



Research article

Interval-valued Pythagorean fuzzy Aczel–Alsina Hamy mean operators for multi-attribute group decision-making with smart healthcare application

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Abstract: Selecting optimal smart healthcare systems is a complex multi-attribute group decision-making (MAGDM) problem characterized by multiple conflicting criteria and inherent uncertainty in expert judgments. Traditional approaches often struggle to adequately model intercriteria relationships and flexible information aggregation under such vagueness. This paper develops a novel MAGDM framework based on interval-valued Pythagorean fuzzy sets (IVPFSs). To effectively aggregate uncertain information, several new aggregation operators are proposed by integrating Aczel–Alsina (AA) operations with the Hamy mean and its dual form. The fundamental mathematical properties of these operators, including idempotency, monotonicity, and boundedness, are rigorously established. A systematic MAGDM methodology is constructed using the proposed operators, where criteria weights are objectively determined using the measurement of alternatives and ranking according to compromise solution (MERECE) method. Its practicality and effectiveness are demonstrated through a real-world case study on smart healthcare system selection. Sensitivity analysis confirms the stability of the ranking results with respect to the AA and Hamy mean parameters. Comparative analysis with existing methods further highlights the superiority of the proposed approach in capturing intercriteria dependencies and providing more balanced and reliable rankings.

Keywords: interval-valued Pythagorean fuzzy sets; Aczel–Alsina norm; Hamy mean operators; group decision-making; smart healthcare system selection

Mathematics Subject Classification: 03E72, 68T35, 90B50

1. Introduction

Decision-making (DeM) constitutes a fundamental and indispensable methodology for solving complex real-world problems, where the primary objective is to select the most suitable course of action from a set of available alternatives [1–3]. The process typically encompasses several critical stages, including problem identification, generation of feasible alternatives, systematic evaluation of options based on multiple criteria, selection of the optimal alternative, implementation of the chosen decision, and post-decision outcome assessment [4]. Within this broader domain, multi-attribute group decision-making (MAGDM) has emerged as a powerful and widely adopted approach. It enables a group of decision-makers (DMs) to systematically evaluate and rank alternatives according to multiple, often conflicting criteria, thereby facilitating more informed, transparent, and consensus-driven decisions.

One of the most pertinent and important forms of complex MAGDM can be observed in the problem of selecting the best smart healthcare system (SHS) for an esteemed public sector hospital of Pakistan. With the objectives of upgrading the service efficiency, and enhancing patient satisfaction, reducing operating costs, and improving clinical performance, the hospital management initiated a decision process to find the best SHS among the five selected candidates. Three qualified DMs with the weights (0.4, 0.4, 0.2) were hired to access and rank the five alternatives based on some established attributes (e.g., cost efficiency, time efficiency, test coverage, resource usage, effectiveness for testing, robustness and scalability, security and integration), where the values of weights of the criteria are completely unknown. Due to the complexity of a healthcare architecture; rapidly evolving technology; and diversity of expectations from patients, caregivers, policymakers, and technicians, the opinions of experts are inherently uncertain and hesitant.

Real-life DeM situations, such as the choice of SHS, tend to be filled with vagueness, ambiguity, and hesitation, making the classical crisp numerical approaches insufficient for accurate modeling and reliable results [5, 6]. In order to cope with these problems, Zadeh [7] presented the notion of fuzzy set (FS) in 1965 that allowed assigning MDs of elements in the closed interval $[0, 1]$. This pioneering approach provided a way of dealing with uncertainty in decision systems. Later on, Bellman and Zadeh [8] generalized FS theory to the case of multi-attribute decision-making (MADM) problems. Further, Yager made a major breakthrough in FS development and created numerous aggregation operators (AOs) [9–11], which significantly enhanced the ability to combine information from multiple sources and improved the operational flexibility of fuzzy models in practical scenarios.

Despite these advancements, standard FSs exhibit notable limitations, particularly in representing the hesitation or non-membership aspect of cognitive information. To resolve this, Atanassov [12] established intuitionistic fuzzy sets (IFSs) wherein every element has membership degree (MD, μ) and non-membership degree (NMD, ν) with the constraint of $\mu + \nu \leq 1$. Even though IFSs provided the scope for a wider representation, the model faced problems in situations where the addition of

MD and NMD results in values above one. In order to tackle this problem, Yager [13] came up with the idea of Pythagorean fuzzy sets (PFSs), where the constraint is relaxed to the orthogonal condition $\mu^2 + \nu^2 \leq 1$. This approach provides much greater flexibility for representing uncertain and hesitant data. A classic illustrative example is a housing society survey in which five residents assign a satisfaction score of 0.5, and the remaining five assign 0.8. Here, $0.5 + 0.8 = 1.3 > 1$ (violates IFS structural bounds), but $0.5^2 + 0.8^2 = 0.89 < 1$ (satisfies the PFS condition). This clearly demonstrates the practical superiority of PFS in real human judgment scenarios. Consequently, numerous extensions and applications of PFS have since been actively developed [14–18].

However, in practice, it is impossible for specialists to prescribe crisp (i.e., single-point) MD or NMD Pythagorean degrees in various real-world conditions due to the deficit of knowledge, the constraint of time, and the natural vagueness embedded in human psychology. In order to solve this difficulty, Zhang [19] presented interval-valued Pythagorean fuzzy sets (IVPFSs) which generalize the PFS by specifying both the MD and NMD as intervals in $[0, 1]$. IVPFS is an ultra-powerful tool to deal with intricate uncertainty in engineering and social advanced applications. Following this paradigm shift, a rich body of literature has subsequently emerged, proposing various AOs in the IVPFS environment [20]. Peng and Yang [21] investigated the fundamental properties of interval-valued Pythagorean fuzzy information and established basic algebraic structures. Garg [22] proposed innovative accuracy and scoring functions to strengthen the structural evaluation of interval-valued Pythagorean fuzzy arguments. To systematically facilitate practical data fusion, several foundational operators have been explicitly developed, including interval-valued Pythagorean fuzzy weighted averaging (IVPFWA) [23], weighted geometric operators [24], Yager-based parameterized formulations [25], Einstein operators driven by smooth algebraic traits [26], Dombi aggregation methodologies [27], and initial operational frameworks utilizing t -norm foundations [28].

In the context of algebraic operational frameworks, the parameterized triangular norms pioneered by Aczel and Alsina [29] have emerged as exceptionally robust mathematical structures. The Aczel–Alsina (AA) t -norm and t -conorm operations utilize a distinctive parameter $\vartheta > 0$, which empowers decision science models with a highly responsive, tunable control over the information aggregation process. Unlike rigid, non-parametric operational laws, AA operators can actively adjust the degree of compensation and smoothly govern trade-offs between input criteria. This structural adaptability has driven widespread interest among researchers to expand AA operations across multiple complex fuzzy domains. For instance, Senapati et al. [29] formalized algebraic tools based on AA operations for intuitionistic fuzzy systems and subsequently extended this architecture to interval-valued intuitionistic structures [30]. Similarly, Hussain et al. [31] and Naeem et al. [32] leveraged single-valued Pythagorean fuzzy settings to model industrial and agricultural multi-attribute networks using parameterized AA operators. Further integration of AA functions with higher-order symmetric concepts, such as the Heronian mean within complex fuzzy matrices [33] or financial investment models [34], underscores the vast capacity of AA norms to mitigate information erosion. Integrating these features into an interval-valued framework enables DMs to calibrate data fusion according to their specific risk attitudes, making AA operations highly desirable for specialized MAGDM problems.

Concurrently, a critical challenge in advanced decision networks involves capturing the internal structural interrelationships between several input criteria. For this purpose, the Hamy mean (HM) operator, which was initially developed by Hara et al. [35], operates as an extremely effective

symmetric mean to facilitate modeling of higher-order dependency between any number of interaction arguments. Instead of traditional arithmetic or geometric mean-based models, which operate on the assumption of total independence of all input arguments, or the classical Bonferroni or Heronian means with respect to their localized two-argument dependency relationships, HM and its dual counterpart, the dual Hamy mean (DHM), enable calculation of multi-dimensional synergy correlations among all elements of the criteria set at once. Such a special ability to identify complex correlation between the input arguments led to applications of HM and DHM methodologies in the optimization of information fusion processes in various academic disciplines. Wu et al. [36] utilized HM operators to evaluate interrelationships within tourism supply chains in interval valued environments. Liu and Liu [37] generalized HM techniques in the linguistic multi-attributes context, and Li et al. [38] introduced symmetric Hamy models in the single-valued Pythagorean environment. Furthermore, Qin [39] coupled interval type-2 fuzzy environments with HM functions, and Wang et al. [40] successfully developed robust HM decision infrastructure under q -rung orthopair fuzzy structures to verify organizational resource benchmarks.

1.1. Limitations of existing literature

Although the analytical tools developed within the IVPFS domain have vastly enhanced contemporary decision science, a critical assessment of the state-of-the-art aggregation mechanisms uncovers fundamental theoretical and operational bottlenecks:

- (1) Assumption of criteria independence: Conventional aggregation operators under IVPFS (such as standard algebraic or geometric averaging models) assume that the underlying evaluation criteria are strictly independent. In complex socio-technical systems such as the Pakistani SHS framework, attributes exhibit dense, multi-directional interdependencies.
- (2) Inability to model multi-dimensional correlations: Although interaction operators such as the Bonferroni mean or Heronian mean have been proposed to address relationship modeling, they are structurally confined to capturing purely pairwise (two-argument) interactions, leaving multi-variable synergistic or antagonistic networks unaddressed.
- (3) Information loss and low control flexibility: Existing t -norm and t -conorm families operate via rigid, fixed operational laws. They lack tunable parameters capable of reflecting a DM's subjective risk attitude or generating behavioral compromise ranges during data fusion.
- (4) Subjectivity in weight allocation: Most existing IVPFS frameworks assume that criteria weights are completely known a priori or computed using subjective frameworks, which seriously distorts ranking reliability when handling unknown weight structures under deep hesitation.

1.2. Research gap and motivation

The overarching research gap identified in this study stems from a glaring void in aggregation theory: *The concurrent mathematical integration of parameterized structural flexibility and multi-dimensional criteria interrelationship modeling within the interval-valued Pythagorean fuzzy space remains entirely uninvestigated.*

Why does this gap exist? Historically, the triangular norm operations formulated by Aczel and Alsina have emerged as highly flexible tools because they utilize a tunable parameter to smoothly govern trade-offs and minimize information loss based on DMs' risk preferences. Concurrently, the

HM and DHM operators are uniquely engineered to model complex, multi-point correlations among an arbitrary number of interactive arguments through symmetric means. However, AA operations and HM/DHM structural blocks have only been studied in isolation or within simpler fuzzy environments. Their synergistic fusion under an interval-valued Pythagorean space has never been explored due to the mathematical complexities of combining non-linear interactive operational intervals with multi-argument symmetric roots.

Furthermore, a significant methodological gap exists regarding weight determination. Although subjective weights induce cognitive bias, objective weighting methods tailored for interval-valued orthogonal environments are scarce. The method based on the removal effects of criteria (MEREC) is recognized as a highly stable objective weighting algorithm, yet its analytical implementation for handling interval-valued Pythagorean inputs remains completely undeveloped.

Driven by these gaps, this study addresses the following fundamental research questions:

- RQ1.** How can we establish a parameterized operational architecture that simultaneously captures multi-argument criteria interdependencies and mitigates information erosion under IVPFS?
- RQ2.** How can we mathematically formulate interactive aggregation blocks that seamlessly bridge AA operations with both HM and DHM constructs?
- RQ3.** How can objective criteria weights be determined strictly from the intrinsic variance of interval-valued Pythagorean decision matrices without relying on subjective expert inputs?

1.3. Major contributions of this study

To resolve the delineated challenges and answer the proposed research questions, this paper systematically develops an integrated MAGDM framework. The primary original contributions of this research are summarized as follows:

- (1) We introduce four pioneering aggregation tools: the interval-valued Pythagorean fuzzy AA Hamy mean (IVPFAAHM), interval-valued Pythagorean fuzzy AA weighted Hamy mean (IVPFAAWHM), interval-valued Pythagorean fuzzy AA dual Hamy mean (IVPFAADHM), and interval-valued Pythagorean fuzzy AA weighted dual Hamy mean (IVPFAAWDHM) operators. These models systematically unite the parameterized flexibility of AA norms with the multi-point correlation capabilities of HM.
- (2) We present rigorous analytical proofs validating that the proposed aggregation operators strictly satisfy fundamental algebraic properties including idempotency, monotonicity, boundedness, and commutativity.
- (3) We extend the MEREC method to the IVPFS domain, providing a fully objective algorithm to calculate completely unknown criteria weights by assessing the structural removal effects of individual interval attributes.
- (4) The practical execution of the developed framework is demonstrated through an empirical case study centered on SHS selection for a public hospital in Pakistan, proving its feasibility in high-stakes decision contexts.
- (5) A thorough comparative analysis against existing IVPFS operators (e.g., Yager, Dombi, and standard averaging methods) is conducted to confirm the stability, versatility, and structural superiority of the proposed parameterized decision architecture.

The remainder of this paper is organized as follows: Section 2 reviews the fundamental concepts relevant to this study. Section 3 introduces several IVPF AOs based on AA operations and the HM, including the IVPFAAHM, IVPFAAWHM, IVPFAADHM, and IVPFAAWDHM operators, and analyzes their key characteristics and properties. In Section 4, a MAGDM framework is developed using the proposed AOs under an IVPF environment. Section 5 presents a real-world case study on SHS selection to demonstrate the feasibility and effectiveness of the proposed methodology. Section 6 investigates the influence of parameters on decision outcomes via sensitivity analysis, and Section 7 provides a comparative evaluation with existing methods to highlight the advantages of the proposed approach. Finally, Section 8 presents the conclusions of the study.

2. Preliminaries

This section presents the basic concepts related to IVPFSs and AA norms, together with their associated operational laws that will be used in the subsequent development of the study.

Definition 1. [21] An IVPFS P in K is defined as

$$P = \{(k, \delta(k), \theta(k)) | k \in K\}, \quad (2.1)$$

where $\delta(k) = [\delta^l(k), \delta^u(k)] \subseteq [0, 1]$ and $\theta(k) = [\theta^-(k), \theta^+(k)] \subseteq [0, 1]$ describe the membership degrees and nonmembership degrees such that $0 \leq (\delta^+(k))^2 + (\theta^+(k))^2 \leq 1$ for all $k \in K$. For convenience, we call $([\delta^-(k), \delta^+(k)], [\theta^-(k), \theta^+(k)])$ an IVPFN, which is represented by $\Upsilon = ([\delta^-, \delta^+], [\theta^-, \theta^+])$.

Definition 2. [21] Let $\Upsilon_1 = ([\delta_1^-, \delta_1^+], [\theta_1^-, \theta_1^+])$ and $\Upsilon_2 = ([\delta_2^-, \delta_2^+], [\theta_2^-, \theta_2^+])$ denote two IVPFNs and $\psi \geq 0$. Then,

$$\begin{aligned} (1) \quad \Upsilon_1 \oplus \Upsilon_2 &= \left(\left[\sqrt{(\delta_1^-)^2 + (\delta_2^-)^2 - (\delta_1^-)^2 (\delta_2^-)^2}, \sqrt{(\delta_1^+)^2 + (\delta_2^+)^2 - (\delta_1^+)^2 (\delta_2^+)^2} \right], \right. \\ &\quad \left. [(\theta_1^-)(\theta_2^-), (\theta_1^+)(\theta_2^+)] \right); \\ (2) \quad \Upsilon_1 \otimes \Upsilon_2 &= \left(\left[\frac{[(\delta_1^-)(\delta_2^-), (\delta_1^+)(\delta_2^+)]}{\sqrt{(\theta_1^-)^2 + (\theta_2^-)^2 - (\theta_1^-)^2 (\theta_2^-)^2}, \sqrt{(\theta_1^+)^2 + (\theta_2^+)^2 - (\theta_1^+)^2 (\theta_2^+)^2}} \right], \right. \\ &\quad \left. [(\delta_1^-)(\delta_2^-), (\delta_1^+)(\delta_2^+)] \right); \\ (3) \quad \psi \Upsilon_1 &= \left(\left[\sqrt{1 - (1 - (\delta_1^-)^2)^\psi}, \sqrt{1 - (1 - (\delta_1^+)^2)^\psi} \right], [(\theta_1^-)^\psi, (\theta_1^+)^\psi] \right); \\ (4) \quad \Upsilon_1^\psi &= \left([(\delta_1^-)^\psi, (\delta_1^+)^\psi], \left[\sqrt{1 - (1 - (\theta_1^-)^2)^\psi}, \sqrt{1 - (1 - (\theta_1^+)^2)^\psi} \right] \right). \end{aligned}$$

Definition 3. [21] Let $\Upsilon = ([\delta^-, \delta^+], [\theta^-, \theta^+])$ be a IVPFN. The score function $S(\Upsilon)$ and the accuracy function $A(\Upsilon)$ of Υ are defined as follows:

$$\begin{aligned} S(\Upsilon) &= \frac{(\delta^-)^2 + (\delta^+)^2 - (\theta^-)^2 - (\theta^+)^2}{2}, \\ A(\Upsilon) &= \frac{(\delta^-)^2 + (\delta^+)^2 + (\theta^-)^2 + (\theta^+)^2}{2}, \end{aligned} \quad (2.2)$$

where $S(\Upsilon) \in [-1, 1]$, and $A(\Upsilon) \in [0, 1]$.

Definition 4. [41] AA t -norms and AA t -conorms are two useful operations that offer clear advantages due to their flexibility with respect to the parameter.

(1) The family $(L_A^\vartheta)_{\vartheta \in [0, \infty)}$ of AA t -norms is defined as follows:

$$L_A^\vartheta(n_1, n_2) = \begin{cases} L_D(n_1, n_2), & \text{if } \vartheta = 0, \\ \min(n_1, n_2), & \text{if } \vartheta \rightarrow \infty, \\ e^{-((- \ln n_1)^\vartheta + (- \ln n_2)^\vartheta)^{1/\vartheta}}, & \text{otherwise;} \end{cases}$$

(2) The family $(S_A^\vartheta)_{\vartheta \in [0, \infty)}$ of AA t -conorms is defined as follows:

$$S_A^\vartheta(n_1, n_2) = \begin{cases} S_D(n_1, n_2), & \text{if } \vartheta = 0, \\ \max(n_1, n_2), & \text{if } \vartheta \rightarrow \infty, \\ 1 - e^{-((- \ln(1-n_1))^\vartheta + (- \ln(1-n_2))^\vartheta)^{1/\vartheta}}, & \text{otherwise;} \end{cases}$$

where $n_1, n_2 \in [0, 1]$, and $\vartheta > 0$ is a parameter. Here, L_D and S_D represent the drastic t -norm and t -conorm, respectively, which are given by

$$L_D(n_1, n_2) = \begin{cases} n_1, & \text{if } n_2 = 1, \\ n_2, & \text{if } n_1 = 1, \\ 0, & \text{otherwise,} \end{cases} \quad S_D(n_1, n_2) = \begin{cases} n_1, & \text{if } n_2 = 0, \\ n_2, & \text{if } n_1 = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 5. [42] Suppose

$$\Upsilon_1 = ([\delta_1^-, \delta_1^+], [\theta_1^-, \theta_1^+]) \quad \text{and} \quad \Upsilon_2 = ([\delta_2^-, \delta_2^+], [\theta_2^-, \theta_2^+])$$

are IVPFNs. Let $\psi > 0$ and $\vartheta > 0$. Then, the AA operations for IVPFNs are defined as follows:

$$\begin{aligned} (1) \quad \Upsilon_1 \oplus \Upsilon_2 &= \left(\left[\begin{array}{l} \sqrt{1 - e^{-((- \ln(1-\delta_1^-))^\vartheta + (- \ln(1-\delta_2^-))^\vartheta)^{1/\vartheta}}}, \sqrt{1 - e^{-((- \ln(1-\delta_1^{+2}))^\vartheta + (- \ln(1-\delta_2^{+2}))^\vartheta)^{1/\vartheta}}} \\ \sqrt{e^{-((- \ln(\theta_1^{-2}))^\vartheta + (- \ln(\theta_2^{-2}))^\vartheta)^{1/\vartheta}}}, \sqrt{e^{-((- \ln(\theta_1^{+2}))^\vartheta + (- \ln(\theta_2^{+2}))^\vartheta)^{1/\vartheta}}} \end{array} \right] \right); \\ (2) \quad \Upsilon_1 \otimes \Upsilon_2 &= \left(\left[\begin{array}{l} \sqrt{e^{-((- \ln(\delta_1^{-2}))^\vartheta + (- \ln(\delta_2^{-2}))^\vartheta)^{1/\vartheta}}}, \sqrt{e^{-((- \ln(\delta_1^{+2}))^\vartheta + (- \ln(\delta_2^{+2}))^\vartheta)^{1/\vartheta}}} \\ \sqrt{1 - e^{-((- \ln(1-\theta_1^{-2}))^\vartheta + (- \ln(1-\theta_2^{-2}))^\vartheta)^{1/\vartheta}}}, \sqrt{1 - e^{-((- \ln(1-\theta_1^{+2}))^\vartheta + (- \ln(1-\theta_2^{+2}))^\vartheta)^{1/\vartheta}}} \end{array} \right] \right); \\ (3) \quad \psi \Upsilon_1 &= \left(\left[\begin{array}{l} \sqrt{1 - e^{-((\psi(- \ln(1-\delta_1^-))^\vartheta)^{1/\vartheta}}}, \sqrt{1 - e^{-((\psi(- \ln(1-\delta_1^{+2}))^\vartheta)^{1/\vartheta}}} \\ \sqrt{e^{-((\psi(- \ln(\theta_1^{-2}))^\vartheta)^{1/\vartheta}}}, \sqrt{e^{-((\psi(- \ln(\theta_1^{+2}))^\vartheta)^{1/\vartheta}}} \end{array} \right] \right); \end{aligned}$$

$$(4) \Upsilon_1^\psi = \left(\left[\sqrt{\frac{e^{-\left(\psi(-\ln(\delta_1^{-2}))^\theta\right)^{\frac{1}{\theta}}}}{1 - e^{-\left(\psi(-\ln(1-\theta_1^{-2}))^\theta\right)^{\frac{1}{\theta}}}}}, \sqrt{\frac{e^{-\left(\psi(-\ln(\delta_1^{+2}))^\theta\right)^{\frac{1}{\theta}}}}{1 - e^{-\left(\psi(-\ln(1-\theta_1^{+2}))^\theta\right)^{\frac{1}{\theta}}}}} \right] \right).$$

3. Interval-valued Pythagorean fuzzy AA operations-based operators

This section presents the HM and DHM operators for IVPFNs based on AA operational laws.

3.1. AA HM AOs

In this subsection, two types of AOs, namely IVPFAAHM and IVPFAAWHM, are formulated, and their mathematical properties are analyzed.

Definition 6. Let $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$, ($s = 1, 2, \dots, b$) be a collection of IVPFNs. Then, the IVPFAAHM operator is defined as follows:

$$IVPFAAHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \frac{\bigoplus_{1 \leq s_1 < \dots < s_\tau \leq b} \left(\bigotimes_{j=1}^\tau \Upsilon_{s_j} \right)^{\frac{1}{\tau}}}{C_b^\tau}, \quad (3.1)$$

where τ ($\tau = 1, 2, \dots, b$) is the parameter of the operator, and C_b^τ denotes the binomial coefficient subject to the constraint $1 \leq s_1 < \dots < s_\tau \leq b$.

Theorem 1. Let $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$, ($s = 1, 2, \dots, b$) be a collection of IVPFNs. Then, the aggregated result obtained by the IVPFAAHM operator is also an IVPFN, which is expressed as follows:

$$IVPFAAHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\sqrt{\frac{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^\tau \left(-\ln(\delta_{s_j}^{-2})^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{1}{\theta}}}}{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^\tau \left(-\ln(\delta_{s_j}^{+2})^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{1}{\theta}}}}}, \sqrt{\frac{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^\tau \left(-\ln(\theta_{s_j}^{-2})^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{1}{\theta}}}}{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^\tau \left(-\ln(\theta_{s_j}^{+2})^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{1}{\theta}}}}} \right] \right). \quad (3.2)$$

Proof. Based on AA operation laws of IVPFNs, we have

$$\bigotimes_{j=1}^{\tau} \Upsilon_{s_j} = \left(\left[\begin{array}{c} \sqrt{e^{-\left(\sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}}, \sqrt{e^{-\left(\sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}} \\ \sqrt{1 - e^{-\left(\sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{-2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}}, \sqrt{1 - e^{-\left(\sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{+2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}} \end{array} \right] \right)$$

and

$$\left(\bigotimes_{j=1}^{\tau} \Upsilon_{s_j} \right)^{\frac{1}{\tau}} = \left(\left[\begin{array}{c} \sqrt{e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}}, \sqrt{e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}} \\ \sqrt{1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{-2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}}, \sqrt{1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{+2}))^{\frac{1}{\theta}}\right)^{\frac{1}{\theta}}}} \end{array} \right] \right).$$

Then,

$$\bigoplus_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(\bigotimes_{j=1}^{\tau} \Upsilon_{s_j} \right)^{\frac{1}{\tau}} = \left(\left[\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)}}, \sqrt{1 - e^{-\left(\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)}}, \\ \sqrt{e^{-\left(\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)}}, \sqrt{e^{-\left(\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)}} \end{array} \right] \right).$$

Therefore,

$$IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{c} \sqrt[1-\epsilon]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)}, \\ \sqrt[1-\epsilon]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)}, \\ \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)}, \\ \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)} \end{array} \right] \right).$$

Next, we show that $IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b)$ is also an IVPFN. Let

$$\delta^{-} = \sqrt[1-\epsilon]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)}$$

and

$$\theta^{-} = \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)}.$$

Because $0 \leq \delta_s^{-}, \theta_s^{-} \leq 1$, we have

$$\Rightarrow 0 \leq (-\ln(\delta_{s_j}^{-2}))^{\theta} \leq 1 \Rightarrow 0 \leq \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \leq 1$$

$$\Rightarrow 0 \leq \sqrt[1-\epsilon]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \right)} \leq 1;$$

also, $0 \leq (-\ln(1-\theta_{s_j}^{-2}))^{\theta} \leq 1$,

$$\Rightarrow 0 \leq \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\theta}} \right) \right)^{\frac{1}{\theta}} \leq 1$$

$$\Rightarrow 0 \leq \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \leq 1.$$

Therefore $0 \leq \delta^-, \theta^- \leq 1$. Similarly, one can verify that

$$0 \leq \delta^+ = \sqrt[e]{1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \leq 1$$

and

$$0 \leq \theta^+ = \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \leq 1.$$

Now,

$$0 \leq \delta^{+2} + \theta^{+2}$$

$$\begin{aligned} &= \left(\sqrt[e]{1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^2 + \left(\sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^2 \\ &= 1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} + e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \\ &\leq 1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} + e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \\ &= 1. \end{aligned}$$

Thus, the aggregated result is an IVPFN, and hence, the proof of Theorem 1 is completed. $\square$

Theorem 2. Suppose

$$\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+]),$$

where $(s = 1, 2, \dots, b)$ denotes a family of IVPFNs. If $\Upsilon_s = \Upsilon$ for every s , then

$$IVPFAAHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \Upsilon. \quad (3.3)$$

Proof. Since

$$IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{c} \sqrt[\frac{1}{\theta}]{\frac{1}{1-e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \\ \sqrt[\frac{1}{\theta}]{\frac{1}{1-e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \\ \sqrt[\frac{1}{\theta}]{\frac{1}{e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \\ \sqrt[\frac{1}{\theta}]{\frac{1}{e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \end{array} \right] \right).$$

As it is given that $\Upsilon_s = \Upsilon$ for every s , therefore,

$$IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{c} \sqrt[\frac{1}{\theta}]{\frac{1}{1-e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \\ \sqrt[\frac{1}{\theta}]{\frac{1}{1-e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \\ \sqrt[\frac{1}{\theta}]{\frac{1}{e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \\ \sqrt[\frac{1}{\theta}]{\frac{1}{e} \left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1-e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\theta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)}}, \end{array} \right] \right).$$

$$\begin{aligned}
& \left(\left[\frac{\sqrt[1-\frac{1}{C_b^\tau}]{\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(\delta^{-2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}}{\sqrt[1-\frac{1}{C_b^\tau}]{\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(\delta^{+2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}} \right]^{\frac{1}{\theta}}, \right. \\
& \left. \left[\frac{\sqrt[e^{-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(1-\theta^{-2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}}]{\sqrt[e^{-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(1-\theta^{+2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}} \right]^{\frac{1}{\theta}} \right) \right] \\
& = \left(\left[\frac{\sqrt[1-\frac{1}{C_b^\tau}]{\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(\delta^{-2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}}{\sqrt[1-\frac{1}{C_b^\tau}]{\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(\delta^{+2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}} \right]^{\frac{1}{\theta}}, \right. \\
& \left. \left[\frac{\sqrt[e^{-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(1-\theta^{-2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}}]{\sqrt[e^{-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1-e^{-\left((-\ln(1-\theta^{+2}))^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}}} \right]^{\frac{1}{\theta}} \right) \right] = \Upsilon.
\end{aligned}$$

This completes the proof. $\square$

Theorem 3. Suppose

$$\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+]) \quad (s = 1, 2, \dots, b)$$

and

$$\widehat{\Upsilon}_s = ([\widehat{\delta}_s^-, \widehat{\delta}_s^+], [\widehat{\theta}_s^-, \widehat{\theta}_s^+]) \quad (s = 1, 2, \dots, b)$$

are two families of IVPFNs. If $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , then

$$IVPF\AA HM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPF\AA HM^\tau(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b). \quad (3.4)$$

Proof. Since $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , it follows that

$$\begin{aligned}
& \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta} \geq \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}, \\
& \Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}} \leq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}},
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \geq 1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \\
&\Rightarrow \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \\
&\leq \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \\
&\Rightarrow -\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}} \\
&\geq -\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}} \\
&\Rightarrow \sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}}{1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}} \right)} \\
&\leq \sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}}{1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}} \right)} .
\end{aligned}$$

Similarly,

$$\sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\delta_{s_j}^{+2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}}{1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{+2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}} \right)} \leq \sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{+2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}}{1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\delta}_{s_j}^{+2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}} \right)}$$

and

$$\begin{aligned}
&\sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{-2}))^{\theta} \geq \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\theta}_{s_j}^{-2}))^{\theta} \\
&\Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}} \leq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\theta}_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}} \\
&\Rightarrow 1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \theta_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)} \geq 1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\theta}_{s_j}^{-2}))^{\theta}\right)^{\frac{1}{\theta}}\right)}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow -\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\theta_{s_j}^{-2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \leq -\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \\
&\Rightarrow \frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\theta_{s_j}^{-2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \\
&\leq \frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \\
&\Rightarrow \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\theta_{s_j}^{-2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}} \\
&\geq \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}.
\end{aligned}$$

Similarly,

$$\sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\theta_{s_j}^{+2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}} \geq \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1-\widehat{\theta}_{s_j}^{+2}) \right)^{\frac{\theta}{\theta}} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \right)^{\frac{1}{\theta}}}.$$

Thus, we have $IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAAHM^{\tau}(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b)$. $\square$

Theorem 4. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) is a family of IVPFNs. Let $\Upsilon^- = \min\{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b\}$ and $\Upsilon^+ = \max\{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b\}$, then

$$\Upsilon^- \leq IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+. \quad (3.5)$$

Proof. From Theorem 2, we have

$IVPFAAHM^{\tau}(\Upsilon^-, \Upsilon^-, \dots, \Upsilon^-) = \Upsilon^-$, $IVPFAAHM^{\tau}(\Upsilon^+, \Upsilon^+, \dots, \Upsilon^+) = \Upsilon^+$. According to Theorem 3, we get $\Upsilon^- \leq IVPFAAHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+$. $\square$

Definition 7. Let $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) be a family of IVPFNs. Let $\chi = (\chi_1, \chi_2, \dots, \chi_b)^T$ denote the corresponding weight vector, where $\chi_s \in [0, 1]$, and

$$\sum_{s=1}^b \chi_s = 1.$$

If

$$IVPFAAWHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \frac{\bigoplus_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(\bigotimes_{j=1}^{\tau} \chi_{s_j} \Upsilon_{s_j} \right)^{\frac{1}{\tau}}}{C_b^{\tau}}, \quad (3.6)$$

then, the operator is referred to as the IVPFAAWHM operator.

Theorem 5. Let $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$, ($s = 1, 2, \dots, b$) be a collection of IVPFNs. Then, the aggregated result obtained by the IVPFAAWHM operator is also an IVPFN, which is expressed as follows:

$$IVPFAAWHM^r(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{l} \sqrt[1 - e^{-\left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(\delta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right)^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \\ \sqrt[1 - e^{-\left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right)^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \\ \sqrt[e^{-\left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(1 - \theta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right)^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \\ \sqrt[e^{-\left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(1 - \theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right)^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}} \end{array} \right] \right). \quad (3.7)$$

Proof. Based on the AA operation laws of IVPFNs, we have

$$\chi_{s_j} \Upsilon_{s_j} = \left(\left[\begin{array}{l} \sqrt[1 - e^{-\left(\chi_{s_j} (-\ln(1 - \delta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \sqrt[1 - e^{-\left(\chi_{s_j} (-\ln(1 - \delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \\ \left[\sqrt[e^{-\left(\chi_{s_j} (-\ln(\theta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \sqrt[e^{-\left(\chi_{s_j} (-\ln(\theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}} \end{array} \right],$$

$$\bigotimes_{j=1}^r \chi_{s_j} \Upsilon_{s_j} = \left(\left[\begin{array}{l} \sqrt[e^{-\left(\sum_{j=1}^r \chi_{s_j} (-\ln(\delta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \sqrt[e^{-\left(\sum_{j=1}^r \chi_{s_j} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \\ \left[\sqrt[1 - e^{-\left(\sum_{j=1}^r \chi_{s_j} (-\ln(1 - \theta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \sqrt[1 - e^{-\left(\sum_{j=1}^r \chi_{s_j} (-\ln(1 - \theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}} \end{array} \right)$$

and

$$\left(\bigotimes_{j=1}^r \chi_{s_j} \Upsilon_{s_j} \right)^{\frac{1}{r}} = \left(\left[\begin{array}{l} \sqrt[e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(\delta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \sqrt[e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(\delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \\ \left[\sqrt[1 - e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(1 - \theta_{s_j}^-))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}}, \sqrt[1 - e^{-\left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} (-\ln(1 - \theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)} \right]^{\frac{1}{\theta}} \end{array} \right).$$

Then,

$$\bigoplus_{1 \leq s_1 < \dots < s_r \leq b} \left(\bigotimes_{j=1}^r \chi_{s_j} \Upsilon_{s_j} \right)^{\frac{1}{r}} = \left(\left[\begin{array}{c} \sqrt[1-e]{- \left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(\delta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[1-e]{- \left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(\delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[e]{- \left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(1-\theta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[e]{- \left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(1-\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}} \end{array} \right] \right)^{\frac{1}{r}}.$$

Therefore,

$$IVPFAAWHM^r(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{c} \sqrt[1-e]{- \left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(\delta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[1-e]{- \left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(\delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[e]{- \left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(1-\theta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[e]{- \left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(1-\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}} \end{array} \right] \right)^{\frac{1}{r}}.$$

Next, we show that $IVPFAAWHM^r(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b)$ is also an IVPFN. Let

$$\delta^- = \sqrt[1-e]{- \left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(\delta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}$$

and

$$\theta^- = \sqrt[e]{- \left(\frac{1}{C_b^r} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(- \ln \left(1-e^{- \left(\frac{1}{r} \sum_{j=1}^r \chi_{s_j} \left(- \ln(1-\theta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right) \right)^{\frac{\theta}{1-\theta}} \right) \right)^{\frac{1}{1-\theta}}}.$$

Because

$$0 \leq \delta_s^-, \quad \theta_s^- \leq 1,$$

we have

$$\begin{aligned} \Rightarrow 0 &\leq \left(-\ln(\delta_{s_j}^{-2})\right)^\vartheta \leq 1, \\ \Rightarrow 0 &\leq \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\delta_{s_j}^{-2})\right)^\vartheta\right)^{\frac{1}{\vartheta}}}\right) \leq 1, \\ \Rightarrow 0 &\leq \sqrt[1 - e^{-\left(\frac{1}{C_b^{\frac{\tau}{\vartheta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\delta_{s_j}^{-2})\right)^\vartheta\right)^{\frac{1}{\vartheta}}}\right)\right)^\vartheta\right)^{\frac{1}{\vartheta}}}]{} \leq 1. \end{aligned}$$

Also, $0 \leq \left(-\ln(1 - \theta_{s_j}^{-2})\right)^\vartheta \leq 1,$

$$\begin{aligned} \Rightarrow 0 &\leq \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2})\right)^\vartheta\right)^{\frac{1}{\vartheta}}}\right)\right)^\vartheta \leq 1 \\ \Rightarrow 0 &\leq \sqrt[e^{-\left(\frac{1}{C_b^{\frac{\tau}{\vartheta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2})\right)^\vartheta\right)^{\frac{1}{\vartheta}}}\right)\right)^\vartheta\right)^{\frac{1}{\vartheta}}}]{} \leq 1. \end{aligned}$$

Therefore,

$$0 \leq \delta^-, \quad \theta^- \leq 1.$$

Similarly, one can verify that

$$0 \leq \delta^+ = \sqrt[1 - e^{-\left(\frac{1}{C_b^{\frac{\tau}{\vartheta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\delta_{s_j}^{+2})\right)^\vartheta\right)^{\frac{1}{\vartheta}}}\right)\right)^\vartheta\right)^{\frac{1}{\vartheta}}}]{} \leq 1$$

and

$$0 \leq \theta^+ = \sqrt[e^{-\left(\frac{1}{C_b^{\frac{\tau}{\vartheta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{+2})\right)^\vartheta\right)^{\frac{1}{\vartheta}}}\right)\right)^\vartheta\right)^{\frac{1}{\vartheta}}}]{} \leq 1.$$

Now,

$$0 \leq \delta^{+2} + \theta^{+2}$$

$$\begin{aligned} &= \left(\sqrt{1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^\theta \right)^\frac{1}{\theta}} \right)} \right)^2 + \left(\sqrt{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^\theta \right)^\frac{1}{\theta}} \right)} \right)^2 \\ &= 1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \chi_{s_j} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^\theta \right)^\frac{1}{\theta}} \right)} + e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \chi_{s_j} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^\theta \right)^\frac{1}{\theta}} \right)} \\ &\leq 1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^\theta \right)^\frac{1}{\theta}} \right)} + e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right)^\theta \right)^\frac{1}{\theta}} \right)} = 1. \end{aligned}$$

Thus, the aggregated result is an IVPFN, and hence, the proof of Theorem 5 is completed. $\square$

Example 1. Let

$$\Upsilon_1 = ([0.2, 0.4], [0.1, 0.5]), \quad \Upsilon_2 = ([0.1, 0.3], [0.2, 0.5]), \quad \text{and} \quad \Upsilon_3 = ([0.2, 0.5], [0.1, 0.4])$$

be the given IVPFNs, with the associated weight vector $\chi = (0.4, 0.4, 0.2)$. For the parameter settings $\vartheta = 3$ and $\tau = 2$, the aggregated result Υ is derived by applying Eq (3.7) as follows:

$$\begin{aligned} \delta^- &= \sqrt{1 - e^{-\left(\left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} \left(0.4(-\ln(0.2^2))^3 + 0.4(-\ln(0.1^2))^3 \right)^{\frac{1}{3}}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} + \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} \left(0.4(-\ln(0.2^2))^3 + 0.2(-\ln(0.2^2))^3 \right)^{\frac{1}{3}}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} + \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} \left(0.4(-\ln(0.1^2))^3 + 0.2(-\ln(0.2^2))^3 \right)^{\frac{1}{3}}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)} = 0.2944, \\ \delta^+ &= \sqrt{1 - e^{-\left(\left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} \left(0.4(-\ln(0.4^2))^3 + 0.4(-\ln(0.3^2))^3 \right)^{\frac{1}{3}}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} + \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} \left(0.4(-\ln(0.4^2))^3 + 0.2(-\ln(0.5^2))^3 \right)^{\frac{1}{3}}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} + \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} \left(0.4(-\ln(0.3^2))^3 + 0.2(-\ln(0.5^2))^3 \right)^{\frac{1}{3}}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)^\frac{1}{3}} \right)} = 0.5145, \end{aligned}$$

$$\theta^- = \sqrt[e]{\left(\left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((1-0.1^2)))^3 + 0.4(-\ln((1-0.2^2)))^3 \right)^{\frac{1}{3}}} \right) \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((1-0.1^2)))^3 + 0.2(-\ln((1-0.1^2)))^3 \right)^{\frac{1}{3}}} \right) \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((1-0.2^2)))^3 + 0.2(-\ln((1-0.1^2)))^3 \right)^{\frac{1}{3}}} \right) \right)^3 \right)^{\frac{1}{3}} \right)^{\frac{1}{3}} = 0.1194,$$

$$\theta^+ = \sqrt[e]{\left(\left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((1-0.5^2)))^3 + 0.4(-\ln((1-0.5^2)))^3 \right)^{\frac{1}{3}}} \right) \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((1-0.5^2)))^3 + 0.2(-\ln((1-0.4^2)))^3 \right)^{\frac{1}{3}}} \right) \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((1-0.5^2)))^3 + 0.2(-\ln((1-0.4^2)))^3 \right)^{\frac{1}{3}}} \right) \right)^3 \right)^{\frac{1}{3}} \right)^{\frac{1}{3}} = 0.4111.$$

Theorem 6. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) and $\widehat{\Upsilon}_s = ([\widehat{\delta}_s^-, \widehat{\delta}_s^+], [\widehat{\theta}_s^-, \widehat{\theta}_s^+])$ ($s = 1, 2, \dots, b$) are two families of IVPFNs. If $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , then

$$IVPFAAWHM^r(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAAWHM^r(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b). \quad (3.8)$$

Proof. Since $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$ and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , it follows that

$$\begin{aligned} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\delta_{s_j}^{-2}))^{\vartheta} &\geq \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\vartheta} \\ &\Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\delta_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}} \leq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}} \\ &\Rightarrow 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\delta_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}}} \geq 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}}} \\ &\Rightarrow \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\delta_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}}} \right) \right)^{\vartheta} \\ &\leq \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\widehat{\delta}_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}}} \right) \right)^{\vartheta} \\ &\Rightarrow -\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\delta_{s_j}^{-2}))^{\vartheta} \right)^{\frac{1}{\vartheta}}} \right) \right)^{\vartheta} \right)^{\frac{1}{\vartheta}} \end{aligned}$$

$$\begin{aligned}
&\geq -\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\delta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}} \right)} \right) \right)^\theta \right)^{\frac{1}{\theta}} \\
&\Rightarrow \sqrt[1-e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\delta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}} \right)} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \\
&\leq \sqrt[1-e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\delta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}} \right)} \right) \right)^\theta \right)^{\frac{1}{\theta}}} .
\end{aligned}$$

Similarly,

$$\sqrt[1-e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\delta_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}} \right)} \right) \right)^\theta \right)^{\frac{1}{\theta}}} \leq \sqrt[1-e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\delta}_{s_j}^{+2}) \right)^\theta \right)^{\frac{1}{\theta}} \right)} \right) \right)^\theta \right)^{\frac{1}{\theta}}}$$

and

$$\begin{aligned}
&\sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2}) \right)^\theta \geq \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\theta}_{s_j}^{-2}) \right)^\theta \\
&\Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}} \leq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\theta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}} \\
&\Rightarrow 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \geq 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\theta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \\
&\Rightarrow -\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \leq -\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\theta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \\
&\Rightarrow \frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \\
&\leq \frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\theta}_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \\
&\Rightarrow \sqrt[1-e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \theta_{s_j}^{-2}) \right)^\theta \right)^{\frac{1}{\theta}}} \right) \right)^\theta \right)^{\frac{1}{\theta}}}
\end{aligned}$$

$$\geq \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln \left(1 - \widehat{\theta}_{s_j}^{-2} \right) \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}}.$$

Similarly,

$$\sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln \left(1 - \theta_{s_j}^{+2} \right) \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \geq \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln \left(1 - \widehat{\theta}_{s_j}^{+2} \right) \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}}.$$

Thus, we have $IVPFAAWHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAAWHM^\tau(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b)$. $\square$

Theorem 7. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) is a family of IVPFNs. Let $\Upsilon^- = \min \{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_s\}$ and $\Upsilon^+ = \max \{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_s\}$; then

$$\Upsilon^- \leq IVPFAAWHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+. \quad (3.9)$$

Proof. From Theorem 5, we have

$$IVPFAAWHM^\tau(\Upsilon^-, \Upsilon^-, \dots, \Upsilon^-) = \Upsilon^-, \quad IVPFAAWHM^\tau(\Upsilon^+, \Upsilon^+, \dots, \Upsilon^+) = \Upsilon^+.$$

According to Theorem 6, we get

$$\Upsilon^- \leq IVPFAAWHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+.$$

This completes the proof. $\square$

3.2. AA DHM AOs

In this subsection, two types of AOs, namely IVPFAADHM and IVPFAAWDHM, are formulated, and their mathematical properties are analyzed.

Definition 8. Let $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$, ($s = 1, 2, \dots, b$) be a collection of IVPFNs. Then, the IVPFAADHM operator is defined as follows:

$$IVPFAADHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\bigotimes_{1 \leq s_1 < \dots < s_\tau \leq b} \left(\frac{\bigoplus_{j=1}^{\tau} \Upsilon_{s_j}}{\tau} \right) \right)^{\frac{1}{C_b^\tau}}, \quad (3.10)$$

where τ ($\tau = 1, 2, \dots, b$) is the parameter of the operator, and C_b^τ denotes the binomial coefficient subject to the constraint $1 \leq s_1 < \dots < s_\tau \leq b$.

Theorem 8. Let $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$, ($s = 1, 2, \dots, b$) be a collection of IVPFNs. Then, the aggregated result obtained by the IVPFAADHM operator is also an IVPFN, which is expressed as follows:

$$IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{c} \sqrt[e]{-\frac{1}{C_b^{\frac{\tau}{\theta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right)^{\theta} \right)^{\frac{1}{\theta}}}, \\ \sqrt[e]{-\frac{1}{C_b^{\frac{\tau}{\theta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right)^{\theta} \right)^{\frac{1}{\theta}}}, \\ \sqrt[1-e]{-\frac{1}{C_b^{\frac{\tau}{\theta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right)^{\theta} \right)^{\frac{1}{\theta}}}, \\ \sqrt[1-e]{-\frac{1}{C_b^{\frac{\tau}{\theta}}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right)^{\theta} \right)^{\frac{1}{\theta}}} \end{array} \right] \right). \quad (3.11)$$

Proof. Based on the AA operation laws of IVPFNs, we have

$$\bigoplus_{j=1}^{\tau} \Upsilon_{s_j} = \left(\left[\begin{array}{c} \sqrt[1-e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}}, \sqrt[1-e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \\ \sqrt[e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}}, \sqrt[e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \end{array} \right] \right)$$

and

$$\left(\frac{\bigoplus_{j=1}^{\tau} \Upsilon_{s_j}}{\tau} \right) = \left(\left[\begin{array}{c} \sqrt[1-e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}}, \sqrt[1-e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \\ \sqrt[e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}}, \sqrt[e]{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \end{array} \right] \right).$$

Then,

$$\bigotimes_{1 \leq s_1 < \dots < s_\tau \leq b} \left(\frac{\bigoplus_{j=1}^{\tau} \Upsilon_{s_j}}{\tau} \right) = \left[\begin{array}{c} \sqrt[e]{-\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[e]{-\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[1-e]{-\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[1-e]{-\sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}} \end{array} \right].$$

Therefore,

$$IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left[\begin{array}{c} \sqrt[e]{-\frac{1}{C_b^{\frac{\tau}{1-\theta}}} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[e]{-\frac{1}{C_b^{\frac{\tau}{1-\theta}}} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[1-e]{-\frac{1}{C_b^{\frac{\tau}{1-\theta}}} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}, \\ \sqrt[1-e]{-\frac{1}{C_b^{\frac{\tau}{1-\theta}}} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}} \end{array} \right].$$

Next, we show that $IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b)$ is also an IVPFN. Let

$$\delta^{-} = \sqrt[e]{-\frac{1}{C_b^{\frac{\tau}{1-\theta}}} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}$$

and

$$\theta^{-} = \sqrt[1-e]{-\frac{1}{C_b^{\frac{\tau}{1-\theta}}} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\frac{1}{\theta}} \right)^{\frac{\theta}{1-\theta}}} \right) \right)^{\frac{1}{1-\theta}}}.$$

Because $0 \leq \delta_s^-, \theta_s^- \leq 1$, we have

$$\begin{aligned} 0 &\leq \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\frac{\theta}{\vartheta}} \leq 1, \\ \Rightarrow 0 &\leq \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}} \leq 1, \\ \Rightarrow 0 &\leq \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}}\right)} \leq 1. \end{aligned}$$

Also,

$$\begin{aligned} 0 &\leq \left(-\ln(\theta_{s_j}^{-2})\right)^{\frac{\theta}{\vartheta}} \leq 1, \\ \Rightarrow 0 &\leq \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\theta_{s_j}^{-2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right) \leq 1, \\ \Rightarrow 0 &\leq \sqrt[1-e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\theta_{s_j}^{-2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}}\right)} \leq 1. \end{aligned}$$

Thus, $0 \leq \delta^-, \theta^- \leq 1$. Similarly, one can verify that

$$0 \leq \delta^+ = \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \delta_{s_j}^{+2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}}\right)} \leq 1,$$

and

$$0 \leq \theta^+ = \sqrt[1-e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\theta_{s_j}^{+2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}}\right)} \leq 1.$$

Now,

$$0 \leq \delta^{+2} + \theta^{+2}$$

$$= \left(\sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(1 - \delta_{s_j}^{+2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}}\right)} \right)^2 + \left(\sqrt[1-e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \left(-\ln(\theta_{s_j}^{+2})\right)^{\frac{\theta}{\vartheta}}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\frac{\theta}{\vartheta}}\right)} \right)^2$$

$$\begin{aligned}
&= e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} + 1 - e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \\
&\leq e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} + 1 - e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}} \\
&= 1.
\end{aligned}$$

Thus, the aggregated result is an IVPFN, and hence, the proof of Theorem 8 is completed. $\square$

Theorem 9. Suppose

$$\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+]),$$

where $(s = 1, 2, \dots, b)$ denotes a family of IVPFNs. If

$$\Upsilon_s = \Upsilon$$

for every s , then

$$IVPFAADHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \Upsilon. \quad (3.12)$$

Proof. Since

$$IVPFAADHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left[\begin{array}{c} \sqrt[\tau]{e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}}}, \\ \sqrt[\tau]{e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}}}, \\ \sqrt[\tau]{1 - e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}}}, \\ \sqrt[\tau]{1 - e^{\left(-\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^\theta \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}}}} \end{array} \right].$$

It is given that $\Upsilon_s = \Upsilon$ for every s , and therefore,

$$\begin{aligned}
 IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) &= \left[\begin{array}{c} \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\delta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1-\delta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}} \end{array} \right] \\
 &= \left[\begin{array}{c} \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(1-\delta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(1-\delta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(\theta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(\theta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}} \end{array} \right] \\
 &= \left[\begin{array}{c} \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(1-\delta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(1-\delta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(\theta^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[\frac{1}{C_b^{\tau}}]{\sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left((-\ln(\theta^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta}}} \end{array} \right] = \Upsilon.
 \end{aligned}$$

This completes the proof. $\square$

Theorem 10. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) and $\widehat{\Upsilon}_s = ([\widehat{\delta}_s^-, \widehat{\delta}_s^+], [\widehat{\theta}_s^-, \widehat{\theta}_s^+])$ ($s = 1, 2, \dots, b$) are two families of IVPFNs. If $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , then

$$IVPFAADHM^\tau(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAADHM^\tau(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b). \quad (3.13)$$

Proof. Since $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , it follows that

$$\begin{aligned} & \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\vartheta} \leq \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{-2}))^{\vartheta} \\ & \Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}} \geq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}} \\ & \Rightarrow 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}} \leq 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}, \\ & \Rightarrow -\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right) \geq -\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right) \\ & \Rightarrow \frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\vartheta} \\ & \geq \frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\vartheta} \\ & \Rightarrow \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\vartheta}\right)^{\frac{1}{\vartheta}}} \\ & \leq \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\vartheta}\right)^{\frac{1}{\vartheta}}}. \end{aligned}$$

Similarly,

$$\sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \delta_{s_j}^{+2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\vartheta}\right)^{\frac{1}{\vartheta}}} \leq \sqrt[e]{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_\tau \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(1 - \widehat{\delta}_{s_j}^{+2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}}\right)\right)^{\vartheta}\right)^{\frac{1}{\vartheta}}}$$

and

$$\sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\vartheta} \leq \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{-2}))^{\vartheta}$$

$$\begin{aligned}
& \Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}} \geq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}} \\
& \Rightarrow 1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \leq 1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \\
& \Rightarrow \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta} \\
& \geq \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta} \\
& \Rightarrow -\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta}\right)^{\frac{1}{\vartheta}} \\
& \leq -\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta}\right)^{\frac{1}{\vartheta}} \\
& \Rightarrow \sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta}\right)}]{1 - e} \\
& \leq \sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{-2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta}\right)}]{1 - e}.
\end{aligned}$$

Similarly,

$$\sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\theta_{s_j}^{+2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta}\right)}]{1 - e} \leq \sqrt[1 - e^{\left(-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} (-\ln(\widehat{\theta}_{s_j}^{+2}))^{\vartheta}\right)^{\frac{1}{\vartheta}}\right)} \right) \right)^{\vartheta}\right)}]{1 - e}.$$

Thus, we have $IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAADHM^{\tau}(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b)$. $\square$

Theorem 11. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) is a family of IVPFNs. Let $\Upsilon^- = \min\{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_s\}$ and $\Upsilon^+ = \max\{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_s\}$; then

$$\Upsilon^- \leq IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+. \quad (3.14)$$

Proof. From Theorem 9, we have

$$IVPFAADHM^{\tau}(\Upsilon^{-}, \Upsilon^{-}, \dots, \Upsilon^{-}) = \Upsilon^{-}, IVPFAADHM^{\tau}(\Upsilon^{+}, \Upsilon^{+}, \dots, \Upsilon^{+}) = \Upsilon^{+}.$$

According to Theorem 10, we get

$$\Upsilon^{-} \leq IVPFAADHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^{+}.$$

This completes the proof. $\square$

Definition 9. Let $\Upsilon_s = ([\delta_s^{-}, \delta_s^{+}], [\theta_s^{-}, \theta_s^{+}])$ ($s = 1, 2, \dots, b$) be a family of IVPFNs. Let

$$\chi = (\chi_1, \chi_2, \dots, \chi_b)^T$$

denote the corresponding weight vector, where $\chi_s \in [0, 1]$, and

$$\sum_{s=1}^b \chi_s = 1.$$

If

$$IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\bigotimes_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(\frac{\bigoplus_{j=1}^{\tau} \chi_{s_j} \Upsilon_{s_j}}{\tau} \right)^{\frac{1}{C_b^{\tau}}} \right)^{\frac{1}{C_b^{\tau}}}, \quad (3.15)$$

then the operator is referred to as the IVPFAAWDHM operator.

Theorem 12. Let $\Upsilon_s = ([\delta_s^{-}, \delta_s^{+}], [\theta_s^{-}, \theta_s^{+}])$, ($s = 1, 2, \dots, b$) be a collection of IVPFNs. Then, the aggregated result obtained by the IVPFAAWDHM operator is also an IVPFN, which is expressed as follows:

$$IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left(\left[\begin{array}{c} \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2}) \right)^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}} \right]} \\ \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2}) \right)^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}} \right]} \\ \sqrt[1-e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2}) \right)^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}} \right]} \\ \sqrt[1-e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\theta}}} \right]} \end{array} \right]. \quad (3.16)$$

Proof. Based on the AA operation laws of IVPFNs, we have

$$\chi_{s_j} \Upsilon_{s_j} = \begin{pmatrix} \left[\sqrt{1 - e^{-\left(\chi_{s_j}(-\ln(1-\delta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}}}, \sqrt{1 - e^{-\left(\chi_{s_j}(-\ln(1-\delta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}}} \right], \\ \left[\sqrt{e^{-\left(\chi_{s_j}(-\ln(\theta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}}}, \sqrt{e^{-\left(\chi_{s_j}(-\ln(\theta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}}} \right] \end{pmatrix},$$

$$\bigoplus_{j=1}^{\tau} \chi_{s_j} \Upsilon_{s_j} = \begin{pmatrix} \left[\sqrt{1 - e^{-\left(\sum_{j=1}^{\tau} \chi_{s_j}(-\ln(1-\delta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}}}, \sqrt{1 - e^{-\left(\sum_{j=1}^{\tau} \chi_{s_j}(-\ln(1-\delta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}}} \right], \\ \left[\sqrt{e^{-\left(\sum_{j=1}^{\tau} \chi_{s_j}(-\ln(\theta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}}}, \sqrt{e^{-\left(\sum_{j=1}^{\tau} \chi_{s_j}(-\ln(\theta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}}} \right] \end{pmatrix}$$

and

$$\left(\frac{\bigoplus_{j=1}^{\tau} \chi_{s_j} \Upsilon_{s_j}}{\tau} \right) = \begin{pmatrix} \left[\sqrt{1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(1-\delta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}}}, \sqrt{1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(1-\delta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}}} \right], \\ \left[\sqrt{e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(\theta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}}}, \sqrt{e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(\theta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}}} \right] \end{pmatrix}.$$

Then,

$$\bigotimes_{1 \leq s_1 < \dots < s_r \leq b} \left(\bigoplus_{j=1}^{\tau} \chi_{s_j} \Upsilon_{s_j} \right)^{\frac{1}{\tau}} = \begin{pmatrix} \left[\sqrt{e^{-\left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(1-\delta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}} \right) \right)^\theta} \right)^{\frac{1}{\theta}}}, \sqrt{e^{-\left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(1-\delta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}} \right) \right)^\theta} \right)^{\frac{1}{\theta}}}} \right], \\ \left[\sqrt{1 - e^{-\left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(\theta_{s_j}^{-2}))^\theta\right)^{\frac{1}{\theta}}} \right) \right)^\theta} \right)^{\frac{1}{\theta}}}}, \sqrt{1 - e^{-\left(\sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j}(-\ln(\theta_{s_j}^{+2}))^\theta\right)^{\frac{1}{\theta}}} \right) \right)^\theta} \right)^{\frac{1}{\theta}}}} \right] \end{pmatrix}.$$

Therefore,

$$IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) = \left[\begin{array}{c} \sqrt[\frac{1}{\theta}]{\sqrt[e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(1-\delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(1-\delta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[1-e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}}, \\ \sqrt[\frac{1}{\theta}]{\sqrt[1-e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\theta_{s_j}^{+2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}} \end{array} \right].$$

Next, we show that $IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b)$ is also an IVPFN. Let

$$\delta^{-} = \sqrt[\frac{1}{\theta}]{\sqrt[e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(1-\delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}}.$$

and

$$\theta^{-} = \sqrt[\frac{1}{\theta}]{\sqrt[1-e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}}.$$

Because $0 \leq \delta_s^{-}, \theta_s^{-} \leq 1$, we have

$$\begin{aligned} 0 &\leq \left(-\ln(1 - \delta_{s_j}^{-2}) \right)^{\theta} \leq 1 \\ \Rightarrow 0 &\leq \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(1-\delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)^{\theta} \leq 1 \\ \Rightarrow 0 &\leq \sqrt[\frac{1}{\theta}]{\sqrt[e]{-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(1-\delta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \right)}} \leq 1. \end{aligned}$$

Also,

$$0 \leq \left(-\ln(\theta_{s_j}^{-2}) \right)^{\theta} \leq 1 \Rightarrow 0 \leq \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} (-\ln(\theta_{s_j}^{-2}))^{\theta} \right)^{\frac{1}{\theta}}} \right) \leq 1$$

$$\Rightarrow 0 \leq \sqrt[1-e]{1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \leq 1.$$

Thus, $0 \leq \delta^-, \theta^- \leq 1$. Similarly, one can verify that

$$0 \leq \delta^+ = \sqrt[e]{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \leq 1$$

and

$$0 \leq \theta^+ = \sqrt[1-e]{1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \leq 1.$$

Now,

$$\begin{aligned} 0 &\leq \delta^{+2} + \theta^{+2} \\ &= \left(\sqrt[e]{e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \right)^2 + \left(\sqrt[1-e]{1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \right)^2 \\ &= e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} + 1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \\ &\leq e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} + 1 - e^{-\left(\frac{1}{C_b^\tau} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right) \right) \right)^{\frac{1}{\theta}}} \\ &= 1. \end{aligned}$$

Thus, the aggregated result is an IVPFN, and hence, the proof of Theorem 12 is completed. $\square$

Example 2. Let $\Upsilon_1 = ([0.2, 0.4], [0.1, 0.5])$, $\Upsilon_2 = ([0.1, 0.3], [0.2, 0.5])$, and $\Upsilon_3 = ([0.2, 0.5], [0.1, 0.4])$ be the given IVPFNs with the associated weight vector $\chi = (0.4, 0.4, 0.2)$. For the parameter settings

$\vartheta = 3$ and $\tau = 2$, the aggregated result Υ is derived by applying Eq (3.16) as follows:

$$\begin{aligned}\delta^- &= \sqrt[3]{e^{\left[-\frac{1}{3} \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln(1-0.2^2))^3 + 0.4(-\ln(1-0.1^2))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln(1-0.2^2))^3 + 0.2(-\ln(1-0.2^2))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln(1-0.1^2))^3 + 0.2(-\ln(1-0.2^{-2}))^3 \right)^{\frac{1}{3}}} \right)^3 \right) \right]}} = 0.1511, \\ \delta^+ &= \sqrt[3]{e^{\left[-\frac{1}{3} \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln(1-0.4^2))^3 + 0.4(-\ln(1-0.3^2))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln(1-0.4^2))^3 + 0.2(-\ln(1-0.5^2))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln(1-0.3^2))^3 + 0.2(-\ln(1-0.5^{-2}))^3 \right)^{\frac{1}{3}}} \right)^3 \right) \right]}} = 0.3478, \\ \theta^- &= \sqrt[3]{1 - e^{\left[-\frac{1}{3} \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((0.1^2)))^3 + 0.4(-\ln((0.2^2)))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((0.1^2)))^3 + 0.2(-\ln((0.1^2)))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((0.2^2)))^3 + 0.2(-\ln((0.1^2)))^3 \right)^{\frac{1}{3}}} \right)^3 \right) \right]}} = 0.2496, \\ \theta^+ &= \sqrt[3]{1 - e^{\left[-\frac{1}{3} \left(\left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((0.5^2)))^3 + 0.4(-\ln((0.5^2)))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((0.5^2)))^3 + 0.2(-\ln((0.4^2)))^3 \right)^{\frac{1}{3}}} \right)^3 + \left(-\ln \left(1 - e^{-\left(\frac{1}{2} (0.4(-\ln((0.5^2)))^3 + 0.2(-\ln((0.4^2)))^3 \right)^{\frac{1}{3}}} \right)^3 \right) \right]}} = 0.5950.\end{aligned}$$

Theorem 13. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) and $\widehat{\Upsilon}_s = ([\widehat{\delta}_s^-, \widehat{\delta}_s^+], [\widehat{\theta}_s^-, \widehat{\theta}_s^+])$ ($s = 1, 2, \dots, b$) are two families of IVPFNs. If $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , then

$$IVPFAAWDHM^r(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAAWDHM^r(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b). \quad (3.17)$$

Proof. Since $\delta_s^- \leq \widehat{\delta}_s^-$, $\delta_s^+ \leq \widehat{\delta}_s^+$, $\theta_s^- \geq \widehat{\theta}_s^-$, and $\theta_s^+ \geq \widehat{\theta}_s^+$ for all s , it follows that

$$\sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2}) \right)^{\vartheta} \leq \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{-2}) \right)^{\vartheta}$$

$$\begin{aligned}
&\Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}} \geq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}} \\
&\Rightarrow 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}} \leq 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}} \\
&\Rightarrow -\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \geq -\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \\
&\Rightarrow \frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta} \\
&\geq \frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta} \\
&\Rightarrow \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta}\right)^{\frac{1}{\theta}}} \\
&\leq \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta}\right)^{\frac{1}{\theta}}}.
\end{aligned}$$

Similarly,

$$\sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \delta_{s_j}^{+2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta}\right)^{\frac{1}{\theta}}} \leq \sqrt[e]{-\left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(1 - \widehat{\delta}_{s_j}^{+2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta}\right)^{\frac{1}{\theta}}}$$

and

$$\begin{aligned}
&\sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2})\right)^{\theta} \leq \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2})\right)^{\theta} \\
&\Rightarrow -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}} \geq -\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}} \\
&\Rightarrow 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}} \leq 1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}} \\
&\Rightarrow \sum_{1 \leq s_1 < \dots < s_{\tau} \leq b} \left(-\ln\left(1 - e^{-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2})\right)^{\theta}\right)^{\frac{1}{\theta}}}\right) \right)^{\theta}
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \\
&\Rightarrow - \left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \\
&\leq - \left(\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \\
&\Rightarrow \sqrt[1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}]{1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}} \\
&\geq \sqrt[1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}]{1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{-2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}}.
\end{aligned}$$

Similarly,

$$\sqrt[1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}]{1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\theta_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}} \geq \sqrt[1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}]{1 - e^{\left(-\frac{1}{C_b^{\tau}} \sum_{1 \leq s_1 < \dots < s_r \leq b} \left(-\ln \left(1 - e^{\left(-\left(\frac{1}{\tau} \sum_{j=1}^{\tau} \chi_{s_j} \left(-\ln(\widehat{\theta}_{s_j}^{+2}) \right)^{\frac{1}{\theta}} \right)^{\frac{1}{\theta}} \right)} \right) \right)^{\frac{1}{\theta}} \right)}}.$$

Thus, we have

$$IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq IVPFAAWDHM^{\tau}(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2, \dots, \widehat{\Upsilon}_b).$$

This completes the proof. $\square$

Theorem 14. Suppose $\Upsilon_s = ([\delta_s^-, \delta_s^+], [\theta_s^-, \theta_s^+])$ ($s = 1, 2, \dots, b$) is a family of IVPFNs. Let $\Upsilon^- = \min \{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b\}$, and $\Upsilon^+ = \max \{\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b\}$; then

$$\Upsilon^- \leq IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+. \quad (3.18)$$

Proof. From Theorem 12, we have

$$IVPFAAWDHM^{\tau}(\Upsilon^-, \Upsilon^-, \dots, \Upsilon^-) = \Upsilon^-, IVPFAAWDHM^{\tau}(\Upsilon^+, \Upsilon^+, \dots, \Upsilon^+) = \Upsilon^+.$$

According to Theorem 13, we get $\Upsilon^- \leq IVPFAAWDHM^{\tau}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_b) \leq \Upsilon^+$. $\square$

4. Decision algorithm for MAGDM problems

In this section, a group DeM approach is developed based on the proposed AOs, incorporating the MEREC method to handle interval-valued Pythagorean fuzzy information.

Let $H = \{H_1, H_2, \dots, H_z\}$ denote the set of alternatives, and let $L = \{L_1, L_2, \dots, L_y\}$ represent the set of attributes. The weight associated with attribute L_g is denoted by χ_g , where $\chi_g > 0$ and

$$\sum_{g=1}^y \chi_g = 1.$$

Furthermore, let $E = \{E_1, E_2, \dots, E_p\}$ be the set of experts. The weight assigned to expert E_q is denoted by $\mathfrak{S}_q$, satisfying $\mathfrak{S}_q > 0$, and

$$\sum_{q=1}^p \mathfrak{S}_q = 1.$$

Suppose that $R^q = [\Upsilon_{ig}^q]_{z \times y}$ represents the decision matrix provided by the expert E_q , where each element Υ_{ig}^q is a IVPFNs defined as

$$\Upsilon_{ig}^q = ([\delta_{ig}^{-q}, \delta_{ig}^{+q}], [\theta_{ig}^{-q}, \theta_{ig}^{+q}]), \quad q = 1, 2, \dots, p, i = 1, 2, \dots, z \quad \text{and} \quad g = 1, 2, \dots, y.$$

The proposed MAGDM procedure is summarized as follows:

Step 1: The decision-maker organizes the experts evaluations into separate decision matrices

$$R^q = [\Upsilon_{ig}^q]_{z \times y}.$$

Step 2: The decision matrix

$$E^q = [\Upsilon_{ig}^q]_{z \times y}$$

is converted into its normalized form

$$\widehat{E}^q = [\widehat{\Upsilon}_{ig}^q]_{z \times y},$$

where

$$\widehat{\Upsilon}_{ig}^q = \begin{cases} ([\delta_{ig}^{-q}, \delta_{ig}^{+q}], [\theta_{ig}^{-q}, \theta_{ig}^{+q}]), & L_g \text{ is benefit criteria,} \\ ([\theta_{ig}^{-q}, \theta_{ig}^{+q}], [\delta_{ig}^{-q}, \delta_{ig}^{+q}]), & L_g \text{ is cost criteria.} \end{cases} \quad (4.1)$$

Step 3: Compute the objective weights χ_g ($g = 1, 2, \dots, y$) of the attributes by applying the MEREC method [43]. This is carried out through the following sub-steps using the normalized decision matrices:

Step 3.1: Develop the score matrix $Z = [Q_{ig}]_{z \times y}$ by converting every IVPFN Q_{ig} into a crisp score value via the score function given in [44]. Here, z and y indicate the total number of alternatives and attributes, respectively.

Step 3.2: Assess the overall performance of each alternative by employing a logarithmic aggregation function that treats all attributes with equal significance. The overall performance score M_i for alternative H_i is obtained as:

$$M_i = \ln \left(1 + \frac{1}{y} \sum_{g=1}^y |\ln(Q_{ig})| \right), \quad i = 1, 2, \dots, z.$$

Step 3.3: Examine the influence of individual attributes on the overall performance by successively removing one attribute at a time and recalculating the performance value for each alternative. The modified performance score after excluding the g th attribute is computed as:

$$M'_{ig} = \ln \left(1 + \frac{1}{y} \sum_{\substack{k=1 \\ k \neq g}}^y |\ln(Q_{ik})| \right).$$

Step 3.4: Quantify the impact of each attribute through the absolute deviation between the original and modified performance scores across all alternatives:

$$X_g = \sum_{i=1}^z |M'_{ig} - M_i|.$$

Step 3.5: Normalize the deviation values to derive the final objective weight of each attribute:

$$\chi_g = \frac{X_g}{\sum_{g=1}^y X_g}.$$

Step 4: All individual decision matrices R^q (where $q = 1, 2, \dots, p$) are aggregated to construct the collective decision matrix R through the application of IVPFAAWHM or IVPFAAWDHM operators, as shown below.

$$\Upsilon_{ig} = IVPFAAWHM^{\tau}(\Upsilon_{ig}^1, \Upsilon_{ig}^2, \dots, \Upsilon_{ig}^q) \quad (4.2)$$

or

$$\Upsilon_{ig} = IVPFAAWDHM^{\tau}(\Upsilon_{ig}^1, \Upsilon_{ig}^2, \dots, \Upsilon_{ig}^q). \quad (4.3)$$

Step 5: The attribute values Υ_{ig} ($g = 1, 2, \dots, y$) are aggregated to form the comprehensive value Υ_i by applying the IVPFAAWHM or IVPFAAWDHM operators, as illustrated below.

$$\Upsilon_i = IVPFAAWHM^{\tau}(\Upsilon_{i1}, \Upsilon_{i2}, \dots, \Upsilon_{iy}) \quad (4.4)$$

or

$$\Upsilon_i = IVPFAAWDHM^{\tau}(\Upsilon_{i1}, \Upsilon_{i2}, \dots, \Upsilon_{iy}). \quad (4.5)$$

Remark 1. In practical engineering or medical applications, the selection between the IVPFAAWHM and IVPFAAWDHM operators in Steps 4 and 5 is determined based on the following two core choice principles:

- (1) The IVPFAAWHM operator is based on the standard Hamy mean, which represents an optimistic or benefit-oriented decision attitude. It is highly suitable when experts want to emphasize alternatives with outstanding positive performance in key features. Conversely, the IVPFAAWDHM operator is based on the DHM, representing a conservative or pessimistic attitude, which is ideal when risk mitigation and safety margins are the primary goals.
- (2) In smart healthcare planning, the IVPFAAWHM operator is selected when the priority is maximizing clinical performance and system efficiency (advancing cutting-edge AI or operational features). On the other hand, the IVPFAAWDHM operator is preferred when safety-critical limitations, strict budget boundaries, or data privacy rules dictate a defensive and risk-averse selection path.

Step 6: Compute the score values Υ_i for $i = 1, 2, \dots, z$ according to Eq (4.6):

$$S(\Upsilon_i) = \frac{(\delta_i^-)^2 + (\delta_i^+)^2 - (\theta_i^-)^2 - (\theta_i^+)^2}{2}. \quad (4.6)$$

Step 7: Rank the alternatives Υ_i for $i = 1, 2, \dots, z$ based on their score values and select the best one.

The pseudo-code of the proposed MAGDM method is presented in Figure 1.

Algorithm	MAGDM Algorithm based on IVPFAAWHM / IVPFAAWDHM operators
→ Input:	Decision matrices $R^q = [\Upsilon_{ig}^q]_{z \times y}$ for each expert $q = 1$ to p
→ Output:	Ranking of alternatives and the best alternative
1	Step 1: Organize expert evaluations into individual decision matrices R^q
2	Step 2: Normalize each decision matrix for $g = 1$ to p for $i = 1$ to z for $g = 1$ to y Apply normalization (benefit or cost) as per Eq. (23)
3	Step 3: Compute objective attribute weights using MEREC method Convert normalized matrices to score matrix Z for $i = 1$ to z Calculate overall performance score M_i for $g = 1$ to y for $i = 1$ to z Compute modified performance M'_{ig} after removing attribute g Calculate deviation $X_g = \sum_{i=1}^z M'_{ig} - M_i $ Normalize weights: $\chi_g = \frac{X_g}{\sum X_g}$, for $g = 1$ to y
4	Step 4: Aggregate expert matrices into collective decision matrix R for $i = 1$ to z for $g = 1$ to y $\Upsilon_{ig} \leftarrow \text{IVPFAAWHM}^r$ (or IVPFAAWDHM^r) of $\{\Upsilon_{ig}^1, \dots, \Upsilon_{ig}^p\}$
5	Step 5: Compute comprehensive values for each alternative for $i = 1$ to z $\Upsilon_i \leftarrow \text{IVPFAAWHM}^r$ (or IVPFAAWDHM^r) of $\{\Upsilon_{i1}, \Upsilon_{i2}, \dots, \Upsilon_{iy}\}$ using weights χ
6	Step 6: Calculate score values for $i = 1$ to z $S(\Upsilon_i) = \frac{(\delta_i^-)^2 + (\delta_i^+)^2 - (\theta_i^-)^2 - (\theta_i^+)^2}{2}$
7	Step 7: Rank the alternatives Sort alternatives in descending order of $S(\Upsilon_i)$ and select the best one

Figure 1. Pseudo-code of the proposed MAGDM method.

5. Case study: smart healthcare system selection

SHS implementation aims to enhance the efficiency, transparency, and quality of medical services in public hospitals by strategically integrating advanced technologies, digital platforms, and data-driven solutions. It encompasses a wide range of activities, including electronic health record management, intelligent appointment scheduling, AI-assisted diagnostics, and automated resource allocation systems that address operational challenges in healthcare institutions. With the rapid population growth in developing countries, public hospitals are experiencing increased patient inflow, limited infrastructure capacity, and inefficient manual processes. This situation places considerable pressure on medical staff, financial resources, and service delivery systems. As a result, there is a growing demand for digital transformation in healthcare management. By implementing smart healthcare technologies, hospitals can optimize resource utilization, reduce patient waiting times, enhance diagnostic accuracy, improve administrative transparency, and promote sustainable healthcare delivery. The importance of SHS selection lies in its potential to create a more resilient, accessible, and patient-centered healthcare environment. However, selecting the most appropriate system requires careful evaluation of multiple technical, financial, and operational criteria. Therefore, a MAGDM framework is adopted to determine the most suitable alternative.

5.1. Criteria description

SHS selection within the MAGDM framework requires the consideration of various criteria to ensure a comprehensive and collaborative DeM process. A detailed explanation is provided as follows:

- (1) **Cost effectiveness (L_1):** Cost effectiveness is a fundamental criteria emphasizing financial sustainability and optimal resource utilization. DMs evaluate both the initial investment and long-term operational expenses associated with each alternative. This includes hardware acquisition, software licensing, maintenance cost, and staff training requirements. A thorough financial assessment ensures that the selected system delivers high performance without placing excessive burden on hospital budgets.
- (2) **Data security and privacy (L_2):** DMs focus data security and privacy when looking at patient data safety, and how private they keep patients' private information. They study cyber security mechanisms, encryption methods, access controls and healthcare data compliant standards. Due to the sensitive nature of patient data, protecting the data from unauthorized access is important to maintain trust.
- (3) **System reliability (L_3):** System reliability refers to the operational stability and performance stability of the system. The hospital management system's system reliability is mainly reflected in downtime, system recovery time, and error occurrence rate. A reliable hospital management system can ensure continuous operation, uninterrupted patient care, and support hospital work.
- (4) **Implementation feasibility (L_4):** Implementation feasibility checks the ease of implementing the system in the hospital setting. The factors the DMs evaluate include technical difficulty, compatibility with existing systems, time necessary for training, and implementation timeframe. The adopted system should ideally should be integrated seamlessly, avoiding operational issues.
- (5) **Scalability and flexibility (L_5):** Scalability and flexibility refer to how well a system can

support future growth and innovation. As patient loads increase, new technologies arise, and healthcare trends shift, hospitals need systems that are able to accommodate those changes. Scalable systems help minimize the number of times that technology must be replaced or a facility will be remodeled as demand increases.

- (6) **User acceptance and useability (L_6):** User acceptance measures how effectively doctors, nurses and administrative staff can adopt, and operate the system. This includes assessment of the simplicity, ease of access, and usability of the user interface. Systems that meet the needs of the users are known to bring about increased productivity and better quality of service.

5.2. Alternatives description

In the MAGDM framework, selecting the right SHS involves a thorough evaluation of multiple alternatives. This systematic and collaborative DeM process helps hospital authorities identify the most suitable solution that aligns well with their operational needs, infrastructure, and future vision.

- (1) **Cloud-based healthcare system (H_1):** The cloud-based healthcare system stores hospital data on remote servers, enabling real time access, automated backup and scalable storage capacity. This alternative enhances accessibility and reduces the need for on site hardware infrastructure. However, it depends heavily on stable internet connectivity and requires strong cybersecurity measures to mitigate potential data risks.
- (2) **On-premises healthcare management system (H_2):** The on-premises system stores all data within the hospital's servers, providing greater control over customization and security than cloud-based systems. It allows institutions to manage data access independently, and maintain strict internal monitoring. However, it involves substantial initial investment, maintenance expenses and technical management responsibilities.
- (3) **Hybrid healthcare system (H_3):** The hybrid system combines cloud-based and local storage infrastructures. Critical medical data are stored on site, whereas less sensitive information is maintained on the cloud. This balanced approach enhances flexibility and risk management but may require advanced technical expertise for system integration and coordination.
- (4) **AI-integrated healthcare system (H_4):** The AI-integrated system incorporates artificial intelligence tools for predictive analysis, clinical decision support, and workflow optimization. It improves diagnostic accuracy, patient risk assessment, and operational efficiency. Despite its innovative advantages this, alternative involves higher implementation and training costs.
- (5) **Modular scalable healthcare system (H_5):** The modular system is designed with expandable components that can be implemented gradually based on hospital needs. It offers adaptability and long-term strategic flexibility. However, integrating multiple modules over time may present technical compatibility challenges.

5.3. Numerical illustration

In an effort to modernize healthcare management and improve service quality, the government of a major public hospital in Pakistan invited three DMs $E = \{E_1, E_2, E_3\}$ with the weight vector $(0.4, 0.4, 0.2)^T$ to address the complex MAGDM problem associated with SHS selection. These DMs shortlisted five alternatives $H = \{H_1, H_2, H_3, H_4, H_5\}$, which were evaluated based on the six different attributes $L = \{L_1, L_2, L_3, L_4, L_5, L_6\}$ mentioned above. The weight vector of these criteria,

denoted by

$$\chi = (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6)^T,$$

is unknown. The DMs provided their evaluation information in the form of IVPFNs to effectively capture the uncertainty and hesitation inherent in their assessments.

5.3.1. Experimental results by IVPFAAWHM operator

Step 1: Based on the inputs provided by the DMs, the evaluation decision matrices are presented in Tables 1–3.

Table 1. Decision matrix R^1 .

	L_1	L_2	L_3
H_1	([0.2, 0.4], [0.1, 0.5])	([0.3, 0.5], [0.2, 0.6])	([0.1, 0.3], [0.2, 0.4])
H_2	([0.1, 0.4], [0.2, 0.6])	([0.2, 0.5], [0.1, 0.3])	([0.3, 0.6], [0.2, 0.4])
H_3	([0.3, 0.5], [0.2, 0.4])	([0.2, 0.6], [0.1, 0.5])	([0.1, 0.4], [0.2, 0.3])
H_4	([0.2, 0.4], [0.1, 0.3])	([0.3, 0.5], [0.2, 0.6])	([0.1, 0.4], [0.2, 0.5])
H_5	([0.3, 0.6], [0.1, 0.4])	([0.2, 0.5], [0.2, 0.6])	([0.1, 0.3], [0.1, 0.5])
	L_4	L_5	L_6
H_1	([0.4, 0.6], [0.3, 0.5])	([0.2, 0.5], [0.1, 0.4])	([0.3, 0.6], [0.2, 0.5])
H_2	([0.2, 0.4], [0.3, 0.5])	([0.1, 0.3], [0.2, 0.6])	([0.4, 0.6], [0.1, 0.5])
H_3	([0.3, 0.6], [0.2, 0.5])	([0.2, 0.5], [0.1, 0.4])	([0.1, 0.3], [0.2, 0.6])
H_4	([0.2, 0.6], [0.1, 0.4])	([0.3, 0.5], [0.2, 0.3])	([0.1, 0.4], [0.2, 0.6])
H_5	([0.4, 0.6], [0.2, 0.4])	([0.2, 0.4], [0.1, 0.6])	([0.3, 0.5], [0.2, 0.3])

Table 2. Decision matrix R^2 .

	L_1	L_2	L_3
H_1	([0.1, 0.3], [0.2, 0.5])	([0.2, 0.4], [0.1, 0.6])	([0.3, 0.5], [0.2, 0.4])
H_2	([0.2, 0.5], [0.1, 0.3])	([0.3, 0.6], [0.2, 0.4])	([0.1, 0.4], [0.2, 0.5])
H_3	([0.3, 0.6], [0.1, 0.4])	([0.2, 0.4], [0.2, 0.5])	([0.1, 0.3], [0.1, 0.6])
H_4	([0.1, 0.4], [0.2, 0.6])	([0.2, 0.5], [0.1, 0.4])	([0.3, 0.6], [0.2, 0.3])
H_5	([0.2, 0.5], [0.2, 0.6])	([0.3, 0.6], [0.1, 0.3])	([0.1, 0.4], [0.2, 0.5])
	L_4	L_5	L_6
H_1	([0.2, 0.6], [0.3, 0.5])	([0.1, 0.4], [0.2, 0.6])	([0.3, 0.5], [0.1, 0.4])
H_2	([0.2, 0.5], [0.1, 0.6])	([0.3, 0.6], [0.2, 0.3])	([0.1, 0.4], [0.2, 0.5])
H_3	([0.4, 0.6], [0.2, 0.4])	([0.2, 0.5], [0.1, 0.3])	([0.3, 0.6], [0.2, 0.5])
H_4	([0.2, 0.4], [0.1, 0.5])	([0.1, 0.3], [0.2, 0.6])	([0.4, 0.6], [0.1, 0.4])
H_5	([0.2, 0.5], [0.1, 0.6])	([0.3, 0.6], [0.2, 0.4])	([0.1, 0.3], [0.1, 0.5])

Table 3. Decision matrix R^3 .

	L_1	L_2	L_3
H_1	$([0.2, 0.5], [0.1, 0.4])$	$([0.3, 0.6], [0.2, 0.3])$	$([0.1, 0.4], [0.2, 0.6])$
H_2	$([0.3, 0.6], [0.2, 0.4])$	$([0.1, 0.3], [0.1, 0.5])$	$([0.2, 0.5], [0.2, 0.6])$
H_3	$([0.1, 0.4], [0.2, 0.3])$	$([0.2, 0.5], [0.1, 0.6])$	$([0.3, 0.6], [0.2, 0.4])$
H_4	$([0.2, 0.5], [0.1, 0.6])$	$([0.3, 0.6], [0.2, 0.3])$	$([0.1, 0.4], [0.2, 0.5])$
H_5	$([0.3, 0.6], [0.2, 0.5])$	$([0.1, 0.4], [0.1, 0.6])$	$([0.2, 0.5], [0.2, 0.4])$
	L_4	L_5	L_6
H_1	$([0.2, 0.5], [0.1, 0.3])$	$([0.3, 0.6], [0.2, 0.5])$	$([0.1, 0.4], [0.1, 0.6])$
H_2	$([0.3, 0.6], [0.1, 0.3])$	$([0.1, 0.4], [0.2, 0.5])$	$([0.2, 0.5], [0.1, 0.6])$
H_3	$([0.1, 0.3], [0.1, 0.5])$	$([0.2, 0.5], [0.2, 0.6])$	$([0.3, 0.6], [0.1, 0.4])$
H_4	$([0.2, 0.5], [0.1, 0.4])$	$([0.3, 0.6], [0.2, 0.6])$	$([0.1, 0.3], [0.1, 0.5])$
H_5	$([0.3, 0.6], [0.1, 0.3])$	$([0.1, 0.3], [0.2, 0.5])$	$([0.2, 0.5], [0.1, 0.6])$

Step 2: Since all the criteria are of the benefit type, normalization is not required.

Step 3: Following the procedure of the proposed MEREC method, the criteria weight vector is obtained as $\chi = (0.2207, 0.0953, 0.2377, 0.2178, 0.1212, 0.1073)$.

Remark 2. Note that while deriving the criteria weights through the MEREC approach, we utilized the score formulation given in Definition 3. Generally, one should adopt the standard formulation provided in the literature [44]. This is because the formulation in Definition 3 ranges between -1 and 1, which can cause mathematical issues or failures if the values become zero or negative. However, since all calculated values for the current data in Table 4 remain strictly above zero, no such issues occurred in this case.

Step 4: By utilizing the IVPFAAWHM operator (taking $\vartheta = 3$, and $\tau = 2$), as depicted in Table 4, all individual decision matrices R^q (for $q = 1, 2, 3$) are combined to form a single collective matrix R .

Table 4. Aggregated decision matrix via IVPFAAWHM operator.

	L_1	L_2	L_3
H_1	$([0.2944, 0.5145], [0.1194, 0.4111])$	$([0.4010, 0.6028], [0.1511, 0.4942])$	$([0.2741, 0.5337], [0.1667, 0.4009])$
H_2	$([0.3154, 0.6028], [0.1511, 0.3825])$	$([0.3198, 0.5878], [0.1194, 0.3478])$	$([0.3423, 0.6203], [0.1667, 0.4335])$
H_3	$([0.3639, 0.6196], [0.1511, 0.3249])$	$([0.3309, 0.6203], [0.1194, 0.4525])$	$([0.2387, 0.5234], [0.1511, 0.3825])$
H_4	$([0.2944, 0.5498], [0.1194, 0.4703])$	$([0.4010, 0.6364], [0.1511, 0.3993])$	$([0.2741, 0.5859], [0.1667, 0.3879])$
H_5	$([0.4010, 0.6720], [0.1511, 0.4370])$	$([0.3198, 0.6196], [0.1194, 0.4703])$	$([0.2298, 0.5145], [0.1511, 0.4111])$
	L_4	L_5	L_6
H_1	$([0.3942, 0.6797], [0.2400, 0.4072])$	$([0.3154, 0.6028], [0.1511, 0.4370])$	$([0.3639, 0.6196], [0.1194, 0.4335])$
H_2	$([0.3523, 0.6028], [0.1436, 0.4486])$	$([0.2741, 0.5587], [0.1667, 0.4266])$	$([0.3720, 0.6203], [0.1194, 0.4525])$
H_3	$([0.3941, 0.6240], [0.1594, 0.3981])$	$([0.3309, 0.6202], [0.1149, 0.3839])$	$([0.3808, 0.6298], [0.1594, 0.4517])$
H_4	$([0.3309, 0.6203], [0.0832, 0.3744])$	$([0.3808, 0.5784], [0.1667, 0.4703])$	$([0.2867, 0.5469], [0.1194, 0.4370])$
H_5	$([0.4414, 0.6720], [0.1194, 0.3993])$	$([0.3198, 0.5469], [0.1511, 0.4370])$	$([0.3423, 0.5614], [0.1194, 0.4273])$

Step 5: By employing the IVPFAAWHM operator, as illustrated in Table 5, all attribute values Υ_{ig} ($g = 1, 2, 3, 4, 5, 6$) are combined to derive the comprehensive value Υ_i .

Table 5. Comprehensive value Υ_i by IVPFAAWHM operator.

Alternative	IVPF value
H_1	([0.5596, 0.7532], [0.1205, 0.3232])
H_2	([0.5486, 0.7577], [0.1103, 0.3174])
H_3	([0.5617, 0.7623], [0.1086, 0.2999])
H_4	([0.5469, 0.7477], [0.1026, 0.3203])
H_5	([0.5652, 0.7582], [0.1022, 0.3235])

Step 6: The score values for the aforementioned results are calculated using Eq (4.6) as follows:

$$S(\Upsilon_1) = 0.3808, \quad S(\Upsilon_2) = 0.3810, \quad S(\Upsilon_3) = 0.3974, \quad S(\Upsilon_4) = 0.3725, \quad S(\Upsilon_5) = 0.3896.$$

Step 7: Based on the computed score values for the alternatives, the ultimate ranking is $H_3 > H_5 > H_2 > H_1 > H_4$, with H_3 identified as the superior option.

5.4. Experimental results by IVPFAAWDHM operator

Steps 1–3: These are the same as those presented in Subsection 5.3.1.

Step 4: By utilizing the IVPFAAWDHM operator (taking $\vartheta = 3$, and $\tau = 2$), as depicted in Table 6, all individual decision matrices R^q (for $q = 1, 2, 3$) are combined to form a single collective matrix R .

Table 6. Aggregated decision matrix via IVPFAAWDHM operator.

	L_1	L_2	L_3
H_1	([0.1511, 0.3478], [0.2496, 0.5950])	([0.2302, 0.4335], [0.2944, 0.6240])	([0.1436, 0.3510], [0.3309, 0.5595])
H_2	([0.1867, 0.4335], [0.2944, 0.5587])	([0.1980, 0.4486], [0.2496, 0.5145])	([0.1880, 0.4370], [0.3309, 0.6028])
H_3	([0.2400, 0.4517], [0.2944, 0.5042])	([0.1667, 0.4370], [0.2496, 0.6364])	([0.1408, 0.3839], [0.2944, 0.5587])
H_4	([0.1511, 0.3646], [0.2496, 0.6298])	([0.2302, 0.4525], [0.2944, 0.5469])	([0.1436, 0.4106], [0.3309, 0.5614])
H_5	([0.2302, 0.4869], [0.2944, 0.6203])	([0.1980, 0.4517], [0.2496, 0.6298])	([0.1149, 0.3478], [0.2944, 0.5950])
	L_4	L_5	L_6
H_1	([0.2356, 0.5005], [0.3639, 0.5635])	([0.1867, 0.4335], [0.2944, 0.6203])	([0.2400, 0.4517], [0.2496, 0.6028])
H_2	([0.1966, 0.4335], [0.2741, 0.5878])	([0.1436, 0.3825], [0.3309, 0.6000])	([0.2143, 0.4370], [0.2496, 0.6364])
H_3	([0.2795, 0.4942], [0.2901, 0.5878])	([0.1667, 0.4244], [0.2298, 0.5234])	([0.2272, 0.4703], [0.2901, 0.6196])
H_4	([0.1667, 0.4370], [0.2062, 0.5635])	([0.2272, 0.4273], [0.3309, 0.6298])	([0.1580, 0.3993], [0.2496, 0.6203])
H_5	([0.2680, 0.4869], [0.2496, 0.5469])	([0.1980, 0.3993], [0.2944, 0.6203])	([0.1880, 0.3879], [0.2496, 0.5784])

Step 5: By employing the IVPFAAWDHM operator, as illustrated in Table 7, all attribute values Υ_{ig} ($g = 1, 2, 3, 4, 5, 6$) are combined to derive the comprehensive value Υ_i .

Table 7. Comprehensive value Υ_i by IVPFAAWDHM operator.

Alternative	IVPF value
H_1	$([0.1530, 0.3235], [0.5167, 0.7527])$
H_2	$([0.1419, 0.3256], [0.5090, 0.7441])$
H_3	$([0.1566, 0.3386], [0.4954, 0.7367])$
H_4	$([0.1372, 0.3170], [0.4996, 0.7499])$
H_5	$([0.1560, 0.3291], [0.4937, 0.7559])$

Step 6: The score values for the aforementioned results are calculated using Eq (4.6) as follows:

$$S(\Upsilon_1) = -0.3527, \quad S(\Upsilon_2) = -0.3433, \quad S(\Upsilon_3) = -0.3245, \quad S(\Upsilon_4) = -0.3463, \quad S(\Upsilon_5) = -0.3412.$$

Step 7: Based on the computed score values for the alternatives, the ultimate ranking is $H_3 > H_5 > H_2 > H_4 > H_1$, with H_3 identified as the best option.

6. Sensitivity analysis

In this section, a sensitivity analysis is conducted to investigate the stability of the proposed approach under different settings of the AA and Hamy parameters in the IVPFAAWHM operator.

6.1. AA parameter perturbation analysis

This subsection examines the influence of the AA parameter ϑ on the aggregation outcomes and the stability of the final ranking. The sensitivity analysis was performed by varying ϑ from 2 to 20, while keeping the HM parameter $\tau = 2$ fixed. The results are shown in Table 8 and Figure 2.

Table 8. Results achieved by various values of AA parameter ϑ .

ϑ	$S(H_1)$	$S(H_2)$	$S(H_3)$	$S(H_4)$	$S(H_5)$	Ranking
2	0.5583	0.5591	0.5684	0.5530	0.5663	$H_3 > H_5 > H_2 > H_1 > H_4$
3	0.3808	0.3810	0.3974	0.3725	0.3896	$H_3 > H_5 > H_2 > H_1 > H_4$
5	0.2247	0.2229	0.2526	0.2128	0.2333	$H_3 > H_5 > H_1 > H_2 > H_4$
6	0.1901	0.1855	0.2214	0.1791	0.1981	$H_3 > H_5 > H_1 > H_2 > H_4$
8	0.1397	0.1412	0.1785	0.1322	0.1556	$H_3 > H_5 > H_2 > H_1 > H_4$
10	0.1180	0.1145	0.1596	0.1138	0.1349	$H_3 > H_5 > H_1 > H_2 > H_4$
12	0.1022	0.0992	0.1489	0.1026	0.1235	$H_3 > H_5 > H_4 > H_1 > H_2$
15	0.0915	0.0887	0.1440	0.0949	0.1209	$H_3 > H_5 > H_4 > H_1 > H_2$
18	0.0839	0.0830	0.1421	0.0907	0.1170	$H_3 > H_5 > H_4 > H_1 > H_2$
20	0.0797	0.0797	0.1413	0.0900	0.1142	$H_3 > H_5 > H_4 > H_1 > H_2$

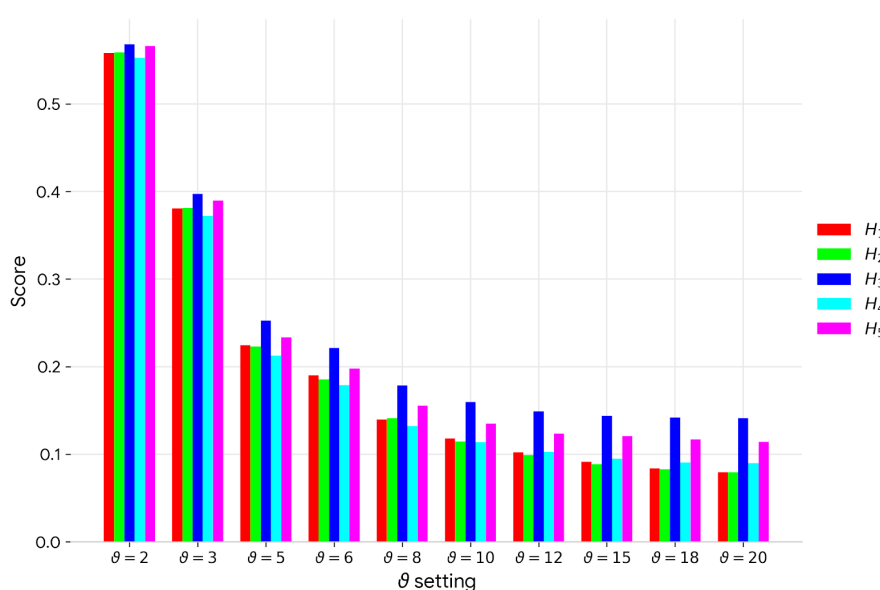


Figure 2. Sensitivity analysis with respect to ϑ .

The sensitivity analysis reveals that the proposed MAGDM framework exhibits robustness with respect to the AA parameter ϑ . For moderate values of ϑ (ranging from 2 to 10), the ranking remains highly stable, with H_3 consistently emerging as the most preferred alternative, followed by H_5 . This stability indicates that the core recommendation of the model is not overly sensitive to moderate changes in the DM's risk attitude.

As ϑ increases beyond 10, minor shifts occur among the lower-ranked alternatives. Specifically, H_4 improves its position relative to H_1 and H_2 for $\vartheta \geq 12$. However, these changes are limited to the lower tier of the ranking. The top two alternatives (H_3 and H_5) maintain their leading positions across all tested values of ϑ , demonstrating the robustness of the proposed approach.

Overall, the consistent dominance of H_3 as the best alternative, even under substantial variation of the parameter ϑ , confirms the stability of the proposed MAGDM framework for SHS selection planning. This robustness enhances confidence in the practical applicability of the obtained results.

6.2. Hamy parameter perturbation analysis

This subsection investigates the impact of changes in the Hamy parameter on the aggregation outcomes, with the aim of evaluating the sensitivity and consistency of the proposed approach.

The parameter τ controls the degree to which interrelationships among multiple input arguments (criteria or DMs) are captured during aggregation. Lower values of τ (e.g., $\tau = 1$) reduce to simpler averaging-like behavior, whereas higher values emphasize higher-order interactions. Analyzing sensitivity with respect to τ is crucial for validating the robustness and stability of the proposed MAGDM approach under varying levels of intercriterion dependency. In the proposed decision algorithm (Section 4), the IVPFAAWHM operator is typically applied in Step 4 (aggregation of individual expert evaluations into a collective decision matrix) and the IVPFAAWDHM operator in Step 5 (aggregation of criteria weights or final ranking scores). For computational efficiency and to systematically assess stability, we fix $\tau = 2$ in Step 4 while varying τ from 1 to 6 in Step 5. This range

covers typical practical scenarios while remaining computationally tractable. The score function values $S(H_i)$ for the five SHS alternatives (H_1 to H_5) and resulting rankings are shown in Table 9 and Figure 3.

Table 9. Results achieved by various values of τ .

τ	$S(H_1)$	$S(H_2)$	$S(H_3)$	$S(H_4)$	$S(H_5)$	Ranking
$\tau = 1$	0.4080	0.3965	0.4170	0.4009	0.4058	$H_3 > H_1 > H_5 > H_4 > H_2$
$\tau = 2$	0.3808	0.3810	0.3974	0.3725	0.3896	$H_3 > H_5 > H_2 > H_1 > H_4$
$\tau = 3$	0.3668	0.3768	0.3868	0.3627	0.3792	$H_3 > H_5 > H_2 > H_1 > H_4$
$\tau = 4$	0.3586	0.3749	0.3797	0.3581	0.3716	$H_3 > H_2 > H_5 > H_1 > H_4$
$\tau = 5$	0.3536	0.3739	0.3745	0.3554	0.3655	$H_3 > H_2 > H_5 > H_4 > H_1$
$\tau = 6$	0.3503	0.3732	0.3704	0.3537	0.3606	$H_2 > H_3 > H_5 > H_4 > H_1$

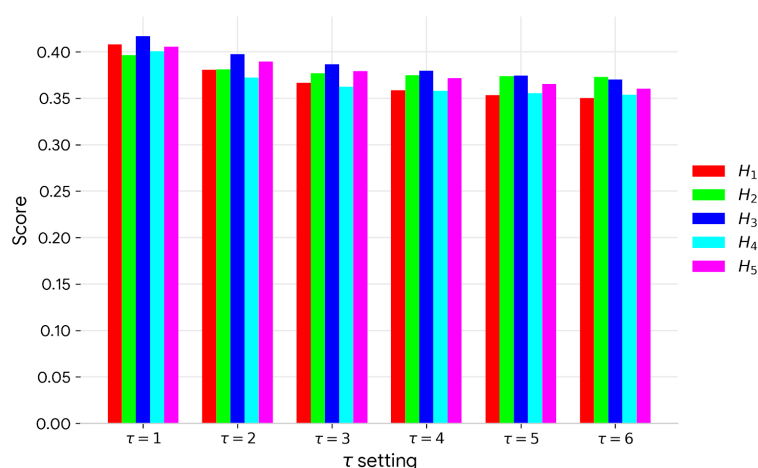


Figure 3. Sensitivity analysis with respect to τ .

The ranking remains consistent across most values of τ , with H_3 (the top-ranked alternative) stable for $\tau = 1$ to 5. Minor shifts occur at higher τ (e.g., $\tau = 6$, where H_2 overtakes H_3), indicating that stronger emphasis on higher-order interrelationships slightly alters preference when criteria dependencies are complex. This behavior is expected and desirable: The HM operator's ability to model multi-argument interactions increases with τ , providing nuanced insights without causing rank reversal in most cases.

7. Comparative analysis

This section presents a comprehensive comparative analysis of the proposed IVPFAAHM and IVPFAAWDHM operators against several well-established AOs within the IVPFS environment [23–25, 27, 42]. To ensure objectivity and robustness in the evaluation process, the criteria weights are determined using the MEREC method, which effectively captures the inherent

conflict and informational contribution of each criterion. The comparison is carried out on the SHS selection problem discussed in Section 5, and the final results are presented in Table 10.

Table 10. Comparison with existing methods.

	$S(H_1)$	$S(H_2)$	$S(H_3)$	$S(H_4)$	$S(H_5)$	Ranking
IVPFYWA [25]	0.3077	0.2949	0.3090	0.2969	0.3137	$H_5 > H_3 > H_1 > H_4 > H_2$
IVPFDWA [27]	0.4630	0.4414	0.4616	0.4460	0.4690	$H_5 > H_1 > H_3 > H_4 > H_2$
IVPFAAWA [42]	0.3325	0.3060	0.2963	0.3012	0.3207	$H_1 > H_5 > H_2 > H_4 > H_3$
IVPFWA [23]	0.4449	0.4352	0.4502	0.4367	0.4512	$H_5 > H_3 > H_1 > H_4 > H_2$
IVPFWG [24]	0.4368	0.4331	0.4451	0.4332	0.4428	$H_3 > H_5 > H_1 > H_4 > H_2$
Proposed IVPFAAWHM	0.3808	0.3810	0.3974	0.3725	0.3896	$H_3 > H_5 > H_2 > H_1 > H_4$
Proposed IVPFAAWDHM	-0.3527	-0.3433	-0.3245	-0.3463	-0.3412	$H_3 > H_5 > H_2 > H_4 > H_1$

Note: Interval-valued Pythagorean fuzzy Yager weighted average, IVPFDWA: Interval-valued Pythagorean fuzzy Dombi weighted average, IVPFAAWA: Interval-valued Pythagorean fuzzy AA weighted average, IVPFWA: Interval-valued Pythagorean fuzzy weighted average, IVPFWG: Interval-valued Pythagorean fuzzy weighted geometric.

7.1. Discussion on ranking results

From Table 10, several important observations can be drawn regarding the performance and distinctive characteristics of the proposed operators compared to existing methods.

- (i) Comparison with Yager and Dombi-based operators [23,27]. The operators IVPFYWA, IVPFDWA, and IVPFWA all rank H_5 as the most preferred alternative. This shows partial agreement with the proposed IVPFAAWHM and IVPFAAWDHM, which also place H_5 in the second position while ranking H_3 first. The observed difference arises because traditional Yager and Dombi-based operators primarily rely on simpler weighted aggregation mechanisms that tend to emphasize strong individual criterion performance with limited attention to intercriterion dependencies. In contrast, the integration of the HM in the proposed operators enables the modeling of higher-order relationships among multiple criteria simultaneously. This results in a more balanced and realistic ranking that prioritizes H_3 , an alternative offering superior overall integration of critical factors such as cost, security, scalability, and usability in SHS.
- (ii) Comparison with AA operations-based operator [42]. From Figure 4, one can see that the IVPFAAWA operator ranks H_1 as the best alternative, which differs markedly from both proposed operators that consistently select H_3 as the top choice. Although the used benchmark operator relies on AA operators, it doesn't feature the HM structure so that the correlations between criterion measures can be reflected. This drawback is addressed in the newly proposed operators through integrating AA operations (for adjustability and info reduction) and HM (for depicting correlations between measures) together, which can present more adequate aggregate or rankings in practice concerning the interconnected, real smart healthcare decision problem.

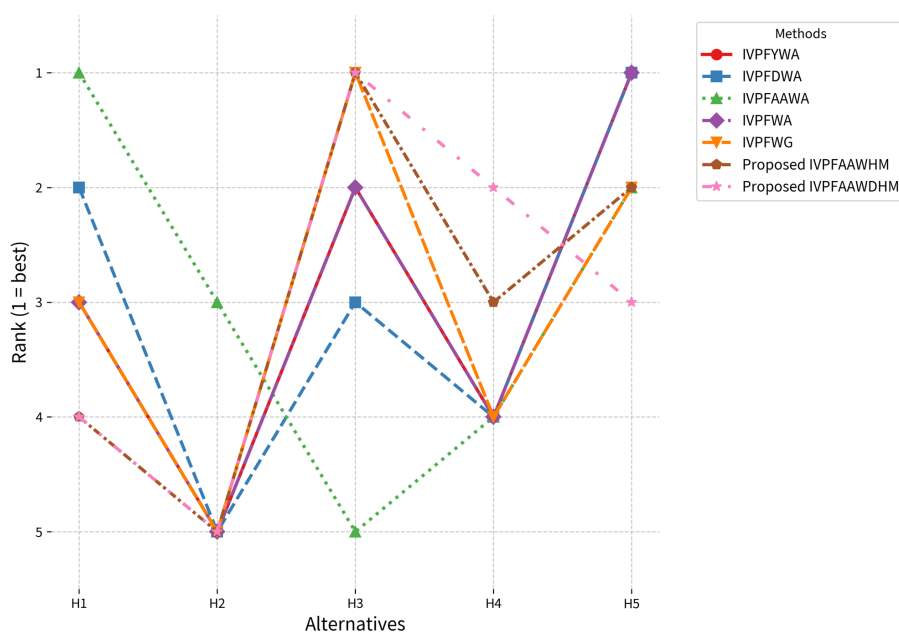


Figure 4. Graphical interpretation of the comparative analysis.

- (iii) Comparison with weighted average and geometric operators [23, 24]. The IVPFWG operator is quite consistent with the proposed approaches because it ranks H_3 to be the best choice and ranks H_5 to be second. Thus, we could be certain about H_3 to be the best SHS. However, the IVPFWA selects H_5 to be the most ideal SHS. We can find that a small fluctuation of ranks might be generated. The reason is that standard weighted average operators take independence of criteria as granted and only adopt the linear compensation method. In comparison, our suggested approaches integrate the HM into the operator so as to include the correlations between criteria and adopt the adjustable AA parameter ϑ so as to have better regulation on the range of compensation, leading to more precise and stable rankings.
- (iv) Overall characteristics of the proposed operators. Both the proposed IVPFAAWHM and IVPFAAWDHM operators exhibit strong mutual consistency, with H_3 clearly emerging as the top-ranked alternative in both cases. This consistency, even across the positive and negative score functions of the Hamy and DHM versions, highlights the robustness and reliability of the developed framework. The core strength lies in the effective integration of AA operations for parametric flexibility and the HM for higher-order inter-criterion relationship modeling.

From these comparisons, the key advantages of the proposed operators readily follow, the details of which are presented in the subsequent section.

7.2. Advantages of the proposed operators

The comparison of result and the structural design indicate that the proposed operators IVPFAAHM and IVPFAAWDHM have the following advantages:

- (1) Effective capture of intercriteria relationships: Combining the HM, the proposed operators take into account interactions between criteria simultaneously, so they break the restriction of

independence among simple average operators and leads to more realistic rankings, as seen in the consistent preference for the well-balanced alternative H_3 .

- (2) Greater flexibility through AA operations: The addition of the tunable parameter ϑ allows greater control over the degrees of information loss and compensation compared to conventional non-parameter operators [23, 24], which has facilitated the improved refined and robust ranking behavior.
- (3) Objective criteria weighting via MEREC: By using the MEREC technique, DMs can make use of data-driven weights that reflect the relative importance and contradiction of criteria for the given set of alternatives in a more robust way compared to methods relying solely on subjective expert weights.
- (4) High ranking stability and practical reliability: The designed operators are robust, as they consistently rank at the top (H_3) at HM and DHM levels under the parameter perturbations, and their performances are more stable than a number of state-of-the-art techniques which are weight sensitive.

These features make the proposed methodology highly suitable for complex uncertain MAGDM problems such as SHS selection.

7.3. Limitations of the proposed method

Despite its strengths, the proposed approach has certain limitations:

- (1) Computational complexity: Due to the combinatorial nature of HM (C_b^{τ}), it can result in higher calculation costs when b or τ is large compared with some elementary average operators.
- (2) Parameter selection challenge: A particular need arises for the optimal tuning of τ and ϑ through methods of sensitivities analysis and/or of specialized knowledge, which is not very straightforward for users who are not specialists.
- (3) Performance on highly sparse data: Extreme uncertainty in input data could propagate through the aggregation process for the method, potentially causing problems with decision matrices of the 'highly' sparse or incomplete variety.
- (4) Reliance on objective weights: Although MEREC provides objectivity, it may not fully capture expert preferences in all scenarios where subjective judgment is valuable.

7.4. Suggestions for future work

Although the operators proposed show satisfying performance, there are still limitations and promising directions for improvements: In future work, we may seek efficient algorithmic approximations to alleviate the high computational complexity that might arise from the combinatorial nature of HM operators. Hybridizing proposed operators with machine learning-based imputation approaches could also be a possible extension for the application where decision matrices are highly sparse or contain lots of missing entries. Moreover, the development of objective weighting schemes for the experts based on novel entropy models or consensus similarity measures specifically designed for interval-valued Pythagorean settings is another relevant direction that could be pursued to automatically obtain decision maker weights without any subjective or prior systemic bias. Additionally, automated parameter optimization strategies (i.e., evolutionary algorithms, deep

learning techniques) would be investigated to find optimal parameters of τ and ϑ given the specific problem structure.

8. Conclusions

This study addressed the challenge of MAGDM under uncertainty, where the accurate representation and aggregation of imprecise information remained a critical issue. To overcome these limitations, an extended framework based on IVPFSs was developed. The main objective of this research was to design flexible and interaction-sensitive AOs capable of effectively handling uncertain and interrelated decision information in complex environments such as SHS selection. To achieve this goal, several novel AA Hamy mean-based operators, including IVPFAAHM, IVPFAAWHM, IVPFAADHM, and IVPFAAWDHM, were proposed. The proposed operators have been rigorously analyzed, and it has been established that they satisfy important structural characteristics, ensuring their mathematical soundness and reliability. These operators not only preserved the advantages of IVPFSs but also incorporated the flexibility of AA operations and the interaction-capturing capability of HM structures. The numerical results obtained from the SHS selection problem demonstrated that the proposed operators produced more stable and discriminative ranking results compared to existing methods. In particular, the proposed approach yielded slightly different ranking orders when compared with traditional AOs, which was mainly due to its ability to capture interdependencies among criteria and to incorporate objective weighting information through the MEREC method. This confirmed that the proposed framework provided a more realistic representation of decision behavior under uncertainty. Moreover, comparative analysis results proved that established methods such as weighted averaging operators or geometrical operators were unable to take the interaction effects of multi-criteria completely into account, but our proposed operators could better realize this effect. Further, sensitivity analyses showed that the method could tolerate parameter changes, thus revealing the stability and reliability of the current method under those practical DeM environments. From a practical perspective, the framed approach has demonstrably improved the quality of the decisions based on better information gathering/integration, minimization of subjective biases by incorporating objective weights, and increased the separation between very similar alternatives. This makes it an effective tool for real-world cases such as healthcare system evaluation, risk assessment, and other complex decision environments.

Author contributions

Amjid Khan: conceptualization, software, resources, original draft preparation, visualization; Jawad Ali: conceptualization, methodology, validation, writing, review and editing; Suhad Ali Osman Abdallah: software, investigation, resources, supervision, project administration; Ioan-Lucian Popa: formal analysis, data curation, writing, review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that Artificial Intelligence (AI) tools were used in the preparation of this article, solely to assist with language polishing and grammar refinement. All scientific content, methodology, analysis, and conclusions were developed and verified by the authors, who take full responsibility for the content of the article.

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Conflict of interest

The authors declare that they have no conflicts of interest regarding this manuscript.

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