delay, *Appl. Anal.*, 2023. <https://doi.org/10.1080/00036811.2023.2196993>
13. B. Feng, C. A. Raposo, C. A. Nonato, A. Soufyane, Analysis of exponential stabilization for Rao-Nakra sandwich beam with time-varying weight and time-varying delay, *Math. Control Relat. Fields*, 2022. <https://doi.org/10.3934/mcrf.2022011>
14. M. Kafini, S. A. Messaoudi, Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay, *Appl. Anal.*, **99** (2020), 530–547. <https://doi.org/10.1080/00036811.2018.1504029>
15. S. A. Messaoudi, A. Fareh, N. Doudi, Well-posedness and exponential stability in a wave

- equation with a strong damping and a strong delay, *J. Math. Phys.*, **57** (2016), 111501. <https://doi.org/10.1063/1.4966551>
16. S. A. Messaoudi, M. M. Al-Gharabli, A general stability result for a nonlinear wave equation with infinite memory, *Appl. Math. Lett.*, **26** (2013), 1082–1086. <https://doi.org/10.1016/j.aml.2013.06.002>
 17. A. M. Al-Mahdi, Stability results of a viscoelastic plate equation with past history and a logarithmic nonlinearity, *Bound. Value Probl.*, **2020** (2020), 1–20. <https://doi.org/10.1186/s13661-020-01382-9>
 18. M. M. Al-Gharabli, A. M. Al-Mahdi, S. A. Messaoudi, General and optimal decay result for a viscoelastic plate equation with a logarithmic nonlinearity, *Math. Method. Appl. Sci.*, **44** (2021), 10646–10658.
 19. A. Peyravi, General stability and exponential growth for a class of semi-linear wave equations with logarithmic source and memory terms, *Appl. Math. Optim.*, **81** (2020), 545–561. <https://doi.org/10.1007/s00245-018-9508-7>
 20. Z. Ullah, J. Hao, L. Bashir, I. Kanwal, Stability and dynamics of fractionally damped piezoelectric beams with time-delayed coupling and logarithmic source, *Commun. Nonlinear Sci. Numer. Simul.*, **163** (2026), 1–20. <https://doi.org/10.1016/j.cnsns.2026.110575>
 21. A. Guesmia, S. A. Messaoudi, A general stability result in a Timoshenko system with infinite memory, *Math. Comput. Model.*, **52** (2010), 705–716.
 22. M. I. Mustafa, Optimal decay rates for the viscoelastic wave equation, *Math. Methods Appl. Sci.*, **41** (2018), 164–170. <https://doi.org/10.1002/mma.4604>
 23. F. Alili, A. Amara, K. Zennir, Existence and stability results for fractional fully hybrid elastic beam equation: Caputo-like formulation, *Math. Method. Appl. Sci.*, **48** (2025), 13839–13853. <https://doi.org/10.1002/mma.11146>
 24. B. Tellab, Existence and stability results for the solution of neutral fractional integro-differential equation with nonlocal conditions, *Tamkang J. Math.*, **53** (2022). <https://doi.org/10.5556/j.tkjm.53.2022.3550>
 25. R. Garra, R. Garrappa, The Prabhakar or three-parameter Mittag-Leffler function: Theory and application, *Commun. Nonlinear Sci. Numer. Simul.*, **56** (2018), 314–329. <https://doi.org/10.1016/j.cnsns.2017.08.018>
 26. Z. Ullah, J. Hao, S. T. M. Thabet, T. Moulahi, Exponential stabilization and finite time blow-up in a fractional thermal piezoelectric beam with delay, *Sci. Rep.*, **16** (2026), 6479. <https://doi.org/10.1038/s41598-026-37381-6>
 27. Z. Ullah, J. Hao, L. Bashir, M. F. Aslam, Nonlinear, damped and strongly delayed wave equation, study of exponential decay and blow-up phenomena, *Differ. Equ. Dyn. Syst.*, **2025**, 1–20. <https://doi.org/10.1007/s12591-025-00740-2>
 28. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud., Elsevier, Amsterdam, **204** (2006).
 29. A. M. Mathai, H. J. Haubold, *Fractional and multivariable calculus: Model building and optimization problems*, Springer, 2017. <https://doi.org/10.1007/978-3-319-59993-9>

30. A. A. Kilbas, M. Saigo, R. K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators, *Integr. Transf. Spec. F.*, **15** (2004), 31–43. <https://doi.org/10.1080/10652460310001600717>
31. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer, Berlin, 2014. <https://doi.org/10.1007/978-3-662-43930-2>
32. R. Garra, R. Gorenflo, F. Mainardi, G. Tomovski, Fractional relaxation and time-fractional diffusion of distributed order, *Fract. Calc. Appl. Anal.*, **17** (2014), 431–442.
33. M. Kirane, R. Aounallah, L. Jlali, General decay and blowing-up solutions of a nonlinear wave equation with nonlocal in time damping and infinite memory, *Math. Methods Appl. Sci.*, 2025. <https://doi.org/10.1002/mma.10777>
34. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, 2 Eds., Academic Press, Amsterdam, 2003.
35. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011. <https://doi.org/10.1007/978-0-387-70914-7>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)