



Research article

Estimation of time delay and mirror efficiency in distributed collector systems

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Abstract: Renewable energy, particularly solar energy, is vital in addressing current energy demands by providing a clean and sustainable alternative to traditional sources. This work focuses on enhancing the distributed collector system (DCS), a solar energy technology that uses parabolic mirrors to concentrate sunlight onto an absorber pipe. High-temperature thermal oil flows through the pipe, utilizing heat for electricity generation or industrial use. This paper enhances the mathematical model of DCS by incorporating spatio-temporal variations in solar irradiance, mirror optical efficiency, and time delay between input irradiance and output fluid temperature. Advanced estimation techniques, the modulating function method (MFM), and the Gauss-Newton method (GNM), are applied to estimate unknown parameters, improving the accuracy and robustness of solar thermal modeling.

Keywords: distributed collector system; time delay; parameter estimation; soft sensing; renewable energy optimization; modulating function method

Mathematics Subject Classification: 60G35, 76-10, 93-10, 93C43, 93E10

1. Introduction

Renewable energy, particularly solar energy, is crucial to developing sustainable solutions that meet current energy demands by harnessing the abundant resources of the Sun. Substantial efforts are underway globally to utilize sunlight for solar power generation, aligning with worldwide sustainability goals. A notable technology is the distributed collector system (DCS), which employs parabolic mirrors as trough collectors for primary energy collection. These collectors are designed to focus sunlight onto a blackened absorber pipe at the focal line. To facilitate efficient heat transfer, high-temperature oil circulates through the absorber pipe, which is enclosed by a transparent tube to minimize convection losses [1] (see Figure 1). Solar energy has become the fastest-growing renewable energy source worldwide. In 2023, global solar generation reached about 1,624 TWh, contributing significantly to the total renewable electricity production of 8,928 TWh, with an annual growth of

approximately 25%. By the end of 2024, installed solar capacity surpassed 1,866 GW, leading the global renewable capacity of 4,443 GW. These trends underline the increasing role of high-efficiency solar technologies, such as DCS, in meeting worldwide clean energy demand [2].

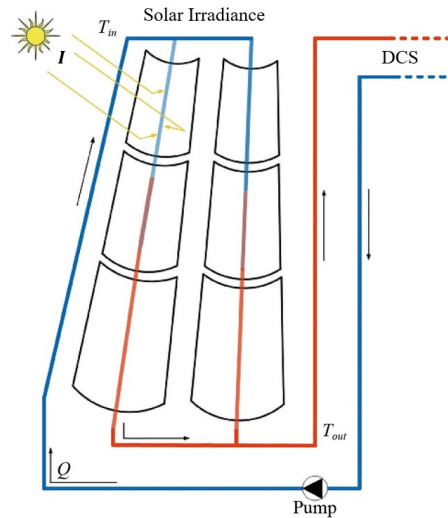


Figure 1. Schematic of DCS [3].

The heat transport within the DCS is represented by the fluid temperature, which is governed by a partial differential equation (PDE) [4]. This model depends on the solar irradiance received and the efficiency parameters of the mirrors. Camacho et al. simplified this model by converting it into an ordinary differential equation (ODE). However, PDEs are preferred in this context because they capture variations in both space and time, providing a more accurate representation of the system's dynamic behavior [5]. Subsequently, many studies [3,4] developed control and estimation algorithms based on this ODE formulation. In [3], Asiri modified the ODE model by incorporating the time delay between solar irradiance (input) and fluid temperature (output). However, to the best of our knowledge, the existing PDE model does not consider the time delay between solar irradiance and fluid temperature, which introduces a dynamic aspect that has yet to be considered.

This delay can arise, for example, due to the thermal inertia of the receiver tube and the transport time of the heat transfer fluid along the collector loop. When solar irradiance increases, it takes time for the receiver and fluid to absorb and reach higher temperatures. Similarly, when the irradiance decreases, the fluid temperature responds after a delay. Therefore, considering this time delay in the PDE model of the DCS is essential to improve the accuracy of the system's dynamic representation and enhance the performance of model-based control strategies. However, the value of the time delay remains uncertain or unknown due to variations in several physical parameters, such as the inlet temperature, fluid flow rate, and mirror reflectivity. This highlights the need to estimate the time delay based on the available measurements during system operation.

Moreover, the presence of dust on specific mirror components can significantly impact their efficiency, subsequently affecting the overall performance of the system. Since the optical efficiency of the mirrors is usually unknown or uncertain and cannot be measured directly, estimating it becomes essential.

Consequently, the incorporation of soft sensing techniques becomes crucial for the accurate estimation of time delay and mirror efficiency. These soft sensing methods play a significant role in mitigating the challenges posed by intermittent cloud cover and dust accumulation, contributing to the robustness and reliability of the heat transport model in the DCS. To address these challenges and improve the overall modeling and estimation framework, advanced estimation techniques are employed. For this purpose, the modulating function method (MFM) is applied to estimate mirror efficiency. Furthermore, the MFM is combined with the Gauss-Newton method (GNM) to estimate the time delay. The MFM depends directly on the governing PDE model, does not require knowledge of initial conditions, avoids the differentiation of noisy measurements, and transforms the estimation of coefficients and source terms into a system of algebraic equations that is linear in the unknown parameters. These approaches provide robust and efficient estimations and enhance the understanding and optimization of heat transport dynamics in the DCS [6]. The symbols used throughout the paper are listed in Table 1.

Table 1. List of symbols and their descriptions.

Symbol	Description	Unit
A_s	Cross-sectional area	m^2
a	Degrees of freedom	—
c	Fluid specific heat capacity	$\text{J}/(^{\circ}\text{C}\cdot\text{kg})$
$C^{(l)}$	Space of functions with l continuous derivatives	—
δ	Time delay	s
$\hat{\delta}$	Estimated time delay	s
Δt	Sampling time	s
Δx	Sampling space	m
G	Mirror optical aperture	m
γ	Basis coefficients	—
Γ	Vector of basis coefficients	—
I	Solar irradiance	W/m^2
J	Number of basis functions	—
l	Order of modulating functions	—
L	Length of the space interval	m
M	Number of modulating functions	—
$Q(t)$	Pump volumetric flow rate	m^3/s
ρ	Fluid density	kg/m^3
σ_1, σ_2	Standard deviations	—
t	Time variable	s
t_f	Final time	s
T	Fluid temperature	$^{\circ}\text{C}$
$\tilde{T}$	Noisy fluid temperature	$^{\circ}\text{C}$
$\partial T/\partial x, T_x$	Heat flux	$^{\circ}\text{C}/\text{m}$
$\tilde{T}_x$	Noisy heat flux	$^{\circ}\text{C}/\text{m}$
T_{in}	Boundary condition (inlet temperature)	$^{\circ}\text{C}$
T_o	Initial condition	$^{\circ}\text{C}$

Symbol	Description	Unit
h	Window length	s
v_0	Mirror optical efficiency	dimensionless
$\hat{v}_0$	Estimated mirror optical efficiency	dimensionless
w_k	Window index	—
ω	Zero-mean white Gaussian random noise	—
x	Space variable	m
ξ	Basis functions	—
ϕ	Modulating function	—
η	Inverse-derivative norm bound (Theorem 5.3)	—
κ_1	Lipschitz constant of $\partial W/\partial \delta$	—
κ_2	Interval bound for local convergence	s
r	Local convergence radius	s
κ_{1I}	Lipschitz constant of $I(L, \cdot)$	—

2. Preliminaries

In this section, we have outlined the key concepts and methods that form the foundation of the framework and methodology employed in this paper. In the following definition, we introduce the concept of a modulating function (MF) along with the necessary conditions it must satisfy.

Definition 2.1. [7] A function $\phi(t) \neq 0$ is considered an MF of order l on $[0, t_f]$ if it satisfies the following conditions:

$$\begin{cases} \phi(t) \in C^{(l)}([0, t_f]), \\ \phi^{(q)}(0) = \phi^{(q)}(t_f) = 0, \quad q = 0, 1, \dots, l-1, \end{cases} \quad (2.1)$$

where $t_f > 0$ and l represents the order of the derivative.

The main steps of the MFM are as follows:

- (1) Multiply the differential equation by an MF.
- (2) Integrate the resulting differential equations over the time interval to filter and attenuate noise [8].
- (3) Apply the integration by parts formula on the integral equations to transfer derivatives from noisy measurements to the MF, which is known exactly. Hence, we avoid differentiating noisy data and maintain the stability of the solution [9].
- (4) Formulate the obtained equations into an algebraic linear system, the solution of which provides an estimate for unknown parameters.

3. Mathematical model

By applying energy balance principles, the solar collector field can be modeled as the heat transport of the circulating fluid. Furthermore, neglecting heat losses leads to the following model (see [4] and

the references therein):

$$\frac{\partial T(x, t)}{\partial t} + \frac{Q(t)}{A_s} \frac{\partial T(x, t)}{\partial x} = \frac{v_0 G I(t)}{\rho c A_s}, \quad (3.1)$$

where x is space (m), t is time (s), T is temperature ($^{\circ}\text{C}$), $Q(t)$ is pump volumetric flow rate (m^3/s), A_s is cross-sectional area (m^2), v_0 is mirror optical efficiency (dimensionless), G is mirror optical aperture (m), I is solar irradiance (W/m^2), ρ is fluid density (kg/m^3), and c is fluid specific heat capacity ($\text{J}/(^{\circ}\text{C}\cdot\text{kg})$).

The existence, uniqueness, and regularity of the solution of Eq (3.1) are studied in [10].

Equation (3.1) does not consider the time delay between the solar irradiance and the fluid temperature, which is a dynamical aspect that still needs to be addressed. In addition, solar irradiance can be affected by atmospheric conditions such as cloud movement or dust in the air, which may lead to temporary fluctuations in the received energy. Therefore, in addition to this time delay, we consider the spatial dependency and therefore $I(x, t - \delta)$ instead of $I(t)$, where δ refers to the time delay. Additionally, dust accumulation on certain areas of the mirrors can significantly reduce their efficiency, which in turn affects the overall performance of the system. Thus, we modify v_0 to be space-time dependent, $v_0(x, t)$. The proposed model is given by

$$\begin{cases} \frac{\partial T(x, t)}{\partial t} + \frac{Q(t)}{A_s} \frac{\partial T(x, t)}{\partial x} = \frac{v_0(x, t) G I(x, t - \delta)}{\rho c A_s}, \\ T(0, t) = T_{in}(t), \\ T(x, 0) = T_o(x), \end{cases} \quad \begin{matrix} t > 0 \\ 0 < x \leq L. \end{matrix} \quad (3.2)$$

Here, $T_{in}(t)$ denotes the inlet oil temperature, and $T_o(x)$ represents the initial temperature distribution along the pipe.

The objective of this study is to solve the following two estimation problems:

EP1: Given the output temperature $T(L, t)$ and the heat flux $\frac{\partial T}{\partial x}(x, t)$ at $x = L$, find the mirror optical efficiency $v_0(L, t)$.

EP2: Given the output temperature $T(L, t)$ and the heat flux $\frac{\partial T}{\partial x}(x, t)$ at $x = L$, find the delay δ .

4. Mirror efficiency estimation using MFM

The DCS relies on mirror reflections to achieve the high temperatures necessary for efficient thermal energy conversion. Inaccurate mirror performance can result in significant energy losses, thus reducing overall system efficiency. Since direct measurement of the mirror efficiency v_0 is often impractical, and because environmental factors such as dust accumulation and surface degradation can impact performance over time, robust estimation of v_0 is essential. Our objective in this section is to solve EP1. To achieve this, we propose, in the next theorem, applying the MFM [11].

Theorem 4.1. Let $\sum_{j=1}^J \xi_j(t) \gamma_j$ be a basis expansion of the unknown mirror efficiency at the outlet $v_0(L, t)$ in (3.2), where $\xi_j(t)$ and γ_j , for $j = 1, \dots, J$, are basis functions and basis coefficients, respectively. Let $\phi_m(t)$, $m = 1, \dots, M$ be a class of linearly independent MFs that satisfy (2.1) with order $l \geq 1$ and $M \geq J$. Then, the unknown coefficients γ_j , $j = 1, 2, \dots, J$, can be estimated by solving the system

$$A\Gamma = K, \quad (4.1)$$

where the components of the $M \times J$ matrix A have the form

$$A_{mj} = \int_0^{t_f} \frac{\phi_m(t) \xi_j(t) I(L, t - \delta)}{A_s \rho c} dt, \quad (4.2)$$

for $m = 1, \dots, M$ and $j = 1, \dots, J$; the components of the vector $K \in \mathbb{R}^{M \times 1}$ are given by

$$K_m = \frac{1}{A_s} \int_0^{t_f} Q(t) \frac{\partial T(L, t)}{\partial x} \phi_m(t) dt - \int_0^{t_f} T(L, t) \phi'_m(t) dt, \quad (4.3)$$

and Γ is the vector of unknowns γ_j , where $j = 1, \dots, J$.

Proof. To derive this result, we proceed as follows:

(1) Fix x in (3.2) at the outlet such that $x = L$:

$$\frac{\partial T(L, t)}{\partial t} + \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} = \frac{v_0(L, t) G I(L, t - \delta)}{\rho c A_s}. \quad (4.4)$$

(2) Multiply Eq (4.4) by an MF $\phi_m(t)$:

$$\frac{\partial T(L, t)}{\partial t} \phi_m(t) + \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} \phi_m(t) = \frac{v_0(L, t) G I(L, t - \delta)}{\rho c A_s} \phi_m(t). \quad (4.5)$$

(3) Integrate Eq (4.5) over $[0, t_f]$:

$$\begin{aligned} & \int_0^{t_f} \frac{\partial T(L, t)}{\partial t} \phi_m(t) dt + \int_0^{t_f} \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} \phi_m(t) dt \\ &= \int_0^{t_f} \frac{v_0(L, t) G I(L, t - \delta)}{\rho c A_s} \phi_m(t) dt. \end{aligned} \quad (4.6)$$

(4) Apply integration by parts to the first term in (4.6):

$$\int_0^{t_f} \frac{\partial T(L, t)}{\partial t} \phi_m(t) dt = \phi_m(t) T(L, t) \Big|_0^{t_f} - \int_0^{t_f} T(L, t) \phi'_m(t) dt. \quad (4.7)$$

By applying the properties of the MF given in (2.1), the first term in the right-hand side (RHS) of (4.7) is canceled; hence we get

$$\int_0^{t_f} \frac{\partial T(L, t)}{\partial t} \phi_m(t) dt = - \int_0^{t_f} T(L, t) \phi'_m(t) dt. \quad (4.8)$$

(5) Therefore, (4.6) can be written as

$$\begin{aligned} & - \int_0^{t_f} T(L, t) \phi'_m(t) dt + \int_0^{t_f} \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} \phi_m(t) dt \\ &= \int_0^{t_f} \frac{v_0(L, t) G I(L, t - \delta)}{\rho c A_s} \phi_m(t) dt. \end{aligned} \quad (4.9)$$

(6) Express v_0 in terms of its basis expansion:

$$\begin{aligned} & - \int_0^{t_f} T(L, t) \phi'_m(t) dt + \int_0^{t_f} \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} \phi_m(t) dt \\ & = \sum_{j=1}^J \gamma_j \int_0^{t_f} \frac{\xi_j(t) G I(L, t - \delta)}{\rho c A_s} \phi_m(t) dt. \end{aligned} \quad (4.10)$$

(7) By considering M modulating functions, we can formulate (4.10) into the linear system in (4.1) with components as in (4.2) and (4.3). □

Remark 4.1. A necessary condition for the solvability of system (4.1) is that the modulating functions be linearly independent.

Remark 4.2. The linear system (4.1) admits a unique least-squares solution provided the matrix A has full column rank, which can be obtained by an appropriate selection of the type and number of basis and modulating functions.

Remark 4.3. Notice that in step 4, Eq (4.8), the differentiation has been transformed from the measurement T to the MF $\phi_m(t)$, which is known exactly. Hence, we avoid differentiating noisy data and maintain the stability of the solution. The integral also serves as a filter improving the accuracy [9].

5. Time delay estimation

Time delay is a critical factor that influences the dynamics and performance of dynamic systems. In the context of the DCS, understanding and accurately estimating the time delay between input (solar irradiance) and output (fluid temperature) is essential for effective control and optimization. Engineers adjust oil pumping rates based on irradiance measurements; without accounting for a delay δ , they risk over- or under-pumping, causing energy losses and potential pipe damage [1].

Here, we focus on online estimation of time delay δ . Unlike offline methods that process collected data post hoc, our approach continuously updates δ in real time using a moving window strategy.

Theorem 5.1. Let $\phi_m(t)$, $m = 0, 1, \dots, M - 1$ be a class of linearly independent MFs of order $l \geq 1$ on $[t - h, t]$, where $0 < h < t_f$. Then, the unknown time delay δ can be estimated by solving the equation

$$C(\delta) = D, \quad (5.1)$$

where $C(\delta), D \in \mathbb{R}^{M \times 1}$, such that their m -th entry is defined as follows: for $m = 0, 1, \dots, M - 1$, $\forall t \in [h, t_f]$,

$$C_m(\delta) = \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta)}{\rho c A_s} \phi_m(\tau) d\tau, \quad (5.2)$$

and

$$D_m = - \int_{t-h}^t T(L, \tau) \phi'_m(\tau) d\tau + \int_{t-h}^t \frac{Q(\tau)}{A_s} \frac{\partial T(L, \tau)}{\partial x} \phi_m(\tau) d\tau. \quad (5.3)$$

Proof. We arrive at Eq (5.1) by following the steps below:

(1) Fix x in (3.2) at the outlet such that $x = L$:

$$\frac{\partial T(L, t)}{\partial t} + \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} = \frac{v_0(L, t) G I(L, t - \delta)}{\rho c A_s}. \quad (5.4)$$

(2) Multiply Eq (5.4) by $\phi_m(t)$:

$$\frac{\partial T(L, t)}{\partial t} \phi_m(t) + \frac{Q(t)}{A_s} \frac{\partial T(L, t)}{\partial x} \phi_m(t) = \frac{v_0(L, t) G I(L, t - \delta)}{\rho c A_s} \phi_m(t). \quad (5.5)$$

(3) Integrate Eq (5.5) over the window $[t - h, t]$:

$$\begin{aligned} & \int_{t-h}^t \frac{\partial T(L, \tau)}{\partial \tau} \phi_m(\tau) d\tau + \int_{t-h}^t \frac{Q(\tau)}{A_s} \frac{\partial T(L, \tau)}{\partial x} \phi_m(\tau) d\tau \\ &= \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta)}{\rho c A_s} \phi_m(\tau) d\tau. \end{aligned} \quad (5.6)$$

(4) Apply integration by parts to the first term in (5.6):

$$\int_{t-h}^t \frac{\partial T(L, \tau)}{\partial \tau} \phi_m(\tau) d\tau = \phi_m(\tau) \frac{\partial T(L, \tau)}{\partial \tau} \Big|_{t-h}^t - \int_{t-h}^t T(L, \tau) \phi'_m(\tau) d\tau. \quad (5.7)$$

By applying the properties of the MF given in (2.1), the first term in the RHS of (5.7) is canceled; hence we get

$$\int_{t-h}^t \frac{\partial T(L, \tau)}{\partial \tau} \phi_m(\tau) d\tau = - \int_{t-h}^t T(L, \tau) \phi'_m(\tau) d\tau.$$

(5) Substituting this result in Eq (5.6) gives:

$$\begin{aligned} & - \int_{t-h}^t T(L, \tau) \phi'_m(\tau) d\tau + \int_{t-h}^t \frac{Q(\tau)}{A_s} \frac{\partial T(L, \tau)}{\partial x} \phi_m(\tau) d\tau \\ &= \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta)}{\rho c A_s} \phi_m(\tau) d\tau. \end{aligned} \quad (5.8)$$

The left-hand side (LHS) of (5.8) is the m^{th} component of D , and the RHS of (5.8) is the m^{th} component of $C(\delta)$. $\square$

System (5.1) is a nonlinear one; hence, it cannot be solved directly. We propose to apply the GNM. First, we write Eq (5.1) as follows:

$$W(\delta) = C(\delta) - D = 0. \quad (5.9)$$

In the next theorem, the nonlinear equation (5.9) is solved using the first-order GNM, inspired by the work in [7] (see the main steps of the approach in the flowchart in Figure 2 and Algorithm 1).

Theorem 5.2. *Starting with an initial guess for the delay δ_0 , the GNM to solve (5.9) proceeds by the iterations:*

$$\delta_{w+1} = \delta_w - \left[\frac{\partial W(\delta_w)}{\partial \delta_w} \right]^+ * W(\delta_w), \quad w = 0, 1, \dots, \quad (5.10)$$

where $W(\delta_w), \frac{dW(\delta_w)}{d\delta_w} \in \mathbb{R}^{M \times 1}$ with entries, for $m = 0, \dots, M-1$,

$$W_m(\delta_w) = \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi_m(\tau) d\tau + \int_{t-h}^t T(L, \tau) \phi'_m(\tau) d\tau - \int_{t-h}^t \frac{Q(\tau)}{A_s} \frac{\partial T(L, \tau)}{\partial x} \phi_m(\tau) d\tau, \quad (5.11)$$

$$\frac{\partial W_m(\delta_w)}{\partial \delta_w} = \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi'_m(\tau) d\tau, \quad (5.12)$$

and ϕ_m is a classical MF of at least first order.

Proof. The components of $W(\delta_w)$ given in (5.11) can be easily obtained by substituting (5.2) and (5.3) into (5.9). Now, to find $\frac{\partial W(\delta_w)}{\partial \delta_w}$, we differentiate the column vector $W(\delta_w)$ given in (5.9) with respect to δ_w , which leads to the following:

$$\frac{\partial W(\delta_w)}{\partial \delta_w} = \frac{\partial C(\delta_w)}{\partial \delta_w} - \frac{\partial D}{\partial \delta_w}.$$

Since D does not depend on δ , $\frac{\partial D}{\partial \delta_w} = 0$. Hence, for $m = 0, \dots, M-1$,

$$\begin{aligned} \frac{\partial W_m(\delta_w)}{\partial \delta_w} &= \frac{\partial C_m(\delta_w)}{\partial \delta_w} = \frac{\partial}{\partial \delta_w} \left[\int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi_m(\tau) d\tau \right] \\ &= \int_{t-h}^t \frac{\partial}{\partial \delta_w} \left(\frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \right) \phi_m(\tau) d\tau. \end{aligned}$$

Now use $\frac{\partial}{\partial \delta_w} \left(\frac{v_0(L, t) G I(L, t - \delta_w)}{\rho c A_s} \right) = -\frac{\partial}{\partial t} \left(\frac{v_0(L, t) G I(L, t - \delta_w)}{\rho c A_s} \right)$ to get

$$\frac{\partial W_m(\delta_w)}{\partial \delta_w} = - \int_{t-h}^t \frac{\partial}{\partial \tau} \left[\frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi_m(\tau) d\tau \right]. \quad (5.13)$$

Apply integration by parts to the RHS of (5.13):

$$\begin{aligned} \int_{t-h}^t \frac{\partial}{\partial \tau} \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi_m(\tau) d\tau &= \phi_m(\tau) \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \Big|_{t-h}^t \\ &\quad - \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi'_m(\tau) d\tau. \end{aligned} \quad (5.14)$$

By applying the properties of MFs given in (2.1), the first term in the RHS of (5.14) is canceled; hence we get

$$\int_{t-h}^t \frac{\partial}{\partial \tau} \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi_m(\tau) d\tau = - \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta_w)}{\rho c A_s} \phi'_m(\tau) d\tau;$$

therefore, Eq (5.14) leads to (5.12). $\square$

*. + represents the left inverse (pseudo-inverse). If $A \in \mathbb{R}^{m \times n}$ has full rank ($\text{rank}(A) = \min\{m, n\}$), its left inverse is given by: $A^+ = (A^T A)^{-1} A^T$.

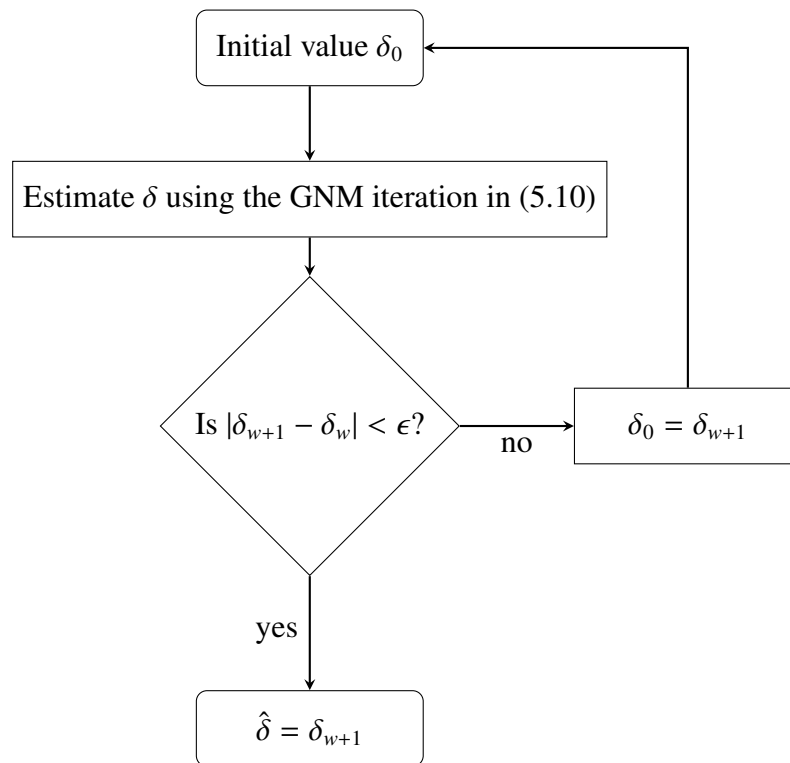


Figure 2. Combination of MFM and GNM (MFM-GNM) in Theorem 5.2.

Algorithm 1 Online time delay estimation with moving windows

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1: Fix window length  $h$  and sampling step  $\Delta t$ 
2: Compute  $\phi_m$  on the local grid  $[0, h]$  once; reuse for every window
3: Set the initial guess  $\delta_0$ 
4: for each window  $w_k = [t - h, t]$ , sliding forward by  $\Delta t$  do
5:   if  $k = 1$  then
6:      $\delta_{\text{init}} \leftarrow \delta_0$ 
7:   else
8:      $\delta_{\text{init}} \leftarrow \hat{\delta}$  from window  $w_{k-1}$ 
9:   end if
10:   $w \leftarrow 0$ ,  $\delta_w \leftarrow \delta_{\text{init}}$ 
11:  repeat
12:    Compute  $C(\delta_w)$ ,  $D$  from the measurements in  $w_k$ 
13:     $W(\delta_w) \leftarrow C(\delta_w) - D$ 
14:    Compute  $\partial W(\delta_w)/\partial \delta_w$ 
15:     $\delta_{w+1} \leftarrow \delta_w - [\partial W(\delta_w)/\partial \delta_w]^+ W(\delta_w)$ 
16:     $w \leftarrow w + 1$ 
17:  until  $|\delta_w - \delta_{w-1}| < \epsilon$ 
18:   $\hat{\delta}^{(k)} \leftarrow \delta_w$ 
19: end for
  
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Remark 5.1. One of the advantages of using the MFM is that the entries of $\frac{\partial W(\delta_w)}{\partial \delta_w}$ are given in closed-form integral expressions, as shown in (5.12). This is particularly beneficial compared to traditional gradient-based methods, where a significant portion of the computational effort is spent on computing the gradient at each step. With the MFM, we obtain an analytical gradient expression, which enhances numerical stability and reduces computational cost [12].

Remark 5.2. For the first window, δ_0 is set based on prior knowledge of the system; for subsequent windows, the estimate from the previous window is used as the initial guess, since δ varies smoothly between consecutive windows.

Next, we prove the local convergence of the GNM iteration in the following theorems.

Theorem 5.3. [13] Let $\delta^* \in \mathbb{R}_+$ be the solution to (5.9), and let $W : \mathbb{R}_+ \rightarrow \mathbb{R}^{M \times 1}$ be a continuously differentiable function such that $\frac{\partial W}{\partial \delta}$ is injective. Define

$$\eta := \left\| \left[\frac{\partial W(\delta)}{\partial \delta}(\delta^*) \right]^+ \right\|. \quad (5.15)$$

Suppose there exist positive constants $\kappa_1, \kappa_2 > 0$ such that the Jacobian satisfies the Lipschitz condition

$$\left\| \frac{\partial W(\delta)}{\partial \delta} \Big|_{\delta=y} - \frac{\partial W(\delta)}{\partial \delta} \Big|_{\delta=z} \right\|_{\infty} \leq \kappa_1 |y - z|, \quad \forall y, z \in \mathcal{B}(\delta^*, \kappa_2)^\dagger. \quad (5.16)$$

Set $r := \min \left\{ \kappa_2, \frac{2}{3\kappa_1\eta} \right\}$. Then, for any initial point $\delta_0 \in \mathcal{B}(\delta^*, r) \setminus \{\delta^*\}$, the GNM iteration generated by (5.10) is well-defined and converges to δ^* , which is the unique zero of $W(\delta)$ in $\mathcal{B}(\delta^*, \frac{2}{\eta\kappa_1})$.

Proof. Refer to Theorem 18 and Corollary 19 in [13] for the detailed proof. $\square$

Theorem 5.4. Assume there exists $\kappa_2 > 0$ such that $I(L, \cdot) \neq 0$ is Lipschitz continuous on $\mathcal{B}(\delta^*, \kappa_2)$ with Lipschitz constant $\kappa_{1I} > 0$. Let $\{\phi_m\}_{m=0}^{M-1}$ be a set of classical modulating functions of at least second order chosen such that

$$\int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta)}{\rho c A_s} \phi_m''(\tau) d\tau \neq 0. \quad (5.17)$$

Then, the sequence generated by (5.10) is well-defined and converges locally to δ^* .

Proof. We establish convergence by applying Theorem 5.3 to (5.9) through two main steps:

(1) Lipschitz continuity of $\frac{\partial W(\delta)}{\partial \delta}$: For any $y, z \in \mathcal{B}(\delta^*, \kappa_2)$, we have

$$\begin{aligned} \left\| \frac{\partial W(\delta)}{\partial \delta} \Big|_{\delta=y} - \frac{\partial W(\delta)}{\partial \delta} \Big|_{\delta=z} \right\|_{\infty} &= \max_m \left| \int_{t-h}^t \frac{v_0(L, \tau) G}{\rho c A_s} [I(L, \tau - y) - I(L, \tau - z)] \phi_m'(\tau) d\tau \right| \\ &\leq \max_m \int_{t-h}^t \left| \frac{G}{\rho c A_s} \right| |v_0(L, \tau)| |I(L, \tau - y) - I(L, \tau - z)| |\phi_m'(\tau)| d\tau \\ &\leq \kappa_1 |y - z|, \end{aligned} \quad (5.18)$$

where $\kappa_1 := h \kappa_{1I} \frac{G}{\rho c A_s} d_v \max_m d_{\phi_m} > 0$, with $d_v := \sup_{\tau \in [0, t_f]} |v_0(L, \tau)|$ and $d_{\phi_m} := \sup_{\tau \in [0, t_f]} |\phi_m'(\tau)|$ for $m = 0, \dots, M-1$.

$^\dagger \mathcal{B}(\delta^*, \kappa_2) := \{\tau \in \mathbb{R}_+ : |\tau - \delta^*| < \kappa_2\}.$

- (2) Injectivity of $\frac{\partial W}{\partial \delta}$: The function $\frac{\partial W}{\partial \delta}(\delta)$ is continuously differentiable, with the second-order derivative given for $m = 0, \dots, M-1$ by

$$\frac{\partial^2 W_m(\delta)}{\partial \delta^2} = \int_{t-h}^t \frac{v_0(L, \tau) G I(L, \tau - \delta)}{\rho c A_s} \phi_m''(\tau) d\tau, \quad (5.19)$$

which is obtained via the same steps used to derive (5.12). By condition (5.17), Eq (5.19) is not zero. Consequently, according to the inverse function theorem, $\frac{\partial W}{\partial \delta}$ is injective.

□

The numerical results are given in the next section.

6. Numerical simulation

In this section, we present numerical simulations to demonstrate the impact of incorporating time delay into the modified model and to evaluate the effectiveness of the proposed methods in estimating mirror efficiency $v_0(L, t)$ and time delay δ . The simulations are designed to assess the robustness of the estimation algorithm against different levels of noisy measurements. Table 2 outlines the parameters used in the simulations.

Table 2. The values of the parameters used in the numerical simulation of (3.2) [4].

Parameter	Value
ρ	903
c	1820
A_s	0.0006
G	1.82
L	172
$Q(t)$	$(0.001)(0.01t + 1)$

The space and time discretization, basis functions, modulating functions, and noisy measurements are chosen as follows:

Space and time discretization. Consider the partition $\{x_k\}$ of the interval $[0, L]$, where $0 = x_0 < x_1 < \dots < x_{N-1} < x_N = L$ and Δx represents the sub-interval length ($\Delta x = x_k - x_{k-1}$). Here, we set $\Delta x = 0.001$ and $L = 172$. Similarly, for the time interval, we use $\Delta t = 0.00001$ and $t_f = 10$.

Basis functions. There are different types of basis functions, such as polynomials, Gaussians, splines, and radial basis functions. In this work, we employ polynomial basis functions of the form

$$\xi_j(t) = t^{j-1}, \quad j = 1, 2, \dots, J,$$

as they are straightforward and provide good results.

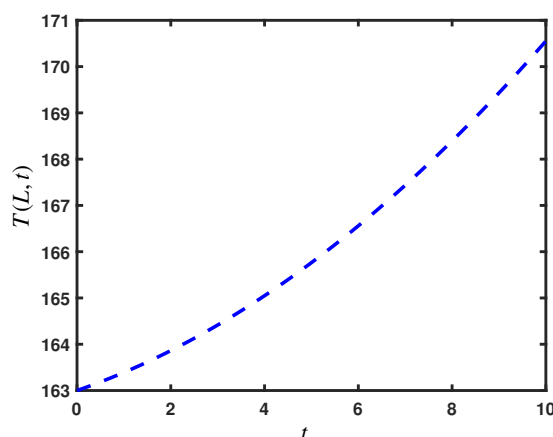
Modulating functions. There are different types of modulating functions, such as polynomial, sinusoidal, Hermite, and Poisson modulating functions; see Table 3.1 in [14]. We utilize the following polynomial MFs: $\forall \tau \in [t-h, t]$,

$$\phi_m(\tau) = (t - \tau)^{a+m}(\tau - t + h)^{a+M+1-m}, \quad m = 0, 1, \dots, M-1,$$

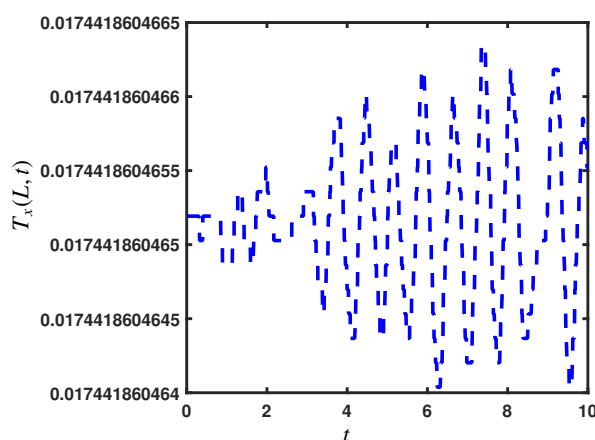
where M is the number of MFs and $a > 0$ is a degree of freedom.

Integration. The integrals appearing in Theorems 4.1 and 5.2 are approximated using the trapezoidal numerical integration.

Exact measurements. We obtain our measurements by solving model (3.2) numerically using finite difference methods, which is detailed in another paper in which the existence and stability are discussed. In Figure 3, we present the outlet temperature measurement $T(L, t)$ and the corresponding heat flux $\frac{\partial T}{\partial x}(L, t)$, which we call exact measurements or noise-free data.



(a) Temperature T against t at $x = L$.



(b) Flux T_x against t at $x = L$.

Figure 3. Exact measurements in the noise-free case.

Noisy measurement. In practice, measurements are usually noisy (not perfect). To simulate this, we add zero-mean white Gaussian noise to the exact measurements as follows:

$$\tilde{T}(L, t) = T(L, t) + \sigma_1 \omega(t),$$

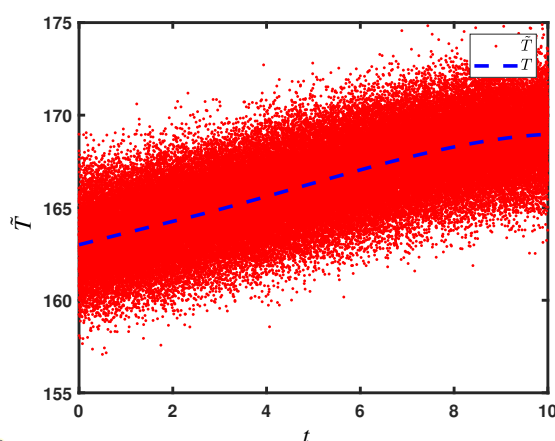
$$\tilde{T}_x(L, t) = T_x(L, t) + \sigma_2 \omega(t),$$

where $\tilde{T}$ and $\tilde{T}_x$ refer to the noisy data, $\omega(t)$ is a zero-mean white Gaussian random noise, and the coefficients σ_1 and σ_2 are standard deviations adjusted such that the selected noise level is obtained;

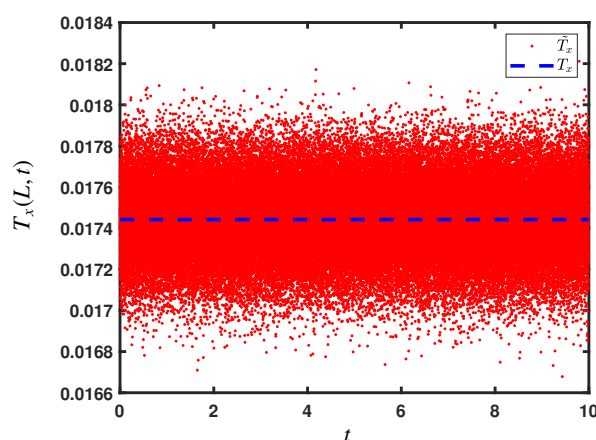
that is,

$$\sigma_1 = \frac{\text{noise level} \times \|T\|_2}{\|\omega\|_2 \times 100}, \quad \sigma_2 = \frac{\text{noise level} \times \|T_x\|_2}{\|\omega\|_2 \times 100}.$$

These noisy measurements are then utilized to assess the method's capability to perform effectively in the presence of noise. Figures 4 and 5 show the exact and noisy profiles for the measurements with 1% ($\sigma_1 = 1.6540$) and 2% ($\sigma_2 = 0.00017366$) noise levels, respectively.

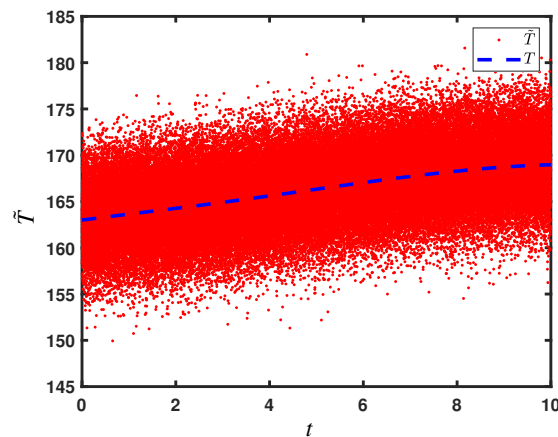


(a) The exact temperature T (blue dashed line) and the noisy temperature $\tilde{T}$ (red dots) after adding 1% noise, where $\sigma_1 = 1.6540$.

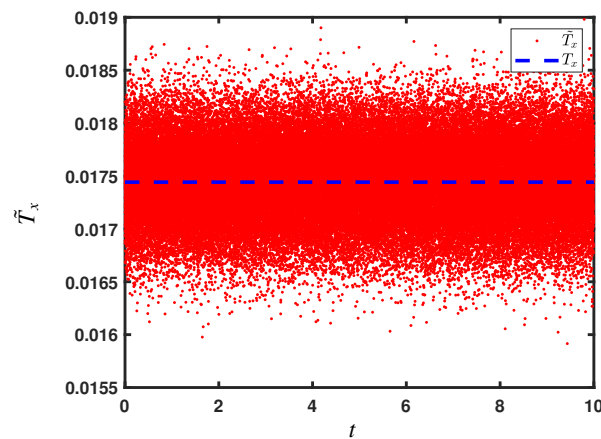


(b) The exact flux T_x (blue dashed line) and the noisy flux $\tilde{T}_x$ (red dots) after adding 1% noise, where $\sigma_2 = 0.00017366$.

Figure 4. Measurements with 1% noise level.



(a) Exact temperature T (blue dashed line) and estimated temperature $\tilde{T}$ (red dots) with 2% noise, where $\sigma_1 = 3.3079$.

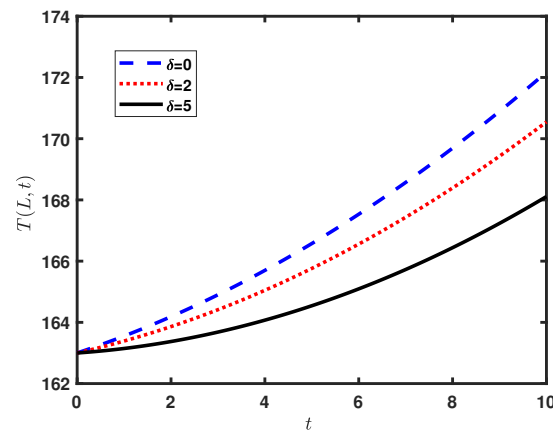


(b) Exact flux T_x (blue dashed line) and estimated flux $\tilde{T}_x$ (red dots) with 2% noise, where $\sigma_2 = 0.00034733$.

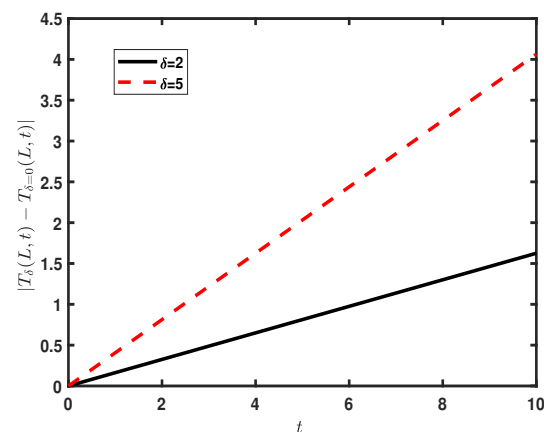
Figure 5. Measurements with 2% noise level.

6.1. Modified mathematical model

Figure 6 illustrates the outlet temperature both when the time delay is disregarded ($\delta = 0$) and when it is incorporated into the model. This figure highlights the influence of time delay on the performance of the DCS.



(a) Outlet temperature with respect to different time delay values.



(b) The effect of varying the delay time on the outlet temperature in absolute terms.

Figure 6. Outlet temperature profiles for different delay times and the corresponding absolute effects.

6.2. Mirror efficiency estimation

This subsection presents the estimated values of the mirror efficiency $\hat{v}_0(L, t)$ using the MFM. The exact mirror efficiency v_0 is given by $v_0(x, t) = 0.9 - 0.008t^2$ while the exact solar irradiance is defined as $I(x, t - \delta) = 60t + 400$.

(a) Noise-free data

Figure 7 represents the estimated mirror efficiency using the MFM in the noise-free case. As shown in the figure, the estimated values closely align with the exact curve throughout the entire time interval, demonstrating the high accuracy of the estimation algorithm.

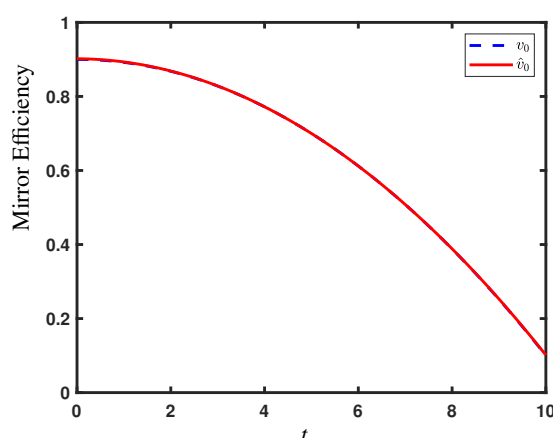


Figure 7. Exact (blue dashed line) and estimated mirror efficiency (red solid line) for the noise-free case. MFM parameters were set as $a = 9$, $M = 5$, and $J = 3$.

(b) Noisy data

The exact v_0 and the estimated one versus different noise levels are shown in Figures 8 and 9, and the corresponding relative errors are presented in Table 3. The relative error reported in Table 3 is defined as

$$\text{Relative Error}(v_0) = \frac{\|\hat{v}_0 - v_0\|_2}{\|v_0\|_2} \times 100\%.$$

Table 3. Relative errors in the estimation of mirror efficiency $v_0(t)$ with respect to different noise levels.

Noise Level	Relative Error
Noise-Free	0.1031%
1%	0.3564%
2%	0.6342%

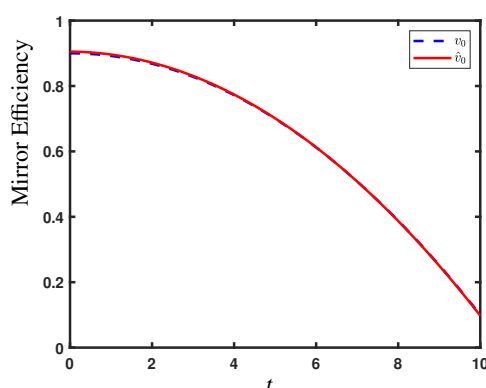


Figure 8. Exact (blue dashed line) and estimated mirror efficiency $v_0(t)$ (red solid line) with 1% noise. The MFM parameters were set as $a = 4$, $M = 5$, and $J = 3$.

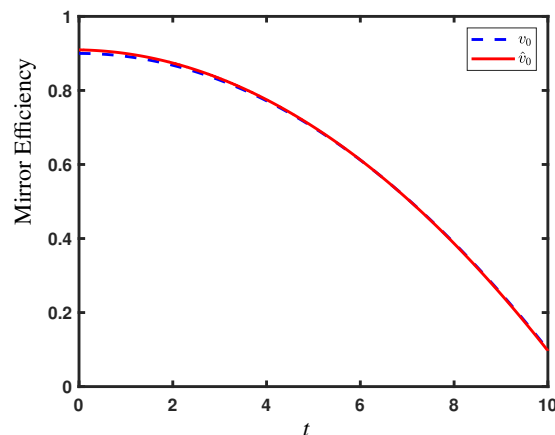


Figure 9. Exact (blue dashed line) and estimated mirror efficiency (red solid line) with 2% noise. The MFM parameters were set as $a = 3$, $M = 5$, and $J = 3$.

From Table 3 and Figures 7–9, it is obvious that the estimation accuracy of the mirror efficiency $v_0(L, t)$ remains consistently high, even with noisy measurements. These findings underscore the robustness of the MFM, which effectively maintains stable and reliable estimates even with varying noise levels.

6.3. Time delay estimation

In this subsection, we present the estimated values of the delay parameter $\hat{\delta}$ corresponding to different noise levels. The exact mirror efficiency v_0 is given by $v_0(x, t) = 0.73$, while the exact solar irradiance is defined as $I(x, t - \delta) = 60(t - \delta) + 400$, where $\delta = 2$. The results are organized for noise-free and noisy cases, with parameters consistently chosen as $h = 5$, $a = 5$, and $M = 6$ across all cases. The interval is divided into windows, denoted as w_k , where k represents the index of each window. Each window has a fixed length of $h = 5$ and moves forward by one sampling time. For each window, the MFM is applied, and iterations of the GNM are performed to estimate the time delay within that window. In each subsequent window, we use the estimated value of δ from the previous window as an initial guess for the next window.

(a) Noise-free data

Figure 10 depicts the estimated time delay, $\hat{\delta} = 2.0000$, obtained using the MFM combined with GNM in the noise-free case. The results indicate a strong agreement between the estimated and exact values throughout the entire time interval.

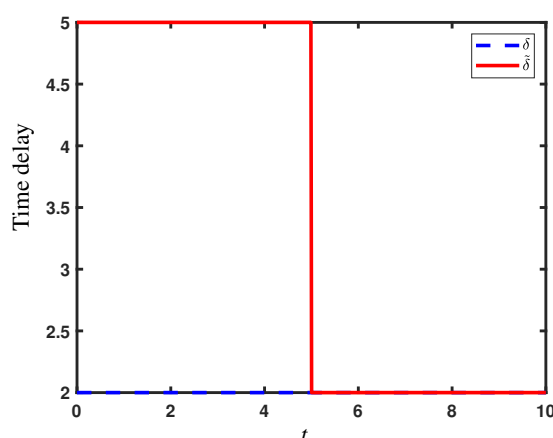


Figure 10. Exact (blue dashed line) and estimated time delay δ (red solid line) for the noise-free case.

(b) Noisy data

Figures 11 and 12 illustrate the exact and the estimated time delays obtained using the MFM and GNM at different noise levels; see Table 4. For Case 1, the estimated time delay is $\hat{\delta} = 2.0008$. For Case 2, the estimated time delay is $\hat{\delta} = 2.0022$. These results show that, despite the increase in noise levels, the estimated values maintain a reasonable agreement with the exact values. The corresponding relative errors for each scenario are presented in Table 5, where

$$\text{Relative Error}(\delta) = \frac{|\hat{\delta} - \delta|}{|\delta|} \times 100\%.$$

Table 4. Two noise levels for time-delay estimation.

Case	σ_1	σ_2
Case 1	0.1	0.00017366
Case 2	0.3	0.0003473

From Table 5 and Figures 10–12, the estimation remains fairly accurate even in noisy cases.

Table 5. Relative errors in the estimation of time delay δ with respect to different noise levels.

Case	$\hat{\delta}$	Relative Error
Noise-Free Case	2.0000	0.0025%
With Noise (Case 1)	2.0008	0.5173%
With Noise (Case 2)	2.0022	1.5556%

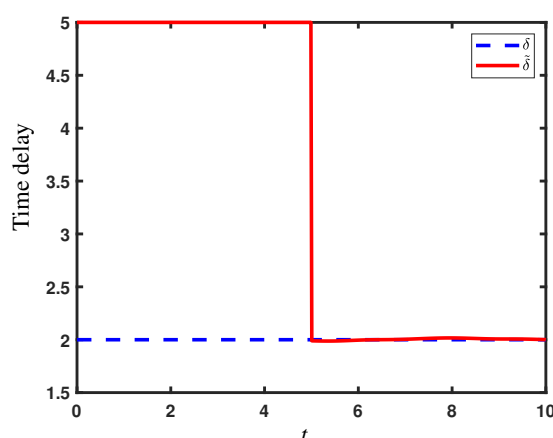


Figure 11. Exact (blue dashed line) and estimated time delay δ (red solid line) for Case 1.

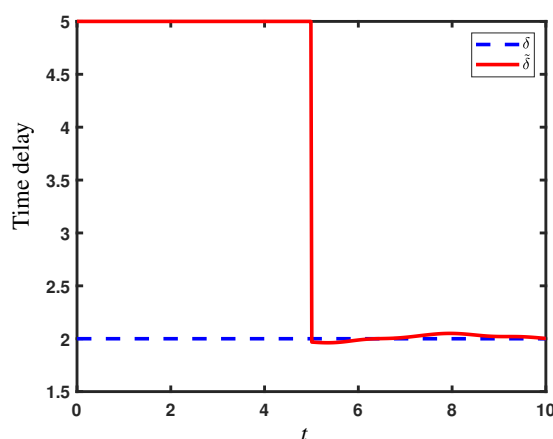


Figure 12. Exact (blue dashed line) and estimated time delay δ (red solid line) for Case 2.

In our earlier work [7], MFM-GNM was directly compared with the algebraic method proposed by Belkoura et al. [15] for delay estimation in a general linear ODE model. The MFM-GNM approach achieved faster convergence and higher accuracy. On the other hand, the algebraic method does not require an initial guess for the delay.

7. Conclusions

This study addressed the mathematical modeling and estimation techniques for the DCS, a key technology in solar thermal energy applications. The study improved the existing model by including spatial and temporal variations in solar irradiance, mirror optical efficiency, and the time delay between solar input and the resulting fluid temperature response. For this purpose, we estimated the mirror efficiency using the MFM based on boundary measurements. We also estimated the time delay online by combining the MFM and GNM, demonstrating the reliability and accuracy of both approaches through numerical simulations. The primary limitation of the GNM lies in the selection of an

appropriate initial guess, which can significantly impact the convergence and accuracy of the results.

Future work could compare the proposed MFM-GNM approach numerically against Kalman-filter-based methods. In addition, it could explore real-time adaptive estimation under varying weather conditions and its integration with predictive control algorithms.

Author contributions

Sharefa Asiri: Conceptualization, methodology, supervision, validation, writing – review & editing; Shumokh AlJedani: Investigation, methodology, software, writing – original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors used AI solely to proofread, rephrase, and refine the language of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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